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Research article

Bilinear θ-type Calderón-Zygmund operators and its commutators on generalized variable exponent Morrey spaces

Bo Xu*

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

* **Correspondence:** Email: 18093478672@163.com; Tel: 18093478672.

Abstract: In this paper, we discuss the boundedness of bilinear θ-type Calderón-Zygmund operators on the generalized variable exponent Morrey spaces. In addition, the corresponding results of commutators generated by bilinear θ-type Calderón-Zygmund operators with BMO functions on these spaces is also obtained.

Keywords: bilinear θ-type Calderón-Zygmund operator; commutator; generalized variable exponent Morrey spaces; BMO($\mathbb{R}^n$)
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1. Introduction

θ-type Calderón-Zygmund operators, which used to study certain classes of pseudo-differential operators, was introduced by Peng [1] in 1985. Firstly, Yang and Tao obtained the boundedness of θ-type Calderón-Zygmund operators on Variable Exponents Herz space [2] and Morrey-Herz-type Hardy spaces with variable exponents [3].

Guliyev further proved that the Calderón-Zygmund operators with kernels of Dini's type are bounded on generalized weighted variable exponent Morrey spaces (see [4]). Besides, Maldonado and Naibo developed a theory of the bilinear Calderón-Zygmund operators of type $\omega(t)$ in 2009 and generalized the results of Yabuta [5]. For comprehensive bilinear θ-type Calderón-Zygmund operators references, interested readers may refer to Zheng [6, 7] and Lu [8].

Variable exponent function spaces play a vital role in the fluid dynamics, elasticity dynamics, and differential equations with nonstandard growth, and thus have received a plenty of attention from researchers. For more details, one may refer to [9–12]. More specially, variable exponent Lebesgue spaces were studied in [13–19], Morrey spaces with variable exponent were studied in [3, 20–22], generalized Morrey spaces with variable exponent were studied in [23–27] and local "complementary" generalized variable exponent Morrey space were studied in [28, 29].

Inspired by the work above, this paper devotes to studying the boundedness of bilinear θ-type Calderón-Zygmund operator and its commutators on generalized variable exponent Morrey spaces.

Suppose that θ is a non-negative and non-decreasing function on $\mathbb{R}^+ = (0, \infty)$ satisfying

$$\int_0^1 \frac{\theta(t)}{t} \mathrm{d}t < \infty. \tag{1.1}$$

A continuous function $K(\cdot, \cdot, \cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y_1, y_2) : x = y_1 = y_2\}$ is said to be a bilinear θ-type Calderón-Zygmund kernel if it satisfies: for all $(x, y_1, y_2) \in \mathbb{R}^n$ with $x \neq y_i$, $i = 1, 2$,

$$| K(x, y_1, y_2) | \leq C(\sum_{i=1}^{2} | x - y_i |)^{-2n}, \tag{1.2}$$

and for all $x, z, y_1, y_2 \in \mathbb{R}^n$ with $2 | x - z | < \max\{| x - y_1 |, | x - y_2 |\}$, then exists a positive constant C such that

$$| K(x, y_1, y_2) - K(z, y_1, y_2) | \leq C\theta\Big(\frac{| x - z |}{\sum_{i=1}^{2} | x - y_i |}\Big)\Big[\sum_{i=1}^{2} | x - y_i |\Big]^{-2n}. \tag{1.3}$$

Now we state the definition of bilinear θ-type Calderón-Zygmund operator as follows.

Let T_θ be a linear operator from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ into its dual $\mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{S}$ denotes the Schwartz class. One can say that T_θ is a bilinear θ-type Calderón-Zygmund operator with kernel K satisfying (1.2) and (1.3), for all $f_1, f_2 \in L_c^\infty(\mathbb{R}^n)$ (the space of compactly supported bounded functions on $\mathbb{R}^n$) and $x \notin supp f_1 \cap supp f_2$,

$$T_\theta(f_1, f_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y_1, y_2) f_1(y_1) f_2(y_2) \mathrm{d}y_1 \mathrm{d}y_2, \tag{1.4}$$

where θ satisfies (1.1).

It is easy to see that the classical bilinear Calderón-Zygmund operator T with standard kernel is a special case of T_θ as $\theta(t) = t^\delta$ with $0 < \delta \leq 1$. Let b_1 and b_2 be locally integrable functions, the commutator generated by b_1, b_2 and T_θ is defined by

$$\begin{aligned}[b_1, b_2, T_\theta](f_1, f_2)(x) \ := \ & b_1(x)b_2(x)T_\theta(f_1, f_2)(x) - b_1(x)T_\theta(f_1, b_2 f_2)(x) \\ & -b_2(x)T_\theta(b_1 f_1, f_2)(x) + T_\theta(b_1 f_1, b_2 f_2)(x).\end{aligned}$$

Also, $[b_1, T_\theta]$ and $[b_2, T_\theta]$ are defined by

$$[b_1, T_\theta](f_1, f_2)(x) = b_1(x)T_\theta(f_1, f_2)(x) - T_\theta(b_1 f_1, f_2)(x),$$

and

$$[b_2, T_\theta](f_1, f_2)(x) = b_2(x)T_\theta(f_1, f_2)(x) - T_\theta(f_1, b_2 f_2)(x),$$

respectively.

Due to the singularity of commutators generated by bilinear θ-type Calderón-Zygmund operators with BMO function is stronger than that of bilinear θ-type Calderón-Zygmund operators. Thus, we need to strength the condition of θ in (1.1). Let θ be a non-negative and non-decreasing function on $(0, \infty)$ such that

$$\int_0^1 \frac{\theta(t)}{t} \mid logt \mid^2 \mathrm{d}t < \infty. \tag{1.5}$$

Furthermore, the commutators of bilinear θ-type Calderón-Zygmund operator are defined by

$$[b_1, b_2, T_\theta](f_1, f_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \prod_{i=1}^{2} (b_i(x) - b_i(y_i)) K(x, y_1, y_2) f_1(y_1) f_2(y_2) \mathrm{d}y_1 \mathrm{d}y_2,$$

where θ satisfies (1.5).

For a measurable subset $E \subseteq \mathbb{R}^n$, we define $\mathcal{P}^0(E)$ to be the set of measurable functions $p(\cdot) : E \to (0, \infty)$ such that

$$p_- = ess \inf_{x \in E} p(x) > 0, \quad p_+ = ess \sup_{x \in E} p(x) < \infty.$$

Define $\mathcal{P}(E)$ to be the set of measurable functions $p(\cdot) : E \to [1, \infty)$ such that

$$p_- = ess \inf_{x \in E} p(x) > 1, \quad p_+ = ess \sup_{x \in E} p(x) < \infty.$$

Define $\mathcal{P}^1(E)$ to be the set of measurable functions $p(\cdot) : E \to [1, \infty)$ such that

$$p_- = ess \inf_{x \in E} p(x) \geq 1, \quad p_+ = ess \sup_{x \in E} p(x) < \infty.$$

By $p^{'}(x) = \frac{p(x)}{p(x)-1}$, we denote the conjugate exponent of $p(x)$.

Let $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{\mid B \mid} \int_B \mid f(y) \mid \mathrm{d}y,$$

where the supremum is taken over all balls B containing x. Let $\mathcal{B}(E)$ be the set of $p(\cdot) \in \mathcal{P}(E)$ such that M is bounded on $L^{p(\cdot)}(E)$.

A subset of $\mathcal{B}(\mathbb{R}^n)$ is the class of globally log-Hölder continuous functions $p(\cdot) \in LH(\mathbb{R}^n)$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Recall that $p(\cdot) \in LH(\mathbb{R}^n)$, if $p(\cdot)$ satisfies

$$\mid p(x) - p(y) \mid \leq \frac{C}{-log \mid x - y \mid}, \ x, y \in \mathbb{R}^n, \mid x - y \mid \leq \frac{1}{2}, \tag{1.6}$$

and

$$\mid p(x) - p_\infty \mid \leq \frac{C_\infty}{log(e + \mid x \mid)}, \quad \mid y \mid \geq \mid x \mid, \tag{1.7}$$

where $p_\infty = \lim_{x \to \infty} p(x) > 1$.

Definition 1.1. [13] Given an open set $E \subset \mathbb{R}^n$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ denotes the set of measurable functions f on E such that

$$I_{p(\cdot)}(f) = \int_E \mid f(x) \mid^{p(x)} \mathrm{d}x < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\| f \|_{L^{p(\cdot)}(E)}= \inf\Big\{\lambda > 0 : \ I_{p(\cdot)}(\frac{f}{\lambda}) \le 1\Big\}.$$

For all compact subsets $E \subset \Omega$, the space $L^{p(\cdot)}_{loc}(\Omega)$ is defined by

$$L^{p(\cdot)}_{loc}(\Omega) = \Big\{f \ is \ measurable : \ f \in L^{p(\cdot)}(E)\Big\}.$$

Definition 1.2. [25] Let $p(\cdot) \in \mathcal{P}^1(\mathbb{R}^n)$, φ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$. The generalized variable exponent Morrey space $\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)$ is defined by

$$\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n) := \Big\{f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n) : \ \| f \|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)}< \infty\Big\},$$

where

$$\begin{aligned}&\| f \|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)}\\ &= \sup_{x\in\mathbb{R}^n, \ t>0} \varphi(x, t)^{-1} t^{-\theta_p(x,t)} \| f \|_{L^{p(\cdot)}(B(x,t))}\\ &= \sup_{x\in\mathbb{R}^n, \ t>0} \varphi(x, t)^{-1} t^{-\theta_p(x,t)} \| f\chi_{B(x,t)} \|_{L^{p(\cdot)}(\mathbb{R}^n)} .\end{aligned}$$

We recall the definition of space of $BMO(\mathbb{R}^n)$.

Definition 1.3. [30] Suppose that $b \in L^1_{loc}(\mathbb{R}^n)$, and let

$$\| b \|_{BMO(\mathbb{R}^n)}= \sup_{x\in\mathbb{R}^n,t>0} \frac{1}{| B(x, t) |} \int_{B(x,t)} | b(y) - b_{B(x,t)} | \mathrm{d}y < \infty,$$

where

$$b_{B(x,t)} = \frac{1}{| B(x, t) |} \int_{B(x,t)} b(y)\mathrm{d}y.$$

Define

$$BMO(\mathbb{R}^n) = \Big\{b \in L^1_{loc}(\mathbb{R}^n) : \ \| b \|_{BMO(\mathbb{R}^n)}< \infty\Big\}.$$

Definition 1.4. [17] The $BMO_{p(\cdot)}(\mathbb{R}^n)$ space is the set of all locally integrable functions b with finite norm

$$\| b \|_{BMO_{p(\cdot)}(\mathbb{R}^n)}= \sup_{x\in\mathbb{R}^n,t>0} \frac{\| (b(\cdot) - b_{B(x,t)})\chi_{B(x,t)} \|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\| \chi_{B(x,t)} \|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

The rest of this paper is organized as follows. Section 2 recalls some basic lemmas that will be used in the sequel. Section 3 demonstrates the boundedness of bilinear θ-type Calderón-Zygmund operators on generalized variable exponent Morrey spaces. Finally, the corresponding results of its commutators are made in Section 4.

2. Preliminaries

The following notions will be encountered often throughout the text. C is denoted by a positive constant which is independent of the main parameters, but it may vary from line to line. $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$. $A \approx B$ means that

$A \gtrsim CB$ and $A \lesssim CB$. $B(x_0, r) = \{x \in \mathbb{R}^n :| x - x_0 |< r\}$ denotes the open ball with center $x_0 \in \mathbb{R}^n$ and radius $r > 0$. Let $\mathbb{B} = \{B(x_0, r) : x_0 \in \mathbb{R}^n, r > 0\}$.

Lemma 2.1. [31] Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, Then there exists a positive constant C such that

$$\| \chi_{B(x,t)}(\cdot) \|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq Ct^{\theta_p(x,t)}, \ \ x \in \mathbb{R}^n, \ t > 0,$$

where

$$\theta_p(x,t) = \begin{cases} \frac{n}{p(x)}, & 0 < t \leq 1, \\ \frac{n}{p(\infty)}, & t \geq 1, \end{cases}$$

and $p_\infty = \lim\limits_{x\to\infty} p(x)$.

Lemma 2.2. [32] Let k be a positive integer. Then one has that, for all $b \in BMO(\mathbb{R}^n)$ and all $i, j \in \mathbb{Z}$ with $j > i$,

$$C^{-1} \| b \|^k_{BMO(\mathbb{R}^n)} \leq \sup_B \frac{1}{\| \chi_B \|_{L^{p(\cdot)}(\mathbb{R}^n)}} \| (b - b_B)^k \chi_B \|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \| b \|^k_{BMO(\mathbb{R}^n)},$$

$$\| (b - b_{B_i})^k \chi_{B_j} \|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C(j - i) \| b \|_{BMO(\mathbb{R}^n)} \| \chi_{B_j} \|_{L^{p(\cdot)}(\mathbb{R}^n)} .$$

Lemma 2.3. [33] Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, there exists a positive constant C such that

$$\frac{1}{| B |} \| \chi_B \|_{L^{p(\cdot)}(\mathbb{R}^n)} \| \chi_B \|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 2.4. [27] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} | f(x)g(x) | \, dx \leq r_p \| f \|_{L^{p(\cdot)}(\mathbb{R}^n)} \| g \|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $r_p = 1 + \frac{1}{p_-} - \frac{1}{p_+}$. This inequality is named the generalized Hölder inequality with respect to the variable Lebesgue spaces.

Lemma 2.5. [34] Let $p(\cdot), \ p_1(\cdot), \ p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, so that $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. Then the inequality

$$\| f_1 f_2 \|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \| f_1 \|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \| f_2 \|_{L^{p_2(\cdot)}(\mathbb{R}^n)}$$

holds for any $f_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$ and $i = 1, 2$.

We will use the following two Lemmas on the boundedness of weighted Hardy operator

$$H_\omega g(s) := \int_s^\infty g(t)\omega(t)\mathrm{d}t, \quad H^*_\omega g(s) := \int_s^\infty (1 + \frac{t}{s})g(t)\omega(t)\mathrm{d}t, \ \ 0 < s < \infty,$$

where ω is a weight.

Lemma 2.6. [35] Let v_1, v_2 and ω be weights on $(0, \infty)$ and $v_1(s)$ be bounded outside a neighborhood at the origin. The inequality

$$\sup_{s>0} v_2(s) H_\omega g(s) \leq C \sup_{s>0} v_1(s) g(s)$$

holds for some $C > 0$ for all non-negative and non-decreasing functions g on $(0, \infty)$ if and only if

$$B := \sup_{s>0} v_2(s) \int_s^\infty \frac{\omega(t)\mathrm{d}t}{\underset{t<r<\infty}{ess\inf} \, v_1(r)} < \infty.$$

Lemma 2.7. [36] Let v_1, v_2 and ω be weights on $(0,\infty)$ and $v_1(s)$ be bounded outside a neighborhood at the origin. The inequality

$$\sup_{s>0} v_2(s) H_\omega^* g(s) \leq C \sup_{s>0} v_1(s) g(s)$$

holds for some $C > 0$ for all non-negative and non-decreasing functions g on $(0,\infty)$ if and only if

$$B := \sup_{s>0} v_2(s) \int_s^\infty (1+\frac{t}{s}) \frac{\omega(t)\mathrm{d}t}{\mathop{ess\inf}\limits_{t<r<\infty} v_1(r)} < \infty.$$

Lemma 2.8. [17] Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$. Then $\|\cdot\|_{BMO_{p(\cdot)}} \approx \|\cdot\|_{BMO}$.

Lemma 2.9. [18] Let T be a bilinear Calderón-Zygmund operators. If $p(\cdot), p_1(\cdot), p_2(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, then for all $f_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$, $i = 1,2$, we have

$$\| T(f_1, f_2) \|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \| f_1 \|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \| f_2 \|_{L^{p_2(\cdot)}(\mathbb{R}^n)} .$$

Lemma 2.10. [21] Let T be a bilinear Calderón-Zygmund operators, $b_1,\ b_2 \in BMO(\mathbb{R}^n)$. If $p(\cdot), p_1(\cdot), p_2(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, then for all $f_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$, $i = 1,2$, we have

$$\| [b_1, b_2, T](f_1, f_2) \|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \| b_1 \|_{BMO(\mathbb{R}^n)} \| b_2 \|_{BMO(\mathbb{R}^n)} \| f_1 \|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \| f_2 \|_{L^{p_2(\cdot)}(\mathbb{R}^n)} .$$

3. Bilinear θ-type Calderón-Zygmund operators on $\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$

The main results of this section are stated as follows.

Lemma 3.1. Let T_θ be a bilinear θ-type Calderón-Zygmund operator defined by (1.4) with θ satisfies (1.1). Suppose $p_1(\cdot),\ p_2(\cdot),\ q(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{q(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. Then for all $f_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$, $i = 1,2$, we have

$$\| T_\theta(f_1, f_2) \|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C \| f_1 \|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \| f_2 \|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \tag{3.1}$$

with the constant $C > 0$ independent of f_1 and f_2.

The above result can be proved by using a similar proof method with that of Lemma 2.9, which is omitted here for brevity.

We are now ready to extend the definition of $T_\theta(f_1, f_2)$ when $f_i \in \mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$ $(i = 1,2)$ and T_θ be a bilinear θ-type Calderón-Zygmund operator defined by (1.4) with θ satisfies (1.1).

Definition 3.2. Let T_θ be a bilinear θ-type Calderón-Zygmund operator defined by (1.4) with θ satisfies (1.1). If $q(\cdot), p_1(\cdot), p_2(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{q(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\varphi, \varphi_1, \varphi_2$ satisfy the condition

$$\int_{\frac{3}{2}r}^\infty \frac{\mathop{ess\inf}\limits_{t<s<\infty}[\varphi_1(x_0,s)\varphi_2(x_0,s)s^{\theta_q(x_0,s)}]}{t^{\theta_q(x_0,t)+1}} dt \leq C\varphi(x_0, r), \tag{3.2}$$

and denote $\varphi(x_0, r) = \varphi_1(x_0, r)\varphi_2(x_0, r)$, where C does not depend on r. Suppose that T_θ is a bounded linear operator on $L^{q(\cdot)}(\mathbb{R}^n)$. For any $f_i \in \mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$, $i = 1,2$, and $x \in B = B(x_0, r) \in \mathbb{B}$, we define

$$\begin{aligned}
&\mathcal{T}_\theta(f_1, f_2)(x) \\
&= T_\theta(f_1\chi_{B(x_0,2r)}, f_2\chi_{B(x_0,2r)})(x) + T_\theta(f_1\chi_{B(x_0,2r)}, f_2\chi_{\mathbb{R}^n\setminus B(x_0,2r)})(x) \\
&\quad + T_\theta(f_1\chi_{\mathbb{R}^n\setminus B(x_0,2r)}, f_2\chi_{B(x_0,2r)})(x) + T_\theta(f_1\chi_{\mathbb{R}^n\setminus B(x_0,2r)}, f_2\chi_{\mathbb{R}^n\setminus B(x_0,2r)})(x) \\
&=: E_1 + E_2 + E_3 + E_4.
\end{aligned} \tag{3.3}$$

We need to show that $\mathcal{T}_\theta(f_1, f_2)$ is well defined. That is, the above definition is independent of the selection of $B(x_0, r)$. Its proof is similar to the Theorem 3.1 in [37].

Theorem 3.3. Let T_θ be a bilinear θ-type Calderón-Zygmund operator defined by (1.4) with θ satisfies (1.1). If $q(\cdot), p_1(\cdot), p_2(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{q(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\varphi, \varphi_1, \varphi_2$ satisfy the condition (3.2) and $\varphi = \varphi_1\varphi_2$. If T_θ is a bounded linear operator on $L^{q(\cdot)}(\mathbb{R}^n)$, then $\mathcal{T}_\theta$ is a well defined linear operator on $\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$.

Since $\mathcal{T}_\theta$ is well defined on $\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$, we are allowed to study the boundedness of $\mathcal{T}_\theta$ on $\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$.

Theorem 3.4. Let T_θ be a bilinear θ-type Calderón-Zygmund operator defined by (1.4) with θ satisfies (1.1). Suppose $p_1(\cdot)$, $p_2(\cdot)$, $q(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{q(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. Then for any ball $B = B(x_0, r)$ and $f_i \in L^{p_i(\cdot)}_{loc}(\mathbb{R}^n)$, $i = 1, 2$, the following inequality

$$\begin{aligned}
&\| \mathcal{T}_\theta(f_1, f_2) \|_{L^{q(\cdot)}(B(x_0,r))} \\
&\le C r^{\theta_{p_1}(x_0,r)+\theta_{p_2}(x_0,r)} \int_{2r}^{\infty} \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))}\, t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1} \mathrm{d}t
\end{aligned} \tag{3.4}$$

holds, where the constant $C > 0$ independent of f_1 and f_2.

Now, we present the boundedness of bilinear θ-type Calderón-Zygmund operators on the generalized variable exponent Morrey spaces based on Lemma 3.1 and Theorem 3.4.

Theorem 3.5. Let T_θ be a bilinear θ-type Calderón-Zygmund operator defined by (1.4) with θ satisfies (1.1). Suppose $p_1(\cdot)$, $p_2(\cdot)$, $q(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{q(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\varphi, \varphi_1, \varphi_2$ satisfy the condition (3.2) and $\varphi = \varphi_1\varphi_2$. Then $\mathcal{T}_\theta$ is bounded from the place $\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times \mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$ to the place $\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$.

Collory 3.6. Let T be a classical bilinear Calderón-Zygmund operators. If $q(\cdot), p_1(\cdot), p_2(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{q(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\varphi, \varphi_1, \varphi_2$ satisfy the condition (3.2) and $\varphi = \varphi_1\varphi_2$. Then $\mathcal{T}$ is bounded from the place $\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times \mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$ to the place $\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$.

Proof of Theorem 3.3. Let $f_i \in \mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$, $i = 1, 2$. As T_θ is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, E_1 is well defined.

Noting that $| x - y_1 | + | x - y_2 | \approx | x_0 - y_2 |$ for $x \in B(x_0, r)$, $y_1 \in 2B$ and $y_2 \in (2B)^c$. Applying Lemma 2.1, Lemma 2.4 and Lemma 2.3, E_2 can be estimated as

$$\begin{aligned}
E_2 &\le \int_{2B}\int_{(2B)^c} \frac{| f_1(y_1) || f_2(y_2) |}{(\sum\limits_{i=1}^{2} | x - y_i |)^{2n}} \mathrm{d}y_1 \mathrm{d}y_2 \\
&\le C \int_{2B} | f_1(y_1) | \mathrm{d}y_1 \int_{(2B)^c} \frac{| f_2(y_2) |}{| x_0 - y_2 |^{2n}} \mathrm{d}y_2 \\
&\le C \int_{2B} | f_1(y_1) | \mathrm{d}y_1 \sum_{k=1}^{\infty} \int_{2^{k+1}B \backslash 2^k B} \frac{| f_2(y_2) |}{| x_0 - y_2 |^{2n}} \mathrm{d}y_2 \\
&\le C \| f_1 \|_{L^{p_1(\cdot)}(2B)} \| 1 \|_{L^{p_1'(\cdot)}(2B)} \sum_{k=1}^{\infty} (2^k r)^{-2n} \| f_2 \|_{L^{p_2(\cdot)}(2^{k+1}B)} \| 1 \|_{L^{p_2'(\cdot)}(2^{k+1}B)} \\
&\le C \| f_1 \|_{L^{p_1(\cdot)}(2B)} \| 1 \|_{L^{p_1'(\cdot)}(2B)} \\
&\quad \times \Big(\sum_{k=1}^{\infty} \int_{2^{k+1}r}^{2^{k+2}r} (2^k r)^{-2n} (2^{k+1} r)^{-1} \| f_2 \|_{L^{p_2(\cdot)}(2^{k+1}B)} \| 1 \|_{L^{p_2'(\cdot)}(2^{k+1}B)} \mathrm{d}t \Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq C2^{4n} \parallel f_1 \parallel_{L^{p_1(\cdot)}(2B)} \parallel 1 \parallel_{L^{p_1'(\cdot)}(2B)} \\
&\quad \times \Big(\sum_{k=1}^{\infty} \int_{2^{k+1}r}^{2^{k+2}r} \parallel f_2 \parallel_{L^{p_2(\cdot)}(B(x_0,t))} t^{-2n-1} \parallel 1 \parallel_{L^{p_2'(\cdot)}(B(x_0,t))} \mathrm{d}t \Big) \\
&\leq C \parallel f_1 \parallel_{L^{p_1(\cdot)}(2B)} \parallel 1 \parallel_{L^{p_1'(\cdot)}(2B)} \int_{2r}^{\infty} \parallel f_2 \parallel_{L^{p_2(\cdot)}(B(x_0,t))} t^{-n-\theta_{p_2}(x_0,t)-1} \mathrm{d}t \\
&\leq C \int_{2r}^{\infty} \parallel f_1 \parallel_{L^{p_1(\cdot)}(B(x_0,t))} \parallel f_2 \parallel_{L^{p_2(\cdot)}(B(x_0,t))} t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1} \mathrm{d}t \\
&\leq C \prod_{i=1}^{2} \parallel f_i \parallel_{\mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)} \int_{2r}^{\infty} \frac{\varphi_1(x_0,t)\varphi_2(x_0,t)}{t} \mathrm{d}t \\
&\leq C\varphi(x_0,r) \prod_{i=1}^{2} \parallel f_i \parallel_{\mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)} .
\end{aligned}
$$

Similar to the estimates for E_2, it is easy to get

$$
\begin{aligned}
E_3 &\leq C \int_{2B} | f_2(y_2) | \mathrm{d}y_2 \int_{(2B)^c} \frac{| f_1(y_1) |}{| x_0 - y_1 |^{2n}} \mathrm{d}y_1 \\
&\leq C \parallel f_2 \parallel_{L^{p_2(\cdot)}(2B)} \parallel 1 \parallel_{L^{p_2'(\cdot)}(2B)} \int_{2r}^{\infty} \parallel f_1 \parallel_{L^{p_1(\cdot)}(B(x_0,t))} t^{-2n-1} \parallel 1 \parallel_{L^{p_1'(\cdot)}(B(x_0,t))} \mathrm{d}t \\
&\leq C \int_{2r}^{\infty} \parallel f_1 \parallel_{L^{p_1(\cdot)}(B(x_0,t))} \parallel f_2 \parallel_{L^{p_2(\cdot)}(B(x_0,t))} t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1} \mathrm{d}t \\
&\leq C \prod_{i=1}^{2} \parallel f_i \parallel_{\mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)} \int_{2r}^{\infty} \frac{\varphi_1(x_0,t)\varphi_2(x_0,t)}{t} \mathrm{d}t \\
&\leq C\varphi(x_0,r) \prod_{i=1}^{2} \parallel f_i \parallel_{\mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)} .
\end{aligned}
$$

For E_4. Noting that $| x - y_1 | + | x - y_2 | \approx | x_0 - y_1 | \approx | x_0 - y_2 |$ for $x \in B(x_0, r)$ and $y_1, y_2 \in (2B)^c$. By applying Lemma 2.1, Lemma 2.4 and Lemma 2.3, it follows that

$$
\begin{aligned}
E_4 &\leq \int_{(2B)^c} \int_{(2B)^c} \frac{| f_1(y_1) || f_2(y_2) |}{(\sum_{i=1}^{2} | x - y_i |)^{2n}} \mathrm{d}y_1 \mathrm{d}y_2 \\
&\leq \int_{(2B)^c} \int_{(2B)^c} \frac{| f_1(y_1) || f_2(y_2) |}{| x_0 - y_1 |^n | x_0 - y_2 |^n} \mathrm{d}y_1 \mathrm{d}y_2 \\
&\leq C \sum_{j=1}^{\infty} \prod_{i=1}^{2} (2^j r)^{-n} \int_{2^{j+1}B \setminus 2^j B} | f_i(y_i) | \mathrm{d}y_i \\
&\leq C \sum_{j=1}^{\infty} (2^j r)^{-2n} \parallel f_1 \parallel_{L^{p_1(\cdot)}(2^{j+1}B)} \parallel 1 \parallel_{L^{p_1'(\cdot)}(2^{j+1}B)} \parallel f_2 \parallel_{L^{p_2(\cdot)}(2^{j+1}B)} \parallel 1 \parallel_{L^{p_2'(\cdot)}(2^{j+1}B)} \\
&= C \sum_{j=1}^{\infty} \int_{2^{j+1}r}^{2^{j+2}r} (2^j r)^{-2n} (2^{j+1} r)^{-1} \parallel f_1 \parallel_{L^{p_1(\cdot)}(2^{j+1}B)}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big(\| 1 \|_{L^{p_1'(\cdot)}(2^{j+1}B)} \| f_2 \|_{L^{p_2(\cdot)}(2^{j+1}B)} \| 1 \|_{L^{p_2'(\cdot)}(2^{j+1}B)} \, \mathrm{d}t\Big) \\
\leq\ & C \sum_{j=1}^{\infty} \int_{2^{j+1}r}^{2^{j+2}r} \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} \, t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1} \mathrm{d}t \\
\leq\ & C \int_{2r}^{\infty} \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} \, t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1} \mathrm{d}t \\
\leq\ & C \int_{2r}^{\infty} \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} \, t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1} \mathrm{d}t \\
\leq\ & C \prod_{i=1}^{2} \| f_i \|_{\mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)} \int_{2r}^{\infty} \frac{\varphi_1(x_0,t)\varphi_2(x_0,t)}{t} \mathrm{d}t \\
\leq\ & C\varphi(x_0,r) \prod_{i=1}^{2} \| f_i \|_{\mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)} \ .
\end{aligned}
$$

Therefore, on the right hand side (3.3) is well defined.

Finally, it remains to show that the definition is independent of $B(x_0, r) \in \mathbb{B}$. That is, for any $x \in B(x_0, r) \cap B(\omega, R)$ with $B(x_0, r), B(\omega, R) \in \mathbb{B}$ and $B(x_0, r) \cap B(\omega, R) \neq \emptyset$, we have

$$
\begin{aligned}
& T_\theta(f_1\chi_{B(x_0,2r)}, f_2\chi_{B(x_0,2r)})(x) + T_\theta(f_1\chi_{B(x_0,2r)}, f_2\chi_{\mathbb{R}^n\backslash B(x_0,2r)})(x) \\
& + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(x_0,2r)}, f_2\chi_{B(x_0,2r)})(x) + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(x_0,2r)}, f_2\chi_{\mathbb{R}^n\backslash B(x_0,2r)})(x) \\
& = T_\theta(f_1\chi_{B(\omega,2R)}, f_2\chi_{B(\omega,2R)})(x) + T_\theta(f_1\chi_{B(\omega,2R)}, f_2\chi_{\mathbb{R}^n\backslash B(\omega,2R)})(x) \\
& + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(\omega,2R)}, f_2\chi_{B(\omega,2R)})(x) + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(\omega,2R)}, f_2\chi_{\mathbb{R}^n\backslash B(\omega,2R)})(x).
\end{aligned}
$$

Suppose $B(S, M) \in \mathbb{B}$ be selected so that $B(x_0, 2r) \cap B(\omega, 2R) \subset B(S, M)$. According to the estimate of E_1, E_2, E_3 and E_4, for any $x \in B(x_0, r) \cap B(\omega, R)$, we can get

$$
T_\theta(f_1\chi_{B(S,M)\backslash B(x_0,2r)}, f_2\chi_{B(S,M)\backslash B(x_0,2r)}) = \int_{B(S,M)\backslash B(x_0,2r)} \int_{B(S,M)\backslash B(x_0,2r)} K(x, y_1, y_2) f(y_1) f(y_2) \mathrm{d}y_1 \mathrm{d}y_2 < \infty,
$$

$$
T_\theta(f_1\chi_{B(S,M)\backslash B(\omega,2R)}, f_2\chi_{B(S,M)\backslash B(\omega,2R)}) = \int_{B(S,M)\backslash B(\omega,2R)} \int_{B(S,M)\backslash B(\omega,2R)} K(x, y_1, y_2) f(y_1) f(y_2) \mathrm{d}y_1 \mathrm{d}y_2 < \infty.
$$

Because of $\chi_{B(x_0,2r)} f_i$, $\chi_{B(S,M)\backslash B(x_0,2r)} f_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$, where $i = 1, 2$, the linearity of T_θ on $L^{q(\cdot)}(\mathbb{R}^n)$ implies that

$$
\begin{aligned}
& T_\theta(f_1\chi_{B(x_0,2r)}, f_2\chi_{B(x_0,2r)}) + T_\theta(f_1\chi_{B(x_0,2r)}, f_2\chi_{\mathbb{R}^n\backslash B(x_0,2r)}) \\
& + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(x_0,2r)}, f_2\chi_{B(x_0,2r)}) + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(x_0,2r)}, f_2\chi_{\mathbb{R}^n\backslash B(x_0,2r)}) \\
= & T_\theta(f_1\chi_{B(x_0,2r)}, f_2\chi_{B(x_0,2r)}) + T_\theta(f_1\chi_{B(x_0,2r)}, f_2\chi_{B(S,M)\backslash B(x_0,2r)}) \\
& + T_\theta(f_1\chi_{B(x_0,2r)}, f_2\chi_{\mathbb{R}^n\backslash B(S,M)}) + T_\theta(f_1\chi_{B(S,M)\backslash B(x_0,2r)}, f_2\chi_{B(x_0,2r)}) \\
& + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(S,M)}, f_2\chi_{B(x_0,2r)}) + T_\theta(f_1\chi_{B(S,M)\backslash B(x_0,2r)}, f_2\chi_{B(S,M)\backslash B(x_0,2r)}) \\
& + T_\theta(f_1\chi_{B(S,M)\backslash B(x_0,2r)}, f_2\chi_{\mathbb{R}^n\backslash B(S,M)}) + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(S,M)}, f_2\chi_{B(S,M)\backslash B(x_0,2r)}) \\
& + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(S,M)}, f_2\chi_{\mathbb{R}^n\backslash B(S,M)}) \\
= & T_\theta(f_1\chi_{B(S,M)}, f_2\chi_{B(S,M)}) + T_\theta(f_1\chi_{B(S,M)}, f_2\chi_{\mathbb{R}^n\backslash B(S,M)}) \\
& + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(S,M)}, f_2\chi_{B(S,M)}) + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(S,M)}, f_2\chi_{\mathbb{R}^n\backslash B(S,M)}).
\end{aligned}
$$

Similarly, we also get

$$\begin{aligned}&T_\theta(f_1\chi_{B(\omega,2R)}, f_2\chi_{B(\omega,2R)}) + T_\theta(f_1\chi_{B(\omega,2R)}, f_2\chi_{\mathbb{R}^n\backslash B(\omega,2R)})\\&\quad + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(\omega,2R)}, f_2\chi_{B(\omega,2R)}) + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(\omega,2R)}, f_2\chi_{\mathbb{R}^n\backslash B(\omega,2R)})\\&= T_\theta(f_1\chi_{B(S,M)}, f_2\chi_{B(S,M)}) + T_\theta(f_1\chi_{B(S,M)}, f_2\chi_{\mathbb{R}^n\backslash B(S,M)})\\&\quad + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(S,M)}, f_2\chi_{B(S,M)}) + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(S,M)}, f_2\chi_{\mathbb{R}^n\backslash B(S,M)})\end{aligned}.$$

Therefore, $\mathcal{T}_\theta(f_1, f_2)$ is well defined when $f_i \in \mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$, $i = 1, 2$. Obviously, because of (3.3), $\mathcal{T}_\theta$ is a linear operator on $\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$.

When $f \in L^{p_i(\cdot)}(\mathbb{R}^n) \cap \mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$ $(i = 1, 2)$, E_2, E_3 and E_4 guarantee that

$$T_\theta(f_1\chi_{B(x_0,2r)}, f_2\chi_{\mathbb{R}^n\backslash B(x_0,2r)}) = \int_{\mathbb{R}^n\backslash B(x_0,2r)}\int_{B(x_0,2r)} K(x, y_1, y_2) f(y_1) f(y_2) \mathrm{d}y_1 \mathrm{d}y_2 < \infty,$$

$$T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(x_0,2r)}, f_2\chi_{B(x_0,2r)}) = \int_{B(x_0,2r)}\int_{\mathbb{R}^n\backslash B(x_0,2r)} K(x, y_1, y_2) f(y_1) f(y_2) \mathrm{d}y_1 \mathrm{d}y_2 < \infty,$$

$$T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(x_0,2r)}, f_2\chi_{\mathbb{R}^n\backslash B(x_0,2r)}) = \int_{\mathbb{R}^n\backslash B(x_0,2r)}\int_{\mathbb{R}^n\backslash B(x_0,2r)} K(x, y_1, y_2) f(y_1) f(y_2) \mathrm{d}y_1 \mathrm{d}y_2 < \infty.$$

Consequently, $\chi_{\mathbb{R}^n\backslash B(x_0,2r)} f_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$ $(i = 1, 2)$ and the linearity of T_θ on $L^{q(\cdot)}(\mathbb{R}^n)$ implies that

$$\begin{aligned}&\mathcal{T}_\theta(f_1, f_2)(x)\\&= T_\theta(f_1\chi_{B(x_0,2r)}, f_2\chi_{B(x_0,2r)})(x) + T_\theta(f_1\chi_{B(x_0,2r)}, f_2\chi_{\mathbb{R}^n\backslash B(x_0,2r)})(x)\\&\quad + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(x_0,2r)}, f_2\chi_{B(x_0,2r)})(x) + T_\theta(f_1\chi_{\mathbb{R}^n\backslash B(x_0,2r)}, f_2\chi_{\mathbb{R}^n\backslash B(x_0,2r)})(x)\\&= T_\theta(f_1, f_2)(x).\end{aligned}$$

That is, $\mathcal{T}_\theta$ reduces to T_θ on $L^{q(\cdot)} \cap \mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$. Therefore, $\mathcal{T}_\theta$ is an extension of T_θ.

So, we can get the precise definition of bilinear θ-type Calderón-Zygmund operators on generalized variable exponent Morrey spaces.

Proof of Theorem 3.4. For arbitrary ball $B = B(x_0, r)$, we represent f as $f_i = f_i^1 + f_i^2$ for $i = 1, 2$, where $f_i^1 = f_i\chi_{2B}$ and $f_i^2 = f_i\chi_{\mathbb{R}^n\backslash 2B}$.

Then, it can be rewritten as

$$\begin{aligned}&\| \mathcal{T}_\theta(f_1, f_2) \|_{L^{q(\cdot)}(B(x_0,r))}\\&\leq \| T_\theta(f_1^1, f_2^1) \|_{L^{q(\cdot)}(B(x_0,r))} + \| T_\theta(f_1^1, f_2^2) \|_{L^{q(\cdot)}(B(x_0,r))}\\&\quad + \| T_\theta(f_1^2, f_2^1) \|_{L^{q(\cdot)}(B(x_0,r))} + \| T_\theta(f_1^2, f_2^2) \|_{L^{q(\cdot)}(B(x_0,r))}\\&=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

According to Lemma 3.1, we conclude that

$$\begin{aligned}I_1 &\leq C \| f_1 \|_{L^{p_1(\cdot)}(2B)} \| f_2 \|_{L^{p_2(\cdot)}(2B)}\\&\leq Cr^{\theta_q(x_0,r)} \| f_1 \|_{L^{p_1(\cdot)}(2B)} \| f_2 \|_{L^{p_2(\cdot)}(2B)} \int_{2r}^{\infty} t^{-\theta_q(x_0,t)-1}\mathrm{d}t\\&\leq Cr^{\theta_q(x_0,r)} \int_{2r}^{\infty} \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))}\, t^{-\theta_q(x_0,t)-1}\mathrm{d}t\end{aligned}$$

$$= \quad Cr^{\theta_{p_1}(x_0,r)+\theta_{p_2}(x_0,r)} \int_{2r}^{\infty} \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} \, t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1} \mathrm{d}t,$$

where

$$\theta_q(x_0,t) = \begin{cases} \frac{n}{q(x_0)} = \frac{n}{p_1(x_0)} + \frac{n}{p_2(x_0)} = \theta_{p_1}(x_0,t) + \theta_{p_2}(x_0,t), & 0 < t \le 1, \\ \frac{n}{q(\infty)} = \frac{n}{p_1(\infty)} + \frac{n}{p_2(\infty)} = \theta_{p_1}(x_0,t) + \theta_{p_2}(x_0,t), & t \ge 1. \end{cases}$$

According to Lemma 2.1 and the estimate of E_2, I_2 can be estimated as

$$\begin{aligned} I_2 &\le \Big\| \int_{2B} \int_{(2B)^c} \frac{| f_1(y_1) \| f_2(y_2) |}{(\sum_{i=1}^{2} | \cdot - y_i |)^{2n}} \mathrm{d}y_1 \mathrm{d}y_2 \Big\|_{L^{q(\cdot)}(B(x_0,r))} \\ &\le C \int_{2B} | f_1(y_1) | \, \mathrm{d}y_1 \Big\| \int_{(2B)^c} \frac{| f_2(y_2) |}{| x_0 - y_2 |^{2n}} \mathrm{d}y_2 \Big\|_{L^{q(\cdot)}(B(x_0,r))} \\ &\le Cr^{\theta_q(x_0,r)} \int_{2B} | f_1(y_1) | \, \mathrm{d}y_1 \sum_{k=1}^{\infty} \int_{2^{k+1}B \backslash 2^k B} \frac{| f_2(y_2) |}{| x_0 - y_2 |^{2n}} \mathrm{d}y_2 \\ &\le Cr^{\theta_{p_1}(x_0,r)+\theta_{p_2}(x_0,r)} \int_{2r}^{\infty} \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} \, t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1} \mathrm{d}t. \end{aligned}$$

Similar to the estimates for I_2, it is easy to get

$$\begin{aligned} I_3 &\le C \int_{2B} | f_2(y_2) | \, \mathrm{d}y_2 \Big\| \int_{(2B)^c} \frac{| f_1(y_1) |}{| x_0 - y_1 |^{2n}} \mathrm{d}y_1 \Big\|_{L^{q(\cdot)}(B(x_0,r))} \\ &\le Cr^{\theta_q(x_0,r)} \| f_2 \|_{L^{p_2(\cdot)}(2B)} \| 1 \|_{L^{p_2'(\cdot)}(2B)} \int_{2r}^{\infty} \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \, t^{-2n-1} \| 1 \|_{L^{p_1'(\cdot)}(B(x_0,t))} \, \mathrm{d}t \\ &\le Cr^{\theta_{p_1}(x_0,r)+\theta_{p_2}(x_0,r)} \int_{2r}^{\infty} \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} \, t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1} \mathrm{d}t. \end{aligned}$$

For I_4. On the basis of Lemma 2.1 and E_4, it follows that

$$\begin{aligned} I_4 &\le \Big\| \int_{(2B)^c} \int_{(2B)^c} \frac{| f_1(y_1) \| f_2(y_2) |}{(\sum_{i=1}^{2} | \cdot - y_i |)^{2n}} \mathrm{d}y_1 \mathrm{d}y_2 \Big\|_{L^{q(\cdot)}(B(x_0,r))} \\ &\le \Big\| \int_{(2B)^c} \int_{(2B)^c} \frac{| f_1(y_1) \| f_2(y_2) |}{| x_0 - y_1 |^n | x_0 - y_2 |^n} \mathrm{d}y_1 \mathrm{d}y_2 \Big\|_{L^{q(\cdot)}(B(x_0,r))} \\ &\le Cr^{\theta_q(x_0,r)} \sum_{j=1}^{\infty} \prod_{i=1}^{2} (2^j r)^{-n} \int_{2^{j+1}B \backslash 2^j B} | f_i(y_i) | \, \mathrm{d}y_i \\ &\le Cr^{\theta_{p_1}(x_0,r)+\theta_{p_2}(x_0,r)} \int_{2r}^{\infty} \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} \, t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1} \mathrm{d}t. \end{aligned}$$

On account of the above estimates for I_1, I_2, I_3 and I_4, (3.4) is obtained.

Proof of Theorem 3.5. Let $f_i \in \mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$, $i = 1, 2$. According to Theorem 3.4, Lemma 2.6 and (3.2), we get

$$\| \mathcal{T}_\theta(f_1, f_2) \|_{\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)}$$

$$= \sup_{x_0\in\mathbb{R}^n,r>0} \varphi(x_0,r)^{-1} r^{-\theta_q(x_0,r)} \parallel T_\theta(f_1,f_2) \parallel_{L^{q(\cdot)}(B(x_0,r))}$$
$$\le C \sup_{x_0\in\mathbb{R}^n,r>0} \varphi(x_0,r)^{-1} \int_{2r}^{\infty} \parallel f_1 \parallel_{L^{p_1(\cdot)}(B(x_0,t))}\parallel f_2 \parallel_{L^{p_2(\cdot)}(B(x_0,t))} t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1} dt$$
$$\le C \prod_{i=1}^{2} \parallel f_i \parallel_{\mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)} \sup_{x_0\in\mathbb{R}^n,r>0} \varphi(x_0,r)^{-1} \int_{2r}^{\infty} \frac{\varphi_1(x_0,t)\varphi_2(x_0,t)}{t} dt$$
$$\le C \prod_{i=1}^{2} \parallel f_i \parallel_{\mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)},$$

thus, the proof of the Theorem 3.5 is completed.

4. Commutators of bilinear θ-type Calderón-Zygmund operators on $\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$

Now we formulate the main results of this section.

Lemma 4.1. Let T_θ be a bilinear θ-type Calderón-Zygmund operator defined by (1.4) with θ satisfies (1.5). Suppose $p_1(\cdot)$, $p_2(\cdot)$, $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, b_1, $b_2 \in BMO(\mathbb{R}^n)$. Then for all $f_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$, $i = 1, 2$, we have

$$\parallel [b_1, b_2, T_\theta](f_1, f_2) \parallel_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \prod_{i=1}^{2} \parallel b_i \parallel_{BMO(\mathbb{R}^n)}\parallel f_1 \parallel_{L^{p_1(\cdot)}(\mathbb{R}^n)}\parallel f_2 \parallel_{L^{p_2(\cdot)}(\mathbb{R}^n)} \tag{4.1}$$

with the constant $C > 0$ independent of f_1 and f_2.

As this result can be proved in a way similar to Lemma 2.10, we do not present the proof here for the sake of brevity.

We now ready to study the boundedness of the commutator $[b_1, b_2, T_\theta]$ on $\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$ by (3.3).

Definition 4.2. Let T_θ be a bilinear θ-type Calderón-Zygmund operator defined by (1.4) with θ satisfies (1.5). Suppose $p_1(\cdot)$, $p_2(\cdot)$, $p(\cdot)$, $q(\cdot)$, $s_1(\cdot)$, $s_2(\cdot)$, $s(\cdot) \in LH(\mathbb{R}^n)\cap\mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{s(\cdot)}$, $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\frac{1}{s(\cdot)} = \frac{1}{s_1(\cdot)} + \frac{1}{s_2(\cdot)}$, b_1, $b_2 \in BMO(\mathbb{R}^n)$, $\varphi, \varphi_1, \varphi_2$ satisfy the condition

$$\int_{\frac{3}{2}r}^{\infty} (1 + log\frac{t}{r})^2 \frac{\operatorname{ess\,inf}_{t<s<\infty}[\varphi_1(x_0,s)\varphi_2(x_0,s)s^{\theta_q(x_0,s)}]}{t^{\theta_q(x_0,t)+1}} dt \le C\varphi(x_0,r), \tag{4.2}$$

and denote $\varphi(x_0,r) = \varphi_1(x_0,r)\varphi_2(x_0,r)$, where C does not on r. Suppose that T_θ and $[b_1, b_2, T_\theta]$ is a bounded linear operator on $L^{p(\cdot)}(\mathbb{R}^n)$. For any $f_i \in \mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$, $i = 1, 2$, and $x \in B = B(x_0, r) \in \mathbb{B}$, we define

$$\begin{aligned} &[b_1, b_2, \mathcal{T}_\theta](f_1, f_2)(x) \\ &= [b_1, b_2, T_\theta](f_1\chi_{B(x_0,2r)}, f_2\chi_{B(x_0,2r)})(x) + [b_1, b_2, T_\theta](f_1\chi_{B(x_0,2r)}, f_2\chi_{\mathbb{R}^n\setminus B(x_0,2r)})(x) \\ &\quad + [b_1, b_2, T_\theta](f_1\chi_{\mathbb{R}^n\setminus B(x_0,2r)}, f_2\chi_{B(x_0,2r)})(x) + [b_1, b_2, T_\theta](f_1\chi_{\mathbb{R}^n\setminus B(x_0,2r)}, f_2\chi_{\mathbb{R}^n\setminus B(x_0,2r)})(x). \end{aligned} \tag{4.3}$$

Theorem 4.3. Let T_θ be a bilinear θ-type Calderón-Zygmund operator defined by (1.4) with θ satisfies (1.5). Suppose $p_1(\cdot)$, $p_2(\cdot)$, $p(\cdot)$, $q(\cdot)$, $s_1(\cdot)$, $s_2(\cdot)$, $s(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{s(\cdot)}$, $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\frac{1}{s(\cdot)} = \frac{1}{s_1(\cdot)} + \frac{1}{s_2(\cdot)}$, b_1, $b_2 \in BMO(\mathbb{R}^n)$, $\varphi, \varphi_1, \varphi_2$ satisfy the condition (4.2) and $\varphi = \varphi_1\varphi_2$. If T_θ and $[b_1, b_2, T_\theta]$ is a bounded linear operator on $L^{p(\cdot)}(\mathbb{R}^n)$, then $[b_1, b_2, \mathcal{T}_\theta]$ is a well defined linear operator on $\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$.

The result can be proved by using a similar proof method with that of Theorem 3.3, which is omitted here for brevity.

Additionally, for any $f \in L^{p_i(\cdot)}(\mathbb{R}^n) \cap \mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$, where $i = 1, 2$, we have $[b_1, b_2, \mathcal{T}_\theta] = [b_1, b_2, T_\theta]$.

Since $[b_1, b_2, \mathcal{T}_\theta]$ is well defined on $\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$, we are allowed to study the boundedness of commutators of bilinear θ-type Calderón-Zygmund operators on $\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$.

Theorem 4.4. Let T_θ be a bilinear θ-type Calderón-Zygmund operator defined by (1.4) with θ satisfies (1.5). Suppose $p_1(\cdot)$, $p_2(\cdot)$, $p(\cdot)$, $q(\cdot)$, $s_1(\cdot)$, $s_2(\cdot)$, $s(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{s(\cdot)}$, $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\frac{1}{s(\cdot)} = \frac{1}{s_1(\cdot)} + \frac{1}{s_2(\cdot)}$, $b_1,\ b_2 \in BMO(\mathbb{R}^n)$. Then for any ball $B = B(x_0, r)$ and $f_i \in L^{p_i(\cdot)}_{loc}(\mathbb{R}^n)$, $i = 1, 2$, the following inequality

$$\begin{aligned}
&\| [b_1, b_2, \mathcal{T}_\theta](f_1, f_2) \|_{L^{q(\cdot)}(B(x_0,r))} \\
&\le C \| b_1 \|_{BMO(\mathbb{R}^n)} \| b_2 \|_{BMO(\mathbb{R}^n)}\, r^{\theta_q(x_0,r)} \\
&\quad \times \Big[\int_{2r}^{\infty} (1 + log\frac{t}{r})^2 \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))}\, t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1} \mathrm{d}t \Big]
\end{aligned} \tag{4.4}$$

holds, where the constant $C > 0$ independent of f_1 and f_2.

Combining Lemma 4.1 with Theorem 4.4, the following Theorem shows the boundedness of commutators of bilinear θ-type Calderón-Zygmund operators on the generalized variable exponent Morrey spaces.

Theorem 4.5. Let T_θ be a bilinear θ-type Calderón-Zygmund operator defined by (1.4) with θ satisfies (1.5). Suppose $p_1(\cdot)$, $p_2(\cdot)$, $p(\cdot)$, $q(\cdot)$, $s_1(\cdot)$, $s_2(\cdot)$, $s(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{s(\cdot)}$, $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\frac{1}{s(\cdot)} = \frac{1}{s_1(\cdot)} + \frac{1}{s_2(\cdot)}$, $b_1,\ b_2 \in BMO(\mathbb{R}^n)$, $\varphi, \varphi_1, \varphi_2$ satisfy the condition (4.2) and $\varphi = \varphi_1\varphi_2$. Then $[b_1, b_2, \mathcal{T}_\theta]$ is bounded from the place $\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times \mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$ to the place $\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$.

Collory 4.6. Let T be a classical bilinear Calderón-Zygmund operators. Suppose $p_1(\cdot)$, $p_2(\cdot)$, $p(\cdot)$, $q(\cdot)$, $s_1(\cdot)$, $s_2(\cdot)$, $s(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{s(\cdot)}$, $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\frac{1}{s(\cdot)} = \frac{1}{s_1(\cdot)} + \frac{1}{s_2(\cdot)}$, $b_1,\ b_2 \in BMO(\mathbb{R}^n)$, $\varphi, \varphi_1, \varphi_2$ satisfy the condition (4.2) and $\varphi = \varphi_1\varphi_2$. Then $[b_1, b_2, \mathcal{T}]$ is bounded from the place $\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times \mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$ to the place $\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)$.

Proof of Theorem 4.4. We decompose the function f_i in the form $f_i = f_i^1 + f_i^2$ in the proof of Theorem 3.4, where $i = 1, 2$. For all $f_i \in L^{p_i(\cdot)}_{loc}(\mathbb{R}^n)$, then

$$\begin{aligned}
&\| [b_1, b_2, \mathcal{T}_\theta](f_1, f_2) \|_{L^{q(\cdot)}(B(x_0,r))} \\
&\le \| [b_1, b_2, T_\theta](f_1^1, f_2^1) \|_{L^{q(\cdot)}(B(x_0,r))} + \| [b_1, b_2, T_\theta](f_1^1, f_2^2) \|_{L^{q(\cdot)}(B(x_0,r))} \\
&\quad + \| [b_1, b_2, T_\theta](f_1^2, f_2^1) \|_{L^{q(\cdot)}(B(x_0,r))} + \| [b_1, b_2, T_\theta](f_1^2, f_2^2) \|_{L^{q(\cdot)}(B(x_0,r))} \\
&=: H_1 + H_2 + H_3 + H_4.
\end{aligned}$$

Let $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{s(\cdot)}$. From Lemma 2.5, Theorem 4.1 and Lemma 2.1, it follows that

$$\begin{aligned}
H_1 &\le \| 1 \|_{L^{s(\cdot)}(B(x_0,r))} \| [b_1, b_2, T_\theta](f_1^1, f_2^1) \|_{L^{p(\cdot)}(B(x_0,r))} \\
&\le C \| b_1 \|_{BMO(\mathbb{R}^n)} \| b_2 \|_{BMO(\mathbb{R}^n)} \| 1 \|_{L^{s(\cdot)}(B(x_0,r))} \| f_1 \|_{L^{p_1(\cdot)}(2B)}
\end{aligned}$$

$$\times\Big(\| f_2 \|_{L^{p_2(\cdot)}(2B)} r^{\theta_p(x_0,r)} \int_{2r}^{\infty} t^{-\theta_p(x_0,t)-1}\mathrm{d}t\Big)$$
$$\leq C\prod_{i=1}^{2} \| b_i \|_{BMO(\mathbb{R}^n)} r^{\theta_q(x_0,r)} \int_{2r}^{\infty} \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))}\| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1}\mathrm{d}t.$$

Now we deal with H_2. H_2 can be decomposed into four terms.

$$\begin{aligned} H_2 &\leq \| \prod_{i=1}^{2}(b_i-(b_i)_B)T_\theta(f_1^1,f_2^2) \|_{L^{q(\cdot)}(B(x_0,r))} \\ &\quad + \| (b_1-(b_1)_B)T_\theta(f_1^1,(b_2-(b_2)_B)f_2^2) \|_{L^{q(\cdot)}(B(x_0,r))} \\ &\quad + \| (b_2-(b_2)_B)T_\theta((b_1-(b_1)_B)f_1^1,f_2^2) \|_{L^{q(\cdot)}(B(x_0,r))} \\ &\quad + \| T_\theta((b_1-(b_1)_B)f_1^1,(b_2-(b_2)_B)f_2^2) \|_{L^{q(\cdot)}(B(x_0,r))} \\ &=: H_{21}+H_{22}+H_{23}+H_{24}. \end{aligned}$$

Noting that $| x-y_1 | + | x-y_2 | \approx | x_0-y_2 |$ for $x\in B(x_0,r)$, $y_1\in 2B$ and $y_2\in(2B)^c$. Let $\frac{1}{p(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}$, $\frac{1}{s(\cdot)}=\frac{1}{s_1(\cdot)}+\frac{1}{s_2(\cdot)}$, $\frac{1}{q(\cdot)}=\frac{1}{p(\cdot)}+\frac{1}{s(\cdot)}$. By Lemma 2.5, Lemma 2.1, lemma 2.8 and Lemma 2.4, one has

$$\begin{aligned} H_{21} &\leq \| \prod_{i=1}^{2}(b_i-(b_i)_B) \|_{L^{s(\cdot)}(B(x_0,r))}\| T_\theta(f_1^1,f_2^2) \|_{L^{p(\cdot)}(B(x_0,r))} \\ &\leq \| b_1-(b_1)_B \|_{L^{s_1(\cdot)}(B)}\| b_2-(b_2)_B \|_{L^{s_2(\cdot)}(B)} \int_{2B} | f_1(y_1) | \mathrm{d}y_1 \\ &\quad \times\Big(\| \int_{(2B)^c} \frac{| f_2(y_2) |}{| x_0-y_2 |^{2n}}\mathrm{d}y_2 \|_{L^{p(\cdot)}(B(x_0,r))} \Big) \\ &\leq C \| b_1 \|_{BMO(\mathbb{R}^n)}\| b_2 \|_{BMO(\mathbb{R}^n)} r^{\theta_s(x_0,r)} \| 1 \|_{L^{p(\cdot)}(B(x_0,r))} \int_{2B} | f_1(y_1) | \mathrm{d}y_1 \\ &\quad \times\Big(\sum_{k=1}^{\infty} \int_{2^{k+1}B\backslash 2^kB} \frac{| f_2(y_2) |}{| x_0-y_2 |^{2n}}\mathrm{d}y_2\Big) \\ &\leq C\prod_{i=1}^{2} \| b_i \|_{BMO(\mathbb{R}^n)} r^{\theta_q(x_0,r)} \int_{2r}^{\infty} \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))}\| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1}\mathrm{d}t. \end{aligned}$$

From Lemma 2.5, Lemma 2.8, Lemma 2.1, Lemma 2.4 and Lemma 2.3, it follows that

$$\begin{aligned} H_{22} &\leq \| b_1-(b_1)_B \|_{L^{s(\cdot)}(B(x_0,r))}\| T_\theta(f_1^1,(b_2-(b_2)_B)f_2^2) \|_{L^{p(\cdot)}(B(x_0,r))} \\ &\leq \| b_1-(b_1)_B \|_{L^{s_1(\cdot)}(B(x_0,r))}\| 1 \|_{L^{s_2(\cdot)}(B(x_0,r))} \\ &\quad \times\Big(\| \int_{2B}\int_{(2B)^c} \frac{| f_1(y_1) \| f_2(y_2) |}{(\sum_{i=1}^{2} | \cdot-y_i |)^{2n}} | b_2(y_2)-(b_2)_B | \mathrm{d}y_2\mathrm{d}y_1 \|_{L^{p(\cdot)}(B(x_0,r))} \Big) \\ &\leq C \| b_1 \|_{BMO(\mathbb{R}^n)}\| 1 \|_{L^{s(\cdot)}(B(x_0,r))} \int_{2B} | f_1(y_1) | \mathrm{d}y_1 \\ &\quad \times\Big(\int_{(2B)^c} \frac{| f_2(y_2) \| b_2(y_2)-(b_2)_B |}{| x_0-y_2 |^{2n}}\mathrm{d}y_2 \|_{L^{p(\cdot)}(B(x_0,r))} \Big) \end{aligned}$$

$$
\begin{aligned}
&\leq C \parallel b_1 \parallel_{BMO(\mathbb{R}^n)} r^{\theta_q(x_0,r)} \int_{2B} | f_1(y_1) | \mathrm{d}y_1 \\
&\quad \times \Big[\int_{(2B)^c} | f_2(y_2) \parallel b_2(y_2) - (b_2)_B | \Big(\int_{|x_0-y_2|}^{\infty} \frac{\mathrm{d}t}{t^{2n+1}} \Big) \mathrm{d}y_2 \Big] \\
&\leq C \parallel b_1 \parallel_{BMO(\mathbb{R}^n)} r^{\theta_q(x_0,r)} \int_{2B} | f_1(y_1) | \mathrm{d}y_1 \\
&\quad \times \Big[\int_{2r}^{\infty} \int_{B(x_0,t)} | f_2(y_2) \parallel b_2(y_2) - (b_2)_B | \mathrm{d}y_2 \frac{\mathrm{d}t}{t^{2n+1}} \Big] \\
&\leq C \parallel b_1 \parallel_{BMO(\mathbb{R}^n)} r^{\theta_q(x_0,r)} \int_{2B} | f_1(y_1) | \mathrm{d}y_1 \\
&\quad \times \Big(\int_{2r}^{\infty} \int_{B(x_0,t)} | f_2(y_2) \parallel b_2(y_2) - (b_2)_{B(x_0,t)} | \mathrm{d}y_2 \frac{\mathrm{d}t}{t^{2n+1}} \Big) \\
&\quad + C \parallel b_1 \parallel_{BMO(\mathbb{R}^n)} r^{\theta_q(x_0,r)} \int_{2B} | f_1(y_1) | \mathrm{d}y_1 \\
&\quad \times \Big(\int_{2r}^{\infty} \int_{B(x_0,t)} | f_2(y_2) \parallel (b_2)_{B(x_0,t)} - (b_2)_B | \mathrm{d}y_2 \frac{\mathrm{d}t}{t^{2n+1}} \Big) \\
&\leq C \parallel b_1 \parallel_{BMO(\mathbb{R}^n)} r^{\theta_q(x_0,r)} \parallel f_1 \parallel_{L^{p_1(\cdot)}(2B)} \parallel 1 \parallel_{L^{p_1'(\cdot)}(2B)} \\
&\quad \times \Big(\int_{2r}^{\infty} \parallel f_2 \parallel_{L^{p_2(\cdot)}(B(x_0,t))} \parallel b_2(\cdot) - (b_2)_{B(x_0,t)} \parallel_{L^{p_2'(\cdot)}(B(x_0,t))} \frac{\mathrm{d}t}{t^{2n+1}} \Big) \\
&\quad + C \parallel b_1 \parallel_{BMO(\mathbb{R}^n)} r^{\theta_q(x_0,r)} \parallel f_1 \parallel_{L^{p_1(\cdot)}(2B)} \parallel 1 \parallel_{L^{p_1'(\cdot)}(2B)} \\
&\quad \times \Big(| (b_2)_{B(x_0,t)} - (b_2)_B | \int_{2r}^{\infty} \parallel f_2 \parallel_{L^{p_2(\cdot)}(B(x_0,t))} \parallel 1 \parallel_{L^{p_2'(\cdot)}(B(x_0,t))} \frac{\mathrm{d}t}{t^{2n+1}} \Big) \\
&\leq C \parallel b_1 \parallel_{BMO(\mathbb{R}^n)} \parallel b_2 \parallel_{BMO(\mathbb{R}^n)} r^{\theta_q(x_0,r)} \\
&\quad \times \Big[\int_{2r}^{\infty} \Big(1 + log\frac{t}{r}\Big) \parallel f_1 \parallel_{L^{p_1(\cdot)}(B(x_0,t))} \parallel f_2 \parallel_{L^{p_2(\cdot)}(B(x_0,t))} t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1} \mathrm{d}t \Big].
\end{aligned}
$$

By applying Lemma 2.5, Lemma 2.1, Lemma 2.8, Lemma 2.4 and Lemma 2.3, we can deduce that

$$
\begin{aligned}
H_{23} &\leq \parallel (b_2 - (b_2)_B) T_\theta((b_1 - (b_1)_B) f_1^1, f_2^2) \parallel_{L^{q(\cdot)}(B(x_0,r))} \\
&\leq \parallel b_2 - (b_2)_B \parallel_{L^{s(\cdot)}(B(x_0,r))} \parallel T_\theta((b_1 - (b_1)_B) f_1^1, f_2^2) \parallel_{L^{p(\cdot)}(B(x_0,r))} \\
&\leq \parallel b_2 - (b_2)_B \parallel_{L^{s_2(\cdot)}(B(x_0,r))} \parallel 1 \parallel_{L^{s_1(\cdot)}(B(x_0,r))} \\
&\quad \times \Big(\parallel \int_{2B} \int_{(2B)^c} \frac{| f_1(y_1) \parallel f_2(y_2) |}{(\sum_{i=1}^{2} | \cdot - y_i |)^{2n}} | b_1(y_1) - (b_1)_B | \mathrm{d}y_2 \mathrm{d}y_1 \parallel_{L^{p(\cdot)}(B(x_0,r))} \Big) \\
&\leq C \parallel b_2 \parallel_{BMO(\mathbb{R}^n)} \parallel 1 \parallel_{L^{s(\cdot)}(B(x_0,r))} \int_{2B} | b_1(y_1) - (b_1)_B \parallel f_1(y_1) | \mathrm{d}y_1 \\
&\quad \times \Big(\parallel \int_{(2B)^c} \frac{| f_2(y_2) |}{| x_0 - y_2 |^{2n}} \mathrm{d}y_2 \parallel_{L^{p(\cdot)}(B(x_0,r))} \Big) \\
&\leq C \parallel b_2 \parallel_{BMO(\mathbb{R}^n)} r^{\theta_q(x_0,r)} \int_{2B} | b_1(y_1) - (b_1)_B \parallel f_1(y_1) | \mathrm{d}y_1
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big[\int_{(2B)^c}|f_2(y_2)|\Big(\int_{|x_0-y_2|}^{\infty}\frac{\mathrm{d}t}{t^{2n+1}}\Big)\mathrm{d}y_2\Big]\\
\le\ & C\|b_2\|_{BMO(\mathbb{R}^n)}r^{\theta_q(x_0,r)}\int_{2B}|b_1(y_1)-(b_1)_B||f_1(y_1)|\mathrm{d}y_1\\
&\times\Big(\int_{2r}^{\infty}\int_{B(x_0,t)}|f_2(y_2)|\mathrm{d}y_2\frac{\mathrm{d}t}{t^{2n+1}}\Big)\\
\le\ & C\|b_2\|_{BMO(\mathbb{R}^n)}r^{\theta_q(x_0,r)}\int_{2B}|b_1(y_1)-(b_1)_{2B}||f_1(y_1)|\mathrm{d}y_1\\
&\times\Big(\int_{2r}^{\infty}\int_{B(x_0,t)}|f_2(y_2)|\mathrm{d}y_2\frac{\mathrm{d}t}{t^{2n+1}}\Big)\\
&+C\|b_2\|_{BMO(\mathbb{R}^n)}r^{\theta_q(x_0,r)}\int_{2B}|(b_1)_{2B}-(b_1)_B||f_1(y_1)|\mathrm{d}y_1\\
&\times\Big(\int_{2r}^{\infty}\int_{B(x_0,t)}|f_2(y_2)|\mathrm{d}y_2\frac{\mathrm{d}t}{t^{2n+1}}\Big)\\
\le\ & C\|b_2\|_{BMO(\mathbb{R}^n)}r^{\theta_q(x_0,r)}\|f_1\|_{L^{p_1(\cdot)}(2B)}\|b_1(\cdot)-(b_1)_{2B}\|_{L^{p_1'(\cdot)}(2B)}\\
&\times\Big(\int_{2r}^{\infty}\|f_2\|_{L^{p_2(\cdot)}(B(x_0,t))}\|1\|_{L^{p_2'(\cdot)}(B(x_0,t))}\frac{\mathrm{d}t}{t^{2n+1}}\Big)\\
&+C\|b_2\|_{BMO(\mathbb{R}^n)}r^{\theta_q(x_0,r)}\|f_1\|_{L^{p_1(\cdot)}(2B)}\|1\|_{L^{p_1'(\cdot)}(2B)}|(b_1)_{2B}-(b_1)_B|\\
&\times\Big(\int_{2r}^{\infty}\|f_2\|_{L^{p_2(\cdot)}(B(x_0,t))}\|1\|_{L^{p_2'(\cdot)}(B(x_0,t))}\frac{\mathrm{d}t}{t^{2n+1}}\Big)\\
\le\ & C\|b_1\|_{BMO(\mathbb{R}^n)}\|b_2\|_{BMO(\mathbb{R}^n)}r^{\theta_q(x_0,r)}\Big(1+log\frac{2r}{r}\Big)\|f_1\|_{L^{p_1(\cdot)}(2B)}\|1\|_{L^{p_1'(\cdot)}(2B)}\\
&\times\Big(\int_{2r}^{\infty}\|f_2\|_{L^{p_2(\cdot)}(B(x_0,t))}t^{-n-\theta_{p_2}(x_0,t)-1}\mathrm{d}t\Big)\\
\le\ & C\|b_1\|_{BMO(\mathbb{R}^n)}\|b_2\|_{BMO(\mathbb{R}^n)}r^{\theta_q(x_0,r)}\\
&\times\Big[\int_{2r}^{\infty}\Big(1+log\frac{t}{r}\Big)\|f_1\|_{L^{p_1(\cdot)}(B(x_0,t))}\|f_2\|_{L^{p_2(\cdot)}(B(x_0,t))}t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1}\mathrm{d}t\Big].
\end{aligned}
$$

Further, according to Lemma 2.1, Lemma 2.4, Lemma 2.8 and Lemma 2.3, we obtain that

$$
\begin{aligned}
H_{24}\ \le\ & \Big\|\int_{2B}\int_{(2B)^c}\frac{|b_1(y_1)-(b_1)_B||b_2(y_2)-(b_2)_B|}{(\sum_{i=1}^{2}|\cdot-y_i|)^{2n}}|f_1(y_1)||f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\Big\|_{L^{q(\cdot)}(B(x_0,r))}\\
\le\ & C\int_{2B}|b_1(y_1)-(b_1)_B||f_1(y_1)|\mathrm{d}y_1\Big\|\int_{(2B)^c}\frac{|f_2(y_2)||b_2(y_2)-(b_2)_B|}{|x_0-y_2|^{2n}}\mathrm{d}y_2\Big\|_{L^{q(\cdot)}(B(x_0,r))}\\
\le\ & Cr^{\theta_q(x_0,r)}\int_{2B}|b_1(y_1)-(b_1)_B||f_1(y_1)|\mathrm{d}y_1\int_{(2B)^c}|f_2(y_2)||b_2(y_2)-(b_2)_B|\\
&\times\Big[\Big(\int_{|x_0-y_2|}^{\infty}\frac{\mathrm{d}t}{t^{2n+1}}\Big)\mathrm{d}y_2\Big]\\
\le\ & Cr^{\theta_q(x_0,r)}\int_{2B}|b_1(y_1)-(b_1)_B||f_1(y_1)|\mathrm{d}y_1
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big(\int_{2r}^{\infty}\int_{B(x_0,t)} | f_2(y_2) \| b_2(y_2)-(b_2)_B | \mathrm{d}y_2 \frac{\mathrm{d}t}{t^{2n+1}}\Big)\\
\leq\ & Cr^{\theta_q(x_0,r)} \| f_1 \|_{L^{p_1(\cdot)}(2B)} \| b_1(\cdot)-(b_1)_B \|_{L^{p_1'(\cdot)}(2B)}\\
&\times\Big(\int_{2r}^{\infty} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} \| b_2(\cdot)-(b_2)_B \|_{L^{p_2'(\cdot)}(B(x_0,t))} \frac{\mathrm{d}t}{t^{2n+1}}\Big)\\
\leq\ & Cr^{\theta_q(x_0,r)} \| f_1 \|_{L^{p_1(\cdot)}(2B)} \Big(\| b_1(\cdot)-(b_1)_{2B} \|_{L^{p_1'(\cdot)}(2B)} + \| (b_1)_{2B}-(b_1)_B \|_{L^{p_1'(\cdot)}(2B)} \Big)\\
&\times\Big[\int_{2r}^{\infty} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} \Big(\| b_2(\cdot)-(b_2)_{B(x_0,t)} \|_{L^{p_2'(\cdot)}(B(x_0,t))}\\
&\quad + \| (b_2)_{B(x_0,t)}-(b_2)_B \|_{L^{p_2'(\cdot)}(B(x_0,t))} \Big)\frac{\mathrm{d}t}{t^{2n+1}}\Big]\\
\leq\ & Cr^{\theta_q(x_0,r)} \| b_1 \|_{BMO(\mathbb{R}^n)} \| b_2 \|_{BMO(\mathbb{R}^n)} \| f_1 \|_{L^{p_1}(2B)} \| 1 \|_{L^{p_1'(\cdot)}(2B)} \Big(1+log\frac{2r}{r}\Big)\\
&\times\Big[\int_{2r}^{\infty}\Big(1+log\frac{t}{r}\Big) \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} \| 1 \|_{L^{p_2'(\cdot)}(B(x_0,t))}\ t^{-2n+1}\mathrm{d}t\Big]\\
\leq\ & C \| b_1 \|_{BMO(\mathbb{R}^n)} \| b_2 \|_{BMO(\mathbb{R}^n)}\ r^{\theta_q(x_0,r)}\\
&\times\Big[\int_{2r}^{\infty}\Big(1+log\frac{t}{r}\Big)^2 \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))}\ t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1}\mathrm{d}t\Big].
\end{aligned}
$$

Which, together with the estimates for H_{21}, H_{22}, H_{23} and H_{24}, we get

$$
\begin{aligned}
H_2 \leq\ & C \| b_1 \|_{BMO(\mathbb{R}^n)} \| b_2 \|_{BMO(\mathbb{R}^n)}\ r^{\theta_q(x_0,r)}\\
&\times\Big[\int_{2r}^{\infty}\Big(1+log\frac{t}{r}\Big)^2 \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))}\ t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1}\mathrm{d}t\Big].
\end{aligned}
$$

Similarly, it is not difficult to obtain

$$
\begin{aligned}
H_3 \leq\ & \| \prod_{i=1}^{2}(b_i-(b_i)_B)T_\theta(f_1^2, f_2^1) \|_{L^{q(\cdot)}(B(x_0,r))}\\
&+ \| (b_1-(b_1)_B)T_\theta(f_1^2,(b_2-(b_2)_B)f_2^1) \|_{L^{q(\cdot)}(B(x_0,r))}\\
&+ \| (b_2-(b_2)_B)T_\theta((b_1-(b_1)_B)f_1^2, f_2^1) \|_{L^{q(\cdot)}(B(x_0,r))}\\
&+ \| T_\theta((b_1-(b_1)_B)f_1^2,(b_2-(b_2)_B)f_2^1) \|_{L^{q(\cdot)}(B(x_0,r))}\\
\leq\ & C \| b_1 \|_{BMO(\mathbb{R}^n)} \| b_2 \|_{BMO(\mathbb{R}^n)}\ r^{\theta_q(x_0,r)}\\
&\times\Big[\int_{2r}^{\infty}\Big(1+log\frac{t}{r}\Big)^2 \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))}\ t^{-\theta_{p_1}(x_0,t)-\theta_{p_2}(x_0,t)-1}\mathrm{d}t\Big].
\end{aligned}
$$

It remains to estimate H_4. Noting that $| x-y_1 | + | x-y_2 | \approx | x-y_1 | \approx | x-y_2 |$ for $x \in B(x_0, r)$ and $y_1, y_2 \in (2B)^c$. By using Lemma 2.1, Lemma 2.4, Lemma 2.8 and Lemma 2.3, we can deduce that

$$
\begin{aligned}
H_4 \leq\ & \| \int_{(2B)^c}\int_{(2B)^c} \frac{| b_1(y_1)-(b_1)_B \| b_2(y_2)-(b_2)_B |}{(\sum_{i=1}^{2} | \cdot - y_i |)^{2n}} | f_1(y_1) \| f_2(y_2) | \mathrm{d}y_1\mathrm{d}y_2 \|_{L^{q(\cdot)}(B(x_0,r))}\\
\leq\ & \| \int_{(2B)^c}\int_{(2B)^c} \frac{| b_1(y_1)-(b_1)_B \| b_2(y_2)-(b_2)_B |}{| x_0-y_1 |^n | x_0-y_2 |^n} | f_1(y_1) \| f_2(y_2) | \mathrm{d}y_1\mathrm{d}y_2 \|_{L^{q(\cdot)}(B(x_0,r))}
\end{aligned}
$$

$$\leq \; Cr^{\theta_q(x_0,r)} \sum_{j=1}^{\infty} \prod_{i=1}^{2} (2^j r)^{-n} \int_{2^{j+1}B\backslash 2^j B} | f_i(y_i) \| b_i(y_i) - (b_i)_B | \mathrm{d}y_i$$

$$\leq \; Cr^{\theta_q(x_0,r)} \sum_{j=1}^{\infty} \prod_{i=1}^{2} (2^j r)^{-n} \int_{2^{j+1}B\backslash 2^j B} | f_i(y_i) \| b_i(y_i) - (b_i)_{2^{j+1}B} | \mathrm{d}y_i$$

$$+Cr^{\theta_q(x_0,r)} \sum_{j=1}^{\infty} \prod_{i=1}^{2} (2^j r)^{-n} \int_{2^{j+1}B\backslash 2^j B} | f_i(y_i) \| (b_i)_{2^{j+1}B} - (b_i)_B | \mathrm{d}y_i$$

$$\leq \; Cr^{\theta_q(x_0,r)} \sum_{j=1}^{\infty} \prod_{i=1}^{2} (2^j r)^{-n} \| f_i \|_{L^{p_i(\cdot)}(2^{j+1}B)} \| b_i(\cdot) - (b_i)_{2^{j+1}B} \|_{L^{p_i'(\cdot)}(2^{j+1}B)}$$

$$+Cr^{\theta_q(x_0,r)} \sum_{j=1}^{\infty} \prod_{i=1}^{2} (2^j r)^{-n} | (b_i)_{2^{j+1}B} - (b_i)_B \| \| f_i \|_{L^{p_i(\cdot)}(2^{j+1}B)} \| 1 \|_{L^{p_i'(\cdot)}(2^{j+1}B)}$$

$$\leq \; C \| b_1 \|_{BMO(\mathbb{R}^n)} \| b_2 \|_{BMO(\mathbb{R}^n)} \, r^{\theta_q(x_0,r)} \sum_{j=1}^{\infty} (2^j r)^{-2n} \Big(1 + log \frac{2^{j+1} r}{r}\Big)^2$$

$$\times \Big(\| f_1 \|_{L^{p_1(\cdot)}(2^{j+1}B)} \| 1 \|_{L^{p_1'(\cdot)}(2^{j+1}B)} \| f_2 \|_{L^{p_2(\cdot)}(2^{j+1}B)} \| 1 \|_{L^{p_2'(\cdot)}(2^{j+1}B)} \Big)$$

$$\leq \; C \| b_1 \|_{BMO(\mathbb{R}^n)} \| b_2 \|_{BMO(\mathbb{R}^n)} \, r^{\theta_q(x_0,r)} \sum_{j=1}^{\infty} \int_{2^{j+1} r}^{2^{j+2} r} t^{-2n-1} \Big(1 + log \frac{t}{r}\Big)^2$$

$$\times \Big(\| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| 1 \|_{L^{p_1'(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} \| 1 \|_{L^{p_2'(\cdot)}(B(x_0,t))} \, \mathrm{d}t \Big)$$

$$\leq \; C \| b_1 \|_{BMO(\mathbb{R}^n)} \| b_2 \|_{BMO(\mathbb{R}^n)} \, r^{\theta_q(x_0,r)}$$

$$\times \Big[\int_{2r}^{\infty} \Big(1 + log \frac{t}{r}\Big)^2 \| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} \, t^{-\theta_{p_1}(x_0,t) - \theta_{p_2}(x_0,t) - 1} \mathrm{d}t \Big].$$

which, combining the estimates of H_1, H_2 and H_3, implies (4.4).

Proof of Theorem 4.5. Let $f_i \in \mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$, $i = 1, 2$. By Theorem 4.4, Lemma 2.7 and (4.2), we obtain

$$\begin{aligned}
&\| [b_1, b_2, \mathcal{T}_\theta](f_1, f_2) \|_{\mathcal{M}^{q(\cdot),\varphi}(\mathbb{R}^n)} \\
&\quad = \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi(x_0, r)^{-1} r^{-\theta_q(x_0,r)} \| [b_1, b_2, T_\theta](f_1, f_2) \|_{L^{q(\cdot)}(B(x_0,r))} \\
&\quad \leq C \prod_{i=1}^{2} \| b_i \|_{BMO(\mathbb{R}^n)} \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi(x_0, r)^{-1} \int_{2r}^{\infty} \Big(1 + log \frac{t}{r}\Big)^2 \\
&\quad \times \Big(\| f_1 \|_{L^{p_1(\cdot)}(B(x_0,t))} \| f_2 \|_{L^{p_2(\cdot)}(B(x_0,t))} \, t^{-\theta_{p_1}(x_0,t) - \theta_{p_2}(x_0,t) - 1} \mathrm{d}t \Big) \\
&\quad \leq C \prod_{i=1}^{2} \| b_i \|_{BMO(\mathbb{R}^n)} \| f_i \|_{\mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)} \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi(x_0, r)^{-1} \int_{2r}^{\infty} \Big(1 + log \frac{t}{r}\Big)^2 \frac{\varphi_1(x_0, t)\varphi_2(x_0, t)}{t} \mathrm{d}t \\
&\quad \leq C \prod_{i=1}^{2} \| b_i \|_{BMO(\mathbb{R}^n)} \| f_i \|_{\mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)},
\end{aligned}$$

which is our desire result.

5. Conclusions

In this paper, I mainly obtain the boundedness of bilinear θ-type Calderón-Zygmund operator T_θ and its commutator $[b_1, b_2, T_\theta]$ on the generalized variable exponent Morrey spaces.

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Conflict of interest

The author declares that he has no conflicts of interest.

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