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Research article

On S-principal right ideal rings

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Abstract: Let S be a multiplicative subset of a ring R. A right ideal A of R is referred to as S-*principal* if there exist an element $s \in S$ and a principal right ideal aR of R such that $As \subseteq aR \subseteq A$. A ring is referred to as an S-*principal right ideal ring* (S-PRIR) if every right ideal is S-principal. This paper examines S-PRIRs, which extend the notion of principal right ideal rings. Various examples, including several extensions of S-PRIRs are investigated, and some practical results are proven. A noncommutative S-PRIR that is not a principal right ideal ring is found, and the S-variants of the Eakin-Nagata-Eisenbud theorem and Cohen's theorem for S-PRIRs are proven.

Keywords: S-principal right ideal ring; S-principal right ideal domain; right S-Noetherian ring; S-principal right ideal; Eakin-Nagata-Eisenbud theorem; Cohen's theorem
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1. Introduction

Anderson and Dumitrescu defined commutative S-Noetherian rings in [2]. Many important and valuable results of commutative S-Noetherian rings are published. Especially, one can see practical results of commutative S-principal ideal rings in [2, Proposition 16], [3, Theorem 2.1], [11, Theorem 2.8] and [13, Theorem 3.2]. One-sided S-Noetherian rings were investigated independently in [4] and [6], and the S-variant of the Hilbert basis theorem [4, Theorem 3.1] and the S-variant of Cohen's theorem [6, Theorem 2.2] were proven. Recently, the S-variant of the Eakin-Nagata-Eisenbud theorem was also proven in [19].

Recall that a submodule N of a right module M over a ring R is called S-*finite* if $Ns \subseteq F \subseteq N$ for some $s \in S$ and finitely generated R-submodule F of M. A right R-module M is called S-*Noetherian* if every R-submodule is S-finite. A right ideal A of a ring R is S-*finite* if A_R is S-finite, and R is right S-*Noetherian* if R_R is S-Noetherian. Clearly, finitely generated submodules of M are S-finite; thus, the notion of right S-Noetherian modules is an extension of the notion of right Noetherian modules.

This concept of S-finite is highly relevant because for an infinitely generated submodule N of M, infinite generators of N (after multiplying $s \in S$) can be controlled by finite generators of some finitely generated submodule F. Surprisingly, many critical properties of one-sided Noetherian modules and rings can be lifted up to one-sided S-Noetherian modules and rings. The readers can refer to [2, 3, 10, 11, 13, 16, 21, 22] for more details on commutative S-Noetherian rings and to [4, 6, 8] for more details on one-sided S-Noetherian rings.

In this paper, we continue to study the S-Noetherian notion when the above finitely generated submodule F is principal. Let us define the followings for rigor.

Definition 1.1. Let S be a multiplicative subset of a ring R and M be a unitary right R-module.

(1) An R-submodule N of M is called *S-principal* if there exist an element $s \in S$ and a cyclic R-submodule P of M such that $Ns \subseteq P \subseteq N$. A right ideal A of R is called *S-principal* if A is S-principal as a right R-submodule.
(2) R is called an *S-principal right ideal ring* (S-PRIR) if every right ideal is S-principal. An S-principal right ideal ring R is called an *S-principal right ideal domain* (S-PRID) if R is a domain.

An S-principal left ideal ring (S-PLIR) and domain (S-PLID) are defined symmetrically. A ring R is called an *S-principal ideal ring* (S-PIR) if R is both an S-PLIR and an S-PRIR, and an S-PIR R is called *S-principal ideal domain* (S-PID) if R is a domain. Of course, neither S-PIR nor S-PID guarantees the commutative condition. Thus, for a commutative ring, we use the respective notations commutative S-PIR and commutative S-PID.

In this paper, one studies various examples including several extensions, and the properties of S-PRIRs. In Section 2, some examples and extensions of S-PRIRs are provided. Among other things, an S-PRID which is not a principal right ideal domain is constructed (Example 2.7). After that, one discover that the S-principal notion can be passed between given rings and some matrix rings (Theorem 2.14), and find a specific condition under which the S-principal concept can be penetrated into Ore localizations (Theorem 2.20). In Section 3, several valuable properties are presented. More precisely, the S-variant of the Eakin-Nagata-Eisenbud theorem for S-PRIRs (Theorem 3.10) and the S-variant of Cohen's theorem for S-PRIRs (Theorem 3.12) are proven. Moreover, one can give some examples and counterexamples for answers to questions that occur naturally in the process.

Throughout this paper, all rings are associative (not necessarily commutative) with unity, and all modules are unitary right modules. A multiplicative (closed) subset of a given ring may not contain the unity element. Without any particular mention, $\mathbb{Z}$, $\mathbb{Z}_n$ and $\mathbb{Q}$ means the integer ring, ring of integers modulo n for a natural number n, and the rational field, respectively. For a fixed natural number n, we denote the $n \times n$ full matrix ring over a ring R by $\mathbb{M}_n(R)$ and the $n \times n$ upper triangular matrix ring over a ring R by $\mathbb{T}_n(R)$. For a given ring R, $Z(R)$ is used for the center of R. We assume that the readers already know the definitions of the abbreviated terms PRIR, PIR, PRID, and PID.

2. Examples of S-principal right ideal rings

In this section, we provide various examples of S-PRIR. Clearly, every PRIR R is an S-PRIR for any multiplicative subset S of R. However, the converse statement is not true in general, when $S \neq \{1\}$. To show this easily, we begin with a trivial result demonstrating that the S-principal concept is natural in ring theory. Recall that a right ideal A of a ring R is called *essential* if, for any nonzero right ideal B of R, $A \cap B \neq \{0\}$ (see [17, Definition 3.26]).

Proposition 2.1. (cf. [2, Proposition 2.(a)]). *If a commutative ring R has an essential ideal A which does not contain any zero-divisor element except* 0*, then R is an S-PIR where $S = A \setminus \{0\}$.*

Proof. Let B be an ideal of R. Because A is essential, there exists an element $s \in B \cap S \neq \{0\}$. Therefore $Bs \subseteq sR \subseteq B$, confirming that R is an S-PIR. □

The next corollary is an immediate consequence of Proposition 2.1.

Corollary 2.2. *If D is a commutative domain and $S = D \setminus \{0\}$, then D is an S-PID.*

Remark 2.3. (1) We can use conveniently the result of Corollary 2.2. For instance, the ring R of [1, Theorem 3.1] is a commutative S-PID, where $S = R \setminus \{0\}$.

(2) (cf. [4, Proposition 2.4]). Let S be a multiplicative subset of right invertible elements of a ring R. If an R-submodule N of an R-module M is S-principal, then N must be principal. In particular, the class of all S-principal right modules over a division ring D coincides with the class of all principal right modules over D.

Commutative domains $\mathbb{Z}[x]$ and $F[x, y]$ are not PIDs, where $\mathbb{Z}[x]$ is the polynomial ring with x over $\mathbb{Z}$ and $F[x, y]$ is the polynomial ring with two variables x, y over a field F. We provide a simple example of a commutative S-PID that is not a PID.

Example 2.4. (see Remark 2.8(3)). Let D be a commutative domain that is not a PID. Clearly, the polynomial ring $D[x_1, \dots, x_n]$ is also a commutative domain but not a PID. Consider a multiplicative subset $S = D[x_1, \dots, x_n] \setminus \{0\}$ of $D[x_1, \dots, x_n]$. Then $D[x_1, \dots, x_n]$ is a commutative S-PID by Corollary 2.2.

Remark 2.5. (see [4, Example 2.12]). If $S \neq D \setminus \{0\}$, then Corollary 2.2 may not be true in general. For instance, let $D = \mathbb{Z}[x]$ and $S = \{1\}$. Then the commutative domain D is not an S-PID.

As seen above, we can find easily an S-PIR by taking a large enough multiplicative subset S of R. However, we must take an S that is as small as possible to control the ideals of S-PIRs efficiently. Here, we provide an example of an S-PIR, but not a PIR, with the smallest multiplicative subset S.

Example 2.6. Let $\mathcal{R} = \mathbb{Z}_2\langle a_1, a_2, s\rangle$ be the free algebra with unity and commuting indeterminates a_1, a_2, s over $\mathbb{Z}_2$. Set $R = \mathcal{R}/\mathcal{I}$ where $\mathcal{I}$ is the ideal of $\mathcal{R}$ generated by the following relations:

$$a_1^2 = a_2^2 = 0,\ a_1 s = a_2 s,\ s^2 = s.$$

We identify $r = r + \mathcal{I}$ for simplicity. Applying Bergman's diamond lemma [5], we can write each element $r \in R$ uniquely in the following reduced form:

$$r = \alpha + \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 s + \alpha_4 a_1 a_2 + \alpha_5 a_1 s,$$

where $\alpha, \alpha_1, \cdots, \alpha_5 \in \mathbb{Z}_2$. Clearly, an ideal of the commutative ring R, $a_1 R + a_2 R$ is not principal and so R is not a PIR. To show that R is an S-PIR, consider a multiplicative subset $S = \{s\}$ of R, and let A be a nontrivial ideal of R. If A is principal, then we are done. So we may assume that A is not principal.

If $s \in A$, then we obtain $As \subseteq sR \subseteq A$. If $s \notin A$, then from the two facts that $(1 + s)Rs = a_1 a_2 Rs = (a_1 + a_2)Rs = \{0\}$ and each element r with $\alpha \neq 0, \alpha_3 = 0$ in the form is a unit element, we only need to check the case $A = a_1R + a_2R$. Because $a_1s = a_2s$, we obtain $As \subseteq a_1sR \subseteq A$. Thus, we can conclude that $As \subseteq P \subseteq A$ for any nonprincipal ideal A of R, where P is $\{0\}$ or sR or a_1sR, as needed.

We study S-PRIRs in earnest. First of all, we construct a noncommutative S-PRID that is not a PRID, capitalizing on [4, Corollary 3.2(1)].

Example 2.7. Let $D = \mathbb{Z}_2\langle a, b\rangle$ be the free algebra with unity and commuting indeterminates a, b, and $S = D\setminus\{0\}$. Then the commutative domain D is an S-PID by Corollary 2.2. Let σ be the automorphism of D such that $\sigma(a) = b, \sigma(b) = a$. Obviously, the skew polynomial ring over D, $R = D[x;\sigma]$ is a noncommutative domain since $ax \neq xa = \sigma(a)x = bx$. Note that a right ideal of R, $A = aR + xR$ is not principal and so R is not a PRID. Now we claim that R is an S-PRID. If B is a nonprincipal right ideal of R, then by [4, Corollary 3.2(1)], there exist an element $s \in S$ and nonzero polynomials $f_i(x) \in R$ such that

$$Bs \subseteq \sum_{i=1}^{m} f_i(x)R \subseteq B.$$

Let $p(x)$ be the smallest degree nonzero polynomial in $\sum_{i=1}^{m} f_i(x)R$. If $p(x) = d \in D$ is a nonzero constant, then we have

$$Bsd\sigma(d) \subseteq \left(\sum_{i=1}^{m} f_i(x)R\right) d\sigma(d) \subseteq d\sigma(d)R \subseteq B$$

since $d\sigma(d) \in Z(R)$. Therefore, we may assume that $p(x) = \sum_{j=0}^{n} p_jx^j$ with $deg(p(x)) = n \geq 1$. From the usual Euclidean algorithm and the minimality of n, we can notice that for each $f_i(x)$, there exist nonzero $g_i(x) \in R$ and $t_i \in S$ such that $f_i(x)t_i = p(x)g_i(x) + q_i(x)$ with $deg(q_i(x)) = n$ or 0. Note that for any polynomial $q(x) = \sum_{k=0}^{n} q_kx^k \in \sum_{i=1}^{m} f_i(x)R$ with $q_n \neq 0$, we have $p(x)q_n + q(x)p_n = 0$ (if n is even) or $p(x)\sigma(q_n) + q(x)\sigma(p_n) = 0$ (if n is odd) by the minimality of n. Thus, we obtain

$$(p(x)R + q(x)R)\, p_nq_n\sigma(p_n)\sigma(q_n) \subseteq q_np(x)R = p_nq(x)R \subseteq p(x)R + q(x)R$$

or

$$(p(x)R + q(x)R)\, p_nq_n\sigma(p_n)\sigma(q_n) \subseteq \sigma(q_n)p(x)R = \sigma(p_n)q(x)R \subseteq p(x)R + q(x)R.$$

This result implies that

$$Bst_1 \cdots t_ms' \subseteq \left(\sum_{i=1}^{m} f_i(x)R\right) t_1 \cdots t_ms' \subseteq s''p(x)R \subseteq \sum_{i=1}^{m} f_i(x)R \subseteq B$$

for some nonzero $s', s'' \in S$. Hence, B is S-principal as desired.

Remark 2.8. (1) (cf. [9, Theorem 2.8]). Based on [4, Theorem 3.1], one may suspect that the Ore extension $R[x;\sigma,\delta]$ over an S-PRIR R is an S-PRIR. However, $\mathbb{Z}[x;\sigma,\delta] = \mathbb{Z}[x]$ eliminates the possibility of the suspicion, where σ is the identity map of $\mathbb{Z}$, δ is the zero map, and $S = \{1\}$.

(2) When $S = \{1\}$, [18, Example 1.25] confirms that there exists an S-PLID but not an S-PRID.

(3) If D is a commutative domain and $S = D \setminus \{0\}$, then the polynomial ring $D[x]$ is an S-PID. Using a very similar argument to that in Example 2.7, one can easily show that if A is a nontrivial ideal of $D[x]$, then $As \subseteq p(x)D[x] \subseteq A$, where $p(x)$ is the nonzero smallest degree polynomial in A and s is the leading coefficient of $p(x)$. Accordingly, one can find a commutative S-PID, but not a PID. In fact, $\mathbb{Z}[x]$ is a commutative S-PID, but not a PID, where $S = \mathbb{Z} \setminus \{0\}$.

To provide various examples of S-PRIRs, we observe several extensions of S-PRIRs. First, we examine the following finite direct sums.

Proposition 2.9. (cf. [4, Proposition 2.8 and Example 2.9]). *For each $i \in \{1, \ldots, n\}$, let S_i be a multiplicative subset of a ring R_i, and M_i be an R_i-module. Then every R_i-submodule of M_i is S_i-principal for each i if and only if every $\left(\bigoplus_{i=1}^n R_i\right)$-submodule of $\bigoplus_{i=1}^n M_i$ is $\left(\bigoplus_{i=1}^n S_i\right)$-principal.*

Proof. ($\Rightarrow$) Let $\mathcal{R} = \bigoplus_{i=1}^n R_i$, $\mathcal{M} = \bigoplus_{i=1}^n M_i$, and $\mathcal{P}$ be an $\mathcal{R}$-submodule of $\mathcal{M}$. Then, we can write $\mathcal{P} = \bigoplus_{i=1}^n P_i$, where each P_i is an R_i-submodule of M_i. By the hypothesis, there exist $p_i \in P_i$ and $s_i \in S_i$ such that $P_i s_i \subseteq p_i R_i \subseteq P_i$ for each i. This implies that $\mathcal{P}(s_1, \ldots, s_n) \subseteq (p_1, \ldots, p_n)\mathcal{R} \subseteq \mathcal{P}$, showing that $\mathcal{P}$ is $\left(\bigoplus_{i=1}^n S_i\right)$-principal.

($\Leftarrow$) Let P_i be an R_i-submodule of M_i for each i, and $\mathcal{P} = \bigoplus_{i=1}^n P_i$ be the $\left(\bigoplus_{i=1}^n R_i\right)$-submodule of $\bigoplus_{i=1}^n M_i$. Since $\mathcal{P}$ is $\left(\bigoplus_{i=1}^n S_i\right)$-principal, there exist $p_i \in P_i$ and $s_i \in S_i$ such that $\mathcal{P}(s_1, \ldots, s_n) \subseteq (p_1, \ldots, p_n) \bigoplus_{i=1}^n R_i \subseteq \mathcal{P}$. This guarantees that $P_i s_i \subseteq p_i R_i \subseteq P_i$ for each i, completing the proof. □

Corollary 2.10. *For each $i \in \{1, \ldots, n\}$, let S_i be a multiplicative subset of a ring R_i. Then each R_i is an S_i-PRIR if and only if the finite direct sum $\bigoplus_{i=1}^n R_i$ is a $\left(\bigoplus_{i=1}^n S_i\right)$-PRIR.*

Proposition 2.11. *Let S be a multiplicative subset of a ring R and M_i be an R-module for each $i \in \{1, \ldots, n\}$. Then every R-submodule of M_i is S-principal for each i if and only if each R-submodule of the R-module $\bigoplus_{i=1}^n M_i$ is S-principal.*

Proof. ($\Rightarrow$) Let $\mathcal{P}$ be an R-submodule of $\mathcal{M} = \bigoplus_{i=1}^n M_i$. For each i, we let $P_i = \mathcal{P} \cap \mathcal{M}_i$, where $\mathcal{M}_i$ is the subset of $\mathcal{M}$ containing only elements in which all entries are 0 except the i-th entry. Then, we can identify each P_i as an R-submodule of M_i. Since every R-submodule of M_i is S-principal, there exist $p_i \in P_i$ and $s_i \in S$ such that $P_i s_i \subseteq p_i R \subseteq P_i$. Thus, we obtain that $\mathcal{P} s_1 \cdots s_n \subseteq (p_1, \ldots, p_n)R \subseteq \mathcal{P}$. Hence, $\mathcal{P}$ is S-principal.

($\Leftarrow$) Let $\mathcal{M} = \bigoplus_{i=1}^n M_i$, and P_i be an R-submodule of M_i for each i. Then $\mathcal{P} = \bigoplus_{i=1}^n P_i$ is an R-submodule of $\mathcal{M}$. Since each R-submodule of $\mathcal{M}$ is S-principal, there exist $p_i \in P_i$ and $s \in S$ such that $\mathcal{P}s \subseteq (p_1, \ldots, p_n)R \subseteq \mathcal{P}$. Consequently, $P_i s \subseteq p_i R \subseteq P_i$ for each i, completing the proof. □

Corollary 2.12. *Let S be a multiplicative subset of a ring R. Then R is an S-PRIR if and only if each R-submodule of the right R-module $\bigoplus_{i=1}^n R$ is S-principal.*

As a corollary of the previous two propositions, we yield the following.

Corollary 2.13. *Let S be a multiplicative subset and e be a central idempotent of a ring R. If the multiplicative subsets eS and $(1-e)S$ do not contain the zero element, then the following statements are equivalent:*

(1) R is an S-PRIR.

(2) *Every right R-submodule of each eR and $(1-e)R$ is S-principal.*
(3) *The ring eR is an eS-PRIR.*
(4) *The ring $(1-e)R$ is a $(1-e)S$-PRIR.*
(5) *The corner ring eRe is an eS-PRIR.*

For a multiplicative subset S of a ring R and a fixed positive integer n, we denote $\mathbb{H}_n(R) = \{[r_{i,j}] \in \mathbb{T}_n(R) \mid r_{1,1} = \cdots = r_{n,n}\}$ and $\mathbb{V}_n(R) = \{[r_{i,j}] \in \mathbb{H}_n(R) \mid r_{u,v} = r_{(u+1),(v+1)}$ for $u = 1, \ldots, n-2$ and $v = 2, \ldots, n-1\}$. Then $\mathbb{H}_n(R)$ is a subring of $\mathbb{T}_n(R)$ and $\mathbb{V}_n(R)$ is a subring of $\mathbb{H}_n(R)$. For the identity matrix I_n, $\mathbb{I}_n(S) = \{sI_n \mid s \in S\}$ means the multiplicative set of all $n \times n$ scalar matrices with entries in S, and $E_{i,j}$ means the matrix unit, which is the matrix with (i, j)-entry 1 and elsewhere 0.

Theorem 2.14. (cf. [4, Proposition 2.16, 2.17 and 2.19] and [23, Proposition 3.4.10]). *Let S be a multiplicative subset of R and fix $n \in \mathbb{N}$. Then the following statements are equivalent:*

(1) *R is an S-PRIR.*
(2) *$\mathbb{M}_n(R)$ is an $\mathbb{I}_n(S)$-PRIR.*
(3) *$\mathbb{T}_n(R)$ is an $\mathbb{I}_n(S)$-PRIR.*
(4) *$\mathbb{H}_n(R)$ is an $\mathbb{I}_n(S)$-PRIR.*
(5) *$\mathbb{V}_n(R)$ is an $\mathbb{I}_n(S)$-PRIR.*

Proof. (1)⇒(2) This proof is nearly identical to the proof of [23, Proposition 3.4.10].

(1)⇐(2) Let A be a right ideal of R and consider the right ideal $\mathbb{A} = \sum_{i=1}^{n} AE_{1,i}$ of $\mathbb{M}_n(R)$. Because $\mathbb{M}_n(R)$ is an S-PRIR, there exist $\sum_{i=1}^{n} a_{1,i}E_{1,i} \in \mathbb{A}$ and $sI_n \in \mathbb{I}_n(S)$ such that $\mathbb{A}sI_n \subseteq (\sum_{i=1}^{n} a_{1,i}E_{1,i})\,\mathbb{M}_n(R) \subseteq \mathbb{A}$. Thus, we have $As \subseteq a_{1,n}R \subseteq A$, confirming that A is S-principal.

(1)⇒(3) Let $\mathbb{A}$ be a right ideal of $\mathbb{T}_n(R)$. By the same argument in [18, Proposition 1.17(2)], $\mathbb{A} = C_1 \oplus \cdots \oplus C_n$, where $C_j = \left(\sum_{i=1}^{j} RE_{i,j}\right) \cap \mathbb{A}$ is an R-submodule of a right R-module $\bigoplus_{i=1}^{j} R$ for each $j \in \{1, \ldots, n\}$. By Corollary 2.12, each C_j is S-principal. Therefore, there exist $\left(c_{1,j}, \ldots, c_{j,j}\right) \in C_j$ and $s_j \in S$ such that $C_j s_j \subseteq \left(c_{1,j}, \ldots, c_{j,j}\right)R \subseteq C_j$ for each j. Now let s be the product of all distinct elements in $\{s_1, \ldots, s_n\}$. Then for each C_j, we have $C_j s \subseteq C_j s_j \cdots s_n \subseteq \left(c_{1,j}, \ldots, c_{j,j}\right)R \cdots s_n \subseteq \left(c_{1,j}, \ldots, c_{j,j}\right)R \subseteq C_j$. This implies that

$$\mathbb{A}sI_n \subseteq \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ 0 & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{n,n} \end{bmatrix} \mathbb{T}_n(R) \subseteq \mathbb{A}.$$

Thus $\mathbb{A}$ is $\mathbb{I}_n(S)$-principal.

(1)⇐(3) Let A be a right ideal of R and $\mathbb{A} = AE_{1,n}$. Since the right ideal $\mathbb{A}$ of $\mathbb{T}_n(R)$ is S-principal, there exist $aE_{1,n} \in \mathbb{A}$ and $sI_n \in \mathbb{I}_n(S)$ such that $\mathbb{A}sI_n \subseteq aE_{1,n}\mathbb{T}_n(R) \subseteq \mathbb{A}$. This guarantees that $As \subseteq aR \subseteq A$ and thus A is S-principal.

Each proof of (1)⇔(4) and (1)⇔(5) is nearly identical to the proof of (1)⇔(3). □

By Theorem 2.14, we can possess the followings.

Corollary 2.15. ([23, Proposition 3.4.10] and [17, Theorem 17.24]). *Fix $n \in \mathbb{N}$. If R is either a PRIR or a PRID, then $\mathbb{M}_n(R)$ is a PRIR.*

Proof. It follows from Theorem 2.14, when $S = \{1\}$. □

For a fixed positive integer n, let $\mathbb{D}_n(R)$ be the set of $n \times n$ diagonal matrices over a ring R. Then $\mathbb{D}_n(R)$ is a Γ-semiring with $\Gamma = \mathbb{D}_n(R)$ under the ternary operation defined by $ABC = AB^TC$ where B^T is the transpose of B, for all $A, B, C \in \mathbb{D}_n(R)$ (see [14, Example 2]).

Corollary 2.16. *Fix $n \in \mathbb{N}$. Then R is an S-PRIR if and only if the Γ-semiring $\mathbb{D}_n(R)$ is an $\mathbb{I}_n(S)$-PRIR.*

According to [20], the map $\rho : \mathbb{V}_n(R) \to R[x]/\langle x^n\rangle$ defined by $\rho(r_{1,1}I_n + r_{1,2}V + \cdots + r_{1,n}V^n) = \sum_{i=1}^{n} r_{1,i} + \langle x^n\rangle$ is a ring isomorphism, where $V = \sum_{i=1}^{n} E_{i,i+1}$ and $\langle x^n\rangle$ is the ideal of $R[x]$ generated by x^n. Note that, for a multiplicative subset S of a ring R, $\rho\left(\mathbb{I}_n(S)\right) = S + \langle x^n\rangle$ is a multiplicative subset of $R[x]/\langle x^n\rangle$.

Corollary 2.17. (cf. [4, Corollary 2.18]). *Fix $n \geq 1$. Then R is an S-PRIR if and only if $R[x]/\langle x^n\rangle$ is an $(S + \langle x^n\rangle)$-PRIR.*

As stated in [4], for a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $\mathcal{T}(R, M) = R \oplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$$

for all $(r_1, m_1), (r_2, m_2) \in \mathcal{T}(R, M)$. It is easy to see that $\mathcal{T}(R, M)$ is (ring) isomorphic to the ring of matrices of the form $\left(\begin{smallmatrix} r & m \\ 0 & r\end{smallmatrix}\right)$, where $r \in R$ and $m \in M$. In particular, $\mathcal{T}(R, R) = \mathbb{H}_2(R) = \mathbb{V}_2(R)$ and so $\mathcal{T}(R, R)$ is an $(S, 0)$-PRIR. However, in general, $\mathcal{T}(R, M)$ need not be an $(S, 0)$-PRIR, even though R is an S-PRIR (see [4, Example 2.22(1)]).

Proposition 2.18. (cf. [4, Proposition 2.21]). *Let S be a multiplicative subset of a ring R and M an (R, R)-bimodule. Then the following statements are equivalent:*

(1) R is an S-PRIR, and every right R-submodule of M is S-principal.
(2) $\mathcal{T}(R, M)$ is an $(S, 0)$-PRIR.

Proof. (1)⇒(2) We first identify $\mathcal{T}(R, M) = \left\{\left[\begin{smallmatrix} r & m \\ 0 & r\end{smallmatrix}\right] \;\middle|\; r \in R, m \in M\right\}$ and $(S, 0) = \left\{sI_2 \;\middle|\; s \in S\right\}$. Let $\mathbb{A}$ be a right ideal of $\mathcal{T}(R, M)$. By the same argument in [18, Proposition 1.17(2)], $\mathbb{A} = C_1 \oplus C_2$, where C_1 is a right ideal of R, and C_2 is a right R-submodule of $M \oplus R$. By Proposition 2.11, C_j is S-principal for each $j \in \{1, 2\}$. Therefore there exist $\left(c_{1,j}, \ldots, c_{j,j}\right) \in C_j$ and $s_j \in S$ such that $C_js_j \subseteq \left(c_{1,j}, \ldots, c_{j,j}\right)R \subseteq C_j$ for each j. Now let $s = s_1s_2$. Then we have $C_js \subseteq \left(c_{1,j}, \ldots, c_{j,j}\right)R \subseteq C_j$ for each C_j, and $c_{1,1} = c_{2,2}$. This implies that

$$\mathbb{A}sI_2 \subseteq \left[\begin{smallmatrix} c_{1,1} & c_{1,2} \\ 0 & c_{1,1}\end{smallmatrix}\right]\mathcal{T}(R, M) \subseteq \mathbb{A}.$$

Thus $\mathbb{A}$ is $(S, 0)$-principal.

(1)⇐(2) We also identify $\mathcal{T}(R, M) = \left\{\left[\begin{smallmatrix} r & m \\ 0 & r\end{smallmatrix}\right] \;\middle|\; r \in R, m \in M\right\}$ and $(S, 0) = \left\{sI_2 \;\middle|\; s \in S\right\}$. Let A be a right ideal of R, and consider the right ideal $\mathbb{A} = \left\{\left[\begin{smallmatrix} a & m \\ 0 & a\end{smallmatrix}\right] \;\middle|\; a \in A, m \in M\right\}$ of $\mathcal{T}(R, M)$. Then there exist $\left[\begin{smallmatrix} p & q \\ 0 & p\end{smallmatrix}\right] \in \mathbb{A}$ and $sI_2 \in (S, 0)$ such that $\mathbb{A}sI_2 \subseteq \left[\begin{smallmatrix} p & q \\ 0 & p\end{smallmatrix}\right]\mathcal{T}(R, M) \subseteq \mathbb{A}$. This shows that $As \subseteq pR \subseteq A$. Lastly, let N be a right R-submodule of M, and consider the right ideal $\mathbb{B} = \left\{\left[\begin{smallmatrix} 0 & n \\ 0 & 0\end{smallmatrix}\right] \;\middle|\; n \in N\right\}$ of $\mathcal{T}(R, M)$. Then there exist $\left[\begin{smallmatrix} 0 & q' \\ 0 & 0\end{smallmatrix}\right] \in \mathbb{B}$ and $s'I_2 \in (S, 0)$ such that $\mathbb{A}s'I_2 \subseteq \left[\begin{smallmatrix} 0 & q' \\ 0 & 0\end{smallmatrix}\right]\mathcal{T}(R, M) \subseteq \mathbb{B}$. This shows that $Ns \subseteq q'R \subseteq N$, completing the proof. □

On the underlying set $\mathbb{Z} \times R$, the *Dorroh extension* of R, denoted by $\mathbb{Z} * R$, is the ring with the usual addition and the following multiplication:

$$(z_1, r_1) * (z_2, r_2) = (z_1 z_2, z_1 r_2 + z_2 r_1 + r_1 r_2)$$

for all $(z_1, r_1), (z_2, r_2) \in \mathbb{Z} \times R$. This ring is associative with unity $(1, 0)$ (see [7]). It was proven by [7, Corollary 2.3] that $\mathbb{Z} * R$ is (ring) isomorphic to $\mathbb{Z} \times R$, even though there exists a right ideal of $\mathbb{Z} * R$ that cannot be written of the form $I * J$, where I is an ideal of $\mathbb{Z}$ and J is a right ideal of R.

Proposition 2.19. *Let S be a multiplicative subset of a ring R. Then R is an S-PRIR if and only if the Dorroh extension $\mathbb{Z} * R$ is a $(0, S)$-PRIR.*

Proof. $(\Rightarrow)$ We first identify $\mathcal{R} = \mathbb{Z} * R = \mathbb{Z} \times R$. Let $\mathcal{A}$ be a right ideal of $\mathcal{R}$. Then we can write $\mathcal{A} = B \times C$ for some ideal B of $\mathbb{Z}$ and right ideal C of R. Since R is an S-PRIR,

$$\mathcal{A}(0, s) = (B \times C)(0, s) \subseteq (0, p)\mathcal{R} \subseteq \mathcal{A}$$

for some $p \in C$ and $s \in S$, which forces that $\mathcal{A}$ is $(0, S)$-principal.
$(\Leftarrow)$ We also identify $\mathcal{R} = \mathbb{Z} * R = \mathbb{Z} \times R$. Let A be a right ideal of R and consider the right ideal $(0, A)$ of $\mathcal{R}$. By the hypothesis, there exist $(0, a) \in (0, A)$ and $(0, s) \in (0, S)$ such that $(0, A)(0, s) \subseteq (0, a)\mathcal{R} \subseteq (0, A)$. This implies that $As \subseteq aR \subseteq A$ as desired. □

Let T be a right denominator set of a ring R. Then a right ring of fractions with respect to T, RT^{-1} exists (see [17, Theorem 10.6]). We close this section with the Ore localizations of S-PRIRs.

Theorem 2.20. (cf. [19, Theorem 3.4]). *Let S be a multiplicative subset of a ring R and T be a right denominator set such that $sT = Ts$ for every $s \in S$. If R is an S-PRIR, then RT^{-1} is an S-PRIR.*

Proof. We first identify $S = \{s/1 \mid s \in S\}$. Note that S is a multiplicative subset of RT^{-1}. Let $\mathcal{A}$ be a right ideal of RT^{-1}. Then, $A = \mathcal{A} \cap R$ is a right ideal of R and $\mathcal{A} = A(RT^{-1}) = AT^{-1}$ by [23, Proposition 2.1.16(3)]. Because A is S-principal, there exist $p \in A$ and $s \in S$ such that $As \subseteq pR \subseteq A$. This implies that $\mathcal{A}s = AT^{-1}s = AsT^{-1} \subseteq pRT^{-1} \subseteq \mathcal{A}$. Thus, $\mathcal{A}$ is S-principal. □

According to [17, (10.17)], a ring R is right Ore if and only if the classical right quotient ring of R, $Q^r_{cl}(R)$ exists.

Corollary 2.21. *Let R be a right Ore ring, S a multiplicative subset of R, and let T be the set of all regular elements in R. If R is an S-PRIR, and $sT = Ts$ for every $s \in S$, then the classical right ring of quotients of R, $Q^r_{cl}(R)$ is an S-PRIR.*

Proof. Since R is right Ore, T is a right denominator set in R. Now, Theorem 2.20 will work. □

Corollary 2.22. *Let T be a right denominator set of a ring R. If R is a PRIR, then RT^{-1} is a PRIR.*

There exists a ring R that is not a PRIR, but RT^{-1} is a PRIR. For instance, let $R = \mathbb{Z}[x]$ be the polynomial ring and $T = R \setminus \{0\}$. Clearly, T is a (right) denominator set of R, and the commutative domain R has a nonprincipal ideal. However, the localization of R by T, $R_T = RT^{-1}$ is a field (see [12, Theorem 3.4.3(3)]). Notice that R is an S-PRIR, where $S = \mathbb{Z} \setminus \{0\}$. Motivated by this, we provide a sufficient condition for the ring RT^{-1} to be a PRIR.

Theorem 2.23. (cf. [19, Theorem 3.6]). *Let T be a right denominator set and S a multiplicative subset of a ring R. If R is an S-PRIR, and $S \subseteq T$, then RT^{-1} is a PRIR.*

Proof. We first identify $S = \{s/1 \mid s \in S\}$. Let $\mathcal{A}$ be a right ideal of RT^{-1}, and consider the right ideal $A = \mathcal{A} \cap R$ of R. Because A is S-principal, there exist $p \in A$ and $s \in S$ such that $As \subseteq pR \subseteq A$. Now, let $b \in \mathcal{A}$. Then, since $\mathcal{A} = A(RT^{-1}) = AT^{-1}$ by [23, Proposition 2.1.16(3)], we can write $b = a/t$ for some $a \in A$ and $t \in T$. Therefore, we obtain

$$b = \frac{a}{t} = \frac{as}{ts} = \frac{pr}{ts} = p\frac{r}{ts} \in pRT^{-1},$$

for some $r \in R$. Thus, $\mathcal{A} \subseteq pRT^{-1} \subseteq \mathcal{A}$ which shows that $\mathcal{A}$ is principal. Hence, RT^{-1} is a PRIR. □

The condition "$S \subseteq T$" is not superfluous as depicted by the following explicit example.

Example 2.24. Let $R = \mathbb{Z}[x, y]$, $T = \mathbb{Z} \setminus \{0\}$ and $S = R \setminus \{0\}$. Clearly, R is a commutative S-PID and T is a (right) denominator set of R. Note that $RT^{-1} = R_T = \mathbb{Q}[x, y]$. However, the ideal $x\mathbb{Q}[x, y] + y\mathbb{Q}[x, y]$ of $\mathbb{Q}[x, y]$ is not principal.

Corollary 2.25. (cf. [19, Corollary 3.9]). *Let R be a right Ore ring, and S be the set of all regular elements in R. If R is an S-PRIR, then $Q^r_{cl}(R)$ is a PRIR.*

Proof. By identifying $S = T$, Theorem 2.23 applies. □

Goldie's theorem states that R is a semiprime right Goldie ring if and only if there exists the semisimple Artinian classical right ring of quotients of R (see [17]).

Corollary 2.26. *Let R be a semiprime right Goldie ring and S be the set of all regular elements in R. If R is an S-PRIR, then $Q^r_{cl}(R)$ is a PRIR.*

3. Properties of S-principal right ideal rings

In this section, we study various properties of S-PRIRs. First, observe that the S-principal condition cannot be passed between subrings and overrings. In Section 2, we already saw that $\mathbb{Z}$ and $\mathbb{Z}[x]T^{-1}$ are S-PIRs but $\mathbb{Z}[x]$ is not an S-PIR, where $S = \{1\}$ and $T = \mathbb{Z}[x]\setminus\{0\}$. Thus, our first goal is to prove the S-variant of the Eakin-Nagata-Eisenbud theorem for S-PRIRs (see [19, Theorem 2.9] and [17, Theorem 3.98]). The proof of the next lemma is very similar to the proof of [19, Proposition 2.2], but we insert it for the sake of completeness.

Lemma 3.1. (cf. [19, Proposition 2.2]). *Let R be a ring, S a multiplicative subset of R, and M an R-module. If R is an S-PRIR and M is S-principal, then every R-submodule of M is S-principal.*

Proof. Suppose to the contrary that there is a non-S-principal R-submodule of M. Let $\mathcal{F}$ be the set of non-S-principal submodules of M. Then $\mathcal{F}$ is a nonempty partially ordered set under inclusion. Let $\{L_\alpha\}_{\alpha\in\Lambda}$ be a chain in $\mathcal{F}$ and let $L = \bigcup_{\alpha\in\Lambda} L_\alpha$. We claim that L is not S-principal: Suppose that L is S-principal. Then there exist an element $s \in S$ and a principal submodule gR of L such that $Ls \subseteq gR$. Since gR is principal, $gR \subseteq L_\beta$ for some $\beta \in \Lambda$; so $L_\beta s \subseteq gR \subseteq L_\beta$. Thus L_β is S-principal, a contradiction, proving the claim. Clearly, L is an upper bound of the chain $\{L_\alpha\}_{\alpha\in\Lambda}$. Thus by Zorn's lemma, we can find a maximal element in $\mathcal{F}$, say N.

Let $P = [N : M] = \{r \in R \,|\, Mr \subseteq N\}$. Then by [19, Lemma 2.1], P is a completely prime ideal of R which is disjoint from S. Since M is S-principal, there exist an element $w \in S$ and a principal submodule fR of M such that $Mw \subseteq fR$; so we have $P = [N : M] \subseteq [N : fR] \subseteq [N : Mw] = (P : w)$, where $(P : w) := \{r \in R \,|\, wr \in P\}$. Since $w \notin P$ and P is completely prime, $(P : w) = P$; so we have

$$P = [N : M] = [N : fR] = [N : Mw] = (P : w).$$

Since P is a proper ideal of R, $f \notin N$. By the maximality of N, $N + fR$ is S-principal; so we can find elements $s_1 \in S$ and $n_1 + fr_1 \in N + fR$ such that $(N + fR)s_1 \subseteq (n_1 + fr_1)R \subseteq N + fR$. Since R is an S-PRIR, there exist $s_2 \in S$ and $t \in R$ such that $[N : fr_1]s_2 \subseteq tR \subseteq [N : fr_1]$.

Now, let $n \in N$ be arbitrary. Then we have $ns_1 = (n_1 + fr_1)x$ for some $x \in R$. Note that $fr_1x = ns_1 - n_1x \in N$; so $x \in [N : fr_1]$. Therefore $xs_2 = ty$ for some $y \in R$. Hence we have

$$ns_1s_2 = (n_1 + fr_1)xs_2 = (n_1 + fr_1)ty \in (n_1t + fr_1t)R.$$

Since n was arbitrarily chosen in N and $fr_1t \in N$, we obtain

$$Ns_1s_2 \subseteq (n_1t + fr_1t)R \subseteq N,$$

which shows that N is S-principal, a contradiction to $N \in \mathcal{F}$. Thus every R-submodule of M is S-principal. □

In the followings, a ring E is called a *ring extension* of a ring R if $R \subseteq E$ and $1_E = 1_R$.

Theorem 3.2. (cf. [19, Theorem 2.3]). *Let S be a multiplicative subset of a ring R, E a ring extension of R, and let M be a right E-module. If every R-submodule of M is S-principal, then every E-submodule of M is S-principal. In particular, if R is an S-PRIR and M is S-principal as a right R-module, then every E-submodule of M is S-principal.*

Proof. Let N be an E-submodule of M. Clearly, N is an R-submodule of M, and so N is S-principal by the hypothesis. Thus, we obtain

$$Ns \subseteq nR \subseteq nE \subseteq N$$

for some $s \in S$ and $n \in N$. This confirms that N is S-principal as a right E-module. The last statement follows from Lemma 3.1. □

The following corollary leads to the conclusion that the one-sided direction of the S-variant of the Eakin-Nagata-Eisenbud theorem for S-PRIRs holds.

Corollary 3.3. (cf. [19, Corollary 2.4]). *Let E be a ring extension of a ring R, and S be a multiplicative subset of R. If R is an S-PRIR and E is S-principal as a right R-module, then E is an S-PRIR.*

Proof. It follows from the last statement of Theorem 3.2, by replacing M with E. □

We next consider the other side direction of the S-variant of the Eakin-Nagata-Eisenbud theorem for S-PRIRs.

Theorem 3.4. (cf. [19, Theorem 2.6]). *Let S be a multiplicative subset of a ring R and E be a ring extension of R such that $sE = Es \subseteq R$ for some $s \in S$. If E is an S-PRIR and M is an S-principal right E-module, then every R-submodule of M is S-principal.*

Proof. By Lemma 3.1, we first note that every E-submodule of M is S-principal. Let N be an R-submodule of M, and consider the E-submodule of M, $NE = \{\sum_{i=1}^{\text{finite}} n_ie_i \mid n_i \in N, e_i \in E\}$. Since NE is S-principal as an E-submodule, there exist $s_1 \in S$ and $\sum_{j=1}^{p} n'_je'_j \in NE$ such that $NEs_1 \subseteq \left(\sum_{j=1}^{p} n'_je'_j\right)E \subseteq NE$. Therefore, we have

$$Ns_1s^2 \subseteq NEs_1s^2 \subseteq \left(\sum_{j=1}^{p} n'_j e'_j\right) Es^2 = \left(\sum_{j=1}^{p} n'_j e'_j\right) sEs \subseteq \left(\sum_{j=1}^{p} n'_j r'_j\right) R \subseteq N,$$

where $r'_j \in R$ with $e'_j s = r'_j$ for each j. Thus, we reach the conclusion. □

Corollary 3.5. (cf. [19, Corollary 2.7]). *Let S be a multiplicative subset of a ring R and E be a ring extension of R such that $sE = Es \subseteq R$ for some $s \in S$. If E is an S-PRIR, then R is an S-PRIR.*

Proof. By Theorem 3.4, every R-submodule of E is S-principal. If A is a right ideal of R, then A is S-principal because A is an R-submodule of E. Thus, R is an S-PRIR. □

By Corollary 3.5, we can find a condition for which the converse of Theorem 2.20 holds.

Corollary 3.6. *Let T be a right denominator set of a ring R and S be a multiplicative subset of R with $sT = Ts$ for every $s \in S$. If RT^{-1} is an S-PRIR and $RT^{-1}(s/1) = (s/1)RT^{-1} \subseteq R/\{1\}$ for some $s/1 \in S/\{1\}$, then R is an S-PRIR.*

If we reduce the condition "$sE = Es \subseteq R$" to "$Es \subseteq R$" in Theorem 3.4 and Corollary 3.5, then we immediately obtain that every R-submodule is S-finite by [19, Theorem 2.6]. However, we failed to show that every S-finite R-module is S-principal. Thus, we leave the following as an open question:

Question 3.7. In Theorem 3.4 and Corollary 3.5, can we reduce the condition "$sE = Es \subseteq R$" to "$Es \subseteq R$"?

Even though we failed to reduce the condition, we provide an interesting example to demonstrate that Theorem 3.4 and Corollary 3.5 are still valid.

Example 3.8. Let $\mathcal{E} = \mathbb{Z}_2\langle a, b, c, s\rangle$ be the free algebra with unity and noncommuting indeterminates a, b, c, s over $\mathbb{Z}_2$. Set $E = \mathcal{E}/\mathcal{I}$, where $\mathcal{I}$ is the ideal of $\mathcal{E}$ generated by the following relations:

$$a^2 = ab = ac = ca = a,\ as = sa = bs = sb,\ ba = b^2 = bc = cb = b,\ c^2 = c,\ cs = sc = s^2 = s.$$

We identify $e = e + \mathcal{I}$ for simplicity. Applying Bergman's diamond lemma [5], we can write each element $e \in E$ uniquely in the following reduced form:

$$e = \alpha + \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 s + \alpha_5 as,$$

where $\alpha, \alpha_1, \ldots, \alpha_5 \in \mathbb{Z}_2$. Clearly, E is noncommutative, and $S = \{s\}$ is a multiplicative subset of E. We first claim that E is an S-PRIR. Let A be a nonprincipal right ideal of E. If $s \in A$, then $As \subseteq sR \subseteq A$ because of $s \in Z(E)$. So we may assume $s \notin A$. From

$$es = (\alpha + \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 s + \alpha_5 as)\, s = (\alpha + \alpha_3 c + \alpha_4 s)\, s + (\alpha_1 a + \alpha_2 b + \alpha_5 as)\, s$$

for every $e \in E$, we obtain that if $As \neq \{0\}$, then A contains either as or $(1 + a)s$. Thus, A must satisfy one of the following: $As \subseteq \{0\} \subseteq A$ or $As \subseteq asE \subseteq A$ or $As \subseteq (1 + a)sE \subseteq A$. Hence, A must be S-principal, yielding that the first claim is true.

Now, let R be the set of all elements of the form $r = \alpha + \alpha_1 a + \alpha_2 b + \alpha_4 s + \alpha_5 as$ in E. Then, R is a subring of E with the same unity. Note that $sE = Es \subseteq R$. Thus, R is an S-PRIR by Corollary 3.5.

Remark 3.9. (1) In Example 3.8, the noncommutative rings E and R are both S-PRIRs, but not both PRIRs. If we let $T = \{1\} \subset R \subset E$ in Example 3.8, then $ET^{-1} \cong E$ and $RT^{-1} \cong R$. Thus, we already exemplified the result of Corollary 3.6.

(2) Let $\mathcal{R}' = \mathbb{Z}_2\langle a, b, s\rangle$ be the free algebra with unity and noncommuting indeterminates a, b, s over $\mathbb{Z}_2$. Set $R' = \mathcal{R}'/\mathcal{J}$, where $\mathcal{J}$ is the ideal of $\mathcal{R}'$ generated by the following relations:

$$a^2 = ab = a,\ as = sa = bs = sb,\ ba = b^2 = b,\ s^2 = s.$$

We identify $r' = r' + \mathcal{J}$ for simplicity. Applying Bergman's diamond lemma [5], we can write each element $r' \in R'$ uniquely in the following reduced form:

$$r' = \alpha' + \alpha'_1 a + \alpha'_2 b + \alpha'_4 s + \alpha'_5 as,$$

where $\alpha', \alpha'_1, \dots, \alpha'_5 \in \mathbb{Z}_2$. Now consider the ring R in Example 3.8. One can easily show that R' is (ring) isomorphic to R. Thus, R' is also an S'-PRIR, where $S' = \{s\}$.

(3) As described previously, the ring $E = \mathbb{Z}[x]T^{-1}$ is an S-PIR, but the subring $R = \mathbb{Z}[x] \cong \mathbb{Z}[x]/\{1\}$ of E is not an S-PIR, where $S = \{1\}$ and $T = \mathbb{Z}[x] \setminus \{0\}$. Hence, we cannot drop the condition "$sE = Es \subseteq R$" in Theorem 3.4, Corollary 3.5 and Corollary 3.6.

By combining Corollary 3.3 and Corollary 3.5, we obtain the S-variant of the Eakin-Nagata-Eisenbud theorem for S-PRIRs.

Theorem 3.10. (cf. [19, Corollary 2.9]). *Let S be a multiplicative subset of a ring R and E be a ring extension of R such that $sE = Es \subseteq R$ for some $s \in S$. Then R is an S-PRIR if and only if E is an S-PRIR.*

Next, we prove the S-variant of Cohen's theorem for S-PRIRs. The S-variant of Cohen's theorem for right S-Noetherian rings was proven in [6, Theorem 2.2] and [19, Theorem 2.11]. According to [15], a proper right ideal P of a ring R is *prime* if for any right ideals A, B of R, $AB \subseteq P$ and $AP \subseteq P$ imply that either $A \subseteq P$ or $B \subseteq P$. A completely prime right ideal of a ring is always a prime right ideal (see [25, p.969]).

Lemma 3.11. (cf. [19, Lemma 2.10]). *Let S be a multiplicative subset of a ring R. If P is maximal among non-S-principal right ideals of R, then P is a prime right ideal of R.*

Proof. Obviously, P is a proper right ideal of R. Suppose to the contrary that P is not a prime right ideal. Then, there exist right ideals A and B of R such that $AB \subseteq P$ and $AP \subseteq P$, but $A \nsubseteq P$ and $B \nsubseteq P$. Let $a \in A \setminus P$ and $b \in B \setminus P$. Then, $aRb \subseteq P$ and $aRP \subseteq P$. Because $P + aR$ is S-principal by the maximality of P, we can find $s_1 \in S$ and $c = p_1 + ar_1 \in P + aR$ such that

$$(P + aR)s_1 \subseteq cR \subseteq P + aR.$$

Set $L = \{r \in R \mid cr \in P\}$. Note that L is a right ideal of R containing b and P. Therefore, L is S-principal by the maximality of P, and so there exist $s_2 \in S$ and $\ell \in L$ such that

$$Ls_2 \subseteq \ell R \subseteq L.$$

Now, let $p \in P$. Then, $ps_1 = (p_1 + ar_1)x \in P$ for some $x \in R$, which indicates that $x \in L$. Thus, we have

$$ps_1s_2 = cxs_2 = c\ell y \in P$$

for some $y \in R$. Accordingly, we obtain $Ps_1s_2 \subseteq c\ell R \subseteq P$, which is a contradiction. Hence, P must be a prime right ideal of R. □

Theorem 3.12. (cf. [2, Proposition 16]). *Let S be a multiplicative subset of a ring R. Then R is an S-PRIR if (and only if) every prime right ideal of R is S-principal.*

Proof. Suppose to the contrary that R is not an S-PRIR. Then, it is easy to show that the set $\mathcal{F}$ of all non-S-principal right ideals of R is a nonempty partially ordered set under inclusion. By Zorn's lemma, there is a non-S-principal right ideal P which is maximal in $\mathcal{F}$. By Lemma 3.11, P is a prime right ideal of R, which contradicts to the hypothesis. Thus, R is an S-PRIR. □

Corollary 3.13. ([15, Theorem 1]). *A ring R is a PRIR if (and only if) every prime right ideal of R is principal.*

Proof. It follows from Theorem 3.12, when $S = \{1\}$. □

A ring R is called *right duo* if every right ideal of R is a two-sided ideal. Clearly, if R is a right duo ring, then I is a prime right ideal of R if and only if I is a prime ideal. So we have:

Corollary 3.14. *Let S be a multiplicative subset of a right duo ring R. Then, R is an S-PRIR if (and only if) every prime ideal of R is S-principal.*

In [25, Theorem 8.5], Reyes proved that a ring R is a PRIR if (and only if) all of the Micher-prime right ideals of R are principal. Based on this fact, we ask whether:

Question 3.15. Let S be a multiplicative subset of a ring R. If every Micher-prime right ideal of R is S-principal, is R an S-PRIR?

One can conveniently use Theorem 3.12 when determining whether a given ring is an S-PRIR. We apply Theorem 3.12 in the following simple example.

Example 3.16. (see Theorem 2.14(3)). Let D be any division ring and $S = \{1\}$. Obviously, $\mathbb{T}_2(D)$ has only the following five right ideals:

$$\{0\}, \mathbb{T}_2(D), \mathcal{D}_1 = \left[\begin{smallmatrix} D & D \\ 0 & 0 \end{smallmatrix}\right], \mathcal{D}_2 = \left[\begin{smallmatrix} 0 & D \\ 0 & D \end{smallmatrix}\right], \mathcal{D}_3 = \left\{\left[\begin{smallmatrix} 0 & d \\ 0 & d \end{smallmatrix}\right] \in \mathbb{T}_2(D) \;\middle|\; d \in D\right\}.$$

Note that $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are all prime right ideals of $\mathbb{T}_2(D)$, and each of them is $(\mathbb{I}_2(S)$-)principal. Thus, $\mathbb{T}_2(D)$ is an $(\mathbb{I}_2(S)$-)PRIR.

Recall that a ring R is said to be *right hereditary* if every right ideal of R is projective as a right R-module (see [17, p.42]). According to [26, Example 2.8.12], every PRID is right hereditary. We partially generalize this fact to S-PRIDs.

Proposition 3.17. *Let S be a multiplicative subset of the center $Z(R)$ of a ring R. If R is an S-PRID and A is a right ideal of R disjoint from S, then A is projective.*

Proof. First, assume that $A = pR$ is principal. Then, the condition that R is a domain guarantees that R is (R-module) isomorphic to pR. Therefore, A is free, implying it is projective by [26, Remark 2.8.2]. So we may assume that A is not principal. Because R is an S-PRIR, there exist $s \in S$ and $q \in A$ such that $As \subseteq qR \subseteq A$. To show that A is projective, consider an R-module homomorphism $f : A \to M$ and an R-epimorphism $g : N \to M$, where N and M are right R-modules. Define maps $h' : A \to qR$ by $h'(a) = as$ for every $a \in A$, and $f' : qR \to M$ by $f'(qr) = f(a)$ for every $qr \in qR$ with $qr = h'(as)$. From the three facts that $s \in Z(R)$, $q \in A$ and R is a domain, each of h' and f' is an R-homomorphism. Note that $f = f' \circ h'$. Since qR is projective, there exists an R-homomorphism $h'' : qR \to N$ such that $f' = g \circ h''$. Thus, the R-homomorphism $h = h'' \circ h'$ satisfies $f = g \circ h$ as needed. □

Corollary 3.18. ([26, Example 2.8.12]). *Every PRID is right hereditary.*

Remark 3.19. (1) As stated in [17, Definition 2.28], a ring R is said to be *right semihereditary* if every finitely generated right ideal of R is projective as a right R-module. A commutative hereditary ring is called a *Dedekind domain* and a commutative semihereditary domain is called a *Prüfer domain*. Obviously, Dedekin domains are Prüfer domains. Based on Proposition 3.17, one may suspect that every commutative S-PID is a Dedekind domain or a Prüfer domain. However, [24, Theorem 7.7] eliminates the possibility of the suspicion.

(2) According to [17, Corollary 2.27], if R is a PRID, then any submodule of a free right R-module is free. Since every free module is projective, every submodule of a free module over a PRID is projective. However, when R is an S-PRID, an R-submodule of a free R-module may not be free nor projective by [24, Theorem 7.7] and [17, Corollary 2.31].

One may suspect that if R is an S-PRID, then a free right R-module is injective. However, [17, Example 3.10A] eliminates the possibility of the suspicion. Next, we find a condition under which a free module over an S-PRID is injective. To do this, we recall the notions of divisible modules and torsion-free modules. Let M be a right module over a ring R. For $m \in M$ and $x \in R$, we call that m is *divisible by* x if $m \in Mx$. Furthermore, M is a *divisible module* if for any $m \in M$ and $x \in R$ such that $Ann_r(x) \subseteq Ann(m)$, m is divisible by x (see [17, Definition 3.16]), where $Ann_r(x) = \{y \in R \mid xy = 0\}$ and $Ann(m) = \{z \in R \mid mz = 0\}$. According to [17, Proposition 3.17], a right R-module M is divisible if and only if for any $r \in R$, any R-homomorphism $f : rR \to M$ extends to an R-homomorphism from R_R to M. We say that M is *torsion-free* if the set $T(M) = \{m \in M \mid mr = 0 \text{ for some regular element } r \in R\}$ is zero (see [17, Exercise 10.19]).

Proposition 3.20. *Let S be a multiplicative subset of the center $Z(R)$ of a ring R, and M be a torsion-free module over R. If R is an S-PRID, then M is injective if and only if M is divisible.*

Proof. ($\Rightarrow$) It follows from [17, Corollary 3.17′].

($\Leftarrow$) To apply Baer's criterion, let A be a right ideal of R and $f : A \to M$ be an R-homomorphism. We need to show that f can be extended to an R-homomorphism from R to M. If $A = pR$, then since M is divisible, f extends to an R-homomorphism from R to M. So we may assume that A is not principal. Because R is an S-PRIR, there exist $s \in S$ and $q \in A$ such that $As \subseteq qR \subseteq A$.

If $s \in A$, then we can identify $q = s$. Let $b \in A$. Then, $Ann_r(b) = \{0\} \subseteq Ann(f(b))$. Therefore, $f(b) \in Mb$ and so $f(b) = m_1 b$ for some $m_1 \in M$. By the same reason, $f(s) = m_2 s$ for some $m_2 \in M$. From $f(bs) = m_1 bs = m_2 bs$, we obtain $m_1 = m_2$ by the torsion-free condition. Thus, there exists an element $m \in M$ such that $f(a) = ma$ for each $a \in A$. Now, define maps $g : A \to sR$ by $g(a) = sa$ for every $a \in A$ and $h_1 : sR \to M$ by $h_1(sr) = mr$ for every $r \in R$. Note that each of g and h_1 is an R-homomorphism, and $f = h_1 \circ g$. Since M is divisible, h_1 can be extended to $h'_1 : R \to M$. This implies that f can be extended to $f' : R \to M$.

If $s \notin A$, then q must be of the form of $\sum_{i=1}^{\text{finite}} b_i r_i s$ for some $r_i \in R$, where $\{b_i\}$ is a (minimal) generating set of A. Now, define maps $g : A \to qR$ by $g(a) = as$ for every $a \in A$ and $h_2 : qR \to M$ by $h_2(qr) = f(a)$ for every $r \in R$ with $qr = as$. Note that each of g and h_2 is an R-homomorphism, and $f = h_2 \circ g$. Since M is divisible, h_2 can be extended to $h'_2 : R \to M$. This implies that f can be extended to $f' : R \to M$ as desired. □

Remark 3.21. (1) It was shown that if R is a PRIR and M is a divisible R-module, then M is injective (see [17, Corollary 3.17′]). However, this fact cannot be extended to S-PRIRs by [17, Example, p.73]. This example also shows that the torsion-free condition in Proposition 3.20 is not superfluous.

(2) Consider the ring $R = D[x; \sigma]$ in Example 2.7 and a new multiplicative subset $S' = \{d\sigma(d) \mid d \in D\}$ of R. Note that $S' \subset Z(R)$ and R is also an S'-PRID. Let $T = R \setminus \{0\}$. Then, T is clearly a right denominator set of R. Since $S' \subset T$ and R is a domain, RT^{-1} is a division ring by [19, Corollary 3.3] or [17, Proposition 10.21]. From the two facts that the polynomial ring $RT^{-1}[y]$ is a torsion-free right R-module and also a divisible R-module, we can conclude that $RT^{-1}[y]$ is an injective right R-module by Proposition 3.20.

Finally, we end this paper with inevitable basic properties of S-PRIRs.

Proposition 3.22. (cf. [4, Remark 2.11 and Lemma 2.14]). *Let S be a multiplicative subset of a ring R and M be a right R-module. Then, the following assertions hold.*

(1) If every R-submodule of M is S-principal and N is an R-submodule of M, then every R-submodule of N is also S-principal.
(2) If every R-submodule of M is S-principal and f is an R-homomorphism, then every R-submodule of $f(M)$ is S-principal.
(3) If every R-submodule of M is S-principal and N is an R-submodule of M, then every R-submodule of M/N is S-principal.
(4) For a short exact sequence of right R-modules, $\{0\} \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow \{0\}$, if every R-submodule of M is S-principal then each R-submodule of M' and M'' is S-principal.
(5) If R is an S-PRIR and M is a cyclic R-module, then every R-submodule of M is S-principal.
(6) If R is an S-PRIR and I is an ideal of R disjoint from S, then R/I is an $(S + I)$-PRIR.

4. Conclusions

In this article, we study the structure of S-principal right ideal rings. Especially, we construct an S-PRID which is not a PRID, and show that the S-principal notion can be passed between based rings and some over rings. Also, we find out a specific condition under which the S-principal concept can be penetrated into Ore localizations. Further, we prove the S-variants of the Eakin-Nagata-Eisenbud theorem and Cohen's theorem for S-PRIRs.

Based on results of this paper, we will focus on the notions of S-injective modules and S-projective modules as further works.

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Conflict of interest

The author declares no conflict of interest.

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