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Research article

Yamabe constant evolution and monotonicity along the conformal Ricci flow

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Abstract: We investigate the Yamabe constant's behaviour in a conformal Ricci flow. For conformal Ricci flow metric $g(t)$, $t \in [0, T)$, the time evolution formula for the Yamabe constant $Y(g(t))$ is derived. It is demonstrated that if the beginning metric $g(0) = g_0$ is Yamabe metric, then the Yamabe constant is monotonically growing along the conformal Ricci flow under some simple assumptions unless g_0 is Einstein. As a result, this study adds to the body of knowledge about the Yamabe problem.

Keywords: Yamabe constant; conformal Ricci flow; Einstein metric; scalar curvature
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1. Introduction

Intrinsic geometric flows such as the Ricci flow, Yamabe flow and their various extensions have been widely studied in the recent time owing to their numerous applications in topology, geometry and physics. The present paper is concerned with a conformally modified Ricci flow introduced by Arthur Fischer in [10] and codenamed conformal Ricci flow as a result of the role played by conformal geometry in the derivation of its equations. Precisely, let (M^n, g_0) be a smooth n-dimensional $(n \geq 3)$ closed connected manifold together with Riemannian metric g_0 of constant scalar curvature R_0. The conformal Ricci flow is defined by a one-parameter family of metric $g(t)$ satisfying the following quasilinear parabolic system

$$\begin{cases} \dfrac{\partial g(t)}{\partial t} = -2\left(Ric(t) - \dfrac{R_0}{n} g(t)\right) - 2p(t)g(t), & (x,t) \in M \times (0,T), \\ R_{g(t)} = R_0, & (x,t) \in M \times [0,T), \end{cases} \tag{1.1}$$

together with the initial condition $g(0) = g_0$ and a family of function $p(t)$, $t \in [0, T)$, where $R_{g(t)}$ is the scalar curvature of the evolving metric $g(t)$. The equation

$$R_{g(t)} = R_0$$

appearing in the above system is known as the constraint equation. Thus, the flow is known to preserve constant scalar curvature of the evolving metric. Indeed, this accounts for naming the function, $p = p(t)$, conformal pressure, since it serves as time-dependent Lagrange multiplier and makes the term $-pg$ acting as the constraint force neccessary to preserve the scalar curvature constraint.

Consequently, $p(t)$ is known to solve a time-dependent elliptic partial differential equation as $g(t)$ evolves. This scenario is completely similar to the case of incompressible Euler or Navier-Stokes equations of fluid dynamics, where the divergence free constraint of the vector field is preserved by the real physical pressure which acts as the Langrage multiplier. However, it can then be clearly shown that (1.1) is equivalently the same as the following system of parabolic-elliptic equations

$$\begin{cases} \dfrac{\partial g}{\partial t} + 2\left(Ric - \dfrac{R_0}{n} g\right) = -2pg, & \text{in } M \times (0, T), \\ (n-1)\Delta p + R_0 p = -\left|Ric - \dfrac{R_0}{n} g\right|^2 & \text{in } M \times [0, T), \end{cases} \tag{1.2}$$

with initial condition $g(0) = g_0$. Considering the role of the conformal pressure, the function $p(t)$ is expected to be zero at an equilibrium point and strictly positive otherwise. Hence, the equilibrium points of the conformal Ricci flow are characterised by Einstein metric, and the term $-(Ric - \frac{R_0}{n} g)$ can then be regarded as a measure of deviation from an equilibrium point.

Since the volume of a Riemannian manifold (M^n, g) is a positive real number and the scalar curvature is a real-valued function on M^n, the constraint on $R_{g(t)}$ is considerably more drastic than the volume constraint of the Hamilton Ricci flow. Thus, the configuration space of the conformal Ricci flow equations is considerably smaller than that of the Hamilton Ricci flow. Obviously, conformal Ricci flow will perform better in searching for certain geometric features since working on a smaller configuration space is more advantageous than working on a larger configuration. More feasible similarities and differences between conformal Ricci flow and Hamilton Ricci flow, as well as some possible applications of conformal Ricci flow to 3-manifold geometry are highlighted in [10].

Arthur Fischer has since 2004 established the short-time existence and uniqueness of the conformal Ricci flow on closed manifolds with negative constant scalar curvature $R_0 < 0$ in [10]. He also observed that Yamabe constant is strictly increasing along the flow on negative Yamabe type closed manifolds. A decade later, Lu, Qing and Zheng [16] extended Fischer's results to the case of conformal Ricci flow on manifolds with nonnegative constant scalar curvature. They applied De-Turck's trick to recover the existence of the flow equation as a strong parabolic-elliptic partial differential equation on closed manifolds as well as on asymptotically flat manifolds. They also proved that Yamabe constant is monotonically increasing along conformal Ricci flow on closed manifolds of nonnegative constant scalar curvature.

Theorem A. (Lu, Qing and Zheng [16]) Assume that the elliptic operator $(n-1)\Delta + R_0$ is invertible. Then there exists a number $0 < T_0 \leq T < \infty$ such that the conformal Ricci flow $g(t)$ exists for $t \in [0, T]$ with $g(0) = g_0$.

Theorem B. (Lu, Qing and Zheng [16]) Assume that $g_0 \in [g_0]$ is the only Yamabe metric in the conformal class with $R_{g_0} = R_0$ and that $(n-1)\Delta + R_0$ is invertible. Then there exists $0 < T_0 \leq T < \infty$ such that the metric, $t \in [0, T_0)$ is a Yamabe metric and the Yamabe constant $Y[g(t)]$ is monotonically increasing for $t \in [0, T_0)$ unless g_0 is Einstein.

Motivated by the results of Fischer [10] and Lu, Qing and Zheng [16] and the tremendous advantages of working on smaller configuration as we have in the case of conformal Ricci flow, this paper is aimed at deriving evolution equation for subcritical Yamabe constant along the conformal Ricci flow on an n-dimensional ($n \geq 3$) closed connected oriented Riemannian manifolds of positive constant scalar curvature. It turns out that evolution equation at the critical point characterises the monotonicity property of the Yamabe constant which is strict unless on the Einstein metric. Note that monotonicities of Yamabe constant proved in [10, 16] are not consequences of evolution equation. Evolution of subcritical Yamabe constant was studied by Chang and Lu [8] along the Hamilton Ricci flow under some technical assumptions. The results were extended to relative subcritical Yamabe constant by [5] under the Ricci flow with boundary when the mean curvature of the boundary vanishes. Various extensions of Chang-Lu's evolution formula are mentioned here for completeness: for instance, see [9] along List extended flow, [3, 11] along Ricci-Burguignon flow and [7] for Cotton tensor flow. Although, our results are procedurally similar to those mentioned above, they are different in the sense that conformal Ricci flow is complementary to the classical Ricci flow and has smaller configuration, whereas, other flows mentioned above are just various extensions of Ricci flow.

The organisation of this paper is as follows: Section 2 gives some basics and preliminary results on the Yamabe constant viz-a-viz Yamabe problem and evolution equations along the conformal Ricci flow. Section 3 is devoted to the main results and their applications, while some examples are constructed in the last section.

2. Preliminaries

2.1. *Yamabe constant*

In 1960, H. Yamabe [27] asked a question, which is now one of the most famous problem in modern differential geometry. Let (M^n, g) be an n-dimensional ($n \geq 3$) closed Riemannian manifold, does there exists a metric $\tilde{g}$ conformal to g which has a constant scalar curvature? Aubin [2], Schoen [24] and Trudinger [25] have solved the problem giving affirmative answer to the question. They proved that an infimum of the normalised Einstein-Hilbert functional is attained in each conformal class of metrics and the infimum is achieved by a metric of constant scalar curvature. Let $[g]$ denote the conformal class of g, consisting of smooth metrics on M pointwise conformal to g and $\mathcal{M}_g$, the set of Riemannian metrics on M. Recall that the normalised Einstein-Hilbert functional $\mathcal{E} : \mathcal{M}_g \to \mathbb{R}$ is given by

$$\mathcal{E}(\tilde{g}) = \frac{\int_M R_{\tilde{g}}\, d\mu_{\tilde{g}}}{Vol(M, \tilde{g})^{\frac{n-2}{n}}}, \tag{2.1}$$

where $\mathcal{E}(\tilde{g})$ is restricted to the class of conformal metric $\tilde{g} \in [g]$. Here R_g, $d\mu_g$ and $Vol(M, g)$ are the scalar curvature, the volume form of metric g and the volume of (M, g), respectively. The quantity $\mathcal{E}(g)$ is well known in literature as Yamabe quotient.

Yamabe problem can be recast in terms of positive solutions of the following nonlinear critical elliptic differential equation

$$\begin{cases} \mathscr{L}_g u = \alpha u^{q_{crit}}, \\ \int_M u^{q_{crit}+1} d\mu_g = 1, \end{cases} \tag{2.2}$$

where α is a constant, $q_{crit} = \frac{n+2}{n-2}$, $\mathscr{L}_g := -a\Delta_g + R_g$ with $a := 4\frac{(n-1)}{(n-2)}$ is the conformal Laplacian (Δ_g is Beltrami Laplacian). It happens that the exponent $q_{crit} = \frac{n+2}{n-2}$ in (2.2) is precisely the critical value, below (subcritical) which the equation is easy to solve and above (supercritical) which may be delicate. The existence of solution to (2.2) follows from direct method in the calculus of variation (cf. [15]). It is also observed that Eq (2.2) is the Euler-Lagrange equation for a minimiser u of the functional

$$\mathcal{Y}(g;u) = \frac{\int_M (a|\nabla u|^2 + R_g u^2) d\mu}{(\int_M u^{\frac{2n}{n-2}} d\mu)^{\frac{n-2}{n}}}, \tag{2.3}$$

where ∇ is the Riemannian connection on M. A function u for which $\mathcal{Y}(g;u)$ get its infimum is called the Yamabe minimiser, while $u^{\frac{4}{n-2}} g$ is called the Yamabe metric and has constant scalar curvature. Thus, for a positive function u satisfying $\tilde{g} = u^{\frac{4}{n-2}} g$, we have infimum in (2.1) and (2.3) being equal, that is,

$$Y(g) = \inf_{0<u\in C^\infty(M)} \mathcal{Y}(g;u) = \inf_{\tilde{g}\in[g]} \mathcal{E}(\tilde{g}). \tag{2.4}$$

Then the infimum in (2.4) is the Yamabe constant of a smooth metric g on closed n-dimensional manifold. For convinience, we assume all metrics have unit volume. The following definition are in order,

Definition 2.1.

(1) Yamabe constant $Y(g)$ is the infimum of Yamabe quotient.
(2) A Riemannian metric g is said to be Yamabe metric if and only if g is the minimising metric for Yamabe functional.
(3) A Riemannian metric g is said to be Einstein metric if and only if its Ricci curvature is a multiple of g, i.e., $Ric - \alpha g = 0$, $0 \neq \alpha \in \mathbb{R}$. That is, g is the minimising metric.

2.2. Evolution equations along conformal Ricci flow

As in the case of other geometric flows, the evolutions of geometric quantities under conformal Ricci flow allow for better understanding of the flow. Following standard techniques (cf. [4, 10, 16]) one can compute directly as follows: Consider nonconstraint conformal Ricci flow equation

$$\frac{\partial g}{\partial t} + 2\left(Ric - \frac{R_0}{n} g\right) = -2pg$$

on closed Riemannian mnaiofld. Then we have

$$\frac{\partial g^{ij}}{\partial t} = 2\left(Ric^{ij} - \frac{R_0}{n} g^{ij}\right) + 2pg^{ij}$$

for inverse metric $g^{ij}(t)$,

$$\begin{aligned}\frac{\partial}{\partial t}R_{g(t)} =&\Delta_g R_{g(t)} + 2\frac{R_0}{n}(R_{g(t)} - R_0) + 2p(R_{g(t)} - R_0)\\ &+ 2[(n-1)\Delta_g + R_0]p + 2\left|Ric - \frac{R_0}{n}g\right|^2\end{aligned}$$

for the scalar curvature $R_{g(t)}$ of $g(t)$, and

$$\frac{\partial}{\partial t}d\mu_{g(t)} = -npd\mu_{g(t)}$$

for the volume form $d\mu_{g(t)}$ of $(M, g(t))$. Heuristically, since $R_{g(t)} = R_0$(constant) one can recover the function $p(t)$ once the operator $(n-1)\Delta_g + R_0$ is invertible, that is,

$$p(t) = [(n-1)\Delta_g + R_0]^{-1}\left(-\left|Ric - \frac{R_0}{n}g\right|^2\right).$$

Similarly, one can compute evolution of $|\nabla u|_g^2$ and $\Delta_g u$ as presented in the following Lemma.

Lemma 2.2. *Let $g(t)$, $t \in [0, T)$ be the conformal Ricci flow. Let $u \in C^\infty(M)$ be a smooth function on $(M, g(t))$. Then we have the following evolutions:*

$$\frac{\partial}{\partial t}|\nabla u|_g^2 = 2\left(Ric^{ij} - \frac{R_0}{n}g^{ij}\right)\nabla_i u\nabla_j u + 2p|\nabla u|_g^2 + 2\nabla_i u\nabla_i u_t, \tag{2.5}$$

$$\frac{\partial}{\partial t}(\Delta_g u) = 2\left(Ric^{ij} - \frac{R_0}{n}g^{ij}\right)\nabla_i\nabla_j u + 2p\Delta_g u + \Delta_g u_t, \tag{2.6}$$

where $u_t = \frac{\partial u}{\partial t}$.

The proof of the above lemma is omitted since the computations follow from standard technique under geometric flow (see [3,4]).

2.3. Evolution of Yamabe quotient and volume

Now, we define Yamabe invariant via the normalised Einstein Hilbert functional. Let $g \in [g]$ conformal class of metric g and the average total scalar curvature is defined by the quantity $AV(R_g) := Vol(M, g)^{-1}\int_M R_g d\mu_g$. Then

$$\mathcal{E}(g) = Vol(M, g)^{\frac{2}{n}} AV(R_g),$$

is saying that Yamabe invariant is a volume-normalised total scalar curvature.

Lemma 2.3. *Let $g(t)$, $t \in [0, T)$, be the conformal Ricci flow on $(M, g(t))$. Let $Vol(M_t)$ and $\mathcal{E}(t)$ be the volume and Yamabe quotient, respectively, along the flow of $g(t)$ with constant scalar curvature. Then*

$$\frac{d}{dt}Vol(M_t) = \frac{n}{R_0}\int_M |Ric - \frac{R_0}{n}g|^2 d\mu_{g(t)} \tag{2.7}$$

and

$$\frac{d}{dt}\mathcal{E}(t) = c_n\int_M |Ric - \frac{R_0}{n}g|^2 d\mu_{g(t)}, \tag{2.8}$$

where c_n is a positive constant depending on the volume and dimension n of M.

Remark 2.4.

(1) Evolution equation (2.7) yields $\frac{d}{dt}Vol(M_t) > 0$ on positive constant scalar curvature unless $Ric - \frac{R_0}{n}g = 0$, meaning that the $Vol(M_t)$ is strictly increasing along the conformal Ricci flow unless the metric is Einstein.

(2) Evolution equation (2.8) yields $\frac{d}{dt}\mathcal{E}(t) > 0$ unless $Ric - \frac{R_0}{n}g = 0$, meaning that the quotient, $\mathcal{E}(t)$, is strictly increasing along the conformal Ricci flow unless the metric is Einstein.

The proof of the Lemma follows from direct computations.

Proof. Recall that $Vol(M_t) = \int_M d\mu_{g(t)}$, $\mathcal{E}(t) = R_0(Vol(M_t))^{\frac{2}{n}}$ and $\frac{\partial}{\partial t}d\mu_{g(t)} = -npd\mu_{g(t)}$. Then

$$\frac{d}{dt}Vol(M_t) = -n\int_M pd\mu_{g(t)} \tag{2.9}$$

and

$$\frac{d}{dt}\mathcal{E}(t) = -2R_0(Vol(M_t))^{\frac{2}{n}-1}\int_M pd\mu_{g(t)}. \tag{2.10}$$

We know that the function p satisfies the elliptic equation, that is, the second equation in system (1.2). Integrating this equation over M using the fact that $\int_M \Delta p d\mu_g = 0$ since M is closed we get

$$R_0\int_M pd\mu_{g(t)} = -\int_M |Ric - \frac{R_0}{n}g|^2 d\mu_{g(t)}.$$

Combining this with (2.9) and (2.10) yields the desired formula (2.7) and (2.8), respectively. □

Considering the behaviours of the Yamabe quotient and the volume along the flow $g(t)$, we therefore expect the conformal Ricci flow to be an effective tool for constructing Einstein metrics. In the next section, we will define Yamabe constant via nonlinear subcritical elliptic differential equation, get the evolution of Yamabe constant along the conformal Ricci flow on smooth manifolds of dimension $n \geq 3$ with nonnegative constant scalar curvature, and then construct Einstein metrics.

3. Main results: Evolution of Yamabe constant

Let $g(t)$ and $p(t)$, $t \in [0, T)$, solve the conformal Ricci flow equations

$$\begin{cases} \dfrac{\partial g}{\partial t} + 2\left(Ric - \dfrac{R_0}{n}g\right) = -2pg, & \text{in } M \times (0, T) \\ (n-1)\Delta p + R_0 p = -|Ric - \dfrac{R_0}{n}g|^2, & \text{in } M \times [0, T), \\ g(0) = g_0 \end{cases} \tag{3.1}$$

on M with nonnegative constant scalar curvature $R_0 \neq 0$.

In order to discuss the Yamabe problem under (3.1), recall the subcritical regularisation of (2.2) for $q \in (1, \frac{n+2}{n-2})$

$$\mathscr{L}_g u = Y_q(g)u^q,$$

$$\int_M u^{q+1} d\mu_g = 1,$$

where $\mathscr{L}_g := -a\Delta + R_g$ with $a = 4\frac{n-1}{n-2}$ and $Y_q(g)$ is a constant. This is the Euler-Lagrange equation for the minimiser of the functional

$$Y_q(g) = \inf_{u>0, u\in C^\infty(M)} \frac{\int_M (a|\nabla u|^2 + R_g u^2) d\mu_g}{(\int_M u^{q+1} d\mu_q)^{\frac{2}{q+1}}}.$$

The existence of solution u of the problem above is due to the direct method in calculation of variation since q is subcritical exponent.

To state the main results we want to consider a regularised constant function of time only. Suppose for $q \in (1, \frac{n+2}{n-2})$, there is a C^1-family of positive smooth function $u(t)$, $t \in [0, T)$, which satisfies

$$\mathscr{L}_{g(t)} u = \widetilde{Y}_q(t) u^q, \quad \int_M u^{q+1} d\mu_{g(t)} = 1, \tag{3.2}$$

where $\mathscr{L}_{g(t)} := -a\Delta + R_{g(t)}$ with $a = 4\frac{n-1}{n-2}$ and $\widetilde{Y}_q(t)$ is a function of t only. Here $\widetilde{Y}_q(t)$ and $Y(g)$ have the same sign, however, $\widetilde{Y}_q(t)$ depends on t, not only on the conformal class. We note $\widetilde{Y}_q(t)$ equals to $Y(g)$ if q attains the critical value $q = \frac{n+2}{n-2}$. It coincides with the principal eigenvalue of the conformal Laplacian, $\mathscr{L}_g$, at $q = 1$.

Proposition 3.1. Let $g(t)$, $t \in [0, T)$, be the conformal Ricci flow, that is, $g(t)$ satisfies (3.1) on the closed oriented Riemannian manifold (M^n, g_0) $(n \geq 3)$ with constant scalar curvature, $R_0 \neq 0$. Then, for any $q \in (1, \frac{n+2}{n-2}]$, assuming that there is a C^1-family of smooth functions $u(t) > 0, t \in [0, T)$ which satisfies (3.2),

$$\begin{aligned} \frac{d}{dt}\widetilde{Y}_q(t) = 2a \int_M \Big[\Big(Ric^{ij} - \frac{R_0}{n} g^{ij}\Big)\nabla_i u \nabla_j u + p|\nabla u|^2\Big] d\mu_{g(t)} \\ - \frac{na}{q+1}\int_M u^2 \Delta p \, d\mu_{g(t)} - \frac{n(q-1)}{q+1}\int_M (a|\nabla u|^2 + R_0 u^2) p \, d\mu_{g(t)}. \end{aligned} \tag{3.3}$$

Proposition 3.2. Let $g(t)$, $t \in [0, T)$, be the conformal Ricci flow, that is, $g(t)$ satisfies (3.1) on the closed oriented Riemannian manifold (M^n, g_0) $(n \geq 3)$ with constant scalar curvature, $R_0 \neq 0$. Then, for any $q \in (1, \frac{n+2}{n-2}]$, assuming that there is a C^1-family of smooth functions $u(t) > 0, t \in [0, T)$ which satisfies (3.2),

$$\frac{d}{dt}\widetilde{Y}_{\frac{n+2}{n-2}}(t) = \frac{8(n-1)}{n-2}\int_M \overline{Ric}^{ij} \nabla_i u \nabla_j u \, d\mu_{g(t)} + 2\int_M |\overline{Ric}|^2 u^2 d\mu_{g(t)}, \tag{3.4}$$

where $\overline{Ric}_{ij} = Ric_{ij} - \frac{R_0}{n} g_{ij}$ is the traceless part of the Ricci tensor under the flow.

Remark 3.3. Note that evolution equation (3.4) can be compared with [8, Eq (10)] under the Ricci flow and with [26, Theorem 1.3]. We remark that (3.4) is better than both since conformal Ricci flow is restricted to a smaller configuration of conformal class and there is no involvement of $\Delta R_{g(t)}$ (or $\frac{\partial}{\partial t} R_{g(t)}$) since $R_{g(t)} = R_0 \neq 0$, a constant.

Proof of Proposition 3.1.

Referring to (3.2), we have for any $q \in (1, \frac{n+2}{n-2}]$ that

$$-a\Delta u + R_g u = \widetilde{Y}_q(t)u^q \tag{3.5}$$

and

$$\int_M u^{q+1} d\mu_g = 1. \tag{3.6}$$

Using (3.5) and (3.6) the problem can be reformulated variationally as

$$\widetilde{Y}_q(t) = \int_M (a|\nabla u|^2 + R_g u^2) d\mu_g. \tag{3.7}$$

Now, differentiating (3.7) with respect to t we get

$$\begin{aligned}\frac{d}{dt}\widetilde{Y}_q(t) = &\int_M 2a\Big[\Big(Ric^{ij} - \frac{R_0}{n}g^{ij}\Big)\nabla_i u \nabla_j u + p|\nabla u|^2 + \nabla_i u \nabla_i u_t\Big] d\mu_g \\ &+ 2\int_M R_g u u_t d\mu_g - n\int_M (a|\nabla u|^2 + R_g u^2) p d\mu_g,\end{aligned}$$

where we have used $(|\nabla u|^2)_t = 2(Ric^{ij} - \frac{R_0}{n}g^{ij})\nabla_i u \nabla_j u + 2p|\nabla u|^2 + 2\nabla_i u \nabla_i u_t$, $(d\mu_g)_t = -npd\mu_g$, and $R_g = R_0$ (see Section 2).

Applying integration by parts, we arrive at

$$\begin{aligned}\frac{d}{dt}\widetilde{Y}_q(t) = &\int_M 2a\Big[\Big(Ric^{ij} - \frac{R_0}{n}g^{ij}\Big)\nabla_i u \nabla_j u + p|\nabla u|^2\Big] d\mu_g \\ &+ 2\int_M (-au_t\Delta u + Ruu_t) d\mu_g - n\int_M (a|\nabla u|^2 + R_g u^2) p d\mu_g.\end{aligned} \tag{3.8}$$

Differentiating the condition (3.6) with respect to t yields

$$(q+1)\int_M u_t u^q d\mu_g = n\int_M p u^{q+1} d\mu_g. \tag{3.9}$$

Multiplying (3.5) by pu and applying integration by parts we obtain

$$\widetilde{Y}_q(t)\int_M p u^{q+1} d\mu_g = -\frac{a}{2}\int_M u^2 \Delta p d\mu_g + \int_M (a|\nabla u|^2 + R_g u^2) p d\mu_g. \tag{3.10}$$

Referring to (3.5), (3.9) and (3.10) we have

$$\begin{aligned}2\int_M (-au_t\Delta u + Ruu_t) d\mu_g &= 2\widetilde{Y}_q(t)\int_M u_t u^q d\mu_g, \quad \text{(by (3.5))} \\ &= \frac{2n}{q+1}\widetilde{Y}_q(t)\int_M p u^{q+1} d\mu_g, \quad \text{(by (3.9))} \\ &= \frac{2n}{q+1}\left[-\frac{a}{2}\int_M u^2\Delta p d\mu_g + \int_M (a|\nabla u|^2 + R_g u^2) p d\mu_g\right].\end{aligned}$$

Combining this with (3.8) leads us to

$$\frac{d}{dt}\widetilde{Y}_q(t) = \int_M 2a\Big[\Big(Ric^{ij} - \frac{R_0}{n}g^{ij}\Big)\nabla_i u\nabla_j u + p|\nabla u|^2\Big]d\mu_g$$
$$- \frac{na}{q+1}\int_M u^2\Delta p d\mu_g + \left(\frac{2n}{q+1} - n\right)\int_M (a|\nabla u|^2 + R_g u^2)p d\mu_g$$

which is the desired evolution equation (3.3) for $\widetilde{Y}_q(t)$.

Proof of Proposition 3.2.

Setting $q = \frac{n+2}{n-2}$ in (3.3) we obtain

$$\frac{d}{dt}\widetilde{Y}_{\frac{n+2}{n-2}}(t) = 2a\int_M \Big(Ric^{ij} - \frac{R_0}{n}g^{ij}\Big)\nabla_i u\nabla_j u d\mu_g - 2\int_M [(n-1)\Delta p + R_0 p]u^2 d\mu_g.$$

By considering the second equation in (3.1) we arrive at

$$\frac{d}{dt}\widetilde{Y}_{\frac{n+2}{n-2}}(t) = 2a\int_M \Big(Ric^{ij} - \frac{R_0}{n}g^{ij}\Big)\nabla_i u\nabla_j u d\mu_g + 2\int_M |Ric - \frac{R_0}{n}g|^2 u^2 d\mu_g.$$

This therefore completes the proof. □

Remark 3.4. Referring to the asssumpton in Propositions 3.1 and 3.2 which says there are functions $u(t)$ such that the metrics $u(t)^{\frac{4}{n-2}}g(t)$ are a C^1-family of smooth metrics which have unit volume and constant scalar curvature (Yamabe metrics), two situations are involved. Namely:

(1) When $\widetilde{Y}(0)$ is nonpositive, there is a unique solution $u(0)$ of the Yamabe problem. It could be shown here, as a consequence of Koiso decomposition [5, 14], that the metrics $u(t)^{\frac{4}{n-2}}g(t)$ is a smooth in t family of metrics for small t.
(2) When $\widetilde{Y}(0)$ is positive, in general it is not certain whether there exists a C^1-family of smooth functions $u(t)$ satisfying the assumption even for a short time since the corresponding Yamabe metric is not unique [8].

It has been observed that Yamabe constant of manifolds of positive Yamabe type can in general behave rather irregularly (see [1, 8, 14, 26]). The evolution equation (3.4) shows a better behaviour since constant scalar curvature is preserved along the flow.

Now, if one assumes that $u(t)^{\frac{4}{n-2}}g(t)$ has unit volume and constant scalar curvature $Y(g(t))$, it is then clear that $\widetilde{Y}(t) = \widetilde{Y}_{\frac{n+2}{n-2}}(t)$ is the scalar curvature of metric $u(t)^{\frac{4}{n-2}}g(t)$. We can therefore prove the following results.

Corollary 3.5. Let (M^n, g_0) be a closed oriented Riemannian manifold of dimension $n \geq 3$ and constant scalar curvature. Let $g(t)$, $t \in [0, T)$ be the conformal Ricci flow with $g(0) = g_0$, where $(n-1)\Delta + R_0$ is an invertible operator. Then

(1) There is a C^1-family of smooth positive functions $u(t)$ on $[0, \delta)$, for some $\delta > 0$, with constant $u(0)$, such that $u(t)^{\frac{4}{n-2}}g(t)$ has unit volume and constant scalar curvature $\widetilde{Y}(t)$.
(2) $\frac{d}{dt}|_{t=0}\widetilde{Y}(t) \geq 0$ and the equality is attained if and only if g_0 is an Einstein metric.

Proof. Since the elliptic operator $(n-1)\Delta + R_0$ is invertible then Koiso's decomposition theorem (Corollary 2.9 in [14] or Theorem 4.44 in [4]) can be applied to conclude as follows: There exists a C^1-family of smooth positive functions $u(t)$ on $[0, \delta)$ for some $\delta > 0$ with $u(0)$ being a constant and satisfying the assumption of Proposition 3.1 with $q = \frac{n+2}{n-2}$. Obviously, $\tilde{Y}(t) = \tilde{Y}_q(t)$ is the scalar curvature of metric $u(t)^{\frac{4}{n-2}} g(t)$.

Now, since $\nabla u(0) = 0$. Hence, we have by Proposition 3.2 that

$$\frac{d}{dt}\tilde{Y}(t)|_{t=0} = 2[u(0)]^2 \int_M |\overline{Ric}(g_0)|^2 d\mu_{g_0} \geq 0.$$

If equality is attained in the last inequality then the traceless Ricci tensor $\overline{Ric}(g_0)$ vanishes identically, and consequently, g_0 must be Einstein. □

4. Example

In this section we construct an example on locally homogeneous 3-geometry of the Heisenberg group. Isenberg and Jackson [12] introduced Hamilton Ricci flow on locally homogeneous 3-manifolds as a way to having more understanding of geometrisation conjecture. Isenberg, Jackson and Lu [13] later studied the Ricci flow on locally homogeneous 4-manifolds (see also Cao and Saloff-Coste [6] for the study of backward Ricci flow in this setting). In [10], Fischer opined that locally homogeneous conformal Ricci flow would provide somewhat an intersection (even equivalence) between Hamilton Ricci flow and conformal Ricci flow (see [10, Section 8]). Hence, conformal Ricci flow has the potential to deepen our understanding of geometrisation of 3-manifolds.

The Heisenberg group is isomorphic to the set of upper-triangle 3×3 matrices

$$\left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

endowed with the matrix multiplication. This group fails to admit any Einstein metric. Meanwhile, the group admits a Milnor frame $\{X_i\}_{i=1}^3$ understood with Lie brackets

$$[X_2, X_3] = X_1, \quad [X_3, X_1] = 0 \quad \text{and} \quad [X_1, X_2] = 0$$

which can diagnolise the initial metric g_0 and the Ricci tensor.

Let $\{\theta^i\}_{i=1}^3$ be the frame dual to the Milnor frame $\{X_i\}_{i=1}^3$ and

$$g_0 = A_0(\theta^1)^2 + B_0(\theta^2)^2 + C_0(\theta^3)^2.$$

We assume that

$$g(t) = A(t)(\theta^1)^2 + B(t)(\theta^2)^2 + C(t)(\theta^3)^2$$

be a solution of the conformal Ricci flow. According to [12] under normalisation $A_0B_0C_0 = 1$ we have

$$Ric_{11} = \frac{1}{2}A^3, \quad Ric_{22} = -\frac{1}{2}A^2B, \quad Ric_{33} = -\frac{1}{2}A^2C, \;\; R = -\frac{1}{2}A^2$$

and

$$\|Ric\|^2 = 3R^2 = \frac{3}{4}A^4.$$

The conformal Ricci flow equations are then written as

$$\begin{cases} \dfrac{dA}{dt} + A^3 + \dfrac{1}{3}A_0^2 A = -2pA, & (a) \\ \dfrac{dB}{dt} - A^2 B + \dfrac{1}{3}A_0^2 B = -2pB, & (b) \\ \dfrac{dC}{dt} - A^2 C + \dfrac{1}{3}A_0^2 C = -2pC, & (c) \\ A = A_0. & (d) \end{cases} \tag{4.1}$$

Substituting (4.1)(d) into (4.1)(a) implies that

$$\frac{4}{3}A_0^2 = -2pA_0 \implies p = -\frac{2}{3}A_0^2. \tag{4.2}$$

Plugging (4.1)(d) and (4.2) into (4.1)(b) we calculate

$$\frac{dB}{dt} = 2A_0^2 B \implies B(t) = B_0 e^{A_0^2 t}. \tag{4.3}$$

Similarly, by plugging (4.1)(d) and (4.2) into (4.1)(c) we obtain

$$C(t) = C_0 e^{A_0^2 t}. \tag{4.4}$$

By this, we can infer (just as in the case of Hamilton Ricci flow [12]) that asymptotic behaviour of conformal Ricci flow in Heisenberg allows two metric coefficients expand, while the third and the scalar curvature remain constant. Clearly, the expanding directions in Heisenberg class conformal Ricci flow generally diverge. However, by (4.3) and (4.4) we obtain

$$B - C = (B_0 - C_0)e^{A_0^2 t}.$$

Hence, generic Heisenberg flows move away from the ones with rotational isotropy, (i.e., those with $B = C$).

Furthermore, recall that $\overline{Ric}_{ij} := Ric_{ij} - \frac{R_0}{3}g_{ij}$ and $|\overline{Ric}|^2 = |Ric|^2 - \frac{R_0}{3}$. Then

$$\overline{Ric}_{11} = \frac{2}{3}A_0^2 g_{11},\ \ \overline{Ric}_{22} = -\frac{1}{3}A_0^2 g_{22},\ \ \overline{Ric}_{33} = -\frac{1}{3}A_0^2 g_{33},\ \text{and } |\overline{Ric}|^2 = \frac{2}{3}A^4.$$

Denote by $\mathbb{H}_3$ the Heisenberg group of 3-geometry. Therefore, for $\widetilde{Y}(t) = \widetilde{Y}_{\frac{n+2}{n-2}}(t)$, we have by definition (3.7) that

$$\widetilde{Y}(t) := \int_{\mathbb{H}_3} (8|\nabla u|^2 - \frac{1}{2}A_0^2 u^2) d\mu_g$$

and by evolution formula (3.4) we have

$$\begin{aligned}\frac{d}{dt}\widetilde{Y} &= 8\int_{\mathbb{H}_3} \overline{Ric}^{ij}\nabla_i u\nabla_j u d\mu_{g(t)} + 2\int_{\mathbb{H}_3} |\overline{Ric}|^2 u^2 d\mu_{g(t)} \\ &\geq -\frac{64}{3}\int_{\mathbb{H}_3} A_0^2|\nabla u|^2 d\mu_{g(t)} + \frac{4}{3}\int_{\mathbb{H}_3} A_0^4 2u^2 d\mu_{g(t)} \\ &\geq -\frac{8}{3}A_0^2\int_{\mathbb{H}_3} (8|\nabla u|^2 - \frac{1}{2}A_0^2u^2)d\mu_g = -\frac{8}{3}A_0^2\widetilde{Y}(t).\end{aligned}$$

This inequality shows that the quantity $\widetilde{Y}(t)e^{\frac{8}{3}A_0^2 t}$ is non-decreasing along the conformal Ricci flow of Heisenberg group of 3-geometry.

5. Conclusions

This paper dealt with evolutionary behaviour of the subcritical Yamabe constant in the setting of conformal Ricci flow on an $n(\geq 3)$-dimensional closed connected oriented Riemannian manifolds of constant scalar curvature. In particular, the time evolution formula for the Yamabe constant was derived and it was demonstrated that if the flow initial metric is Yamabe then the Yamabe constant is monotonically increasing under some assumption unless the metric is Einstein. Recall that conformal Ricci flow is a modified Hamilton Ricci flow with respect to conformal geometry and is well known to preserve constant scalar curvature of the flow metric. The choice of the conformal Ricci flow in this paper came as a result of the quest for better results. Conformal Ricci flow performs better in this respect since it is restricted to a smaller configuration space than Hamilton Ricci flow.

Evolution of Yamabe constant has been studied under Ricci flow in [8] and under Ricci flow with boundary in [5]. We remarked that our evolution formula is better than those in these two papers since ours does not involve time evolution and diffusion of the scalar curvature ($\frac{\partial}{\partial t}R_{g(t)}$ and $\Delta R_{g(t)}$). We also remarked that similar results have been proved in [10] and [16] but not as a consequence of an evolution equation. Finally, an example of conformal Ricci flow was constructed on locally homogeneous 3-geometry of the Heisenberg group in which case we obtained a nondecreasing quantity involving Yamabe constant as a consequence of the evolution formula. In the next work, we plan to study the Yamabe problem and further extension of the results of this paper with singularity theory and submanifolds theory, etc. in [13–28] to obtain new results and theorems.

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Conflict of interest

There is no conflict of interest.

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