AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(7): 12050–12076.
DOI: 10.3934/math.2022670
Received: 22 January 2022
Revised: 25 March 2022
Accepted: 06 April 2022
Published: 21 April 2022

Research article

Generalized exponential function and initial value problem for conformable dynamic equations

Awais Younus[1], **Khizra Bukhsh**[2], **Manar A. Alqudah**[3] **and Thabet Abdeljawad**[4,5,*]

[1] Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan, Pakistan

[2] Department of Mathematics, Ghazi University, Dera Ghazi Khan, Pakistan

[3] Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah Bint Abdulrahman University, P. O. Box 84428, Riyadh 11671, Saudi Arabia

[4] Department of Mathematics and Sciences, Prince Sultan University, Riyadh, Saudi Arabia

[5] Department of Medical Research, China Medical University, Taichung 40402, Taiwan

* **Correspondence:** Email: tabdeljawad@psu.edu.sa.

Abstract: In this article, we define the generalized exponential function on arbitrary time scales in the conformable setting and develop its fundamental characteristics. We address the fundamental theory of a conformable fractional dynamic equation on time scales, subject to the local and non-local initial conditions. We generalized the Grönwall type inequalities in a conformable environment. The generalized exponential function and the Grönwall's inequalities are indispensable for the study of the qualitative aspects of the local initial value problem. We developed some criteria related to global existence, extension and boundedness, as well as stability of solutions.

Keywords: generalized exponential function; conformable dynamic equation; local initial value problem; nonlocal initial value problem; Grönwall's inequalities
Mathematics Subject Classification: 34A08, 34N05

1. Introduction

Fractional calculus generalized the classical calculus to an arbitrary (non-integer) order. The history of this theory goes back to mathematicians like Leibniz (1646–1716), Liouville (1809–1882), Riemann (1826–1866), Letnikov (1837–1888), Grünwald (1838–1920) and others [16, 19]. For the last three centuries, fractional calculus is getting famous, like all fields of science. It is one of the most intensively developed fields of mathematical assessment. Because of its various applications in designing, financial aspects, account, geography, likelihood and measurements, compound designing, physical science, splines, thermodynamics and neural organizations [5, 9, 22].

There are some recent developments in fractional calculus and its applications. In [8], the authors studied the fractional second and third-order nonlinear Schrödinger equations. They studied symmetric and antisymmetric solutions and analyzed the influence of the Lêvy index on different solutions. Lu et al. [14] solved the fractional discrete coupled nonlinear Schrödinger equations on account of the modified Riemann-Liouville fractional derivative and Mittag Leffler function. In [13], Li et al. studied the existence, bifurcation and stability of two-dimensional optical solutions in the framework of fractional nonlinear Schrödinger equation.

From the literature review reader can see several definitions of fractional operators like Riemann-Liouville, Caputo, Grünwald-Letnikov, Weyl, Hadamard, Marchaud and Riesz [12, 15]. These types of derivatives do not satisfy the fundamental formulas of differentiation like the product rule, the quotient rule, the chain rule, etc. Khalil [12] introduced a new well-behaved simple fractional derivative known as the conformable fractional derivative (CFD) based on the derivative's basic limit concept: Suppose $\Phi : [0, \infty) \longrightarrow \mathbb{R}$ be a function, then for all $\upsilon > 0$ and $\alpha \in (0, 1]$,

$$\Phi^{(\alpha)}(\upsilon) = \lim_{\epsilon \to 0} \frac{\Phi\left(\upsilon + \epsilon \upsilon^{1-\alpha}\right) - \Phi(\upsilon)}{\epsilon},$$

where $\Phi^{(\alpha)}(\upsilon)$ is known as the CFD of Φ of order α. The definition of CFD introduced by Khalil retaines all the classical characteristics of the derivative, and satisfy the chain rule. This new definition attracted many researchers, some results have been obtained for the fundamental properties of the CFD in [1].

In 1988, Hilger in his Ph. D. thesis introduced the time scale theory that has recently received a great deal of attraction to integrate and extend the discrete and continuous analysis [10]. The investigation of dynamic equations on arbitrary time scales show such distinction and helps avoid proving outcomes twice: Once for the differential equation and once for the difference equation. The basic idea is to prove a result for a dynamic equation where the domain of the function is a so-called time scale $\mathbb{T}$, which is an arbitrary nonempty closed subset of reals [7].

Bastos in his Ph. D. thesis developed fractional calculus on time [18]. The theory of fractional (non-integer order) calculus on time scales is a topic of great interest for researchers nowadays. In [6], Benkhettou et al. developed a conformable fractional calculus theory on an arbitrary time scale, which extends the conformable fractional calculus and the fundamental techniques for fractional differentiation and integration on time scales.

Ahmed [2] discussed the class of non-local stochastic differential equations involving conformable fractional time derivative operator and the existence of mild solution for the non-local conformable stochastic differential equation. In [3], Ahmed discussed the class of conformable fractional stochastic differential equations driven by the Rosenblatt process. In [4], Ahmed studied a non-instantaneous impulsive conformable fractional stochastic delay integro-differential system driven by the Rosenblatt process.

The exponential function on the time scale introduced by Hilger in [11] and has been devoted to solving the first-order linear dynamic equations, and second-order linear dynamic equations with constant and variable coefficients [11]. Euler-Cauchy dynamic equations on time scales have been solved using an exponential function. Exponential function for conformable fractional calculus has been defined in [17]. The conformable exponential function with respect to operator Δ^{α} has been defined [20]. This definition is implicit. In this paper, we define the generalized conformable fractional exponential function by following the conformable calculus outline and also developing its fundamental characteristics. However, our definition is explicit. The generalized exponential function is consistent with the exponential function on time scales for $\alpha = 1$ and consistent with CF exponential function if $\mathbb{T} = \mathbb{R}$. The generalized exponential function and these theorems are the generalizations of the exponential function on time scales discussed by Bohner et al. [7] and the conformable fractional exponential function in [17].

In [24], the global existence, extension, boundedness and stabilities of solutions have been discussed for the following conformable fractional differential equation:

$$x^{(\alpha)}(t) = f(t, x(t)), t \in [a, \infty), 0 < \alpha < 1,$$

corresponding to the local and non-local initial condition $x(a) = x_a$ and $x(a) + g(x) = x_a$ respectively. We generalized the theory established in [24] to the conformable fractional dynamic equation:

$$\psi^{(\alpha)}(s) = k(s, \psi(s)),\ s \in [0, \infty)_{\mathbb{T}}, 0 < \alpha < 1.$$

We consider the global existence, extension, boundedness and stability of the solutions corresponding to the conformable fractional dynamic equation:

$$\psi^{(\alpha)}(s) = k(s, \psi(s)),\ s \in [0, \infty)_{\mathbb{T}}, 0 < \alpha < 1, \tag{1.1}$$

according to the local initial condition

$$\psi(0) = \psi_0, \tag{1.2}$$

and non-local initial condition

$$\psi(0) + g(\psi) = \psi_0, \tag{1.3}$$

where $\psi^{(\alpha)}(s)$ refers to the CFD of order $\alpha \in (0, 1]$ for functions defined on arbitrary time scales $\mathbb{T}$, $k : [0, \infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$. Also, k is right dense continuous and on a suitable function space g is functional.

We structure the remaining of the paper: Section 2 recalls some essential definitions and results. Section 3 aims to define the generalized exponential function and develop its fundamental characteristics. In Section 4, we generalize the Grönwall type inequalities in the conformable setting. In Section 5, we set some rules for the global existence, extension and boundedness of solutions to the local initial value problem and then discuss the stability of solutions. Section 6 examines the existence of solutions to the nonlocal initial value problem. Finally, we conclude our findings in the last section.

2. Basic notions

Here, we recall some basic definitions and results which are essential to the sequel [21]. Throughout this manuscript, let us denote the time scale by $\mathbb{T}$ and the set of all rd-continuous functions by C_{rd}.

Lemma 2.1. *If $h \in C_{rd}$ and $u \in \mathbb{T}^k$, then*

$$\int_u^{\sigma(u)} h(\tau)\,\Delta\tau = \mu(u)\,h(u).$$

Definition 2.1. *Assume that $k > 0$, the Hilger complex number is defined by*

$$\mathbb{C}_k = \left\{ z \in \mathbb{C} \colon z \neq -\frac{1}{k} \right\}.$$

Definition 2.2. *For $k > 0$, the strip is defined by*

$$\mathbb{Z}_k = \left\{ z \in \mathbb{C} : -\frac{\pi}{k} < \operatorname{Im}(z) < \frac{\pi}{k} \right\}.$$

Definition 2.3. *For $k > 0$, the cylindrical transformation $\xi_k : \mathbb{C}_k \to \mathbb{Z}_k$ is defined by*

$$\xi_k(z) = \tfrac{1}{k} \log(1 + zk),$$

where log *is the principal logarithm function. Notice that*

$$\xi_k^{-1}(z) = \frac{\exp(zk) - 1}{k}.$$

Let us denote by $C^\alpha(\mathbb{T}, \mathbb{R})$ the set of all functions whose conformable fractional differentiable of order α is continuous, where $\alpha \in (0, 1]$. The following lemmas give some meaningful connections concerning the conformable fractional derivative on time scales [6].

Lemma 2.2. *Suppose $l \in C(\mathbb{T}, \mathbb{R})$ and assume $v \in \mathbb{T}^k$. If l is continuous at v where v is right-scattered, implies that $l \in C^\alpha(\mathbb{T}, \mathbb{R})$ at v such that*

$$(l)^{(\alpha)}(v) = \frac{l(\sigma(v)) - l(v)}{\mu(v)} v^{1-\alpha} = l^\Delta(v) v^{1-\alpha},$$

where $(l)^{(\alpha)}(\cdot)$ represents the conformable fractional derivative of order $\alpha \in (0, 1]$.

Lemma 2.3. *Suppose $h, l \in C^\alpha(\mathbb{T}, \mathbb{R})$. Then*

(1) If $h,l \in C^\alpha(\mathbb{T}, \mathbb{R})$, then the product $hl \in C^\alpha(\mathbb{T}, \mathbb{R})$ with

$$(hl)^{(\alpha)} = (h)^{(\alpha)} l + (h \circ \sigma)(l)^{(\alpha)} = (h)^{(\alpha)} (l \circ \sigma) + h(l)^{(\alpha)} = (hl)^\Delta(s) s^{1-\alpha}.$$

(2) If $h \in C^\alpha(\mathbb{T}, \mathbb{R})$, then $\frac{1}{h} \in C^\alpha(\mathbb{T}, \mathbb{R})$ such that

$$\left(\tfrac{1}{h}\right)^{(\alpha)} = -\frac{(h)^{(\alpha)}}{h(h \circ \sigma)} = \left(\tfrac{1}{h}\right)^\Delta(s) s^{1-\alpha},$$

applicable at each points $s \in \mathbb{T}^k$ for which $h(s) h(\sigma(s)) \neq 0$.

(3) If $h,l \in C^\alpha(\mathbb{T}, \mathbb{R})$, then $\frac{h}{l} \in C^\alpha(\mathbb{T}, \mathbb{R})$ such that

$$\left(\tfrac{h}{l}\right)^{(\alpha)} = \frac{(h)^{(\alpha)} l - h(l)^{(\alpha)}}{l(l \circ \sigma)} = \left(\tfrac{h}{l}\right)^\Delta(s) s^{1-\alpha},$$

applicable at each points $s \in \mathbb{T}^k$ for which $l(s) l(\sigma(s)) \neq 0$.

Lemma 2.4. *If $(g(s))^{(\alpha)}$ is continuous on $[c,d]_{\mathbb{T}}$, then*

$$I_\alpha (g(s))^{(\alpha)} = g(s) - g(0),$$

where I_α represents the conformable fractional integral of order $\alpha \in (0,1]$.

Lemma 2.5. *Let $\alpha \in (0,1]$. Then for all right-dense continuous function $g : \mathbb{T} \longrightarrow \mathbb{R}$, a function $G_\alpha : \mathbb{T} \to \mathbb{R}$ exists in such a way that*

$$[(G_\alpha)(s)]^{(\alpha)} = (I_\alpha g(s))^{(\alpha)} = g(s),$$

for each $s \in \mathbb{T}^k$. Function G_α is called an α-antiderivative of g.

3. Generalized exponential function

In this section, we use Definition 2.3 to define a generalized exponential function. Let us generalize the regressive concept in a conformable setting.

Definition 3.1. *A function $l : \mathbb{T} \to \mathbb{R}$ is said to be "α-regressive" provided*

$$1 + \mu(u) l(u) u^{\alpha-1} \neq 0, \ \forall \ u \in \mathbb{T}^k,$$

holds. The set of all α-regressive and right-dense continuous functions $l : \mathbb{T} \to \mathbb{R}$ [7, Definition 1.58] is referred to as $\mathcal{R}^\alpha$.

Definition 3.2. *In $\mathcal{R}^\alpha$ "α-circle plus" addition $\oplus_\alpha$ is defined as below:*

$$(k \oplus_\alpha F)(\omega) = k(\omega) + F(\omega) + \mu(\omega) k(\omega) F(\omega) \omega^{\alpha-1}, \ \forall \ \omega \in \mathbb{T}^k.$$

Definition 3.3. *For $h \in \mathcal{R}^\alpha$, define*

$$(\ominus_\alpha h)(v) = -\frac{h(v)}{1 + \mu(v) h(v) v^{\alpha-1}}, \ \forall \ v \in \mathbb{T}^k.$$

Definition 3.4. *Define "α-circle minus" subtraction $\ominus_\alpha$ on $\mathcal{R}^\alpha$ as below:*

$$(l \ominus_\alpha k)(u) = (l \oplus_\alpha (\ominus_\alpha k))(u), \ \forall \ v \in \mathbb{T}^k.$$

For $k, l \in \mathcal{R}^\alpha$, we have

$$(l \ominus_\alpha k)(u) = \frac{l(u) - k(u)}{1 + \mu(u) k(u) u^{\alpha-1}}.$$

Lemma 3.1. *Show that $(\mathcal{R}^\alpha, \oplus_\alpha)$ is an Abelian group.*

Proof. The proof is straight-forward, so it is left as an exercise. □

The group $(\mathcal{R}^\alpha, \oplus_\alpha)$ is also called the α-regressive group.

Remark 3.1. *For $\alpha = 1$, it becomes regressive group [7].*

Definition 3.5. *The set $\mathcal{R}^{\alpha^+}$ of all positively α-regressive elements of $\mathcal{R}^\alpha$ is defined by*

$$\mathcal{R}^{\alpha^+} = \mathcal{R}^{\alpha^+}(\mathbb{T}, \mathbb{R}) = \left\{ f : f \in \mathcal{R}^{\alpha},\, 1 + \mu(t) f(t) t^{\alpha-1} > 0,\, \forall\, t \in \mathbb{T}^k \right\}.$$

Theorem 3.1. *Suppose* $h, k \in \mathcal{R}^{\alpha}$*. Then*

(1) $(h \ominus_\alpha h)(\tau) = 0,$
(2) $\ominus_\alpha (\ominus_\alpha h)(\tau) = \varphi(\tau),$
(3) $(k \ominus_\alpha h)(\tau) \in \mathcal{R}^{\alpha},$
(4) $\ominus_\alpha (h \ominus_\alpha k)(\tau) = (k \ominus_\alpha h)(\tau),$
(5) $\ominus_\alpha (k \oplus_\alpha h)(\tau) = [(\ominus_\alpha k) \oplus_\alpha (\ominus_\alpha h)](\tau),$
(6) $\left[h \oplus_\alpha \dfrac{k}{1 + \mu h \tau^{\alpha-1}} \right](\tau) = h(\tau) + k(\tau).$

Proof. (1) By using Definitions 3.2–3.4 respectively, it follows that

$$\begin{aligned}
(h \ominus_\alpha h)(\tau) &= (h \oplus_\alpha (\ominus_\alpha h))(\tau) = h(\tau) \oplus_\alpha \left(\frac{-h(\tau)}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} \right) \\
&= h(\tau) - \frac{h(\tau)}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} - \frac{h^2(\tau) \mu(\tau) \tau^{\alpha-1}}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} \\
&= \frac{h(\tau) + h^2(\tau) \mu(\tau) \tau^{\alpha-1} - h(\tau) - h^2(\tau) \mu(\tau) \tau^{\alpha-1}}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} = 0.
\end{aligned}$$

(2) Definition 3.3 yields that

$$\begin{aligned}
\ominus_\alpha (\ominus_\alpha h)(\tau) &= \ominus_\alpha \left(\frac{-h(\tau)}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} \right) = \frac{\frac{h(\tau)}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}}}{1 - \frac{\mu(\tau) h(\tau) \tau^{\alpha-1}}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}}} \\
&= \frac{h(\tau)}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1} - \mu(\tau) h(\tau) \tau^{\alpha-1}} = h(\tau).
\end{aligned}$$

(3) By using Definitions 3.1 and 3.4 respectively, we have

$$\begin{aligned}
1 + \mu(\tau)(k \ominus_\alpha h)(\tau) \tau^{\alpha-1} &= 1 + \mu(\tau) \left(\frac{k(\tau) - h(\tau)}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} \right) \tau^{\alpha-1} \\
&= 1 + \frac{\mu(\tau) k(\tau) \tau^{\alpha-1} - \mu(\tau) h(\tau) \tau^{\alpha-1}}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} \\
&= \frac{1 + \mu(\tau) h(\tau) \tau^{\alpha-1} + \mu(\tau) k(\tau) \tau^{\alpha-1} - \mu(\tau) h(\tau) \tau^{\alpha-1}}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} \\
&= \frac{1 + \mu(\tau) k(\tau) \tau^{\alpha-1}}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} \neq 0.
\end{aligned}$$

We note that $\dfrac{k(\tau) - h(\tau)}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}}$ is right-dense continuous. Therefore, $(k \ominus_\alpha h)(\tau) \in \mathcal{R}^{\alpha}$.

(4) Using Definitions 3.3 and 3.4, it implies that

$$\begin{aligned}
\ominus_\alpha [h \ominus_\alpha k](\tau) &= \frac{-\left[\frac{h(\tau) - k(\tau)}{1 + \mu(\tau) k(\tau) \tau^{\alpha-1}} \right]}{1 + \mu(\tau) \left(\frac{h(\tau) - k(\tau)}{1 + \mu(\tau) k(\tau) \tau^{\alpha-1}} \right) \tau^{\alpha-1}} \\
&= \frac{k(\tau) - h(\tau)}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} = [k \ominus_\alpha h](\tau).
\end{aligned}$$

(5) By using Definitions 3.2 and 3.3 respectively, we get

$$\begin{aligned}
\ominus_\alpha [k \oplus_\alpha h](\tau) &= \ominus_\alpha \left(k(\tau) + h(\tau) + \mu(\tau) k(\tau) h(\tau) \tau^{\alpha-1}\right) \\
&= \frac{-\left[k(\tau) + h(\tau) + \mu(\tau) k(\tau) h(\tau) \tau^{\alpha-1}\right]}{1 + \mu(\tau)\left[k(\tau) + h(\tau) + \mu(\tau) k(\tau) h(\tau) \tau^{\alpha-1}\right] \tau^{\alpha-1}} \\
&= \frac{-\left[k(\tau) + h(\tau) + \mu(\tau) k(\tau) h(\tau) \tau^{\alpha-1}\right]}{1 + \mu(\tau) k(\tau) \tau^{\alpha-1} + \mu(\tau) h(\tau) \tau^{\alpha-1} + \mu^2(\tau) k(\tau) h(\tau) \tau^{2\alpha-2}} \\
&= \frac{-\left[k(\tau) + h(\tau) + \mu(\tau) k(\tau) h(\tau) \tau^{\alpha-1}\right]}{\left[1 + \mu(\tau) k(\tau) \tau^{\alpha-1}\right]\left[1 + \mu(\tau) h(\tau) \tau^{\alpha-1}\right]}.
\end{aligned}$$

On the other side, using Definitions 3.2 and 3.3 respectively, we obtain

$$\begin{aligned}
[(\ominus_\alpha k) \oplus_\alpha (\ominus_\alpha h)](\tau) &= \left(-\frac{k(\tau)}{1 + \mu(\tau) k(\tau) \tau^{\alpha-1}}\right) \oplus_\alpha \left(-\frac{h(\tau)}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}}\right) \\
&= -\frac{k(\tau)}{1 + \mu(\tau) k(\tau) \tau^{\alpha-1}} - \frac{h(\tau)}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} \\
&\quad + \frac{\mu(\tau) k(\tau) h(\tau) \tau^{\alpha-1}}{\left[1 + \mu(\tau) k(\tau) \tau^{\alpha-1}\right]\left[1 + \mu(\tau) h(\tau) \tau^{\alpha-1}\right]} \\
&= \frac{-\left(k(\tau) + h(\tau) + \mu(\tau) k(\tau) h(\tau) \tau^{\alpha-1}\right)}{\left[1 + \mu(\tau) k(\tau) \tau^{\alpha-1}\right]\left[1 + \mu(\tau) h(\tau) \tau^{\alpha-1}\right]}.
\end{aligned}$$

Hence,

$$\ominus_\alpha (k \oplus_\alpha h)(\tau) = [(\ominus_\alpha k) \oplus_\alpha (\ominus_\alpha h)](\tau).$$

(6) From Definition 3.2, we have

$$\begin{aligned}
\left[h \oplus_\alpha \frac{k}{1 + \mu h \tau^{\alpha-1}}\right](\tau) &= h(\tau) + \frac{k(\tau)}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} + \frac{\mu(\tau) h(\tau) k(\tau) \tau^{\alpha-1}}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} \\
&= \frac{h(\tau) + \mu(\tau) h^2(\tau) \tau^{\alpha-1} + k(\tau) + \mu(\tau) h(\tau) k(\tau) \tau^{\alpha-1}}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} \\
&= \frac{h(\tau)\left[1 + \mu(\tau) h(\tau) \tau^{\alpha-1}\right] + k(\tau)\left[1 + \mu(\tau) h(\tau) \tau^{\alpha-1}\right]}{1 + \mu(\tau) h(\tau) \tau^{\alpha-1}} \\
&= h(\tau) + k(\tau).
\end{aligned}$$

□

Next, we define the generalized exponential function.

Definition 3.6. *Let $h \in \mathcal{R}^\alpha$, then the generalized exponential function is defined by*

$$E_h(r, 0) = \exp\left(\int_0^r \xi_{\mu(\tau)}\left(h(\tau) \tau^{\alpha-1}\right) \Delta\tau\right), \ \forall\, 0, r \in \mathbb{T}. \tag{3.1}$$

To be more precise, using the definition for the cylindrical transformation Definition 2.3 we obtain

$$E_h(r, 0) = \exp\left(\int_0^r \frac{1}{\mu(\tau)} \log\left(1 + \mu(\tau) h(\tau) \tau^{\alpha-1}\right) \Delta\tau\right), \ \forall\, 0, r \in \mathbb{T}.$$

Example 3.1. *When $\mathbb{T} = \mathbb{R}$, then*

$$E_h(r,0) = \exp\left(\int_0^r \xi_0\left(h(\tau)\tau^{\alpha-1}\right)d\tau\right) = \exp\left(\int_0^r h(\tau)\, d^\alpha\tau\right).$$

Example 3.2. *When $\mathbb{T} = \mathbb{Z}$, then*

$$\begin{aligned} E_h(r,0) &= \exp\left(\int_0^r \xi_1\left(h(\tau)\tau^{\alpha-1}\right)\Delta\tau\right) \\ &= \exp\left(\sum_{\tau=0}^{r-1}\xi\left(h(\tau)\tau^{\alpha-1}\right)\right) \\ &= \prod_{\tau=0}^{r-1}\left[1+h(\tau)\tau^{\alpha-1}\right]. \end{aligned}$$

Example 3.3. *When $\mathbb{T} = q^{\mathbb{N}}$, $q > 1$, then*

$$\begin{aligned} E_F(r,0) &= \exp\left(\int_0^r \xi_{(q-1)\omega}\left(F(\omega)\omega^{\alpha-1}\right)\Delta\omega\right) \\ &= \exp\left(\sum_{\omega=0}^{r-1}\left[\log\left(1+(q-1)F(\omega)\omega^\alpha\right)\right]\right) \\ &= \prod_{\omega=0}^{r-1}\left[1+(q-1)F(\omega)\omega^\alpha\right]. \end{aligned}$$

Remark 3.2. *The exponential function on arbitrary time scale [7] is obtained by choosing $\alpha = 1$.*

Remark 3.3. *Definition 2.3 in [17] is obtained by choosing $\mathbb{T} = \mathbb{R}$.*

3.1. Characteristics of generalized exponential function

Theorem 3.2. *(Semigroup property) If $f \in \mathcal{R}^\alpha$, then*

$$E_f(t,0)\,E_f(0,r) = E_f(t,r),\ \forall\, t,0,r \in \mathbb{T}.$$

Proof. By using Definition 3.6, we have

$$\begin{aligned} E_f(t,0)\,E_f(0,r) &= \exp\left(\int_0^t \xi_{\mu(\tau)}\left(f(\tau)\tau^{\alpha-1}\right)\Delta\tau\right)\exp\left(\int_r^0 \xi_{\mu(\tau)}\left(f(\tau)\tau^{\alpha-1}\right)\Delta\tau\right) \\ &= \exp\left(\int_r^t \xi_{\mu(\tau)}\left(f(\tau)\tau^{\alpha-1}\right)\Delta\tau\right) = E_f(t,r). \end{aligned}$$

□

Theorem 3.3. *Suppose $k \in \mathcal{R}^\alpha$. Then*

$$E_k^\Delta(v,0) = k(v)\,E_k(v,0)\,v^{\alpha-1}$$

and

$$E_k^{(\alpha)}(v,0) = k(v)\,E_k(v,0).$$

Proof. Let $\sigma(v) > v$. Then

$$\begin{aligned}
E_k^{\Delta}(v,0) &= \frac{E_k(\sigma(v),0) - E_k(v,0)}{\mu(v)} \\
&= \frac{\exp\left(\int_0^v \xi_{\mu(\tau)}\left(k(\tau)\tau^{\alpha-1}\right)\Delta\tau + \int_v^{\sigma(v)} \xi_{\mu(\tau)}\left(k(\tau)\tau^{\alpha-1}\right)\Delta\tau\right) - \exp\left(\int_0^v \xi_{\mu(\tau)}\left(k(\tau)\tau^{\alpha-1}\right)\Delta\tau\right)}{\mu(v)} \\
&= \frac{\left[\exp\left(\int_v^{\sigma(v)} \xi_{\mu(\tau)}\left(k(\tau)\tau^{\alpha-1}\right)\Delta\tau\right) - 1\right] E_k(v,0)}{\mu(v)}.
\end{aligned}$$

From Lemma 2.1, it implies that

$$E_k^{\Delta}(v,0) = \frac{\left[\exp\left(\mu(v)\,\xi_{\mu(v)}(k(v)\,v^{\alpha-1}\right)) - 1\right] E_k(v,0)}{\mu(v)}.$$

From Definition 2.3, it follows that

$$E_k^{\Delta}(v,0) = k(v)\,E_k(v,0)\,v^{\alpha-1}.$$

Hence we have

$$E_k^{(\alpha)}(v,0) = k(v)\,E_k(v,0).$$

□

Corollary 3.1. *Suppose $h \in \mathcal{R}^{\alpha}$. Then $E_h(u,0)$ is a solution of the following Cauchy problem:*

$$x^{(\alpha)}(u) = h(u)\,x(u),\ x(0) = 1. \tag{3.2}$$

Proof. Let $x(\cdot) = E_h(\cdot,0)$ be a solution of Eq (3.2). First note that

$$x(0) = E_h(0,0) = 1.$$

It remains to show that $E_h(u,0)$ satisfies Eq (3.2). By Theorem 3.3, it follows that

$$(E_h(u,0))^{(\alpha)} = h(u)\,E_h(u,0).$$

Therefore,

$$(x(u))^{(\alpha)} = h(u)\,x(u).$$

Hence $E_h(u,0)$ is a solution to the Cauchy problem (3.2). □

Corollary 3.2. *Suppose $h \in \mathcal{R}^{\alpha}$. Then $E_h(v,0)$ is the unique solution of the IVP (3.2).*

Proof. Suppose $x(\cdot)$ be any solution of the IVP (3.2). Then by using part (3) of Lemma 2.3, it follows that

$$\left(\frac{x(v)}{E_h(v,0)}\right)^{(\alpha)} = \frac{x^{(\alpha)}(v)\,E_h(v,0) - x(v)\,E_h^{(\alpha)}(v,0)}{E_h(\sigma(v),0)\,E_h(v,0)}.$$

According to our assumption and Theorem 3.3, it implies that

$$\left(\frac{x(v)}{E_h(v,0)}\right)^{(\alpha)} = \frac{h(v)\,x(v)\,E_h(v,0) - x(v)\,h(v)\,E_h(v,0)}{E_h(\sigma(v),0)\,E_h(v,0)} = 0.$$

Consequently, $x(v) = bE_h(v,0)$, where b is a constant. Thus,

$$1 = x(0) = bE_h(0,0) = b.$$

Hence, $x(v) = E_h(v,0)$ is the unique solution. □

Theorem 3.4. *Suppose $h \in \mathcal{R}^{\alpha}$. Then*

$$E_h(\sigma(u),0) = \left(1 + \mu(u)\,h(u)\,u^{\alpha-1}\right)E_h(u,0).$$

Proof. Since for right-dense points ($\sigma(u) = u$), the case is trivial. And for right-scattered points ($\sigma(u) > u$), by Lemma 2.2, we have

$$E_h(\sigma(u),0) - E_h(u,0) = E_h^{(\alpha)}(u,0)\,\mu(u)\,u^{\alpha-1}.$$

Theorem 3.3 implies that

$$E_h(\sigma(u),0) = \left(1 + \mu(u)\,h(u)\,u^{\alpha-1}\right)E_h(u,0),$$

which proves the desired result. □

Theorem 3.5. *If $l \in \mathcal{R}^{\alpha}$, then*

$$E_l(0,u) = \frac{1}{E_l(u,0)} = E_{\ominus_\alpha l}(u,0).$$

Proof. Let us consider the following IVP:

$$x^{(\alpha)}(u) = \ominus_\alpha l(u)\,x(u),\; x(0) = 1.$$

First note that

$$x(0) = E_{\ominus_\alpha l}(0,0) = 1.$$

Differentiating

$$(x)^{(\alpha)}(u) = \left(\frac{1}{E_l(u,0)}\right)^{(\alpha)} = \left(\frac{1}{E_l(u,0)}\right)^{\Delta} u^{1-\alpha}.$$

By part (2) of Lemma 2.3, Theorems 3.3, 3.4 and Definition 3.3 we obtain

$$(E_{\ominus_\alpha l}(u,0))^{(\alpha)} = \left(\frac{-E_l^{\Delta}(u,0)}{E_l(\sigma(u),0)\,E_l(u,0)}\right)u^{1-\alpha} = (\ominus_\alpha l)(u)\,E_{\ominus_\alpha l}(u,0).$$

Hence,

$$(x)^{(\alpha)}(u) = (\ominus_\alpha l)(u)\,x(u),$$

which proves the required result

$$\frac{1}{E_l(u,0)} = E_{\ominus_\alpha l}(u,0).$$

□

Theorem 3.6. *If $h, p \in \mathcal{R}^{\alpha}$, then*

$$E_h(\omega,0)\,E_p(\omega,0) = E_{h\oplus_\alpha p}(\omega,0).$$

Proof. Let us consider the IVP:

$$(x)^{(\alpha)}(\omega) = (h\oplus_\alpha p)(\omega)\,x(\omega),\ x(0) = 1.$$

We show that its solution is $E_h(\omega,0)\,E_p(\omega,0)$. We have

$$x(0) = E_h(0,0)\,E_p(0,0) = 1.$$

Now by using part (1) of Lemma 2.3 it implies that

$$\begin{aligned}(x)^{(\alpha)}(\omega) &= \left(E_h(\omega,0)\,E_p(\omega,0)\right)^{(\alpha)}\\ &= (E_h(\omega,0))^{(\alpha)}\,E_p(\sigma(\omega),0) + E_h(\omega,0)\left(E_p(\omega,0)\right)^{(\alpha)}.\end{aligned}$$

By Theorems 3.3, 3.4 and Definition 3.2, we have

$$\begin{aligned}(x)^{(\alpha)}(\omega) &= \left\{h(\omega)\left[1+\mu(\omega)\,p(\omega)\,\omega^{\alpha-1}\right] + p(\omega)\right\}E_h(\omega,0)\,E_p(\omega,0)\\ &= (h\oplus_\alpha p)(\omega)\,E_h(\omega,0)\,E_p(\omega,0).\end{aligned}$$

Therefore,

$$(x)^{(\alpha)}(\omega) = (h\oplus_\alpha p)(\omega)\,x(\omega),$$

which proves the desired result. □

Theorem 3.7. *Assume $h, p \in \mathcal{R}^{\alpha}$. Then*

$$\frac{E_h(\omega,0)}{E_p(\omega,0)} = E_{h\ominus_\alpha p}(\omega,0).$$

Proof. By using Theorems 3.5 and 3.6, we have

$$\frac{E_h(\omega,0)}{E_p(\omega,0)} = E_{h\oplus_\alpha(\ominus_\alpha p)}(\omega,0) = E_{h\ominus_\alpha p}(\omega,0).$$

By using Definitions 3.2–3.4, we have

$$(h\oplus_\alpha(\ominus_\alpha p))(\omega) = h(\omega) + (\ominus_\alpha p)(\omega) + \mu(\omega)\,h(\omega)\,(\ominus_\alpha p)(\omega) = (h\ominus_\alpha p)(\omega),$$

which proves the required result. □

Theorem 3.8. *Suppose $h \in \mathcal{R}^{\alpha}$. Then*

$$\left(\frac{1}{E_h(v,0)}\right)^{(\alpha)} = \frac{-h(v)}{E_h(\sigma(v),0)}.$$

Proof. By taking α-conformable fractional derivative on time scales, we have

$$\left(\frac{1}{E_h(v,0)}\right)^{(\alpha)} = \left(\frac{1}{E_h(v,0)}\right)^{\Delta} v^{1-\alpha}.$$

According to Theorems 3.3 and 3.5, it follows that

$$\left(\frac{1}{E_h(v,0)}\right)^{(\alpha)} = (\ominus_\alpha h) E_{\ominus_\alpha h}(v,0).$$

By using Definition 3.3 and Theorems 3.4, 3.5, we can obtain

$$\left(\frac{1}{E_h(v,0)}\right)^{(\alpha)} = \frac{-h(v)}{E_h(\sigma(v),0)},$$

which proves the required result. □

Theorem 3.9. *Let $h, l \in \mathcal{R}^\alpha$. Then*

$$E^{(\alpha)}_{h\ominus_\alpha l}(v,0) = \frac{[h(v) - l(v)] E_h(v,0)}{E_l(\sigma(v),0)}.$$

Proof. By taking α-conformable fractional derivative on time scales, we have

$$E^{(\alpha)}_{h\ominus_\alpha l}(v,0) = \left[E^{\Delta}_{h\ominus_\alpha l}(v,0)\right] v^{1-\alpha}.$$

Theorem 3.7 and using part (3) of Lemma 2.3 implies that

$$E^{(\alpha)}_{h\ominus_\alpha l}(v,0) = \left[\frac{E^{\Delta}_h(v,0) E_l(v,0) - E_h(v,0) E^{\Delta}_l(v,0)}{E_l(v,0) E_l(\sigma(v),0)}\right] v^{1-\alpha}.$$

By applying Theorem 3.3, we obtain

$$E^{(\alpha)}_{h\ominus_\alpha l}(v,0) = \frac{(h(v) - l(v)) E_h(v,0)}{E_l(\sigma(v),0)},$$

which proves the required result. □

Theorem 3.10. *Suppose $h \in \mathcal{R}^\alpha$ and $c, d, 0 \in \mathbb{T}$. Then*

$$(E_h(0,v))^{(\alpha)} = -h(v) E_h(0,\sigma(v))$$

and

$$\int_c^d h(v) E_h(0,\sigma(v)) \Delta^\alpha v = E_h(0,c) - E_h(0,d).$$

Proof. Firstly, according to Theorems 3.4 and 3.5, we obtain

$$h(v) E_h(0,\sigma(v)) = h(v)\left[1 + \mu(v)(\ominus_\alpha h)(v) v^{\alpha-1}\right] E_{\ominus_\alpha h}(v,0).$$

According to Definition 3.3, Theorems 3.3 and 3.5, it follows that

$$E_h^{(\alpha)}(0,v) = -h(v)\,E_h(0,\sigma(v)),$$

which proves our first part.

Now, taking both sides the α-conformable fractional integral, we have

$$\int_c^d h(v)\,E_h(0,\sigma(v))\,\Delta^\alpha v = E_h(0,c) - E_h(0,d),$$

which gives the desired result. □

Theorem 3.11. *Let $k \in \mathcal{R}^\alpha$. Then*

$$E_{\ominus_\alpha k}(\sigma(v),u) = \left(1 + (\ominus_\alpha k)(v)\,\mu(v)\,v^{\alpha-1}\right) E_{\ominus_\alpha k}(v,u).$$

Proof. By Lemma 2.2, it implies that

$$E_{\ominus_\alpha k}(\sigma(v),u) - E_{\ominus_\alpha k}(v,u) = E_{\ominus_\alpha k}^{(\alpha)}(v,u)\,\mu(v)\,v^{\alpha-1}.$$

From Theorem 3.3, we obtain

$$E_{\ominus_\alpha k}(\sigma(v),u) = \left[1 + (\ominus_\alpha k)(v)\,\mu(v)\,v^{\alpha-1}\right] E_{\ominus_\alpha k}(v,u).$$

Similarly, we find

$$E_k(u,\sigma(v)) = \left[1 + k(v)\,\mu(v)\,v^{\alpha-1}\right] E_k(u,v).$$

□

Theorem 3.12. *Let $k \in \mathcal{R}^\alpha$. Then*

$$(E_k(\eta,\omega))^{(\alpha)} = -k(\omega)\,E_k(\eta,\sigma(\omega))$$

and

$$\int_0^\eta k(\eta)\,E_k(\eta,\sigma(\omega))\,\Delta^\alpha\omega = E_k(\eta,0) - 1.$$

Proof. From Theorems 3.5 and 3.11, we obtain

$$k(\omega)\,E_k(\eta,\sigma(\omega)) = k(\omega)\left(1 + \ominus_\alpha k(\omega)\,\mu(\omega)\,\omega^{\alpha-1}\right) E_{\ominus_\alpha k}(\omega,\eta).$$

Definition 3.3, Theorems 3.3 and 3.5 implies that

$$E_k^{(\alpha)}(\eta,\omega) = -k(\omega)\,E_k(\eta,\sigma(\omega)),$$

which proves our first part.

Therefore, conformable α-fractional integration on time scales implies

$$\int_0^\eta k(\eta)\,E_k(\eta,\sigma(\omega))\,\Delta^\alpha\omega = E_k(\eta,0) - 1,$$

which proves the required result. □

4. Grönwall's inequality

Theorem 4.1. *Let* $y, f \in C_{rd}$ *and* $p \in \mathcal{R}^{\alpha^+}$. *Then*

$$(y)^{(\alpha)}(t) \leq p(t)\, y(t) + f(t), \ \ \textit{for all } t \in \mathbb{T}^k,$$

implies

$$y(t) \leq y(0)\, E_p(t,0) + \int_0^t E_p(t, \sigma(\tau))\, f(\tau)\, \Delta^\alpha \tau, \ \forall\, t \in \mathbb{T}.$$

Proof. By using part (1) of Lemma 2.3, Theorems 3.3 and 3.4, we have

$$\left[yE_{\ominus_\alpha p}(t,0)\right]^{(\alpha)}(t) = y^{(\alpha)}(t)\, E_{\ominus_\alpha p}(\sigma(t), 0) + \left(\frac{(\ominus_\alpha p)(t)}{1 + \mu(t)(\ominus_\alpha p)(t)\, t^{\alpha-1}}\right) y(t)\, E_{\ominus_\alpha p}(\sigma(t), 0).$$

Definition 3.3 implies that

$$\left[yE_{\ominus_\alpha p}(t,0)\right]^{(\alpha)}(t) = \left[y^{(\alpha)}(t) - p(t)\, y(t)\right] E_{\ominus_\alpha p}(\sigma(t), 0).$$

Now taking the α-conformable fractional integral on time scales, and by using Lemma 2.4, it implies that

$$y(t)\, E_{\ominus_\alpha p}(t,0) - y(0)\, E_{\ominus_\alpha p}(0,0) = \int_0^t \left[y^{(\alpha)}(\tau) - p(\tau)\, y(\tau)\right] E_{\ominus_\alpha p}(\sigma(\tau), 0)\, \Delta^\alpha \tau.$$

By given assumption and using Theorem 3.5, we obtain

$$y(t) \leq y(0)\, E_p(t,0) + \int_0^t f(\tau)\, \frac{E_{\ominus_\alpha p}(\sigma(\tau), 0)}{E_{\ominus_\alpha p}(t,0)} \Delta^\alpha \tau.$$

And hence the assertion follows by applying Theorem 3.2:

$$y(t) \leq y(0)\, E_p(t,0) + \int_0^t f(\tau)\, E_p(t, \sigma(\tau))\, \Delta^\alpha \tau,$$

which proves the required result. □

Theorem 4.2. *Let* $y, f \in C_{rd}$ *and* $p \in \mathcal{R}^{\alpha^+}$, $p \geq 0$. *Then*

$$y(t) \leq f(t) + \int_0^t y(\tau)\, p(\tau)\, \Delta^\alpha \tau, \ \forall\, t \in \mathbb{T},$$

implies that

$$y(t) \leq f(t) + \int_0^t E_p(t, \sigma(\tau))\, f(\tau)\, p(\tau)\, \Delta^\alpha \tau, \ \forall\, t \in \mathbb{T}.$$

Proof. Define

$$z(t) := \int_0^t y(\tau)\, p(\tau)\, \Delta^\alpha \tau, \ \forall\, t \in \mathbb{T}.$$

Then $z(0) = 0$ and

$$y(t) \leq f(t) + z(t). \tag{4.1}$$

By using Lemma 2.5 and Eq (4.1), we obtain

$$z^{(\alpha)}(t) = y(t)\,p(t) \leq f(t)\,p(t) + p(t)\,z(t).$$

Theorem 4.1 yields

$$z(t) \leq \int_0^t E_p(t, \sigma(\tau))\,p(\tau)\,f(\tau)\,\Delta^\alpha \tau.$$

And hence the claim follows because of Eq (4.1). Therefore,

$$y(t) \leq f(t) + \int_0^t E_p(t, \sigma(\tau))\,f(\tau)\,p(\tau)\,\Delta^\alpha \tau,$$

which completes the proof. □

Theorem 4.3. *(Grönwall's inequality) Let* $y, p \in C_{rd}$, $p \in \mathcal{R}^{\alpha^+}$ *and* $\lambda \geq 0$ *such that*

$$y(t) \leq \lambda + \int_0^t y(\tau)\,p(\tau)\,\Delta^\alpha \tau,\ \forall\ t \in [0, b]_{\mathbb{T}},$$

then

$$y(t) \leq \lambda E_p(t, 0),\ \forall\ t \in [0, b]_{\mathbb{T}}. \tag{4.2}$$

Proof. Let $f(t) = \lambda$. Then by Theorem 4.2, it follows that

$$y(t) \leq \lambda \left[1 + \int_0^t E_p(t, \sigma(\tau))\,p(\tau)\,\Delta^\alpha \tau\right].$$

From Theorem 3.12, we obtain

$$y(t) = \lambda \left[1 + E_p(t, 0) - E_p(t, t)\right] = \lambda E_p(t, 0).$$

Therefore,

$$y(t) \leq \lambda E_p(t, 0),$$

which completes the proof. □

Remark 4.1. *For* $\alpha = 1$ *and* $\mathbb{T} = \mathbb{R}$*, we obtained the Grönwall's inequalities in classical calculus.*

5. Local initial value problems

We will develop some conditions for the global existence, extension and boundedness of solutions related to the local initial value problem (LIVP) in the following section.

The following assumptions will be needed throughout the following section.

Suppose $\Omega = [0, \infty)_{\mathbb{T}} \times \mathbb{R}$.

(H1) The mapping $k : \Omega \to \mathbb{R}$ is right-dense continuous.

(H2) A positive constant $K > 0$ exists in such manner that, for all (τ, x), (τ, x) in Ω,

$$|k(\tau, x) - k(\tau, y)| \leq K|x - y|.$$

(H3) A nonnegative mapping $l \geq 0$ exists in such manner that, for all (τ, x) in Ω,

$$|k(\tau, x)| \leq l(\tau)|x|,$$

for which

$$\int_0^\tau \xi_{\mu(r)}(l(r))\,\Delta^\alpha r$$

is bounded on $[0, \infty)_{\mathbb{T}}$.

(H4) A nonnegative mapping $h \geq 0$ and a positive constant $K > 0$ exists in such manner that, for all (τ, x), (τ, y) in Ω,

$$|k(\tau, x) - k(\tau, y)| \leq h(\tau)|x - y| \leq K|x - y|,$$

for which

$$\int_0^\tau \xi_{\mu(r)}(h(r))\,\Delta^\alpha r$$

is bounded on $[0, \infty)_{\mathbb{T}}$.

Through Lemmas 2.4 and 2.5, the LIVP (1.1) and (1.2) is simply transformed into an Integral Equation (IE).

Lemma 5.1. *If (H1) holds, then a function ψ in $C([0, b]_{\mathbb{T}})$ is a solution of local initial value problem (1.1) and (1.2) if and only if ψ is a continuous solution of the following integral equation:*

$$\psi(\tau) = \psi_0 + \int_0^\tau k(r, \psi(r))\,\Delta^\alpha r,\ \tau \in [0, a]_{\mathbb{T}}. \tag{5.1}$$

We can now demonstrate the existence and uniqueness of the solution to the LIVP (1.1) and (1.2) as a consequence of Definition 3.6.

Theorem 5.1. *The local initial value problem (1.1) and (1.2) has unique solution defined on $[0, a]_{\mathbb{T}}$ whenever the assumptions (H1) and (H2) hold.*

Proof. The claim will be verified via Banach's contraction principle on $C([0, a]_{\mathbb{T}})$. Let $k > 0$ be a constant and $k \in R^{\alpha^+}$ and let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$. Define the interval $[0, a]_{\mathbb{T}}$. Let us denote by $C([0, a]_{\mathbb{T}})$ the space of continuous functions along with a suitable norm. Define the term "TZ-norm"

$$\|\psi\|_k = \sup_{\tau \in [0, \sigma(a)]_{\mathbb{T}}} \frac{\|\psi(\tau)\|}{E_k(\tau, 0)},$$

where $E_k(\tau, 0)$ in Defintion 3.6. The well-known sup-norm

$$\|\psi\|_0 = \sup_{\tau \in [0, \sigma(a)]_{\mathbb{T}}} \|\psi(\tau)\|.$$

It is simple to prove that $\|\cdot\|_k$ is equivalent to $\|\cdot\|_0$. Hence $(C([0, a]_{\mathbb{T}}), \|\cdot\|_k)$ is Banach space.

Define an operator

$$T : (C([0, a]_{\mathbb{T}}), \|\cdot\|_k) \longrightarrow (C([0, a]_{\mathbb{T}}), \|\cdot\|_k)$$

as

$$T(\psi(\tau)) = \psi(0) + \int_0^\tau k(r, \psi(r))\,\Delta^\alpha r.$$

Lemma 5.1 assures that the fixed points of the operator T are the solutions of local IVP (1.1) and (1.2).
For any $\psi, \varphi \in (C([0,a]_{\mathbb{T}}), \|\cdot\|_k)$, then

$$\begin{aligned}\|T(\psi) - T(\varphi)\|_k &= \sup_{\tau\in[0,\sigma(a)]_{\mathbb{T}}} \frac{\|T(\psi(\tau)) - T(\varphi(\tau))\|}{E_k(\tau,0)} \\ &\le \sup_{\tau\in[0,\sigma(a)]_{\mathbb{T}}} \left[\tfrac{1}{E_k(\tau,0)} \int_0^\tau \|k(r,\psi(r)) - k(r,\varphi(r))\| \Delta^\alpha r\right].\end{aligned}$$

By (H2), it follows that

$$\begin{aligned}\|T(\psi) - T(\varphi)\|_k &\le \sup_{\tau\in[0,\sigma(a)]_{\mathbb{T}}} \left[\tfrac{1}{E_k(\tau,0)} \int_0^\tau K\|\psi(r) - \varphi(r)\| \Delta^\alpha r\right] \\ &= K \sup_{\tau\in[0,\sigma(a)]_{\mathbb{T}}} \left[\tfrac{1}{E_k(\tau,0)} \int_0^\tau E_k(r,0)\frac{\|\psi(r) - \varphi(r)\|}{E_k(r,0)} \Delta^\alpha r\right] \\ &\le K \sup_{r\in[0,\sigma(a)]_{\mathbb{T}}} \frac{\|\psi(r) - \varphi(r)\|}{E_k(r,0)} \times \sup_{\tau\in[0,\sigma(a)]_{\mathbb{T}}} \left[\tfrac{1}{E_k(\tau,0)} \int_0^\tau E_k(r,0) \Delta^\alpha r\right] \\ &= K\|\psi - \varphi\|_k \sup_{\tau\in[0,\sigma(a)]_{\mathbb{T}}} \left[\tfrac{1}{E_k(\tau,0)} \int_0^\tau E_k(r,0) \Delta^\alpha r\right].\end{aligned}$$

Thus, it implies that

$$\|T(\psi) - T(\varphi)\|_k < K\|\psi - \varphi\|_k \sup_{\tau\in[0,\sigma(a)]_{\mathbb{T}}} \left[\tfrac{1}{E_k(\tau,0)} \int_0^\tau E_k(r,0) \Delta^\alpha r\right]. \tag{5.2}$$

Now we have to find

$$\int_0^\tau E_k(r,0) \Delta^\alpha r.$$

By Theorem 3.3, it implies that

$$E_k(r,0) = \tfrac{1}{k} E_k^{(\alpha)}(r,0),$$

where $k > 0$ be a positive constant. Then by taking α-conformable fractional integral, it follows that

$$\int_0^\tau E_k(r,0) \Delta^\alpha r = \int_0^\tau \tfrac{1}{k} E_k^{(\alpha)}(r,0) \Delta^\alpha r.$$

By using Lemma 2.4, we obtain

$$\int_0^\tau E_k(r,0) \Delta^\alpha r = \tfrac{1}{k}\left[E_k(\tau,0) - 1\right]. \tag{5.3}$$

Now, using Eq (5.3) in Eq (5.2), it follows that

$$\begin{aligned}\|T(\psi) - T(\varphi)\|_k &< K\|\psi - \varphi\|_k \sup_{\tau\in[0,\sigma(a)]_{\mathbb{T}}} \left[\tfrac{1}{kE_k(\tau,0)}\left[E_k(\tau,0) - 1\right]\right] \\ &< \frac{K}{k}\|\psi - \varphi\|_k \sup_{\tau\in[0,\sigma(a)]_{\mathbb{T}}} \left[1 - \frac{1}{E_k(\tau,0)}\right] \\ &< \frac{K}{k}\|\psi - \varphi\|_k \sup_{\tau\in[0,\sigma(a)]_{\mathbb{T}}} \left[1 - \frac{1}{E_k(\sigma(a),0)}\right].\end{aligned}$$

Therefore,

$$\|T(\psi) - T(\varphi)\|_k < \frac{K}{k}\|\psi - \varphi\|_k.$$

As $0 < \dfrac{K}{k} < 1$, we observe that T is a contractive map and the Banach contraction principle assures that there exists only one solution ψ in $C([0,a]_{\mathbb{T}})$ such that $T(\psi) = \psi$, and therefore the LIVP (1.1) and (1.2) has unique ψ in $C([0,a]_{\mathbb{T}})$. This completes the proof. □

Next, we investigate the expansion to the right of the solutions of LIVP (1.1) and (1.2).

Lemma 5.2. *Suppose $\psi(\tau)$ is a solution to the local IVP (1.1) and (1.2) defined on $[0,\tau^+)_{\mathbb{T}}$ such that $\tau^+ \neq \infty$. If $\lim_{\tau\to\tau^+}\psi(\tau)$ exists, then $\psi(\tau)$ can be expanded to $[0,\tau^+]_{\mathbb{T}}$ provided the hypothesis (H1) holds.*

Proof. Here, τ^+ is a right-dense point. Let $\lim_{\tau\longrightarrow\tau^+}\psi(\tau) = \psi^+$. Now suppose $\mathbf{J} = [0,\tau^+)_{\mathbb{T}}$ and define a function $\tilde{\psi}(\tau)$ by

$$\tilde{\psi}(\tau) = \begin{cases} \psi(\tau), & \tau \in [0,\tau^+)_{\mathbb{T}}, \\ \psi^+, & \tau = \tau^+. \end{cases}$$

By [6, part (i) of Theorem 4] and since $\lim_{\tau\longrightarrow\tau^+}\psi(\tau) = \psi^+$, therefore the function $\tilde{\psi}(\tau)$ is surely continuous on $[0,\tau^+]_{\mathbb{T}}$. We next demonstrate that the function $\tilde{\psi}(\tau)$ is also a solution of the LIVP (1.1) and (1.2) defined on $[0,\tau^+]_{\mathbb{T}}$, and obviously, it is sufficient to prove

$$\tilde{\psi}^{(\alpha)}(\tau^+) = k\left(\tau^+, \tilde{\psi}(\tau^+)\right).$$

Note that the equation

$$\tilde{\psi}^{(\alpha)}(\tau) = k\left(\tau, \tilde{\psi}(\tau)\right), \tau \in [0,\tau^+)_{\mathbb{T}}.$$

And the continuities of $\tilde{\psi}$ and k gives that

$$\lim_{\tau\to\tau^+}\tilde{\psi}^{(\alpha)}(\tau) = k\left(\tau^+, \tilde{\psi}(\tau^+)\right). \tag{5.4}$$

Moreover, using mean value theorem [23, Theorem 15], we see that for all τ in $[0,\tau^+)_{\mathbb{T}}$, $\exists$ a point ζ in $[\tau,\tau^+]_{\mathbb{T}}^k$ such that

$$\tilde{\psi}^{(\alpha)}(\zeta) = \frac{\tilde{\psi}(\tau^+) - \tilde{\psi}(\tau)}{\tau^+ - \tau}\zeta^{1-\alpha}, \ \zeta \in [\tau,\tau^+]_{\mathbb{T}}^k.$$

Now taking the $\lim_{\tau\to\tau^+}$ on both sides and using Eq (5.4), we obtain

$$k\left(\tau^+, \tilde{\psi}(\tau^+)\right) = \left(\lim_{\tau\to\tau^+}\frac{\tilde{\psi}(\tau^+) - \tilde{\psi}(\tau)}{\tau^+ - \tau}\right)\left(\lim_{\tau\to\tau^+}\tau^{1-\alpha}\right) = \tilde{\psi}^{\Delta}(\tau^+)(\tau^+)^{1-\alpha}.$$

By Lemma 2.2, we conclude that

$$\tilde{\psi}^{(\alpha)}(\tau^+) = k\left(\tau^+, \tilde{\psi}(\tau^+)\right).$$

Hence, we have shown that the function $\tilde{\psi}(\tau)$ is also a solution of the LIVP (1.1) and (1.2) defined on $[0,\tau^+]_{\mathbb{T}}$, and it is an extension of the solution $\psi(\tau)$ to $[0,\tau^+]_{\mathbb{T}}$. Therefore, the required result follows. □

Definition 5.1. *Suppose* $\mathbf{I}$ *is the maximal existence interval of the solution* $\psi(\tau)$ *of the LIVP (1.1) and (1.2), then* $\psi(\tau)$ *is said to be come arbitrarily close to the boundary of* $\Omega = [0,\infty)_{\mathbb{T}} \times \mathbb{R}$ *to the right if it is not possible for every closed and bounded domain* Ω_0 *in* Ω*, the point* $(\tau, \psi(\tau))$ *always remains in* Ω_0 *for all* τ *in* $\mathbf{I}$.

Theorem 5.2. *If (H1) and (H2) hold, then the solution of the local initial value problem (1.1) and (1.2) comes arbitrarily close to the boundary of* $\Omega = [0,\infty)_{\mathbb{T}} \times \mathbb{R}$ *to the right.*

Proof. The local IVP (1.1) and (1.2) has a unique solution by Theorem 5.1, and refers to the solution by $\psi(\tau)$. Suppose $\boldsymbol{I}$ refers to the maximal existence interval of $\psi(\tau)$. Again, we conclude that using Theorem 5.1, $\mathbf{I} = [0,\infty)_{\mathbb{T}}$ or $[0,\tau^+)_{\mathbb{T}}$ with $\tau^+ \neq \infty$. The required result is clear when $\mathbf{I} = [0,\infty)_{\mathbb{T}}$. Now, assume the case $\mathbf{I} = [0,\tau^+)_{\mathbb{T}}$ with $\tau^+ \neq \infty$. Conversely, suppose that the desired result is not true. That is, the solution $\psi(\tau)$ of the LIVP (1.1) and (1.2) does not go arbitrarily near to the boundary of $\Omega = [0,\infty)_{\mathbb{T}} \times \mathbb{R}$ to the right implies that $\Omega_0 \subset \Omega$ exists such that Ω_0 is closed and bounded with $(\tau, \psi(\tau)) \in \Omega_0$, $\forall\, \tau \in \mathbf{I}$. Because k on $\Omega_0 \subset \Omega$ is continuous, a positive number C must exist such that

$$|k(\tau, \psi(\tau))| \leq C \text{ for all } \tau \in \mathbf{I}. \tag{5.5}$$

Furthermore, mean value theorem [23] ensures that, for any τ_1, τ_2 in $\mathbf{I}$ with $\tau_1 < \tau_2$, a point ξ exists in $[\tau_1, \tau_2]_{\mathbb{T}}^k$ such that

$$\psi(\tau_2) - \psi(\tau_1) = \left[\xi^{\alpha-1} (k)^{(\alpha)}(\xi)\right](\tau_2 - \tau_1).$$

From Eqs (1.1) and (5.5), it follows that

$$\begin{aligned} |\psi(\tau_2) - \psi(\tau_1)| &= \frac{|k(\xi, \psi(\xi))|}{\xi^{1-\alpha}} [|\tau_2 - \tau_1|] \\ &\leq \frac{C}{\xi^{1-\alpha}} [|\tau_2 - \tau_1|]. \end{aligned}$$

Therefore,

$$\begin{aligned} &|\psi(\tau_2) - \psi(\tau_1)| < \epsilon \text{ whenever } |\tau_2 - \tau_1| < \frac{\epsilon \xi^{1-\alpha}}{C} = \delta, \\ &|\psi(\tau_2) - \psi(\tau_1)| < \epsilon \text{ whenever } |\tau_2 - \tau_1| < \delta, \end{aligned}$$

which shows that $\psi(\tau)$ is uniformly continuous on $\mathbf{I}$, and hence $\lim\limits_{\tau \longrightarrow \tau^+} \psi(\tau)$ exists. And thus by using Lemma 5.2, the solution $\psi(\tau)$ can be extended to the closed interval $[0,\tau^+]_{\mathbb{T}}$, it violates the statement that $[0,\tau^+)_{\mathbb{T}}$ is the maximal existence interval of $\psi(\tau)$. Therefore, it follows the required result. □

By the use of Theorems 5.1, 5.2 and Grönwall's inequality (4.3), we now present a result to ensure that the solution of Eq (1.1) with the local initial condition (1.2) is defined and bounded on $[0,\infty)_{\mathbb{T}}$.

Theorem 5.3. *The solution of the local IVP (1.1) and (1.2) is defined and bounded on* $[0,\infty)_{\mathbb{T}}$ *whenever (H1)–(H3) hold.*

Proof. The LIVP has only one solution, according to Theorem 5.1. Define the solution as $\psi(\tau)$, with $[0,\tau^+)_{\mathbb{T}}$ as its maximal existence interval. Now we have to show that $\tau^+ = \infty$ and $\psi(\tau)$ is bounded on $[0,\tau^+)_{\mathbb{T}}$. Based on assumption (H3), Eq (5.1) follows that

$$|\psi(\tau)| \leq |\psi(0)| + I_\alpha \left[k(\tau) |\psi(\tau)|\right]. \tag{5.6}$$

And so, Eq (5.6) follows using Grönwall's inequality (4.3):

$$\begin{aligned}|\psi(\tau)| &\leq |\psi_0| E_k(\tau, 0), \ \forall \tau \in [0, \tau^+)_{\mathbb{T}}, \\ &\leq |\psi_0| \left[\exp\left(\int_0^{\tau} \xi_{\mu(\nu)}(k(\nu)) \nu^{\alpha-1} \Delta\nu\right)\right].\end{aligned}$$

It should be noted that the hypothesis of boundedness of $\int_0^{\tau} \xi_{\mu(\nu)}(k(\nu)) \nu^{\alpha-1} \Delta\nu$ would imply that a positive number C exists so that $\int_0^{\tau} \xi_{\mu(\nu)}(k(\nu)) \nu^{\alpha-1} \Delta\nu \leq C$. Therefore,

$$|\psi(\tau)| \leq |\psi_0| \left[e^C\right], \ \forall \tau \in [0, \tau^+)_{\mathbb{T}}.$$

Thus $\psi(\tau)$ is bounded on $[0, \tau^+)_{\mathbb{T}}$. When $\tau^+ \neq \infty$, then by Theorem 5.2 clearly follows that

$$\lim_{\tau \to \tau^+} \psi(\tau) = \infty.$$

This is in contradiction with the boundedness of $\psi(\tau)$ on $[0, \tau^+)_{\mathbb{T}}$. Therefore, $\tau^+ = \infty$, and consequently follows the required result. □

Through the Grönwall's inequality (4.3), we are further investigating the stability of the solutions to the LIVP (1.1) and (1.2).

Definition 5.2. *Assume $\psi(\tau)$ be a solution to Eq (1.1) defined on $[0, \infty)_{\mathbb{T}}$ with $\psi(0) = \psi_0$. The solution $\psi(\tau)$ is called stable if for all positive number ϵ, $\exists$ a positive number δ such that each solution $\varphi(\tau)$ with $|\varphi(0) - \psi(0)| < \delta$ holds for any $\tau \geq 0$ and satisfies the inequality*

$$|\varphi(\tau) - \psi(\tau)| < \epsilon, \text{for } \tau \geq 0.$$

Theorem 5.4. *Every solution of the LIVP (1.1) and (1.2) is always stable whenever the hypothesis (H1), (H3) and (H4) holds.*

Proof. By Theorem 5.3, the solution of LIVP (1.1) and (1.2) always exists and is defined on $[0, \infty)_{\mathbb{T}}$. Let $\psi(\tau)$ be a solution with $\psi(0) = \psi_0$ and $\varphi(\tau)$ a solution with $\varphi(0) = \varphi_0$. Then

$$\psi(\tau) = \psi_0 + I_\alpha k(\tau, \psi(\tau))$$

and

$$\varphi(\tau) = \varphi(0) + I_\alpha k(\tau, \varphi(\tau)).$$

By (H4), it follows that

$$|\varphi(\tau) - \psi(\tau)| \leq |\varphi_0 - \psi_0| + I_\alpha \left[h(\tau) |\varphi(\tau) - \psi(\tau)|\right]. \tag{5.7}$$

And Eq (5.7) follows, using Grönwall's inequality (4.3) and the hypothesis of boundedness of $\int_0^{\tau} \xi_{\mu(\nu)}(h(\nu)) \Delta^\alpha \nu$ would imply that a positive number C exists so that $\int_0^{\tau} \xi_{\mu(\upsilon)}(h(\upsilon)) \Delta^\alpha \upsilon \leq C$. Therefore,

$$|\varphi(\tau) - \psi(\tau)| \leq |\varphi_0 - \psi_0| e^C, \ \forall \tau \in [0, \infty)_{\mathbb{T}}.$$

By Definition 5.2, it follows that

$$|\varphi(\tau) - \psi(\tau)| < \epsilon.$$

Hence $\psi(\tau)$ is stable on $[0, \infty)_{\mathbb{T}}$. □

6. Non-local initial value problems

The existence of solutions to the non-local initial value problem (NLIVP) is addressed in this section. Then, in order to prove the main result, we recall a fixed point theorem that will be used in this section.

Lemma 6.1. *(Fixed point theorem) U is an open set in a Banach space B's closed, convex set C. Suppose $0 \in U$. It is also assumed that $A\left(\bar{U}\right)$ is bounded and that $A : \bar{U} \to C$ is given by*

$$A = A_1 + A_2,$$

in which

$$A_1 : \bar{U} \to B \text{ is completely continuous},$$

and

$$A_2 : \bar{U} \to B \text{ is a nonlinear contraction.}$$

That is, a nonnegative nondecreasing function $\phi : [0, \infty)_{\mathbb{T}} \to [0, \infty)_{\mathbb{T}}$ exists satisfying $\phi(z) < z$ for $z > 0$, such that

$$\|A_2(x) - A_2(y)\|_k \leq \phi(\|x - y\|_k),$$

for any $x, y \in \bar{U}$. Then either (C1) A has a fixed point $u \in \bar{U}$; or (C2) there exist a point $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u = \lambda A(u)$, where $\bar{U}$ and ∂U refers to the closure and boundary of U, respectively.

Further we need the following hypothesis.

(H5) k is a right-dense continuous function defined on $[0, a]_{\mathbb{T}} \times \mathbb{R}$.

(H6) There exists a positive constant γ in $(0, 1)$ and a nonnegative and nondecreasing function ϕ in $C([0, \infty)_{\mathbb{T}})$ with $\phi(\varsigma) < \gamma\varsigma$, $\varsigma > 0$ and $|g(\tau) - g(\nu)|_k \leq \phi(\|\tau - \nu\|_k)$ for all τ, ν in $C([0, a]_{\mathbb{T}})$.

(H7) There is a nonnegative function $\varphi \in C([0, a]_{\mathbb{T}})$ such that $\varphi > 0$ on a subinterval of $[0, a]_{\mathbb{T}}$, as well as a nonnegative and nondecreasing function $\Psi \in C([0, \infty)_{\mathbb{T}})$,

$$|k(t, u)| \leq \varphi(t) \Psi(|u|),$$

for each (t, u) in $[0, a]_{\mathbb{T}} \times \mathbb{R}$ and

$$\sup_{r \in (0, \infty)} \frac{r}{|\psi_0| + \Psi(r) I_\alpha \varphi(a)} > \frac{1}{1 - \gamma}.$$

The following lemma is easy to verify by Lemmas 2.4 and 2.5.

Lemma 6.2. *A function ψ in $C([0, a]_{\mathbb{T}})$ satisfies the NLIVP (1.1) and (1.3) provided that assumption (H5) holds iff ψ is a continuous as well as solution of IE:*

$$\psi(\tau) = \psi_0 - g(\psi) + I_\alpha k(\tau, \psi(\tau)), \ \tau \in [0, a]_{\mathbb{T}}. \tag{6.1}$$

We first define some sets of functions in $C([0, a]_{\mathbb{T}})$ and operators in order to use FPT to address the existence of solutions to the NLIVP.

Given a positive number r and $k > 0$ be a positive constant, define the subset u_r of $C([0, a]_{\mathbb{T}})$ by

$$u_r = \{u \in C([0, a]_{\mathbb{T}}) : \|u\|_k < r\}. \tag{6.2}$$

Similarly, we define

$$\bar{u}_r = \{u \in C([0,a]_{\mathbb{T}}) : \|u\|_k \leq r\}. \tag{6.3}$$

Also, define three operators from the space $C([0,a]_{\mathbb{T}})$ to itself, respectively, by

$$A_1\psi(t) = I_\alpha k(t, \psi(t)), \tag{6.4}$$

$$A_2\psi(t) = \psi_0 - g(\psi), \tag{6.5}$$

$$A\psi(t) = A_1\psi(t) + A_2\psi(t). \tag{6.6}$$

Using the standard arguments, the complete continuity of the operator $A_1 : \bar{u}_r \mapsto C([0,a]_{\mathbb{T}})$ can be verified, and it is also easy to check that the operator $A_2 : \bar{u}_r \mapsto C([0,a]_{\mathbb{T}})$ is a nonlinear contraction under the condition (H6). Here we omit their proofs.

Lemma 6.3. *The operator $A_1 : \bar{u}_r \mapsto C([0,a]_{\mathbb{T}})$ is completely continuous provided (H5) holds.*

Lemma 6.4. *The operator $A_2 : \bar{u}_r \mapsto C([0,a]_{\mathbb{T}})$ is a nonlinear contraction provided (H6) holds.*

In this section, we are now presenting the main result.

Theorem 6.1. *There exists at least one solution defined on $[0,a]_{\mathbb{T}}$ of the NLIVP (1.1) and (1.3) whenever (H5)–(H7) holds.*

Proof. A positive number r exists in view of the hypothesis of supremum in (H7), such that

$$\frac{r}{|\psi_0| + \Psi(r) I_\alpha \varphi(a)} > \frac{1}{1-\gamma}. \tag{6.7}$$

And then we define the set u_r by

$$u_r = \{u \in C([0,a]_{\mathbb{T}}) : \|u\|_k < r\}.$$

We first show that the operators A, A_1, A_2 satisfy the corresponding conditions of FPT and owing to Lemmas 6.3 and 6.4, we just need to demonstrate the ßoundedness of $A(\bar{u}_r)$. Infact, for each ψ in $\bar{u}_r$ it implies under the assumptions (H5) and (H6) that

$$|A_1\psi(t)| \leq I_\alpha |k(t, \psi(t))| \leq \frac{a^\alpha}{\alpha} \sup\{|k(t,u)| : t \in [0,a]_{\mathbb{T}},\ |u| \leq r\} \tag{6.8}$$

and

$$|A_2\psi(t)| = |\psi_0 - g(\psi)| \leq |\psi_0| + |g(\psi)|. \tag{6.9}$$

According to Eq (6.3), it implies that

$$|g(\psi)| \leq \phi(\|\psi\|_k) \leq \gamma \|\psi\|_k \leq \gamma r. \tag{6.10}$$

By using Eq (6.10) in Eq (6.9), it follows that

$$|A_2\psi(t)| \leq |\psi_0| + \gamma r. \tag{6.11}$$

Hence, according the definition of the operator A and using Eqs (6.8) and (6.11), we have

$$\|A\psi\|_k \leq |\psi_0| + \gamma r + \frac{a^\alpha}{\alpha} \sup\{|k(t,u)| : t \in [0,a]_{\mathbb{T}},\ |u| \leq r\},$$

which justifies the uniform boundedness of the set $A(\bar{u}_r)$.

Lastly, it remains to be shown that the case (C2) does not occur in the FPT (6.1). We claim, by contradiction. Suppose (C2) holds implies that $\lambda \in (0,1)$ and $\psi \in \partial u_r$ exists with $\psi = \lambda A(\psi)$, that is,

$$\psi(t) = \lambda\left[\psi_0 - g(\psi) + I_\alpha k(t,\psi(t))\right].$$

Under the hypothesis (H6) and (H7), it further follows that

$$|\psi(t)| \leq \lambda\left[|\psi_0| + |g(\psi)| + I_\alpha |k(t,\psi(t))|\right]. \tag{6.12}$$

By using Eq (6.3), it implies that

$$|k(t,\psi(t))| \leq \varphi(t)\,\Psi(|\psi(t)|) \leq \varphi(t)\,\Psi(r). \tag{6.13}$$

Using Eqs (6.10) and (6.13) in Eq (6.12), it becomes

$$|\psi(t)| \leq \lambda\left[|\psi_0| + \gamma r + I_\alpha \varphi(t)\,\Psi(r)\right].$$

Hence,

$$|\psi(t)| \leq \left[|\psi_0| + \gamma r + \Psi(r)\, I_\alpha \varphi(t)\right].$$

But

$$r \leq \sup_{t \in [0,a]_{\mathbb{T}}} \left[|\psi_0| + \gamma r + \Psi(r)\, I_\alpha \varphi(t)\right] \leq |\psi_0| + \gamma r + \Psi(r)\, I_\alpha \varphi(a).$$

This implies that

$$\frac{r}{|\psi_0| + \Psi(r)\, I_\alpha \varphi(a)} \leq \frac{1}{1-\gamma},$$

which is in contradiction with inequality (6.7). We have therefore shown that the operators A, A_1 and A_2 meet all the conditions in FPT (6.1), and hence we deduce that the operator A has at least one fixed point ψ in $\bar{u}_r$, which satisfies the NLIVP. □

Remark 6.1. *For $\alpha = 1$ and $\mathbb{T} = \mathbb{R}$, the classical results corresponding to ordinary differential equations will be yielded.*

Example 6.1. *Assume $\mathbb{T} = \mathbb{R}$, $r_0 = 0$ and $p(s) = -1$, then $E_p(s,s_0) = e^{-\frac{s^\alpha}{\alpha}}$. Also let $\Omega = [0,\infty) \times \mathbb{R}$, $k(s,x) = e^{-\frac{s^\alpha}{\alpha}}(x + \sin x)$, $l(s) = h(s) = 2e^{-\frac{s^\alpha}{\alpha}}$ and $K = 2$.*

(I) Local initial value problems.

For all (s,x), $(s,y) \in \Omega$,

$$|k(s,x) - k(s,y)| = \left|e^{-\frac{s^\alpha}{\alpha}}(x + \sin x) - e^{-\frac{s^\alpha}{\alpha}}(y + \sin y)\right| \leq \left|e^{-\frac{s^\alpha}{\alpha}}\right|\left[|x-y| + |\sin x - \sin y|\right].$$

It is easy to see that $\sin x$ *is Lipschitz:* $|\sin x - \sin y| \leq |x-y|$ *with Lipschitz constant* $L = 1$. *It implies that*

$$|k(s,x) - k(s,y)| \leq \left|e^{-\frac{s^\alpha}{\alpha}}\right|\left[|x-y| + |x-y|\right] = 2\left|e^{-\frac{s^\alpha}{\alpha}}\right||x-y| \leq (1)\left[2|x-y|\right].$$

So

$$|k(s,x) - k(s,y)| \leq K|x-y|,$$

and

$$|k(s,x)| = \left|e^{-\frac{s^\alpha}{\alpha}}(x+\sin x)\right| \leq \left|e^{-\frac{s^\alpha}{\alpha}}\right|(|x| + |\sin x|).$$

Since $|\sin x| \leq |x|$, *therefore,*

$$|k(s,x)| \leq \left|e^{-\frac{s^\alpha}{\alpha}}\right|(|x| + |x|) = 2\left|e^{-\frac{s^\alpha}{\alpha}}\right||x| = l(s)|x|.$$

Thus,

$$|k(s,x)| \leq l(s)|x|.$$

Since $\mathbb{T} = \mathbb{R}$, *therefore* $\mu(s) = 0$ *and* $\xi_{\mu(s)}[l(s)] = l(s)$, *for which*

$$\int_0^t \xi_{\mu(s)}[l(s)]\, d^\alpha s = 2\int_0^t e^{-\frac{s^\alpha}{\alpha}} d^\alpha s.$$

Since we know that

$$\left(e^{-\frac{s^\alpha}{\alpha}}\right)^{(\alpha)} = (-1)\, e^{-\frac{s^\alpha}{\alpha}}.$$

Hence, we can write

$$\int_0^t \xi_{\mu(s)}[l(s)]\, d^\alpha s = -2\int_0^t \left(e^{-\frac{s^\alpha}{\alpha}}\right)^{(\alpha)} d^\alpha s = 2 - 2e^{-\frac{t^\alpha}{\alpha}} = 2 - l(t),$$

which implies that

$$\int_0^t \xi_{\mu(s)}[l(s)]\, d^\alpha s = 2 - l(t) \leq 2.$$

Thus, for the above-mentioned functions and variables, hypotheses (H1)–(H3) in Theorem 5.3 are met, indicating that the solution to the LIVP (1.1) and (1.2) is defined and bounded on $[0,\infty)_{\mathbb{T}}$.

(II) Stabilities.

We observe that, analogous to the case (I), for each (s,x), $(s,y) \in \Omega$,

$$|k(s,x) - k(s,y)| \leq h(s)|x-y| \leq K|x-y|,$$

for which $I_\alpha h(s) \leq 2$. *Hence, all the requirements are satisfied for Theorem 5.4. We can derive that each solution to the LIVP (1.1) and (1.2) is always stable using Theorem 5.4.*

(III) Non-local initial value problems.

Select $[0,a]$ *such that* $2 > e^{\frac{a^\alpha}{\alpha}}$. *Let us define the functions*

$$\varphi(s) = 2e^{-\frac{s^\alpha}{\alpha}},\ \Psi(\tau) = \tau \text{ and } \phi(\tau) = \frac{\gamma}{2}\tau,$$

such that $0 < \gamma < \varphi(a) - 1$. *For* $u \in C([0,a]_{\mathbb{T}})$, *define the functional*

$$g(u) = \frac{\gamma}{2a}\int_0^a u(s)\, ds,$$

and then it's simple to determine whether g is a contraction:

$$|g(u) - g(v)|_k \leq \frac{\gamma}{2a}\int_0^a |u(t) - v(t)|_k \, dt \leq \frac{\gamma}{2a}\int_0^a \|u - v\|_k \, dt \leq \phi(\|u - v\|_k).$$

Moreover, observe that

$$|k(s, x)| \leq \varphi(s)\, \Psi(|x|),$$

for any $(s, x) \in [0, a] \times \mathbb{R}$ *as well as the fact that a direct calculation yields*

$$\sup_{t\in(0,\infty)} \frac{t}{|u_0| + \Psi(t)\, I_\alpha \varphi(a)} = \sup_{t\in(0,\infty)} \frac{t}{t I_\alpha \varphi(a)} = \frac{1}{I_\alpha \varphi(a)} = \frac{1}{2 - \varphi(a)}.$$

As

$$\gamma < \varphi(a) - 1,$$
$$\frac{1}{2 - \varphi(a)} > \frac{1}{1 - \gamma}.$$

Thus,

$$\sup_{t\in(0,\infty)} \frac{t}{|u_0| + \Psi(t)\, I_\alpha \varphi(a)} = \frac{1}{2 - \varphi(a)} > \frac{1}{1 - \gamma}.$$

We can deduce that the corresponding non-local IVP (1.1) and(1.3) has at least one solution defined on $[0, a]$ *because assumptions (H5)–(H7) in Theorem 6.1 are fulfilled for the aforementioned functions, functionals and parameters.*

Remark 6.2. *For* $\alpha = 1$ *and* $\mathbb{T} = \mathbb{R}$*, the classical results corresponding to ordinary differential equations will be yielded.*

7. Conclusions

Based on the theory of conformable fractional calculus on time scales, we defined generalized exponential function. Also, we proved some of its fundamental properties. Furthermore, we introduced Grönwall's type inequalities in the considered frame. Through Grönwall's inequality, we investigate the stability of the solution to the LIVP. In addition, some conditions for the global existence, extension and boundedness of LIVP's solutions as well as their stabilities are established by using conformable fractional calculus on time scales and FPT. Moreover, we obtained the existence result of the nonlocal initial value problem.

Acknowledgments

The author M. A. Alqudah was supported by Princess Nourah bint Abdulrahman University Researchers Supporting Project (No. PNURSP2022R14). The author T. Abdeljawad would like to thank Prince Sultan University for the support through the TAS research lab.

Conflict of interest

The authors declare no conflicts of interest regarding this article.

References

1. T. Abdeljawad, On conformable fractional calculus, *J. Comput. Appl. Math.*, **279** (2015), 57–66. https://doi.org/10.1016/j.cam.2014.10.016
2. H. Ahmed, Sobolev-type nonlocal conformable stochastic differential equations, *Bull. Iran. Math. Soc.*, 2021, 1–15. https://doi.org/10.1007/s41980-021-00615-6
3. H. M. Ahmed, Conformable fractional stochastic differential equations with control function, *Syst. Control Lett.*, **158** (2021), 105062. https://doi.org/10.1016/j.sysconle.2021.105062
4. H. M. Ahmed, Noninstantaneous impulsive conformable fractional stochastic delay integro-differential system with Rosenblatt process and control function, *Qual. Theory Dyn. Syst.*, **21** (2022), 1–22. https://doi.org/10.1007/s12346-021-00544-z
5. D. R. Anderson, E. Camrud, D. J. Ulness, On the nature of the conformable derivative and its applications to physics, *J. Fract. Calc. Appl.*, **10** (2019), 92–135.
6. N. Benkhettou, S. Hassani, D. F. M. Torres, A conformable fractional calculus on arbitrary time scales, *J. King Saud Univ. Sci.*, **28** (2016), 93–98. https://doi.org/10.1016/j.jksus.2015.05.003
7. M. Bohner, A. Peterson, *Dynamic equations on time scales: An introduction with applications*, Boston: Birkhäuser, 2001.
8. Q. H. Cao, C. Q. Dai, Symmetric and anti-symmetric solitons of the fractional second-and third-order nonlinear Schrödinger equation, *Chinese Phys. Lett.*, **38** (2021), 090501.
9. A. R. Carvalho, C. M. Pinto, D. Baleanu, HIV/HCV coinfection model: A fractional-order perspective for the effect of the HIV viral load, *Adv. Differ. Equ.*, **2018** (2018), 1–22. https://doi.org/10.1186/s13662-017-1456-z
10. S. Hilger, Ein maßkettenkalkül mit anwendung auf zentrumsmannigfaltigkeiten, Ph. D. thesis, Universität Würzburg, 1988.
11. S. Hilger, Analysis on measure chains-a unifed approach to continuous and discrete calculus, *Results Math.*, **18** (1990), 18–56. https://doi.org/10.1007/BF03323153
12. R. Khalil, M. A. Horani, A. Yousaf, M. Sababhen, A new definition of fractional derivative, *J. Comput. Appl. Math.*, **264** (2014), 65–70. https://doi.org/10.1016/j.cam.2014.01.002
13. P. F. Li, R. J. Li, C. Q. Dai, Existence, symmetry breaking bifurcation and stability of two-dimensional optical solitons supported by fractional diffraction, *Opt. Express*, **29** (2021), 3193–3209. https://doi.org/10.1364/OE.415028
14. P. H. Lu, Y. Y. Wang, C. Q. Dai, Discrete soliton solutions of the fractional discrete coupled nonlinear Schrödinger equations: Three analytical approaches, *Math. Methods Appl. Sci.*, **44** (2021), 11089–11101. https://doi.org/10.1002/mma.7473
15. R. K. Miller, *Nonlinear Volterra integral equations*, Menlo Park: W. A. Benjamin, 1967.

16. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: Willy, 1993.

17. V. Mohammadnezhad, M. Eslami, H. Rezazadeh, Stability analysis of linear conformable fractional differential equations system with time delays, *Bol. Soc. Parana. Mat.*, **38** (2020), 159–171. https://doi.org/10.5269/bspm.v38i6.37010

18. N. R. de Oliveira Bastos, *Fractional calculus on time scales*, Ph. D. thesis, Universidade de Aveiro, 2012.

19. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.

20. M. R. S. Rahmat, A new definition of conformable fractional derivative on arbitrary time scales, *Adv. Differ. Equ.*, **2019** (2019), 1–16. https://doi.org/10.1186/s13662-019-2294-y

21. S. G. Svetlin, *Integral equations on time scales*, Paris: Atlantis Press, 2016. https://doi.org/10.2991/978-94-6239-228-1

22. N. H. Sweilam, S. M. Al-Mekhlafi, Comparative study for multi-strain tuberculosis (TB) model of fractional order, *J. Appl. Math. Inform. Sci.*, **10** (2016), 1403–1413.

23. Y. N. Wang, J. W. Zhou, Y. K. Li, Fractional Sobolev's spaces on time scales via conformable fractional calculus and their application to a fractional differential equation on time scales, *Adv. Math. Phys.*, **2016** (2016), 1–21. https://doi.org/10.1155/2016/9636491

24. W. Y. Zhong, L. F. Wang, Basic theory of initial value problems of conformable fractional differential equations, *Adv. Differ. Equ.*, **2018** (2018), 1–14. https://doi.org/10.1186/s13662-018-1778-5