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Research article

The norm of continuous linear operator between two fuzzy quasi-normed spaces

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Abstract: In this paper, firstly, we introduce the concepts of continuity and boundedness of linear operators between two fuzzy quasi-normed spaces with general continuous t-norms, prove the equivalence of them, and point out that the set of all continuous linear operators forms a convex cone. Secondly, we establish the family of star quasi-seminorms on the cone of continuous linear operators, and construct a fuzzy quasi-norm of a continuous linear operator.

Keywords: fuzzy quasi-norm; continuous linear operator; bounded linear operator; family of star quasi-seminorms; fuzzy functional analysis
Mathematics Subject Classification: 46B28, 46S40

1. Introduction

In 1984, Katsaras [18] first introduced an idea of fuzzy norm on a linear space. In 1992, Felbin [11] introduced the concept of fuzzy norm on a linear space whose associated metric is in the sense of Kaleva and Seikkala [17]. In 1999, Felbin [12] proposed the concept of fuzzy boundedness of linear operators between two fuzzy normed spaces. On this basis, in 2003, Xiao and Zhu [25] proposed different concepts of fuzzy continuity and fuzzy boundedness of linear operators between two fuzzy normed spaces, and proved the equivalence of them. Meanwhile, they constructed the fuzzy norm of linear operators. In 2008, Bag and Samanta [6] also studied the fuzzy normed linear space defined by Felbin. They defined two types (strong and weak) of fuzzy continuity and fuzzy boundedness of linear operators, and proved the equivalence of them. At the same time, the fuzzy norm of the strongly fuzzy bounded linear operator was constructed.

In 1994, by using a different approach Cheng and Mordeson [8] introduced another type of fuzzy norm on a linear space whose associated metric is similar to Kramosil and Michalek [16]. In 2003, Bag and Samanta [4] also introduced the concept of the fuzzy norm (KM-type fuzzy norm) on a linear space based on the KM-type fuzzy metric. Then, they proposed various types of fuzzy continuity and fuzzy boundedness of linear operators between two fuzzy normed spaces with the continuous t-norm $\wedge$ in [5], and studied some basic results on finite dimensional fuzzy normed spaces in general t-norm setting in [7]. In 2009, Sadeqi and Kia [22] defined the topological continuity of linear operators on fuzzy normed spaces, and proved the equivalence of fuzzy continuity and topological continuity.

With the exception of symmetry of fuzzy norm in [4], Alegre and Romaguera [2] introduced the concept of fuzzy quasi-norm with general continuous t-norm. They gave some results, such as the uniform boundedness theorem in fuzzy quasi-normed space in [3]. Recently, Hussein and Al-Basri [15] studied the completion of quasi-fuzzy normed algebra over fuzzy field. Gao et al. [13] introduced the concept of a family of star quasi-seminorms on a linear space with general continuous t-norm, and gave the decomposition theorem for a fuzzy quasi-norm. Li and Wu [20] studied continuous linear functional on a fuzzy quasi-normed space, and proved the Hahn-Banach extension theorem and separation theorem for convex subsets of fuzzy quasi-normed spaces.

Now, there have been many results about the norm of a linear operator between two fuzzy normed spaces (see [23,24]), but there are few their counterparts in the asymmetric case. This article will go into this topic. In Section 2, we give some basic notions and results about fuzzy quasi-normed spaces used in this article. In Section 3, firstly, we introduce the concepts of continuity and boundedness of linear operators between two fuzzy quasi-normed spaces and show some results about them, then, we construct the fuzzy quasi-norm of a continuous linear operator. Finally, a brief conclusion is given in Section 4.

In this paper, the symbols R, X and θ mean the set of all real numbers, a real linear space and the zero vector.

2. Preliminaries

In this section, let us recall some basic notions and results in fuzzy quasi-normed spaces.

Definition 2.1. ([21]) A binary operation $*$: $[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if it satisfies the following conditions: for all $a,b,c,d\in[0,1]$,

(1) $a*b=b*a$ (commutativity),

(2) $(a*b)*c=a*(b*c)$ (associativity),

(3) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ (monotonicity),

(4) $a*1=a$ (boundary condition),
(5) $*$ is continuous on $[0,1]\times[0,1]$ (continuity).

Proposition 2.1. ([14]) Let $*$ be a continuous t-norm.
(1) If $1\geq r_1\geq r_2$, then exists $r_3\in(0,1)$ such that $r_1*r_3\geq r_2$;
(2) If $r_4\in(0,1)$, then exists $r_5\in(0,1)$ such that $r_5*r_5\geq r_4$.

Definition 2.2. ([2]) A fuzzy quasi-norm on a real linear space X is a pair $(N,*)$ such that $*$ is a continuous t-norm and N is a fuzzy set in $X\times[0,+\infty)$ satisfying the following conditions: for all $x,y\in X$,
(FQN1) $N(x,0)=0$,
(FQN2) $N(x,t)=N(-x,t)=1$ for all $t>0$ if and only if $x=\theta$,
(FQN3) $N(cx,t)=N(x,t/c)$ for all $c,t>0$,
(FQN4) $N(x+y,t+s)\geq N(x,t)*N(y,s)$ for all $s,t>0$,
(FQN5) $N(x,\cdot):(0,+\infty)\to[0,1]$ is left continuous,
(FQN6) $\lim_{t\to+\infty}N(x,t)=1$.

Obviously, the function $N(x,\cdot)$ is increasing for the given $x\in X$; $(N^{-1},*)$ is also a fuzzy quasi-norm, where $N^{-1}(x,t)=N(-x,t)$ for all $x\in X$ and $t\geq 0$, $(N^{-1},*)$ is called the conjugate of $(N,*)$.

Remark 2.1. A fuzzy quasi-norm on a convex cone C is a pair $(N,*)$ such that $*$ is a continuous t-norm and N is a fuzzy set in $C\times[0,+\infty)$ satisfying FQN1, FQN3-FQN6, and FQN2': If $x\in C$, $-x\in C$, then $N(x,t)=N(-x,t)=1$ for all $t>0$ if and only if $x=\theta$.

A fuzzy quasi-norm $(N,*)$ is often denoted by N for simplicity.

Proposition 2.2. ([2]) Each fuzzy quasi-norm $(N,*)$ on X induces a topology τ_N which has as a base the family of open balls

$$\mathcal{B}(x)=\{B_N(x,r,t):r\in(0,1),t>0\}$$

at $x\in X$, where

$$B_N(x,r,t)=\{y\in X:N(y-x,t)>1-r\}.$$

Obviously, τ_N is a T_0 topology on X. Since $x+B_N(\theta,r,t)=B_N(x,r,t)$, the topology τ_N is translation invariant. In fact, τ_N is a paratopology ([1,10]) on X.

Definition 2.4. ([13]) Let X be a linear space and $*$ be a continuous t-norm. $P=\{p_\alpha:X\to[0,+\infty),\alpha\in(0,1)\}$ is called the family of star quasi-seminorms if it satisfies the following conditions: for all $x,y\in X$, $\alpha,\beta\in(0,1)$ and $\lambda>0$,
(*QN1) $p_\alpha(\lambda x)=\lambda p_\alpha(x)$,

(*QN2) $p_{\alpha*\beta}(x+y) \le p_\alpha(x) + p_\beta(y)$.

If P satisfies the following condition:

(*QN3) $p_\alpha(x) = p_\alpha(-x) = 0$ for every $\alpha \in (0,1)$ implies $x = \theta$,

then, P is said to be separating [19].

Proposition 2.3. ([13]) Let $(X, N, *)$ be a fuzzy quasi-normed space and $\alpha \in (0,1)$. The function $\|\cdot\|_\alpha : X \to [0, +\infty)$ is given by

$$\|x\|_\alpha = \inf\{t > 0 : N(x,t) \ge \alpha\}, \tag{2.1}$$

$P_N = \{\|\bullet\|_\alpha : \alpha \in (0,1)\}$. Then,

(1) $\|x\|_\alpha = \sup\{t > 0 : N(x,t) < \alpha\}$ for all $x \in X$ and $\alpha \in (0,1)$;

(2) P_N is increasing, that is, $\|x\|_\alpha$ is increasing with respect to $\alpha \in (0,1)$ for all $x \in X$;

(3) P_N is a separating family of star quasi-seminorms induced by $(N, *)$.

Remark 2.4. If $* = \wedge$, then P_N is a family of quasi-seminorms. The background of the formula (2.1) can also be found in [9] and [19].

Proposition 2.4. Let $(X, N, *)$ be a fuzzy quasi-normed space, $(N^{-1}, *)$ be the conjugate of $(N, *)$, and $P_{N^{-1}} = \{\|\bullet\|_\alpha^\# : \alpha \in (0,1)\}$ be a separating family of star quasi-seminorms induced by $(N^{-1}, *)$. Then, $\|x\|_\alpha^\# = \|-x\|_\alpha$ for each $x \in X$, $\alpha \in (0,1)$.

Proof. For each $x \in X$, we have $N^{-1}(x,t) = N(-x,t)$ for each $t \ge 0$, then

$$\|x\|_\alpha^\# = \inf\{t > 0 : N^{-1}(x,t) \ge \alpha\} = \inf\{t > 0 : N(-x,t) \ge \alpha\} = \|-x\|_\alpha.$$

Proposition 2.5. ([13]) Let X be a linear space, $*$ be a continuous t-norm, and $P_N = \{\|\bullet\|_\alpha : \alpha \in (0,1)\}$ be an increasing family of star quasi-seminorms on X. For each $x \in X$, let

$$U_{P_N}(x) = \{U(x; \alpha_1, \alpha_2, \ldots, \alpha_n; \varepsilon) : \varepsilon > 0; \alpha_1, \alpha_2, \ldots, \alpha_n \in (0,1)\},$$

and so

$$\begin{aligned} U(x; \alpha_1, \alpha_2, \ldots, \alpha_n; \varepsilon) &= \{y \in X : \|y - x\|_{\alpha_i} < \varepsilon, \alpha_i \in (0,1), i = 1, 2, \ldots, n\} \\ &= \bigcap_{i=1}^n \{y \in X : \|y - x\|_{\alpha_i} < \varepsilon, \alpha_i \in (0,1)\} \\ &= \{y \in X : \|y - x\|_{\max\{\alpha_i : 1 \le i \le n\}} < \varepsilon\}. \end{aligned}$$

Then, $U_{P_N}(x)$ is a basis of neighborhoods of x.

The topology taking $U_{P_N}(x)$ as a basis of neighborhoods of x is said to be the topology induced by P_N and denoted by τ_{P_N}.

Theorem 2.1. ([13]) Let $(X,N,*)$ be a fuzzy quasi-normed space and P_N be an increasing family of star quasi-seminorms defined by (2.1). Then the topology τ_{P_N} induced by P_N coincides the topology τ_N induced by N.

Theorem 2.2. ([13]) Let X be a linear space, $*$ be a continuous t-norm, and $P_N=\{\|\bullet\|_\alpha:\alpha\in(0,1)\}$ be an increasing and separating family of star quasi-seminorms on X. For all $x\in X$ and $t\geq 0$, let $N:X\times[0,+\infty)\to[0,1]$ be given by

$$N(x,t)=\begin{cases}\sup\{\alpha\in(0,1):\|x\|_\alpha<t\} & ,\ t>0\\ 0 & ,\ t=0\end{cases}. \tag{2.2}$$

Then $(N,*)$ is a fuzzy quasi-norm on X.

3. The norm of continuous linear operators on fuzzy quasi-normed space

In this section we first define continuity and boundedness of linear operators between two fuzzy quasi-normed spaces.

Let $(X,N_1,*_1)$ and $(Y,N_2,*_2)$ be two fuzzy quasi-normed spaces, $P_{N_1}=\{\|\bullet\|_{\alpha,1}:\alpha\in(0,1)\}$ and $Q_{N_2}=\{\|\bullet\|_{\alpha,2}:\alpha\in(0,1)\}$ be the families of star quasi-seminorms defined by (2.1) corresponding to $(N_1,*_1)$ and $(N_2,*_2)$ respectively, $\tau_{P_{N_1}}$ and $\tau_{Q_{N_2}}$ be the topologies induced by P_{N_1} and Q_{N_2} respectively. The notations $P_{N_1^{-1}}=\{\|\bullet\|_{\alpha,1}^{\#}:\alpha\in(0,1)\}$, $Q_{N_2^{-1}}=\{\|\bullet\|_{\alpha,2}^{\#}:\alpha\in(0,1)\}$, $\tau_{P_{N_1^{-1}}}$ and $\tau_{Q_{N_2^{-1}}}$ are defined similarly.

3.1. The continuity and boundedness of linear operators

Definition 3.1. A linear operator T from $(X,N_1,*_1)$ to $(Y,N_2,*_2)$ is said to be $(\tau_{P_{N_1}},\tau_{Q_{N_2}})$-continuous at $x_0\in X$, if for each open neighborhood $V\in\tau_{Q_{N_2}}$ of $Tx_0\in Y$, there exists an open neighborhood $U\in\tau_{P_{N_1}}$ of x_0 such that $T(U)\subseteq V$.

If T is $(\tau_{P_{N_1}},\tau_{Q_{N_2}})$-continuous at each point of X, then T is said to be $(\tau_{P_{N_1}},\tau_{Q_{N_2}})$-continuous on X.

The set of all $(\tau_{P_{N_1}},\tau_{Q_{N_2}})$-continuous linear operators from $(X,N_1,*_1)$ to $(Y,N_2,*_2)$ is denoted by $LC(X,Y)$, the set of all $(\tau_{P_{N_1^{-1}}},\tau_{Q_{N_2^{-1}}})$-continuous linear operators from $(X,N_1^{-1},*_1)$ to $(Y,N_2^{-1},*_2)$ is denoted by $LC^{-1}(X,Y)$.

Definition 3.2. A linear operator T from $(X,N_1,*_1)$ to $(Y,N_2,*_2)$ is said to be

$\left(P_{N_1}, Q_{N_2}\right)$-bounded, if for all $\beta \in (0,1)$, there exist $\alpha \in (0,1)$ and $M > 0$ such that $\|Tx\|_{\beta,2} \le M \cdot \|x\|_{\alpha,1}$ for all $x \in X$.

Theorem 3.1. Let $(X, N_1, *_1)$ and $(Y, N_2, *_2)$ be two fuzzy quasi-normed spaces and $T: X \to Y$ be a linear operator, then the followings are equivalent:
(1) T is $\left(\tau_{P_{N_1}}, \tau_{Q_{N_2}}\right)$-continuous on X,
(2) T is $\left(\tau_{P_{N_1}}, \tau_{Q_{N_2}}\right)$-continuous at $\theta \in X$,
(3) T is $\left(P_{N_1}, Q_{N_2}\right)$-bounded.

Proof. (1)$\Leftrightarrow$(2) is obvious from the translation invariance of $\tau_{P_{N_1}}$ and $\tau_{Q_{N_2}}$.

(2)$\Rightarrow$(3): By the $\left(\tau_{P_{N_1}}, \tau_{Q_{N_2}}\right)$-continuity of T at $\theta \in X$, for any $\beta \in (0,1)$ and $\delta > 0$, there exist $\alpha \in (0,1)$ and $\varepsilon > 0$ such that $T\left(U(\alpha,\varepsilon)\right) \subseteq V(\beta,\delta)$, where

$$U(\alpha,\varepsilon) = \left\{x \in X : \|x\|_{\alpha,1} < \varepsilon\right\},\ V(\beta,\delta) = \left\{y \in Y : \|y\|_{\beta,2} < \delta\right\}.$$

To prove (2)$\Rightarrow$(3), it is sufficient to prove that $\|Tx\|_{\beta,2} \le \frac{\delta}{\varepsilon}\|x\|_{\alpha,1}$ for all $x \in X$.

If $\|x\|_{\alpha,1} = 0$, then we have $\|\lambda x\|_{\alpha,1} = 0$ for all $\lambda > 0$. Thus $\lambda x \in U(\alpha,\varepsilon)$, and so $\|T\lambda x\|_{\beta,2} < \delta$, i.e., $\lambda\|Tx\|_{\beta,2} < \delta$. Hence $\|Tx\|_{\beta,2} < \frac{\delta}{\lambda}$. It follows from the arbitrariness of λ that $\|Tx\|_{\beta,2} = 0$. Thus, $\|Tx\|_{\beta,2} \le \frac{\delta}{\varepsilon}\|x\|_{\alpha,1}$.

If $\|x\|_{\alpha,1} \ne 0$, then $\frac{\varepsilon x}{\|x\|_{\alpha,1}} \in U(\alpha,\varepsilon)$, so $\left\|T\left(\frac{\varepsilon x}{\|x\|_{\alpha,1}}\right)\right\|_{\beta,2} < \delta$, that is, $\|Tx\|_{\beta,2} < \frac{\delta}{\varepsilon}\|x\|_{\alpha,1}$.

(3)$\Rightarrow$(2): By the boundedness of T on X, for all $x \in X$ and $\beta \in (0,1)$, there exist $\alpha \in (0,1)$ and $M > 0$ such that $\|Tx\|_{\beta,2} \le M \cdot \|x\|_{\alpha,1}$. For any $\delta > 0$, set $\varepsilon = \frac{\delta}{M} > 0$, then $\|Tx\|_{\beta,2} < \delta$ when $\|x\|_{\alpha,1} < \varepsilon$, that is, $T\left(U(\alpha,\varepsilon)\right) \subseteq V(\beta,\delta)$. Therefore, T is $\left(\tau_{P_{N_1}}, \tau_{Q_{N_2}}\right)$-continuous at $\theta \in X$.

Let $L(X,Y)$ be the linear space containing all the linear operators from $(X, N_1, *_1)$ to $(Y, N_2, *_2)$.

Proposition 3.1. Let $(X, N_1, *_1)$ and $(Y, N_2, *_2)$ be two fuzzy quasi-normed spaces, then $LC(X,Y)$ is a convex cone of the linear space $L(X,Y)$.

Proof. From Theorem 3.1, for any $T \in LC(X,Y)$ and $\beta \in (0,1)$, there exist $\alpha \in (0,1)$ and $M > 0$ such that $\|Tx\|_{\beta,2} \le M \cdot \|x\|_{\alpha,1}$ for all $x \in X$. Then, for any $\lambda > 0$, $\|(\lambda T)x\|_{\beta,2} \le (\lambda M) \cdot \|x\|_{\alpha,1}$ holds for all $x \in X$. Therefore $\lambda T \in LC(X,Y)$ for all $\lambda > 0$.

Let $T,S \in LC(X,Y)$, $\beta \in (0,1)$. From Proposition 2.1, there are $\beta_1,\beta_2 \in (0,1)$ such that $\beta_1 *_2 \beta_2 \ge \beta$. By Theorem 3.1, there are $\alpha_1,\alpha_2 \in (0,1)$, $M_1,M_2 > 0$ such that $\|Tx\|_{\beta_1,2} \le M_1 \cdot \|x\|_{\alpha_1,1}$, $\|Sx\|_{\beta_2,2} \le M_2 \cdot \|x\|_{\alpha_2,1}$ for all $x \in X$. Let $M = M_1 + M_2$, $\alpha = \max\{\alpha_1,\alpha_2\}$, then

$$\begin{aligned}\|(T+S)x\|_{\beta,2} &\le \|(T+S)x\|_{\beta_1 *_2 \beta_2,2} \le \|Tx\|_{\beta_1,2} + \|Sx\|_{\beta_2,2} \\ &\le M_1 \cdot \|x\|_{\alpha_1,1} + M_2 \cdot \|x\|_{\alpha_2,1} \le M \cdot \|x\|_{\alpha,1}\end{aligned}$$

for all $x \in X$. Therefore $T+S \in LC(X,Y)$.

The following theorem is obvious from Proposition 2.4.

Theorem 3.2. Let $(X,N_1,*_1)$ and $(Y,N_2,*_2)$ be two fuzzy quasi-normed spaces. Then, $LC(X,Y) = LC^{-1}(X,Y)$.

3.2. The fuzzy quasi-norm of a continuous linear operator

Proposition 3.2. Let $(X,N_1,*_1)$ and $(Y,N_2,*_2)$ be two fuzzy quasi-normed spaces. For any $T \in LC(X,Y)$ and $\beta \in (0,1)$, set

$$\alpha_T(\beta) = \left\{\alpha \in (0,1) : \exists M > 0, \|Tx\|_{\beta,2} \le M \cdot \|x\|_{\alpha,1}, \forall x \in X\right\}, \tag{3.1}$$

$$\alpha_T^{-1}(\beta) = \left\{\alpha \in (0,1) : \exists M > 0, \|Tx\|_{\beta,2}^{\#} \le M \cdot \|x\|_{\alpha,1}^{\#}, \forall x \in X\right\}. \tag{3.2}$$

Then,

(1) $\alpha_T(\beta) = \alpha_T^{-1}(\beta)$,

(2) for any $\beta_1,\beta_2 \in (0,1)$ with $\beta_1 < \beta_2$, $\alpha_T(\beta_2) \subseteq \alpha_T(\beta_1)$,

(3) for any $T,S \in LC(X,Y)$ and $\beta_1,\beta_2 \in (0,1), \alpha_T(\beta_1) \vee \alpha_S(\beta_2) \subseteq \alpha_{T+S}(\beta_1 *_2 \beta_2)$.

Proof. (1) Since $T \in LC(X,Y)$, for any $\alpha \in \alpha_T(\beta)$, there exists $M > 0$ such that $\|Tx\|_{\beta,2} \le M \cdot \|x\|_{\alpha,1}$ for all $x \in X$. From $\|x\|_{\alpha,1}^{\#} = \|-x\|_{\alpha,1}$ and $\|Tx\|_{\beta,2}^{\#} = \|-Tx\|_{\beta,2}$ $= \|T(-x)\|_{\beta,2}$, we get $\|Tx\|_{\beta,2}^{\#} \le M \cdot \|x\|_{\alpha,1}^{\#}$ for all $x \in X$. So $\alpha \in \alpha_T^{-1}(\beta)$, that is, $\alpha_T(\beta) \subseteq \alpha_T^{-1}(\beta)$. Similarly, we have $\alpha_T^{-1}(\beta) \subseteq \alpha_T(\beta)$. Thus $\alpha_T(\beta) = \alpha_T^{-1}(\beta)$.

(2) Since $T \in LC(X,Y)$, for any $\alpha \in \alpha_T(\beta_2)$, there is $M > 0$ such that $\|Tx\|_{\beta_2,2}$ $\le M \cdot \|x\|_{\alpha,1}$ for all $x \in X$. From $\beta_1 < \beta_2$, we have $\|Tx\|_{\beta_1,2} \le \|Tx\|_{\beta_2,2} \le M \cdot \|x\|_{\alpha,1}$ for all $x \in X$. Therefore $\alpha \in \alpha_T(\beta_1)$. Thus $\alpha_T(\beta_2) \subseteq \alpha_T(\beta_1)$.

(3) Since $T \in LC(X,Y)$, for any $\alpha_1 \in \alpha_T(\beta_1)$ and $\alpha_2 \in \alpha_S(\beta_2)$, there exist $M_1,M_2 > 0$ such that $\|Tx\|_{\beta_1,2} \le M_1 \|x\|_{\alpha_1,1}$, $\|Sx\|_{\beta_2,2} \le M_2 \|x\|_{\alpha_2,1}$ for all $x \in X$. Let $M = M_1 + M_2$, then

$$\|(T+S)x\|_{\beta_1 *_2 \beta_2,2} \le \|Tx\|_{\beta_1,2} + \|Sx\|_{\beta_2,2} \le M_1 \|x\|_{\alpha_1,1} + M_2 \|x\|_{\alpha_2,1} \le M \|x\|_{\alpha_1 \vee \alpha_2,1}.$$

Thus $\alpha_1 \vee \alpha_2 \in \alpha_{T+S}\left(\beta_1 *_2 \beta_2\right)$, that is, $\alpha_T\left(\beta_1\right) \vee \alpha_S\left(\beta_2\right) \subseteq \alpha_{T+S}\left(\beta_1 *_2 \beta_2\right)$.

Theorem 3.3. Let $\left(X, N_1, *_1\right)$ and $\left(Y, N_2, *_2\right)$ be two fuzzy quasi-normed spaces. For all $T \in LC\left(X, Y\right)$ and $\beta \in \left(0,1\right)$, define

$$\|T\|_{\alpha,\beta} = \sup_{\substack{x \neq \theta, \\ x \in X}} \frac{\|Tx\|_{\beta,2}}{\|x\|_{\alpha,1}}, \quad \alpha \in \alpha_T\left(\beta\right), \tag{3.3}$$

$$\|T\|_{\beta} = \inf_{\alpha \in \alpha_T(\beta)} \|T\|_{\alpha,\beta}. \tag{3.4}$$

Then,

(1) $\left\{\|\bullet\|_{\beta} : \beta \in \left(0,1\right)\right\}$ is a family of star quasi-seminorms on the cone $LC\left(X, Y\right)$,

(2) $\left\{\|\bullet\|_{\beta} : \beta \in \left(0,1\right)\right\}$ is increasing with respect to $\beta \in \left(0,1\right)$,

(3) $\left\{\|\bullet\|_{\beta} : \beta \in \left(0,1\right)\right\}$ is separating.

Proof. (1) It is obvious that $\|T\|_{\beta} \geq 0$.

(*QN1) For any $\lambda > 0$, we have

$$\|\lambda T\|_{\beta} = \inf_{\alpha \in \alpha_T(\beta)} \|\lambda T\|_{\alpha,\beta} = \inf_{\alpha \in \alpha_T(\beta)} \left(\sup_{\substack{x \neq \theta, \\ x \in X}} \frac{\|\lambda Tx\|_{\beta,2}}{\|x\|_{\alpha,1}} \right) = \inf_{\alpha \in \alpha_T(\beta)} \left(\sup_{\substack{x \neq \theta, \\ x \in X}} \frac{\lambda \|Tx\|_{\beta,2}}{\|x\|_{\alpha,1}} \right) = \lambda \|T\|_{\beta}.$$

(*QN2) For any $T, S \in LC\left(X, Y\right)$, $\beta_1, \beta_2 \in \left(0,1\right)$, $\alpha \in \alpha_{T+S}\left(\beta_1 *_2 \beta_2\right)$,

$$\|T+S\|_{\alpha,\beta_1 *_2 \beta_2} = \sup_{\substack{x \neq \theta, \\ x \in X}} \frac{\|\left(T+S\right)x\|_{\beta_1 *_2 \beta_2, 2}}{\|x\|_{\alpha,1}} \leq \sup_{\substack{x \neq \theta, \\ x \in X}} \frac{\|Tx\|_{\beta_1,2} + \|Sx\|_{\beta_2,2}}{\|x\|_{\alpha,1}} \leq \sup_{\substack{x \neq \theta, \\ x \in X}} \frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha,1}} + \sup_{\substack{x \neq \theta, \\ x \in X}} \frac{\|Sx\|_{\beta_2,2}}{\|x\|_{\alpha,1}},$$

and

$$\|T+S\|_{\beta_1 *_2 \beta_2} = \inf_{\alpha \in \alpha_{T+S}(\beta_1 *_2 \beta_2)} \|T+S\|_{\alpha,\beta_1 *_2 \beta_2} \leq \inf_{\alpha \in \alpha_{T+S}(\beta_1 *_2 \beta_2)} \left(\sup_{\substack{x \neq \theta, \\ x \in X}} \frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha,1}} + \sup_{\substack{x \neq \theta, \\ x \in X}} \frac{\|Sx\|_{\beta_2,2}}{\|x\|_{\alpha,1}} \right).$$

By Proposition 3.2, we have $\alpha_T\left(\beta_1\right) \vee \alpha_S\left(\beta_2\right) \subseteq \alpha_{T+S}\left(\beta_1 *_2 \beta_2\right)$, then

$$\begin{aligned} \inf_{\alpha \in \alpha_{T+S}(\beta_1 *_2 \beta_2)} \left(\sup_{\substack{x \neq \theta, \\ x \in X}} \frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha,1}} \right) &\leq \inf_{\substack{\alpha_1 \in \alpha_T(\beta_1) \\ \alpha_2 \in \alpha_S(\beta_2)}} \left(\sup_{\substack{x \neq \theta, \\ x \in X}} \frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha_1 \vee \alpha_2,1}} \right) \\ &\leq \inf_{\substack{\alpha_1 \in \alpha_T(\beta_1) \\ \alpha_2 \in \alpha_S(\beta_2)}} \left(\sup_{\substack{x \neq \theta, \\ x \in X}} \frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha_1,1}} \right) = \inf_{\alpha_1 \in \alpha_T(\beta_1)} \left(\sup_{\substack{x \neq \theta, \\ x \in X}} \frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha_1,1}} \right), \end{aligned}$$

that is,

$$\inf_{\alpha\in\alpha_{T+S}(\beta_1 *_2 \beta_2)}\left(\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha,1}}\right)\leq\|T\|_{\beta_1}.$$

Similarly, we have

$$\inf_{\alpha\in\alpha_{T+S}(\beta_1 *_2 \beta_2)}\left(\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Sx\|_{\beta_2,2}}{\|x\|_{\alpha,1}}\right)\leq\|S\|_{\beta_2}.$$

Let $\inf_{\alpha\in\alpha_{T+S}(\beta_1 *_2 \beta_2)}\left(\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha,1}}\right)=A$, $\inf_{\alpha\in\alpha_{T+S}(\beta_1 *_2 \beta_2)}\left(\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Sx\|_{\beta_2,2}}{\|x\|_{\alpha,1}}\right)=B$, then $A\leq\|T\|_{\beta_1}$, $B\leq\|S\|_{\beta_2}$. For any $\varepsilon>0$, there exist $\alpha_1,\alpha_2\in\alpha_{T+S}\left(\beta_1 *_2 \beta_2\right)$ such that

$$\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha_1,1}}<A+\frac{\varepsilon}{2},\ \sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Sx\|_{\beta_2,2}}{\|x\|_{\alpha_2,1}}<B+\frac{\varepsilon}{2}.$$

Let $\alpha_3=\alpha_1\vee\alpha_2$. Then $\alpha_3\in\alpha_{T+S}\left(\beta_1 *_2 \beta_2\right)$, and so

$$\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha_3,1}}\leq\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha_1,1}}<A+\frac{\varepsilon}{2},$$

$$\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Sx\|_{\beta_2,2}}{\|x\|_{\alpha_3,1}}\leq\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Sx\|_{\beta_2,2}}{\|x\|_{\alpha_2,1}}<B+\frac{\varepsilon}{2}.$$

Then

$$\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|(T+S)x\|_{\beta_1 *_2 \beta_2,2}}{\|x\|_{\alpha_3,1}}\leq\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha_3,1}}+\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Sx\|_{\beta_2,2}}{\|x\|_{\alpha_3,1}}<A+B+\varepsilon.$$

By the arbitrariness of ε, we have

$$\inf_{\alpha\in\alpha_{T+S}(\beta_1 *_2 \beta_2)}\left(\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha,1}}+\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Sx\|_{\beta_2,2}}{\|x\|_{\alpha,1}}\right)\leq A+B,$$

thus

$$\inf_{\alpha\in\alpha_{T+S}(\beta_1 *_2 \beta_2)}\left(\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|(T+S)x\|_{\beta_1 *_2 \beta_2,2}}{\|x\|_{\alpha,1}}\right)\leq\inf_{\alpha\in\alpha_{T+S}(\beta_1 *_2 \beta_2)}\left(\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha,1}}+\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Sx\|_{\beta_2,2}}{\|x\|_{\alpha,1}}\right)\leq A+B.$$

That is, $\|T+S\|_{\beta_1 *_2 \beta_2}\leq\|T\|_{\beta_1}+\|S\|_{\beta_2}$.

(2) Let $\beta_1,\beta_2\in(0,1)$ with $\beta_1<\beta_2$. Proposition 3.2 implies that $\alpha_T\left(\beta_2\right)\subseteq\alpha_T\left(\beta_1\right)$.

So $\inf_{\alpha\in\alpha_T(\beta_1)}\|T\|_{\alpha,\beta_1} \le \inf_{\alpha\in\alpha_T(\beta_2)}\|T\|_{\alpha,\beta_1}$. Since

$$\inf_{\alpha\in\alpha_T(\beta_2)}\|T\|_{\alpha,\beta_1} = \inf_{\alpha\in\alpha_T(\beta_2)}\left(\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Tx\|_{\beta_1,2}}{\|x\|_{\alpha,1}}\right) \le \inf_{\alpha\in\alpha_T(\beta_2)}\left(\sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Tx\|_{\beta_2,2}}{\|x\|_{\alpha,1}}\right) = \inf_{\alpha\in\alpha_T(\beta_2)}\|T\|_{\alpha,\beta_2},$$

we have

$$\|T\|_{\beta_1} = \inf_{\alpha\in\alpha_T(\beta_1)}\|T\|_{\alpha,\beta_1} \le \inf_{\alpha\in\alpha_T(\beta_2)}\|T\|_{\alpha,\beta_2} = \|T\|_{\beta_2}.$$

Therefore $\left\{\|\bullet\|_\beta : \beta\in(0,1)\right\}$ is increasing with respect to $\beta\in(0,1)$.

(3) Firstly, we prove that for any $T\in LC(X,Y)$, $\beta\in(0,1)$ and $\alpha\in\alpha_T(\beta)$,

$$\|T\|_{\alpha,\beta} = \sup_{x\neq\theta, x\in X}\frac{\|Tx\|_{\beta,2}}{\|x\|_{\alpha,1}} = \sup_{x\in X, \|x\|_{\alpha,1}=1}\|Tx\|_{\beta,2}.$$

For any $y\in X\setminus\{\theta\}$, we have $\left\|\frac{y}{\|y\|_{\alpha,1}}\right\|_{\alpha,1}=1$, then $\left\|T\left(\frac{y}{\|y\|_{\alpha,1}}\right)\right\|_{\beta,2} \le \sup_{x\in X,\|x\|_{\alpha,1}=1}\|Tx\|_{\beta,2}$, therefore $\|T\|_{\alpha,\beta} = \sup_{y\neq\theta, y\in X}\frac{\|Ty\|_{\beta,2}}{\|y\|_{\alpha,1}} \le \sup_{x\in X,\|x\|_{\alpha,1}=1}\|Tx\|_{\beta,2}$. On the other hand, we have

$$\|T\|_{\alpha,\beta} = \sup_{\substack{x\neq\theta,\\ x\in X}}\frac{\|Tx\|_{\beta,2}}{\|x\|_{\alpha,1}} \ge \sup_{x\in X,\|x\|_{\alpha,1}=1}\|Tx\|_{\beta,2}.$$

Thus

$$\|T\|_{\alpha,\beta} = \sup_{x\in X,\|x\|_{\alpha,1}=1}\|Tx\|_{\beta,2}. \tag{3.5}$$

In order to prove that $\left\{\|\bullet\|_\beta : \beta\in(0,1)\right\}$ is separating, we suppose that $\|T\|_\beta = \|-T\|_\beta = 0$, that is, $\inf_{\alpha\in\alpha_T(\beta)}\|T\|_{\alpha,\beta} = \inf_{\alpha\in\alpha_{-T}(\beta)}\|-T\|_{\alpha,\beta} = 0$, for any $\beta\in(0,1)$. Let $\varepsilon>0$, then there exists $\alpha_1\in\alpha_T(\beta)$ such that $\|T\|_{\alpha_1,\beta}<\varepsilon$, and thus $\sup_{x\in X,\|x\|_{\alpha_1,1}=1}\|Tx\|_{\beta,2}<\varepsilon$. Therefore, $\|Tx\|_{\beta,2}<\varepsilon$ for all $x\in\left\{x\in X : \|x\|_{\alpha_1,1}=1\right\}$. By the arbitrariness of ε, we have $\|Tx\|_{\beta,2}=0$. So $\|Tx\|_{\beta,2} = \|x\|_{\alpha_1,1}\left\|T\left(\frac{x}{\|x\|_{\alpha_1,1}}\right)\right\|_{\beta,2} = 0$ for any $x\in X\setminus\{\theta\}$. So, $\|Tx\|_{\beta,2}=0$ for all $x\in X$. Similarly, from $\|-T\|_\beta = \inf_{\alpha\in\alpha_{-T}(\beta)}\|-T\|_{\alpha,\beta}=0$, we have $\|-Tx\|_{\beta,2}=0$ for all $x\in X$. Since $Q_{N_2} = \left\{\|\bullet\|_{\alpha,2} : \alpha\in(0,1)\right\}$ is separating, we have $Tx=\theta$ for all $x\in X$. Thus $T=0$, we have that $\left\{\|\bullet\|_\beta : \beta\in(0,1)\right\}$ is separating.

Let $(X,N_1,*_1)$ and $(Y,N_2,*_2)$ be two fuzzy quasi-normed spaces, and let $(N_1^{-1},*_1)$ and $(N_2^{-1},*_2)$ be conjugate fuzzy quasi-norms of $(N_1,*_1)$ and $(N_2,*_2)$ respectively. By a similar method to the proof of Theorem 3.3, we can show that a family of star quasi-seminorms $\left\{\|\bullet\|_\beta^\#,\beta\in(0,1)\right\}$ on $LC^{-1}(X,Y)$ can be given by

$$\|T\|_{\alpha,\beta}^\# = \sup_{\substack{x\neq\theta,\\ x\in X}} \frac{\|Tx\|_{\beta,2}^\#}{\|x\|_{\alpha,1}^\#},\quad \alpha\in\alpha_T^{-1}(\beta),$$

$$\|T\|_\beta^\# = \inf_{\alpha\in\alpha_T^{-1}(\beta)} \|T\|_{\alpha,\beta}^\#.$$

Proposition 3.3. Let $(X,N_1,*_1)$ and $(Y,N_2,*_2)$ be two fuzzy quasi-normed spaces, $T\in LC(X,Y)=LC^{-1}(X,Y)$ and $\beta\in(0,1)$, then $\|T\|_\beta^\# = \|T\|_\beta$.

Proof. For any $T\in LC(X,Y)=LC^{-1}(X,Y)$ and $\beta\in(0,1)$, we have

$$\|T\|_{\alpha,\beta} = \sup_{\substack{x\neq\theta\\ x\in X}} \frac{\|Tx\|_{\beta,2}}{\|x\|_{\alpha,1}} = \sup_{\substack{x\neq\theta\\ x\in X}} \frac{\|T(-x)\|_{\beta,2}}{\|-x\|_{\alpha,1}} = \sup_{\substack{x\neq\theta\\ x\in X}} \frac{\|Tx\|_{\beta,2}^{\#}}{\|x\|_{\alpha,1}^{\#}} = \|T\|_{\alpha,\beta}^\#.$$

From Proposition 3.2, we have $\alpha_T(\beta)=\alpha_T^{-1}(\beta)$, then $\|T\|_\beta^\# = \|T\|_\beta$.

Theorem 3.4. Let $(X,N_1,*_1)$ and $(Y,N_2,*_2)$ be two fuzzy quasi-normed spaces. $\left\{\|\bullet\|_\beta:\beta\in(0,1)\right\}$ is an increasing and separating family of star quasi-seminorms on $LC(X,Y)$. Let N: $LC(X,Y)\times[0,+\infty)\to[0,1]$ be given by

$$N(T,t)=\begin{cases}\sup\left\{\beta\in(0,1):\|T\|_\beta<t\right\}, & t>0,\\ 0, & t=0,\end{cases} \tag{3.6}$$

then $(N,*_2)$ is a fuzzy quasi-norm on $LC(X,Y)$.

Proof. From Theorem 2.2, it is obvious that $(N,*_2)$ satisfies FQN1, FQN3-FQN6. Now we prove that $(N,*_2)$ satisfies FQN2'.

Let $SLC(X,Y)=\{T\in LC(X,Y):-T\in LC(X,Y)\}$ be a subset of $LC(X,Y)$. It is obvious that $0\in SLC(X,Y)$, thus, $SLC(X,Y)\neq\phi$. If $T\in SLC(X,Y)$ such that $N(T,t)=N(-T,t)=1$ for all $t>0$, from (3.6) we have that $\|T\|_\beta<t$ and $\|-T\|_\beta<t$ for all $\beta\in(0,1)$. By the arbitrariness of t, we get that $\|T\|_\beta=\|-T\|_\beta=0$ for all $\beta\in(0,1)$. Since $\left\{\|\bullet\|_\beta:\beta\in(0,1)\right\}$ is separating, we have $T=0$. Therefore $(N,*_2)$ satisfies FQN2'. Thus, $(N,*_2)$ is a fuzzy quasi-norm on $LC(X,Y)$.

4. Conclusions

In this paper, we focus on the continuity, boundedness and fuzzy quasi-norm of linear operators between two fuzzy quasi-normed spaces with general continuous t-norms, construct a fuzzy quasi-norm of a continuous linear operator. The obtained results demonstrate that the methods proposed in this paper are very useful. On the basis of this paper, one can further the researches about the linear operator theory of fuzzy quasi-normed spaces.

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Conflict of interest

The authors declare that there is no conflict of interest in this paper.

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