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*Research article*

# Two-dimensional pseudo-steady supersonic flow around a sharp corner for the generalized Chaplygin gas

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**Abstract:** In this paper, the expansion problem which arises in a two-dimensional (2D) isentropic pseudo-steady supersonic flow expanding into vacuum around a sharp corner for the generalized Chaplygin gas is studied. This expanding problem catches the interaction of an incomplete centered simple wave with a backward planar rarefaction wave and the interaction of a non-planar simple wave with a rigid wall boundary of the 2D self-similar Euler equations. Using the methods of characteristic decompositions and invariant regions, we get the hyperbolicity in the wave interaction domains and prior $C^1$ estimates of solutions to the two interaction problems. It follows the global existence of the solution up to infinity of the gas expansion problem.

**Keywords:** two-dimensional Euler equations; generalized Chaplygin gas; pseudo-steady supersonic flow; incomplete centered simple wave; interaction of simple wave with rigid wall boundary
**Mathematics Subject Classification:** 35L65, 35J70, 35R35, 35J65

## 1. Introduction

Supersonic flow around a bend or sharp corner, one of the most important elementary flows, is effected by a simple wave. In [5], Courant and Friedrichs constructed these simple waves for the steady flow. Recently, Sheng and You ( [31]) considered an expansion problem which arises in a supersonic flow expanding into vacuum around a sharp corner under the condition that the inclination angle of the rigid wall boundary is larger than a critical one and obtained the global solution to the expansion problem by solving the interaction problem of a complete centered simple wave with a backward planar rarefaction wave. In [29], Sheng and Yao considered the expansion problem when the inclination angle of the rigid wall boundary is smaller than the critical angle. They studied the interaction of an incomplete centered simple wave with a backward planar rarefaction wave and the interaction of a non-planar simple wave with rigid wall boundary and obtained the global existence of the solution of the gas expansion problem for the polytropic gas. In [30], Sheng and Yao constructed the self-similar solution for the supersonic flow around a convex corner and proved that the supersonic flow turns the convex corner by an incomplete centered expansion wave or an incomplete centered compression wave depending on the conditions of the downstream state. In [15], Lai and Sheng obtained the existence of global piecewise smooth (or Lipshitz-continuous) solutions to the problem of pseudosteady flows around a sharp corner for for the polytropic gas when the uniform flow is sonic or subsonic.

In this paper, we consider the problem of a supersonic flow expanding into vacuum around a sharp corner for the generalized Chaplygin gas. Suppose that the sharp corner is made up to a horizontal ground and a sloping straight rigid boundary at a sharp point. At the very beginning, the flow arrives with constant velocity along the straight ground wall up to the sharp point. Further, we assume that the oncoming flow is of constant state in a region adjacent to the part of the wall before the sharp point and is vacuum outside of the region. We want to know how dose the flow turn the corner and expand into vacuum. For this purpose, we study the 2D unsteady isentropic compressible Euler equations for generalized Chaplygin gas

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + \left(\rho u^2 + p\right)_x + (\rho uv)_y = 0, \\ (\rho v)_t + (\rho uv)_x + \left(\rho u^2 + p\right)_y = 0, \end{cases} \tag{1.1}$$

where $\rho$ is the density, $(u, v)$ is the velocity and $p$ is the pressure given by $p(\rho) = -\mathcal{K}\rho^{-\gamma}$ for generalized Chaplygin gas, the constant $\mathcal{K} > 0$ and $\gamma$ is the adiabatic exponent satisfying $0 < \gamma < 1$.

The expansion problem can be seen as a special 2D Riemann problem for Euler equations with boundary. There will be an incomplete centered simple wave emanating from the sharp corner and a planar rarefaction wave spreading from the supersonic oncoming flow to vacuum. The two simple waves will interact with each other. Due to the particularity of the generalized Chaplygin gas, the centered simple wave from the origin is a incomplete one, the interaction will piece together three parts of the flow in the interaction domain. The first part is the interaction of the incomplete centered simple wave with the planar rarefaction wave. The second part is a nonplanar simple wave adjacent to a constant state. And the last one is the interaction of the simple wave with rigid wall boundary.

The 2D Riemann problem for the Euler equations (1.1), which is a special Cauchy problem that the initial data are constants along each ray from the origin, are an interesting and complicated problem. It involves many kinds of wave interactions. The solution configurations conjectured in [35]. Up to now, there is much progress to the theory of system (1.1) or simplified models of it, see [1–3, 7, 8, 17, 20, 24, 25, 27, 33, 34] for polytropic gas and the Chaplygin gas equations, etc.

The expansion problem of a flow into vacuum has been a favorite problem for a long time. It has been interpreted hydraulically as the collapse of a wedge-shaped dam containing water initially with a uniform velocity ( [18]). This is a special 2D Riemann problem, mainly involving the interaction of two 2D planar rarefaction waves. There have been a lot of research results on the problem. A set of interesting explicit solutions were found in [32] for special wedge angle. In [6], Dai and Zhang established the global smooth solution for the expansion problem of pressure gradient system. In [19], Li studied a much more difficult problem for Euler equations of the expansion of a wedge of gas into vacuum and established the existence of a solution to the problem in the hodograph plane by the hodograph transformation. Li and Zheng proved the existence of the classical spolution in the hodograph plane and the non-degeneracy of the hodograph transformation on non-simple wave region, and obtained the existence of classical self-similar solution of the expansion problem by the hodograph transformation and the characteristic decompositions of characteristic inclination angles in [23]. In [21], the authors developed the direct approach to get global classical solution for 2D polytropic Euler equations to avoid the complicated procedure of the hodograph transformation. For more related papers, see [9–14, 16, 36]. In 2010, Sheng etc. in [28] considered an oblique rarefaction wave reflection along a compressive corner. By using the numerical generalized characteristic analysis method, they found a critical transonic shock. A supersonic bubble near the compression corner grows and break through as the rarefaction wave size increases in their results. The supersonic oncoming flow with a constant state $(u_1, 0, \rho_1)$ along the straight wall $AO$ up to the sharp point $O$, expands into the vacuum in the other region of the corner. Assuming that the inclination angle of the sloping straight rigid boundary $OB$ is $-\theta\left(0 < \theta < \frac{\pi}{2}\right)$. For unsteady isentropic compressible generalized Chanpygin gas, this expansion problem can be prescribed as the system (1.1) with the initial data (see Figure 1).

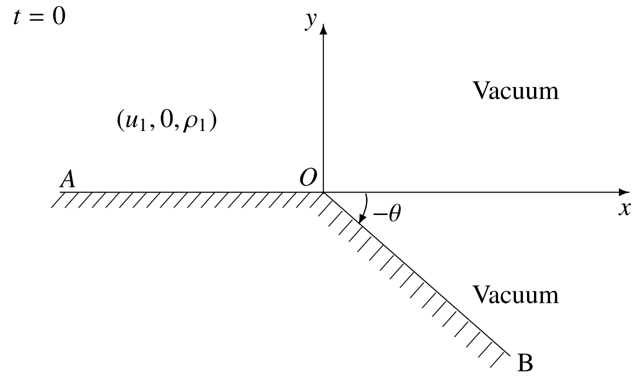


**Figure 1.** Initial-boundary data conditions.

$$(u, v, \rho)(x, y, 0) = \begin{cases} (u_1, 0, \rho_1), & \text{as } (x, y) \in \{x < 0, y > 0\}; \\ \text{Vacuum}, & \text{as } (x, y) \in \{y < 0, y > -x \tan\theta\} \cup \{x > 0, y \geq 0\}, \end{cases} \tag{1.2}$$

and the boundary data

$$\begin{cases} (\rho v)(x, 0, t) = 0, & \text{as } x < 0, y > 0; \\ (\rho v)(x, y, t) = -(\rho u)(x, y, t)\tan\theta, & \text{as } (x, y) \in \{y < 0, y = -x\cot\theta\}, t > 0. \end{cases} \tag{1.3}$$

where $u_1 > 0$ and $\rho_1 > 0$, the Mach angle of the initial oncoming flow is defined by $\alpha_1 = \arcsin\frac{c_1}{u_1}$, $c_1 = \mathcal{K}\gamma\rho_1^{\frac{-\gamma-1}{2}}$ is sonic velocity satisfying $u_1 > c_1$.

For the sake of the discussion below, we use following notations:

$$k = -\frac{1+\gamma}{2}, \quad m = \frac{3+\gamma}{1-\gamma} > 3, \quad \mu^2 = \frac{1+\gamma}{1-\gamma} > 1, \quad \tan^2\hat{\theta} = m, \quad \delta = \frac{\alpha-\beta}{2}, \quad \sigma = \frac{\alpha+\beta}{2},$$

$$\Omega = \frac{\cos 2\delta - k}{(k+1)\cos^2\delta} = m - \tan^2\delta, \quad \partial_- = \partial_\xi + \lambda_-\partial_\eta, \quad \partial_+ = \partial_\xi + \lambda_+\partial_\eta,$$

$$\bar{\partial}_+ = \cos\alpha\partial_\xi + \sin\alpha\partial_\eta, \quad \bar{\partial}_- = \cos\beta\partial_\xi + \sin\beta\partial_\eta, \quad \bar{\partial}_0 = \cos\sigma\partial_\xi + \sin\sigma\partial_\eta.$$

In addition, we introduce $\alpha = g(x)$, which is the inverse function of $\frac{\hat{v}(\alpha)}{\hat{u}(\alpha)} = -\tan x$ in Theorem 4.

The main result of this paper, which will be proved, is in the following theorem.

**Theorem 1.** *Assume that* $0 < \gamma < 1$, $2\hat{\theta} - \frac{\pi}{2} < \alpha_1 < \frac{\pi}{2}$ *and* $0 < \theta < g^{-1}\left(2\hat{\theta} - \frac{\pi}{2}\right)$, *the initial-boundary data problem* (1.1) *with* (1.2) *and* (1.3) *admits a global solution.*

This paper is organized as follows. In Section 2, for the generalized Chaplygion gas, we present some preliminaries, including the inclination angles $(\alpha,\beta)$ of $C_\pm$ characteristic, the characteristic forms of the 2D self-similar isentropic ir-rotational compressible Euler equations and the characteristic decompositions of speed of sound $c$. The characteristic equations and the characteristic decompositions are used to control the hyperbolicity and the priori gradient estimates of the solutions. In Section 3, we obtain the incomplete centered simple wave through the principle part of the centered simple waves and give the expressing of the incomplete centered simple wave. In Section 4, we obtain the global solution of the interaction of the incomplete centered simple wave with the backward planar rarefaction wave. In Section 5, we solve the reflecting problem of the simple wave on the rigid wall by the interaction of two symmetric simple waves. In the last section, we obtain the global existence of the solution of the expansion problem around the sharp corner for generalized Chaplygin gas.

## 2. Systems of 2D pseudo-steady isentropic ir-rotational flow

### 2.1. 2D pseudo-steady isentropic Euler equations

System (1.1) can be written as follows for smooth flow with the self-similar variables $(\xi,\eta) = \left(\frac{x}{t}, \frac{y}{t}\right)$

$$\begin{cases} U\rho_\xi + V\rho_\eta + \rho\left(u_\xi + v_\eta\right) = 0, \\ Uu_\xi + Vu_\eta + \frac{c^2}{\rho}\rho_\xi = 0, \\ Uv_\xi + Vv_\eta + \frac{c^2}{\rho}\rho_\eta = 0, \end{cases} \tag{2.1}$$

where $(U, V) = (u-\xi, v-\eta)$ is the pseudo-flow velocity, $c$ is the speed of sound with $c^2 = p'(\rho) = \mathcal{K}\gamma\rho^{-\gamma-1}$ for the generalized Chaplygin gas.

We further assume that the flow is ir-rotational, i.e. $u_y = v_x$. Then, the system (2.1) can be reduced as

$$\begin{cases} \left(c^2 - U^2\right)u_\xi - UV\left(u_\eta + v_\xi\right) + \left(c^2 - V^2\right)v_\eta = 0, \\ u_\eta - v_\xi = 0, \end{cases} \tag{2.2}$$

supplemented by pseudo-Bernoulli's law:

$$\frac{c^2}{-\gamma-1} + \frac{U^2+V^2}{2} = -\varphi + \text{Const.} \tag{2.3}$$

where $\varphi(\xi,\eta)$ is the pseudo-potential function such that $\varphi_\xi = U$, $\varphi_\eta = V$. The eigenvalues of Eq (2.2) are determined by

$$(\lambda U - V)^2 - c^2\left(1 + \lambda^2\right) = 0, \tag{2.4}$$

which are

$$\lambda_{\pm} = \frac{UV \pm \sqrt{c^2\left(U^2 + V^2 - c^2\right)}}{U^2 - c^2}. \tag{2.5}$$

So, if and only if $U^2 + V^2 > c^2$, the Eq (2.2) are hyperbolic. The integral curves

$$C_{\pm} : \frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \lambda_{\pm} \tag{2.6}$$

are the pseudo-wave $C_{\pm}$ characteristics of (2.2), respectively. The pseudo-flow characteristic is defined by integral curves of

$$C_0 : \frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \lambda_0 = \frac{V}{U}. \tag{2.7}$$

The left-eigenvectors of the eigenvalues $\lambda_{\pm}$ are

$$l_{\pm} = \left(1, \mp\sqrt{c^2\left(U^2 + V^2 - c^2\right)}\right). \tag{2.8}$$

Multiplying (2.2) by $l_{\pm}$ on the left and differentiating the pseudo-Bernoulli's law (2.3), we get the characteristic forms of the system (2.1)

$$\begin{cases} \partial_+ u + \lambda_- \partial_+ v = 0, \\ \partial_- u + \lambda_+ \partial_- v = 0 \\ \partial_{\pm} c^2 = -2k\left(U\partial_{\pm} u + V\partial_{\pm} v\right). \end{cases} \tag{2.9}$$

### 2.2. *Characteristic equations and characteristic decompositions*

Let $\alpha, \beta$ be the inclination angles of $C_{\pm}$ characteristics as in [23], defined by

$$\tan\alpha = \lambda_+, \quad \tan\beta = \lambda_-. \tag{2.10}$$

For the convenience to our problem, we choose

$$\begin{cases} \dfrac{U}{c} = \dfrac{\cos\sigma}{\sin\delta}, \\ \dfrac{V}{c} = \dfrac{\sin\sigma}{\sin\delta}, \end{cases} \quad i.e., \quad \begin{cases} u = \xi + c\dfrac{\cos\sigma}{\sin\delta}, \\ v = \eta + c\dfrac{\sin\sigma}{\sin\delta}, \end{cases} \tag{2.11}$$

where $\sigma = \frac{\alpha+\beta}{2}$ is the inclination angle of pseudo-flow characteristic, and $\delta = \frac{\alpha-\beta}{2}$ is the Mach angle satisfying $q^2 = U^2 + V^2 = \frac{c^2}{\sin^2\delta}$ (see Figure 2).

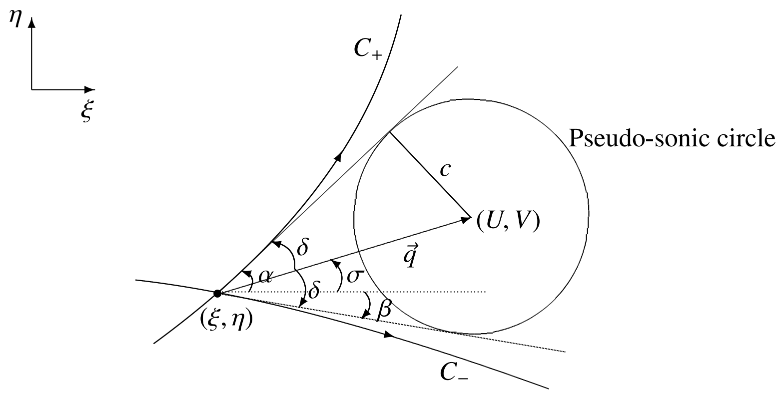


**Figure 2.** Characteristic curves, characteristic angles and pseudo-sonic circle.

The pseudo-Bernoulli's law (2.3) can be written as the function of variables $\alpha, \beta$ and $c$

$$\frac{c^2}{-\gamma-1}+\frac{c^2}{2\sin^2\delta}=-\varphi+\text{Const.}. \tag{2.12}$$

Then, we have

$$\begin{cases} c\bar{\partial}_+\alpha=\Omega\cos^2\delta\left(2\sin^2\delta+c\bar{\partial}_+\beta\right),\\ c\bar{\partial}_-\beta=\Omega\cos^2\delta\left(-2\sin^2\delta+c\bar{\partial}_-\alpha\right),\\ \bar{\partial}_\perp\varphi=0, \end{cases} \tag{2.13}$$

where $\bar{\partial}_\perp=-\sin\sigma\partial_\xi+\cos\sigma\partial_\eta$, the variable $c=c(\delta,\varphi):=\sqrt{-\frac{2(\gamma+1)\varphi\sin^2\delta}{(\gamma+1)-2\sin^2\delta}}$ by letting the constant of the pseudo-Bernoulli's law (2.12) to be 0. Moreover, we have the characteristic equations of $u$, $v$, $\alpha$ and $\beta$ expressed by $\bar{\partial}_\pm c$

$$\begin{cases} \bar{\partial}_+u=\dfrac{\sin\beta}{k}\bar{\partial}_+c,\\ \bar{\partial}_+v=-\dfrac{\cos\beta}{k}\bar{\partial}_+c, \end{cases}\qquad \begin{cases} \bar{\partial}_-u=-\dfrac{\sin\alpha}{k}\bar{\partial}_-c,\\ \bar{\partial}_-v=\dfrac{\cos\alpha}{k}\bar{\partial}_-c, \end{cases} \tag{2.14}$$

$$\begin{cases} c\bar{\partial}_+\alpha=\dfrac{\Omega\sin 2\delta}{2\mu^2}\bar{\partial}_+c,\\ c\bar{\partial}_+\beta=\dfrac{\tan\delta}{\mu^2}\bar{\partial}_+c-2\sin^2\delta, \end{cases}\qquad \begin{cases} c\bar{\partial}_-\alpha=-\dfrac{\tan\delta}{\mu^2}\bar{\partial}_-c+2\sin^2\delta,\\ c\bar{\partial}_-\beta=-\dfrac{\Omega\sin 2\delta}{2\mu^2}\bar{\partial}_-c. \end{cases} \tag{2.15}$$

**Lemma 1.** *The Eqs* (2.9) *or* (2.13) *can be reduced to the following diagonal form*

$$\begin{cases} \bar{\partial}_+r=-H(c,\delta),\\ \bar{\partial}_-s=-H(c,\delta),\\ \bar{\partial}_0\left(c^2\left(1+kM^2\right)\right)=-2ckM \quad or \quad \bar{\partial}_\perp\varphi=0, \end{cases} \tag{2.16}$$

*where*

$$r=r(\alpha,\beta)=\psi(\delta)-\beta,\quad s=s(\alpha,\beta)=\psi(\delta)+\alpha,$$
$$\psi(\delta)=-\frac{1}{2\mu}\ln\left|\frac{\mu\cot\delta-1}{\mu\cot\delta+1}\right|,\ H(c,\delta)=\frac{\sin^2\delta\,(\cos 2\delta-k)}{c\left(k+\sin^2\delta\right)},\ M=\frac{\sqrt{U^2+V^2}}{c}=\frac{1}{\sin\delta}. \tag{2.17}$$

And the characteristic decompositions of $c$ satisfy

**Lemma 2.** *For the variable c, we have*

$$\begin{cases} c\bar{\partial}_+\bar{\partial}_-c=\bar{\partial}_-c\left(\sin 2\delta-\dfrac{1}{2\mu^2\cos^2\delta}\bar{\partial}_-c+\left(1-\dfrac{\Omega\cos 2\delta}{2\mu^2}\right)\bar{\partial}_+c\right),\\ c\bar{\partial}_-\bar{\partial}_+c=\bar{\partial}_+c\left(\sin 2\delta-\dfrac{1}{2\mu^2\cos^2\delta}\bar{\partial}_+c+\left(1-\dfrac{\Omega\cos 2\delta}{2\mu^2}\right)\bar{\partial}_-c\right). \end{cases} \tag{2.18}$$

By the self-similar transformation, the initial-boundary data problem (1.1) with (1.2) and (1.3) (see Figure 1) can be reduced to corresponding boundary data problem of (2.1) at infinity with (2.19) and (2.20) (see Figure 3). The initial condition (1.2) is changed into

$$(u,v,\rho)\rightarrow\begin{cases}(u_1,0,\rho_1), & \xi<0,\ \eta>0,\ \text{and}\ \xi^2+\eta^2\rightarrow\infty,\\ \text{Vacuum}, & \eta>-\xi\tan\theta,\ \xi>0,\ \text{and}\ \xi^2+\eta^2\rightarrow\infty.\end{cases}\tag{2.19}$$

and the boundary condition (1.3) is changed into

$$\begin{cases}(\rho v)(\xi,\eta)=0, & (\xi,\eta)\in\{(\xi,\eta)\,|\xi<0,\eta=0\};\\ (\rho v)(\xi,\eta)=-(\rho u)(\xi,\eta)\tan\theta, & (\xi,\eta)\in\{(\xi,\eta)\,|\eta<0,\eta=-\xi\tan\theta\}\end{cases}\tag{2.20}$$

We will solve the problem in the following.

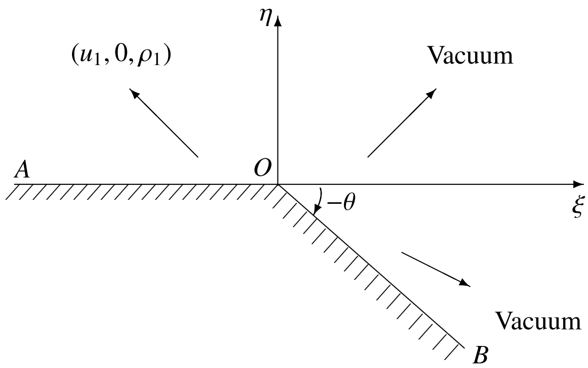


**Figure 3.** Boundary data at infinity.

## 3. Center simple wave around a corner

We have already known that a wave adjacent to a constant state is a simple wave, so we only have to consider the the trivial case (see [13]) in our problem.

### *3.1. The principle part of isentropic irrotational pseudo-steady centered waves*

We discuss the properties of the principal part of the centered simple wave for the system (2.1).

**Definition 1.** Let $\Lambda(t)$ be an angular domain with boundaries (see Figure 4)

$$\Lambda(t):=\{(\xi,\eta)\,|\,0\le\xi\le t,\ \xi\tan\alpha_2\le\eta=\xi\lambda_+\le\xi\tan\alpha_1\},\tag{3.1}$$

A function $(u,v,c)(\xi,\eta)$ is called a $C_+$ type centered simple wave solution for system (2.1) with the origin $(0,0)$ as the center point if the following properties are satisfied (see [13,26,31])
1) $(u,v,c)(\xi,\eta)$ can be determined by $(u,v,c)(\xi,\eta)=(\tilde{u},\tilde{v},\tilde{c})(\xi,\alpha)$ and $\eta=\xi\tan\alpha$ defined on a rectangular domain

$$\tilde{\Lambda}(t):=\{(\xi,\alpha)\,|\,0\le\xi\le t,\ \alpha_2\le\alpha\le\alpha_1\}.$$

Moreover, $(\tilde{u},\tilde{v},\tilde{c})$ belong to $C^1(\tilde{\Lambda}(t))$;
2) The function $(u,v,c)(\xi,\eta)$ defined above satisfies Eq (2.1) on $\Lambda(t)\setminus(0,0)$;

3) For any $\alpha \in [\alpha_2, \alpha_1]$ , $\eta = \xi\lambda_+$ gives the $C_+$ characteristic line passing through the origin $(0,0)$ with the slope $\tan\alpha$ at the origin.

Substituting $\eta = \xi\tan\alpha$ into pseudo-Bernoulli law (2.3), we obtain pseudo-potential function

$$\varphi = \tilde{\varphi}(\xi,\alpha) = \text{Const.} - \frac{1}{2}((\tilde{u}-\xi)^2 + (\tilde{v}-\xi\tan\alpha)^2) - \frac{\tilde{c}^2}{-\gamma-1}.$$

$(\hat{u},\hat{v},\hat{c})(\alpha) = \lim\limits_{\xi\to 0}(\tilde{u},\tilde{v},\tilde{c})(\xi,\alpha)$, $\hat{\varphi}(\alpha) = \lim\limits_{\xi\to 0}\tilde{\varphi}(\xi,\alpha)$ is called the principal part of this $C_+$ type centered simple wave.

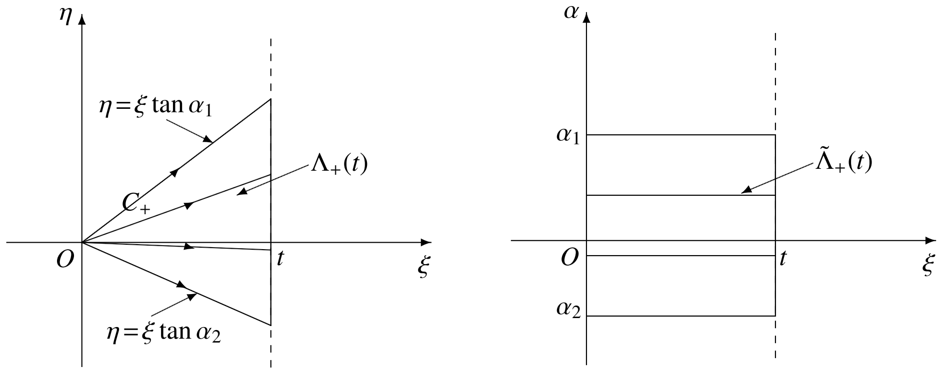


**Figure 4.** The centered simple wave.

**Theorem 2.** *Assume that* $(u,v,c)(\xi,\eta) = (\tilde{u},\tilde{v},\tilde{c})(\xi,\alpha)$, $\varphi(\xi,\eta) = \tilde{\varphi}(\xi,\alpha)$, $\eta = \xi\tan\alpha$, $\xi > 0$, $\alpha_2 \le \alpha \le \alpha_1$ *is the* $C_+$ *type centered simple wave solution of the system* (2.1) *in pseudo-supersonic domain, then the principal part* $(\hat{u},\hat{v},\hat{c})(\alpha)$ *and* $\hat{\varphi}(\alpha)$ *satisfy*

$$\begin{aligned} &\frac{1}{2}\left(\hat{u}^2(\alpha)+\hat{v}^2(\alpha)\right) + \frac{\hat{c}^2(\alpha)}{-\gamma-1} = \text{Const.} - \hat{\varphi}(\tilde{\alpha}_1),\\ &\frac{\mathrm{d}\hat{\varphi}(\alpha)}{\mathrm{d}\alpha} = 0, \quad \frac{\mathrm{d}\hat{u}(\alpha)}{\mathrm{d}\alpha} + \tan\alpha\frac{\mathrm{d}\hat{v}(\alpha)}{\mathrm{d}\alpha} = 0,\\ &\tan\alpha = \frac{\hat{u}(\alpha)\hat{v}(\alpha) + \hat{c}(\alpha)\sqrt{\hat{u}^2(\alpha)+\hat{v}^2(\alpha)-\hat{c}^2(\alpha)}}{\hat{u}^2(\alpha)-\hat{c}^2(\alpha)}. \end{aligned} \tag{3.2}$$

**Theorem 3.** *Assume that the functions* $(\hat{u},\hat{v},\hat{c})(\alpha) = \lim\limits_{\xi\to 0}(\tilde{u},\tilde{v},\tilde{c})(\xi,\alpha)$, $\alpha \in [\alpha_2,\alpha_1]$ *satisfy*

$$\begin{aligned} &\frac{1}{2}\left(\hat{u}^2(\alpha)+\hat{v}^2(\alpha)\right) + \frac{\hat{c}^2(\alpha)}{-\gamma-1} = \text{Const.},\\ &\frac{\mathrm{d}\hat{u}(\alpha)}{\mathrm{d}\alpha} + \tan\alpha\frac{\mathrm{d}\hat{v}(\alpha)}{\mathrm{d}\alpha} = 0,\\ &\tan\alpha = \frac{\hat{u}(\alpha)\hat{v}(\alpha) + \hat{c}(\alpha)\sqrt{\hat{u}^2(\alpha)+\hat{v}^2(\alpha)-\hat{c}^2(\alpha)}}{\hat{u}^2(\alpha)-\hat{c}^2(\alpha)}, \end{aligned} \tag{3.3}$$

*and the values of* $(u,v,c)(\xi,\eta)$ *on the ray* $\eta = \xi\tan\alpha$ *are defined as* $(\hat{u},\hat{v},\hat{c})(\alpha)$, *then* $(u,v,c)(\xi,\eta)$ *is the centered simple wave solution of* (2.1) *with the origin* $(0,0)$ *as the center point.*

*Proof.* (1) Verifying Eq (2.4) is enough to prove that for any $\alpha \in [\tilde{\alpha}_2, \tilde{\alpha}_1]$, the straight line $\lambda = \frac{\eta}{\xi} = \tan\alpha$ is a $C_+$ characteristic line. From the third equation of (3.3), by simple computation,

$$\begin{aligned}
&\lambda\,(u\,(\xi,\eta)-\xi)-(v\,(\xi,\eta)-\eta)\\
&=(\lambda u\,(\xi,\eta)-v\,(\xi,\eta))-(\lambda\xi-\eta)=\lambda\tilde{u}\,(\xi,\alpha)-\tilde{v}\,(\xi,\alpha)\\
&=\hat{u}(\alpha)\tan\alpha-\hat{v}(\alpha)=\frac{\hat{c}(\alpha)}{\cos(\alpha)}=\hat{c}(\alpha)\sqrt{1+\lambda^2}=c(\xi,\eta)\sqrt{1+\lambda^2}.
\end{aligned}\tag{3.4}$$

(2) We want to prove that the expressions of $(u, v, c)\,(\xi, \eta) = (\hat{u}, \hat{v}, \hat{c})\,(\alpha)$ in (3.3) satisfy the characteristic system (2.9). For fixed $\alpha$, along the straight $C_+$ characteristic line $\frac{\eta}{\xi} = \tan\alpha$, $u$ ,$v$, $c$ are constant, we have $\bar{\partial}_+u = \bar{\partial}_+v = \bar{\partial}_+c = 0$ and immediately we have $\bar{\partial}_+u+\lambda_-\bar{\partial}_+v = 0$. By the second equation of (3.3), we have

$$\bar{\partial}_-u+\lambda_+\bar{\partial}_-v=(\hat{u}'(\alpha)+\lambda_+\hat{v}'(\alpha))\,\bar{\partial}_-\alpha=0.\tag{3.5}$$

(3) The pseudo-Bernoulli's law (2.12) is also satisfied under the expressions of $(u, v, c)\,(\xi, \eta) = (\hat{u}, \hat{v}, \hat{c})\,(\alpha)$ in (3.3). Because of $\bar{\partial}_+u = \bar{\partial}_+v = \bar{\partial}_+c = 0$, it is obviously that along the direction of $C_+$ characteristic line, we have

$$\begin{aligned}
&\bar{\partial}_+\left(\frac{(u-\xi)^2+(v-\eta)^2}{2}+\frac{c^2}{-\gamma-1}+\varphi\right)\\
&=(u-\xi)\,\bar{\partial}_+u+(v-\eta)\,\bar{\partial}_+v+\frac{2c}{-\gamma-1}\bar{\partial}_+c=0.
\end{aligned}\tag{3.6}$$

And along the direction of $C_-$ characteristic, we have

$$\begin{aligned}
&\bar{\partial}_-\left(\frac{(u-\xi)^2+(v-\eta)^2}{2}+\frac{c^2}{-\gamma-1}+\varphi\right)\\
&=(u-\xi)\,\bar{\partial}_-u+(v-\eta)\,\bar{\partial}_-v+\frac{2c}{-\gamma-1}\bar{\partial}_-c\\
&=\left(\hat{u}(\alpha)\hat{u}'(\alpha)+\hat{v}(\alpha)\hat{v}'(\alpha)+\frac{2\hat{c}(\alpha)}{-\gamma-1}\hat{c}'(\alpha)-\xi\hat{u}'(\alpha)-\xi\tan\alpha\hat{v}'(\alpha)\right)\bar{\partial}_-\alpha\\
&=-\xi\,(\hat{u}'(\alpha)+\tan\alpha\hat{v}'v)\,\bar{\partial}_-\alpha\overset{(3.3)}{=}0.
\end{aligned}\tag{3.7}$$

□

### *3.2. Center simple wave around a corner*

**Theorem 4.** *For the oncoming supersonic flow $(u_1, 0, c_1)$, near the corner O, the problem* (2.1) *with* (2.19) *and* (2.20) *admits a local solution consisting of constant states* $\mathrm{I}(u_1, 0, c_1)$, $\mathrm{II}(u_2, v_2, c_2)$

*and an incomplete centered simple wave determined by*

$$R_1:\begin{cases}\hat{u}(\alpha)=e^{-\mu(\alpha-\alpha_1)}\left(\left(d_1+d_2e^{2\mu(\alpha-\alpha_1)}\right)\sin\alpha+\dfrac{1}{\mu}\left(d_1-d_2e^{2\mu(\alpha-\alpha_1)}\right)\cos\alpha\right)\\ \hat{v}(\alpha)=e^{-\mu(\alpha-\alpha_1)}\left(-\left(d_1+d_2e^{2\mu(\alpha-\alpha_1)}\right)\cos\alpha+\dfrac{1}{\mu}\left(d_1-d_2e^{2\mu(\alpha-\alpha_1)}\right)\sin\alpha\right)\\ \hat{c}(\alpha)=-\check{c}(\alpha)=e^{-\mu(\alpha-\alpha_1)}\left(d_1+d_2e^{2\mu(\alpha-\alpha_1)}\right)\\ \alpha=\arctan\dfrac{\eta}{\xi}\end{cases}, \tag{3.8}$$

*where* $\alpha_2<\alpha<\alpha_1$, $d_1=\frac{c_1+\mu\sqrt{u_1^2-c_1^2}}{2}$, $d_2=\frac{c_1-\mu\sqrt{u_1^2-c_1^2}}{2}$. *The flow arrives at the constant states* $(u_2,v_2,c_2)$ *when* $\alpha\le\alpha_2$, *and* $\frac{\hat{v}(\alpha_2)}{\hat{u}(\alpha_2)}=-\tan\theta$. *From* $\frac{\hat{v}(\alpha)}{\hat{u}(\alpha)}=-\tan x$, *we get* $\frac{\mathrm{d}\theta}{\mathrm{d}\alpha}<0$. *So we obtain the inverse function denoted by* $\alpha=g(x)$ *(see Figure 5).*

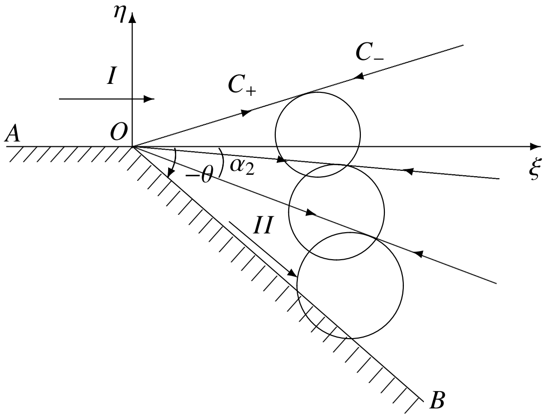


**Figure 5.** Incomplete centered simple wave in the flow around a sharp corner.

*Proof.* The principle part of the $C_+$ incomplete centered simple wave is

$$(u,v,c)(\xi,\eta)=(\hat{u},\hat{v},\hat{c})(\alpha),$$

where $\alpha$ is the inclining angle of characteristic line $\eta=\xi\tan\alpha$

From Theorem 3.2, we have

$$\cos\alpha\frac{\mathrm{d}\hat{u}}{\mathrm{d}\alpha}+\sin\alpha\frac{\mathrm{d}\hat{v}}{\mathrm{d}\alpha}=0. \tag{3.9}$$

Project pseudo-velocity $(U,V)=(\hat{u}(\alpha)-\xi,\hat{v}(\alpha)-\xi\tan\alpha)$ on direction $(-\sin\alpha,\cos\alpha)$ and the direction of $C_+$ characteristics $(\cos\alpha,\sin\alpha)$, respectively. Then, we obtain $(\check{c}(\alpha),g(\xi,\alpha))$ satisfying:

$$\begin{cases}\check{c}(\alpha)=-\hat{c}(\alpha)=-(\hat{u}(\alpha)-\xi)\sin\alpha+(\hat{v}(\alpha)-\xi\tan\alpha)\cos\alpha\\ \qquad\qquad=-\hat{u}(\alpha)\sin\alpha+\hat{v}(\alpha)\cos\alpha,\\ g(\xi,\alpha)=(\hat{u}(\alpha)-\xi)\cos\alpha+(\hat{v}(\alpha)-\xi\tan\alpha)\sin\alpha.\end{cases} \tag{3.10}$$

Let $\xi\to 0$, we have

$$\begin{cases}\check{c}(\alpha)=-\hat{u}(\alpha)\sin\alpha+\hat{v}(\alpha)\cos\alpha,\\ \check{g}(\alpha)=\lim\limits_{\xi\to 0}g(\xi,\alpha)=\hat{u}(\alpha)\cos\alpha+\hat{v}(\alpha)\sin\alpha.\end{cases} \tag{3.11}$$

We obtain
$$\breve{g}^2(\alpha) + \breve{c}^2(\alpha) = \hat{u}^2(\alpha) + \hat{v}^2(\alpha), \tag{3.12}$$
and
$$\begin{cases} \hat{u}(\alpha) = -\breve{c}(\alpha)\sin\alpha + \breve{g}(\alpha)\cos\alpha, \\ \hat{v}(\alpha) = \breve{c}(\alpha)\cos\alpha + \breve{g}(\alpha)\sin\alpha. \end{cases} \tag{3.13}$$
Deriving with respect to $\alpha$ on both side, we obtain
$$\begin{cases} \dfrac{\mathrm{d}\hat{\mathrm{u}}(\alpha)}{\mathrm{d}\alpha} = -\breve{c}'(\alpha)\sin\alpha - \breve{c}(\alpha)\cos\alpha + \breve{g}'(\alpha)\cos\alpha - \breve{g}(\alpha)\sin\alpha, \\ \dfrac{\mathrm{d}\hat{\mathrm{v}}(\alpha)}{\mathrm{d}\alpha} = \breve{c}'(\alpha)\cos\alpha - \breve{c}(\alpha)\sin\alpha + \breve{g}'(\alpha)\sin\alpha + \breve{g}(\alpha)\cos\alpha. \end{cases} \tag{3.14}$$
Inserting (3.14) into (3.9), we have
$$\breve{g}'(\alpha) = \breve{c}(\alpha). \tag{3.15}$$
By virture of Theorem 2, the Bernoulli's law may be changed to
$$\mu^2\left(\hat{u}^2(\alpha) + \hat{v}^2(\alpha)\right) - \left(1+\mu^2\right)\hat{c}^2(\alpha) = C.$$
From (3.12), accordingly,
$$C + \breve{c}^2(\alpha) = \mu^2\left(\hat{u}^2(\alpha) + \hat{v}^2(\alpha) - \breve{c}^2(\alpha)\right) = \mu^2\breve{g}^2(\alpha). \tag{3.16}$$
Differentiating the equation above with respect to $\alpha$ on both side, and because of (3.15)
$$2\breve{c}(\alpha)\breve{c}'(\alpha) = 2\mu^2\breve{g}(\alpha)\breve{g}'(\alpha) = 2\mu^2\breve{g}(\alpha)\breve{c}(\alpha). \tag{3.17}$$
we have
$$\breve{c}'(\alpha) = \mu^2\breve{g}(\alpha). \tag{3.18}$$
Combining (3.15) with (3.18)
$$\breve{c}''(\alpha) - \mu^2\breve{c}(\alpha) = 0. \tag{3.19}$$
Solving Eq (3.19) with the initial data $\breve{c}(\alpha_1) = -c_1$ and $\breve{g}^2(\alpha_1) = u_1^2 - \breve{c}^2(\alpha_1)$, we obtain (3.8) immediately. □

**Theorem 5.** *The incomplete centered simple wave $R_1$ defined in Theorem 3 may not expand to vacuum, i.e., the centered simple wave $R_1$ connects constant state $(u_1, 0, c_1)$ and another constant state $(u_2, v_2, c_2)$, where $c_2 < \infty$.*

*Proof.* For any $-\theta < \alpha < \alpha_1$, we have $e^{-\mu(\alpha-\alpha_1)} > 1$, $e^{\mu(\alpha-\alpha_1)} < 1$,
$$\hat{c}'(\alpha) = -\breve{c}'(\alpha) = \mu\left(-d_1 e^{-\mu(\alpha-\alpha_1)} + d_2 e^{\mu(\alpha-\alpha_1)}\right) < 0. \tag{3.20}$$
$\hat{c}(\alpha)$ increases as $\alpha$ decreases, so the speed of sonic $\hat{c}(\alpha) = d_1 e^{-\mu(\alpha-\alpha_1)} + d_2 e^{\mu(\alpha-\alpha_1)}$ can not tend to infinity. □

## 4. Interaction of the incomplete centered simple wave with the planar rarefaction wave

### 4.1. Goursat problem

According to [2], the backward planar rarefaction wave $R_2$ expanding the supersonic oncoming flow $(u_1, 0, c_1)$ to the vacuum is determined by

$$R_2: \begin{cases} \xi = u_1 + \dfrac{1-\gamma}{1+\gamma}\sqrt{\mathcal{K}\gamma}\rho^{-\frac{1+\gamma}{2}} - \dfrac{2}{1+\gamma}\sqrt{\mathcal{K}\gamma}\rho_1^{-\frac{1+\gamma}{2}} = u - c, \\ u = u_1 + \dfrac{2}{1+\gamma}\sqrt{\mathcal{K}\gamma}\rho^{-\frac{1+\gamma}{2}} - \dfrac{2}{1+\gamma}\sqrt{\mathcal{K}\gamma}\rho_1^{-\frac{1+\gamma}{2}}, \\ v = 0, \end{cases} \quad 0 \le \rho \le \rho_1. \tag{4.1}$$

The two waves $R_1$ and $R_2$ interact firstly at the point $P\left(u_1 - c_1, c_1\sqrt{\frac{u_1-c_1}{u_1+c_1}}\right)$ and then form a interaction region $\Sigma_1$ separated from $R_1$, $R_2$ by $PD$, $PC_\infty$, which are the $C_\mp$ cross characteristics curves determined by $R_1$ and $R_2$, respectively (see Figure 6). It is easy to know that the cross characteristic curve $PC_\infty$ is determined by

$$\begin{cases} \xi = u_1 + \dfrac{1-\gamma}{1+\gamma}\sqrt{\mathcal{K}\gamma}\rho^{-\frac{1+\gamma}{2}} - \dfrac{2}{1+\gamma}\sqrt{\mathcal{K}\gamma}\rho_1^{-\frac{1+\gamma}{2}}, \\ \eta = \rho^{\frac{1-\gamma}{4}}\left(\dfrac{\mathcal{K}\gamma(1-\gamma)}{3+\gamma}\rho^{-\frac{3+\gamma}{2}} + \mathcal{K}\gamma\left(\dfrac{u_1-c_1}{u_1+c_1} - \dfrac{1-\gamma}{3+\gamma}\right)\rho_1^{-\frac{3+\gamma}{2}}\right)^{1/2}, \end{cases} \quad 0 \le \rho \le \rho_1. \tag{4.2}$$

Therefore, we can get the expression of $PC_\infty$, which is

$$\eta = \left(\mu^4\frac{1-\gamma}{3+\gamma}\left(\xi - u_1 + \frac{2c_1}{1+\gamma}\right)^2 + c_1^2\rho_1^{\frac{-1+\gamma}{2}}\left(\frac{u_1-c_1}{u_1+c_1} - \frac{1-\gamma}{3+\gamma}\right)\left(\xi - u_1 + \frac{2c_1}{1+\gamma}\right)^{-\frac{1-\gamma}{1+\gamma}}\left(\frac{\mu^2}{\sqrt{\mathcal{K}\gamma}}\right)^{-\frac{1-\gamma}{1+\gamma}}\right)^{1/2} \tag{4.3}$$

where $u_1 - c_1 \le \xi$.

We define $\Sigma_1$ more precisely to contain the boundaries $PD$, $PC_\infty$ as well as the $C_+$ cross characteristic $DE_\infty^1$. The solution of the boundary data problem (2.1) with (2.19) in the outside region of $\Sigma_1$ consists of constant state $(u_1, 0, c_1)$, vacuum at infinity, incomplete centered simple wave $R_1$ and the backward rarefaction wave $R_2$, (see Figure 6). The solution of the boundary data problem inside the wave interaction region $\Sigma_1$ can be reduced to a Goursat problem, namely, the system (2.1) with the boundary data

$$(u, v, \rho)(\xi, \eta) = \begin{cases} (u_+, v_+, \rho_+)(\xi, \eta), & \text{on } PC_\infty, \\ (u_-, v_-, \rho_-)(\xi, \eta), & \text{on } PD, \end{cases} \tag{4.4}$$

where $(u_\pm, v_\pm, \rho_\pm)(\xi, \eta)$ is determined by rarefaction wave $R_2(R_1)$.

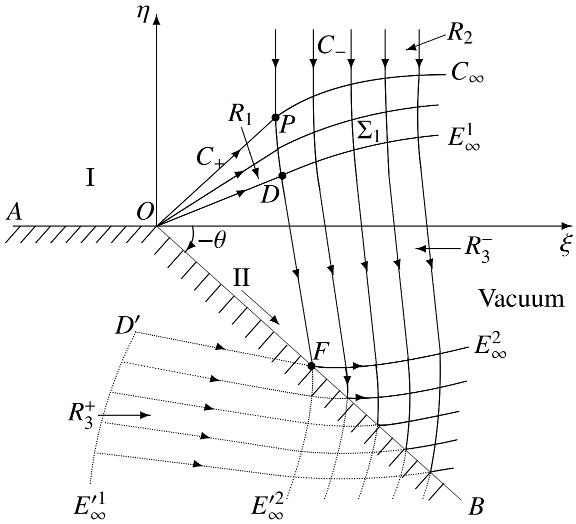


**Figure 6.** Interactions of waves and global solution of (2.1) with (2.19).

**Lemma 3.** (Local existence) *There is a $h > c_1$, $h$ is sufficiently close to $c_1$ such that the Goursat problem* (2.1) *with* (4.4) *admits a unique $C^1$ solution on the angular domain $\Sigma_h$ bounded by $PC_h$, $PD_h$ and level curve $c(\xi,\eta) = h$, which intersects with $PC_\infty$ at $C_h$ and $PD$ at $D_h$.*

*Proof.* From [26] we know that for sufficiently small $\varepsilon > 0$ the Goursat problem (2.1) with (4.4) admits a unique local $C^1$ solution on the angular domain closed by $PC_\infty$, $PD$ and the straight line $\xi = u_1 - c_1 + \varepsilon$. By $\bar{\partial}_- c|_{PC_\infty} > 0$, $\bar{\partial}_+ c|_{PD} > 0$ and the characteristic decompositions (2.18), we have that the solution satisfies $\bar{\partial}_\pm c > 0$. Let

$$h = \inf_{\xi = u_1 - c_1 + \varepsilon} c(\xi, \eta),$$

Then, the Goursat problem has a $C^1$ solution on $\Sigma_h$. □

Boundary data estimates

**Lemma 4.** 1) *On the $C_+$ cross characteristic curve $PC_\infty$, we have*

$$0 < \hat{\theta} \le \delta \le \frac{\alpha_1 + \frac{\pi}{2}}{2},\ \beta = -\frac{\pi}{2},\ \bar{\partial}_+ \alpha < 0. \tag{4.5}$$

2) *Let $\beta = \beta(\alpha)$ ($\alpha_2 \le \alpha \le \alpha_1$) be the image curve of $PD$ on the $(\alpha,\beta)$ planar, then this curve can be one of the following three situations* (see Figure 7): (*i*) *$\beta = \beta(\alpha)$ passes through the line $\alpha - \beta = 2\hat{\theta}$ and has unique intersection point with it, such that $\beta'(\alpha) > 0$ above the line, $\beta'(\alpha) = 0$ at the intersection point and $\beta'(\alpha) < 0$ below the line;*

(*ii*) *$\beta = \beta(\alpha)$ doesn't pass through $\alpha - \beta = 2\hat{\theta}$ and has unique intersection point with it, such that $\beta'(\alpha) = 0$ at the intersection point and $\beta'(\alpha) < 0$ below the line;*

(*iii*) *$\beta = \beta(\alpha)$ is located below $\alpha - \beta = 2\hat{\theta}$, such that $\beta'(\alpha) < 0$.*

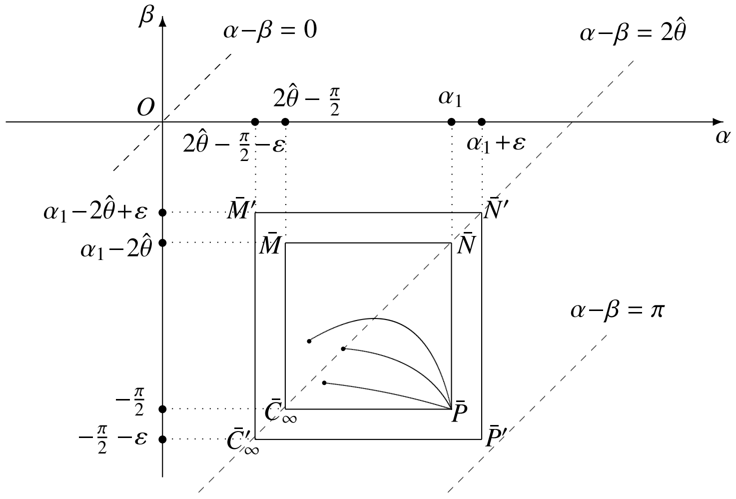


**Figure 7.** Invariant region of $(\alpha,\beta)$.

*Proof.* 1) According to (4.3), we get that (4.5) is valid on $PC_\infty$.

2) From the incomplete centered simple wave $R_1$, we obtain $\bar\partial_- c > 0$ and $\bar\partial_-\alpha < 0$ on $PD$. The image point $\bar P$ of $P$ is below the line $\alpha-\beta = 2\hat\theta$ because of $(\alpha-\beta)(P) = \alpha_1 + \frac{\pi}{2} > 2\hat\theta$. From the last equation of (2.15), we get $\bar\partial_-\beta(P) > 0$. Combining with $\bar\partial_-\alpha(P) < 0$, we have

$$\beta'(\alpha)\Big|_{\bar P} = \frac{\bar\partial_-\beta}{\bar\partial_-\alpha}\bigg|_P < 0.$$

Hereinafter, we consider the property of $\beta=\beta(\alpha)$ along $PD$. Before $\beta=\beta(\alpha)$ arrives at $\alpha-\beta = 2\hat\theta$, we get $\bar\partial_-\beta > 0$ by (2.15). Combining with $\bar\partial_-\alpha < 0$, we have $\beta'(\alpha) < 0$. At the intersection point of $\beta=\beta(\alpha)$ with $\alpha-\beta = 2\hat\theta$, we get $\bar\partial_-\beta = 0$, which combines with $\bar\partial_-\alpha < 0$ to give $\beta'(\alpha) = 0$. If $(\alpha,\beta)(D)$ is located above the straight line $\alpha-\beta = 2\hat\theta$, $\beta=\beta(\alpha)$ must pass through the straight line . After $\beta=\beta(\alpha)$ passes through $\alpha-\beta = 2\hat\theta$, we have $\bar\partial_-\beta < 0$ by (2.15), which combines with $\bar\partial_-\alpha < 0$ to give $\beta'(\alpha) > 0$. Moreover, $\bar\partial_-\alpha < 0$ implies that there is only one intersection point of $\beta=\beta(\alpha)$ with $\alpha-\beta = 2\hat\theta$. Then, we have that, the case (ii) holds, if $(\alpha,\beta)(D)$ is on the line $\alpha-\beta = 2\hat\theta$. The case (iii) holds, if $(\alpha,\beta)(D)$ is located below the straight line. □

**Lemma 5.** (Invariant region) *Suppose that the Goursat problem* (2.1) *with* (4.4) *admits a $C^1$ solution on $\Sigma_h$ ($h > c_1$), and the assumptions of Theorem 1 hold, there exists a positive constant $\varepsilon_0$ such that for any $\varepsilon \in (0,\varepsilon_0)$,*

$$(\alpha,\beta)(\xi,\eta) \in \Lambda_\varepsilon, \quad \forall\,(\xi,\eta)\in\Sigma_h, \tag{4.6}$$

*where* $\Lambda_\varepsilon = \left(2\hat\theta - \frac{\pi}{2} - \varepsilon, \alpha_1+\varepsilon\right) \times \left(-\frac{\pi}{2}-\varepsilon, \alpha_1 - 2\hat\theta + \varepsilon\right)$.

*Proof.* Because of $\alpha_1 < \frac{\pi}{2}$ and $\frac{\pi}{2} < 4\hat\theta - \frac{\pi}{2}$, we have the square domain $\Lambda = \left(2\hat\theta-\frac{\pi}{2}, \alpha_1\right)\times\left(-\frac{\pi}{2}, \alpha_1 - 2\hat\theta\right) \subset \Gamma$, where $\Gamma$ denotes the open strip region between the line $\alpha-\beta=0$ and the line $\alpha-\beta=\pi$. By computations, for sufficient small constant $\varepsilon_0 > 0$, the open square domain $\Lambda_\varepsilon \subset \Gamma$ for any $\varepsilon\in(0,\varepsilon_0)$. The images of $PC_\infty$ and $PD$ are both in $\Lambda_\varepsilon$ from $\alpha_2 > 2\hat\theta - \frac{\pi}{2}$ in Theorem 1. According to the local existence in Lemma 3 and $\bar P \in \Lambda$, we have $(\alpha,\beta)(\xi,\eta) \in \Lambda_\varepsilon$ for any $(\xi,\eta)\in\Sigma_h$ when $h$ is sufficiently close to $c_1$. Referring to Figure 7, the four vertexes of $\Lambda_\varepsilon$ are $\bar{M}'(2\hat{\theta}-\frac{\pi}{2}-\varepsilon,\alpha_1-2\hat{\theta}+\varepsilon)$, $\bar{N}'(\alpha_1+\varepsilon,\alpha_1-2\hat{\theta}+\varepsilon)$, $\bar{P}'(\alpha_1+\varepsilon,-\frac{\pi}{2}-\varepsilon)$ and $\bar{C}'_\infty(2\hat{\theta}-\frac{\pi}{2}-\varepsilon,-\frac{\pi}{2}-\varepsilon)$.

Then, due to $\bar{\partial}_-c|_{PC_\infty}>0$, $\bar{\partial}_+c|_{PD}>0$ and the characteristic decompositions (2.18), we have that

$$\bar{\partial}_\pm c>0 \quad \text{in} \quad \Sigma_h. \tag{4.7}$$

If the conclusion of Lemma 5 is invalid, by the method of continuity, there must exist a point $H\in\Sigma_h$, such that $(\alpha,\beta)(H)\in\bigcup_{i=4}^{4}l_i$ and $(\alpha,\beta)(\xi,\eta)\in\Lambda_\varepsilon$ for all $(\xi,\eta)\in\Sigma_H\setminus\{H\}$, where $\Sigma_H$ is the closed domain bounded by characteristic curves $PH_+$, $PH_-$, $H_+H$ and $H_-H$, and $H_+$ $(H_-)$ is the intersection point of the $C_+$ $(C_-)$ characteristic curve passing through $H$ with $PD$ $(PC_\infty)$, (see Figure 8).

1) If $(\alpha,\beta)(H)\in\bar{M}'\bar{N}'\setminus\{\bar{N}'\}$, we have

$$c\bar{\partial}_-\beta(H)=-\frac{\sin 2\delta}{2\mu^2}\left(\tan^2\hat{\theta}-\tan^2\delta\right)\bar{\partial}_-c|_H<0$$

which contradicts to $\bar{\partial}_-\beta(H)\geq 0$.

2) If $(\alpha,\beta)(H)\in\bar{N}'\bar{P}'\setminus\{\bar{N}'\}$, we have

$$c\bar{\partial}_+\alpha(H)=\frac{\sin 2\delta}{2\mu^2}\left(\tan^2\hat{\theta}-\tan^2\delta\right)\bar{\partial}_+c|_H<0$$

which contradicts to $\bar{\partial}_+\alpha(H)\geq 0$.

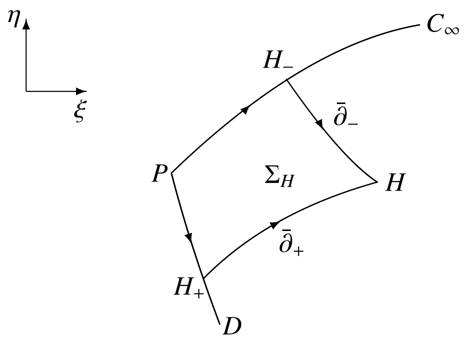


**Figure 8.** Domain $\Sigma_H$.

3) If $(\alpha,\beta)(H)\in\bar{P}'\bar{C}'_\infty\setminus\{\bar{C}'_\infty\}$ or $(\alpha,\beta)(H)\in\bar{C}'_\infty\bar{M}'\setminus\{\bar{C}'_\infty\}$, we obtain the conclusions similarly.

4) If $(\alpha,\beta)(H)=\bar{N}'$, we define $\tilde{\alpha}$ on $H_+H$ such that

$$\begin{cases} c\bar{\partial}_+\tilde{\alpha}=\dfrac{\sin 2\delta}{2\mu^2}\left(\tan^2\hat{\theta}-\tan^2\dfrac{\tilde{\alpha}-\left(\bar{\alpha}_1-2\hat{\theta}+\varepsilon\right)}{2}\right)\bar{\partial}_+c, \quad \text{along } H_+H, \\ \tilde{\alpha}(H_+)=\alpha(H_+). \end{cases} \tag{4.8}$$

Then we have

$$\tilde{\alpha}(H_+)<\bar{\alpha}_1+\varepsilon.$$

Combining (4.8) with the first equation of (2.15), we get

$$\begin{cases} c\bar{\partial}_+(\alpha-\tilde{\alpha})=\dfrac{\sin 2\delta}{2\mu^2}\left(\tan^2\dfrac{\tilde{\alpha}-\left(\bar{\alpha}_1-2\hat{\theta}+\varepsilon\right)}{2}-\tan^2\dfrac{\alpha-\beta}{2}\right)\bar{\partial}_+c, \quad \text{along } H_+H, \\ (\alpha-\tilde{\alpha})(H_+)=0. \end{cases} \tag{4.9}$$

Substituting $\alpha(H_+) = \tilde{\alpha}(H_+)$ into the first equation of (4.9), we get $\bar{\partial}_+(\alpha - \tilde{\alpha})(H_+) < 0$. We assert that $(\alpha - \tilde{\alpha})(\tilde{H}) < 0$, $\tilde{H} \in H_+H$. If not, there exists a point $H_1 \in H_+H \setminus \{H_+, H\}$ such that $(\alpha - \tilde{\alpha})(H_1) = 0$ and $(\alpha - \tilde{\alpha})(\xi, \eta) < 0$, $(\xi, \eta) \in H_+H_1 \setminus \{H_+, H_1\}$. Therefore, $\bar{\partial}_+(\alpha - \tilde{\alpha})(H_1) \geq 0$. But, according to hypothesis $2\hat{\theta} - \frac{\pi}{2} - \varepsilon < \alpha(H_1) < \bar{\alpha}_1 + \varepsilon$, $-\frac{\pi}{2} - \varepsilon < \beta(H_1) < \bar{\alpha}_1 - 2\hat{\theta} + \varepsilon$ and the Eq (4.9), we get $\bar{\partial}_+(\alpha - \tilde{\alpha})(H_1) < 0$, which leads to a contradiction. Hence, we get

$$\alpha(H) < \tilde{\alpha}(H) < \bar{\alpha}_1 + \varepsilon,$$

which contradicts to the hypothesis $(\alpha,\beta)(H) = \bar{N}'$. Similarly, we have that $(\alpha,\beta)(H) = \bar{C}'_\infty$ is also impossible. □

Let $\varepsilon \to 0$, we can get the following theorem according to Lemma 5.

**Theorem 6.** (Hyperbolicity) *Assume that the Goursat problem* (2.1) *with* (4.4) *admits a $C^1$ solution on $\Sigma_h$ $(h > c_1)$, and the assumptions of Theorem 1.1 hold. Then there holds*

$$0 < 2\hat{\theta} - \frac{\pi}{4} - \frac{\alpha_1}{2} \leq \delta \leq \frac{\alpha_1}{2} + \frac{\pi}{4} < \pi$$

*and*

$$0 < 2\hat{\theta} - \frac{\pi}{2} \leq \alpha \leq \alpha_1, \ \ -\frac{\pi}{2} \leq \beta \leq \alpha_1 - 2\hat{\theta}$$

*on $\Sigma_h$.*

**Lemma 6.** ($C^0$ estimates) *Assume that the Goursat problem* (2.1) *with* (4.4) *admits a $C^1$ solution on $\Sigma_h$ $(h > c_1)$, and the assumptions of Theorem 1.1 hold. Then there exists a function $\mathcal{M}(h)$, such that*

$$\| (u, v, c) \|_{C^0(\Sigma_h)} < \mathcal{M}(h).$$

*Proof.* We choose the maximum $\xi := \xi(h)$ on the level curve $c(\xi, \eta) = h$, and for any $u$ on the domain $\Sigma_h$, we have

$$u = \xi + c\frac{\cos\sigma}{\sin\delta} \leq \xi(h) + Ch := \mathcal{M}(h).$$

we can estimates $v$ similarly. □

**Lemma 7.** *Assume that the Goursat problem* (2.1) *with* (4.4) *admits a $C^1$ solution on $\Sigma_h$ $(h > c_1)$, and the assumptions of Theorem 1.1 hold. Then $\frac{\bar{\partial}_\pm c}{c^3}$ are uniformly bounded in $\Sigma_h$, that is $(\frac{\bar{\partial}_+ c}{c^3}, \frac{\bar{\partial}_- c}{c^3}) \in (0, \mathcal{N}) \times (0, \mathcal{N})$, where $\mathcal{N} = 2\mu^2$.*

*Proof.* Since $\frac{\bar{\partial}_+ c}{c^3} > 0$ on $PC_\infty$ and $\frac{\bar{\partial}_- c}{c^3} > 0$ on $PD$, we can get $\frac{\bar{\partial}_\pm c}{c^3} > 0$ for all $(\xi, \eta) \in \Sigma_h$ from characteristic decompositions (2.18). As $h$ is sufficiently close to $c_1$, it is obvious that $\|\frac{\bar{\partial}_\pm c}{c^3}\|_{C^0} < \mathcal{N}$.

We use the method of contradiction to prove the uniform boundedness. Suppose there exists a interior point $T$ such that $\frac{\bar{\partial}_- c}{c^3} = \mathcal{N}$ or $\frac{\bar{\partial}_+ c}{c^3} = \mathcal{N}$ and $(\frac{\bar{\partial}_+ c}{c^3}, \frac{\bar{\partial}_- c}{c^3}) \in (0, \mathcal{N}) \times (0, \mathcal{N})$ on $\Sigma_T \setminus \{T\}$, where $\Sigma_T$ is the closed domain bounded by $PT_-$, $PT_+$, $TT_-$, $TT_+$ (the $C_-$ characteristic curve passing through $T$ intersects with $PC_\infty$ at point $T_-$, the $C_+$ characteristic curve passing through $T$ intersects with $PD$ at point $T_+$). Without loss of generality, we assume that $\frac{\bar{\partial}_- c}{c^3} = \mathcal{N}$ at the point $T$. So we get $\bar{\partial}_+\left(\frac{\bar{\partial}_- c}{c^3}\right)(T) \geq 0$. Because of the first equation of the characteristic decompositions (2.18), we get

$$c\bar{\partial}_+\left(\frac{\bar{\partial}_- c}{c^3}\right) = \frac{\bar{\partial}_- c}{c^3}\left(\sin 2\delta - \frac{c^3}{2\mu^2\cos^2\delta}\frac{\bar{\partial}_- c}{c^3} - c^3\left(2 + \frac{\Omega\cos 2\delta}{2\mu^2}\right)\frac{\bar{\partial}_+ c}{c^3}\right).$$

Substituting $\frac{\bar{\partial}_- c}{c^3}(T) = \mathcal{N}$ into the above equation, we have that

$$c\bar{\partial}_+\left(\frac{\bar{\partial}_- c}{c^3}\right)(T) = \mathcal{N}\left(\sin 2\delta - \frac{c^3}{2\mu^2\cos^2\delta}\mathcal{N} - c^3\left(2 + \frac{\Omega\cos 2\delta}{2\mu^2}\right)\frac{\bar{\partial}_+ c}{c^3}\right) < 0,$$

which leads to a contradiction. So we complete the proof. □

From Lemma 7, (2.14) and the identities

$$\partial_\xi = -\frac{\sin\beta\bar{\partial}_+ - \sin\alpha\bar{\partial}_-}{\sin(2\delta)}, \quad \partial_\eta = \frac{\cos\beta\bar{\partial}_+ - \cos\alpha\bar{\partial}_-}{\sin(2\delta)},$$

we have the following lemma.

**Lemma 8.** *Assume that the Goursat problem* (2.1) *with* (4.4) *admits a $C^1$ solution on $\Sigma_h$ ($h > c_1$), and the assumptions of Theorem 1.1 hold. Then there exists a constant $\mathcal{P}$, which is independent of h, such that*

$$\| (Du, Dv, Dc) \|_{C^0(\Sigma_h)} \leq \frac{\mathcal{P}h^4}{\varepsilon(h)}.$$

*4.2. Global solution*

**Theorem 7.** *Assume that the assumptions of Theorem 1.1 hold. Then the Goursat problem* (2.1) *and* (4.4) *admits a unique $C^1$ solution on the region $\Sigma$ bounded by $PC_\infty$ and $PD$.*

*Proof.* The Goursat problem (2.1) with (4.4) admits a $C^1$ solution on $\Sigma_h$. Similar to the proof in Theorem 4.12 in [31], we obtain that the level curve $c(\xi,\eta) = h$ is Lipschitz continuous, and then it is rectifiable. See Figure 9, let $Q'$ and $Q$ are two arbitrary different points on the level curve $c(\xi,\eta) = h$. The $C_+$ characteristic passing through $Q'$ intersects with the $C_-$ characteristic passing through $Q$ at a point $J$. The level curve $c(\xi,\eta) = h$ is a non-characteristic because of $\bar{\partial}_\pm c > 0$, so $J \neq Q'$ and $J \neq Q$. By the method in Theorem 4.12 in [31], it can be proved that the 'small' Goursat problem (2.1) with the $C_\pm$ characteristic boundaries $JQ'$ and $JQ$ admits a unique $C^1$ solution if the $|Q'Q|$ is sufficiently small. Then, by the Heine-Borel Theorem, we can extend the existence region of $C^1$ solution from $\Sigma_h$ to $\Sigma_{\bar{h}}$ ($\bar{h} > h$) by solving finite number of 'small' Goursat problem. □

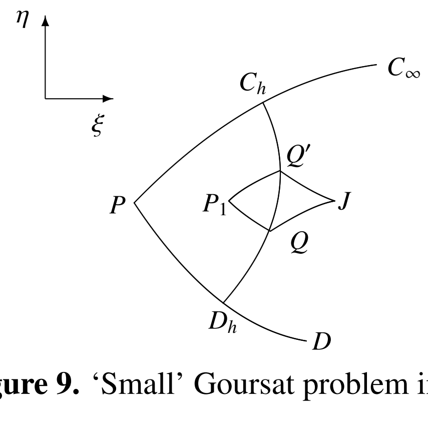


**Figure 9.** 'Small' Goursat problem in $(\xi,\eta)$ plane.

## 5. Interaction of a non-planar simple wave with the solid boundary

After the interaction of the incomplete centered simple wave $R_1$ with the backward planar rarefaction wave $R_2$ in the wave interaction region $\Sigma_1$, a simple wave denoted by $R_3^-$, which is adjacent to the constant state $(u_2, v_2, c_2)$, emits from the $C_+$ cross characteristic curve $DE_\infty^1$ (see Figure 6). The characteristic decompositions (2.18) follow that $\bar{\partial}_+ c|_{DE_\infty^1} > 0$, and then $\bar{\partial}_+ c > 0$ in $R_3^-$. Therefore $R_3^-$ is an expand wave. Using (2.18), the two equations in the middle of (2.15) and $\bar{\partial}_+ c > 0$ in $R_3^-$, we know that the straight characteristics of $R_3^-$ can not intersect with each other. Then $R_3^-$ will touch the rigid wall and interact with each other. The straight characteristic $DF$ touches the boundary firstly and reflects a $C_+$ characteristics.

The outcome of the reflecting of the simple wave $R_3^-$ on a rigid wall simply corresponds to the interaction of two symmetric simple waves with the wall as the axis of symmetry. So the reflecting problem above can be solved by the interaction of two symmetric simple waves $R_3^-$ and $R_3^+$, which is the symmetric part of $R_3^-$ with respect to $OB$. It forms a new Goursat problem, the two characteristic boundaries of which are $C_+$ characteristics $FE_\infty^2$ and $C_-$ characteristics $FE_\infty'^2$, see Figure 6. The solution of the reflecting problem is the half part of the Goursat problem.

### 5.1. Interaction of two symmetrical simple waves

We may study a new symmetric Goursat problem for convenience because of the invariance of the system in coordinates translation and rotation, (see Figure 10).

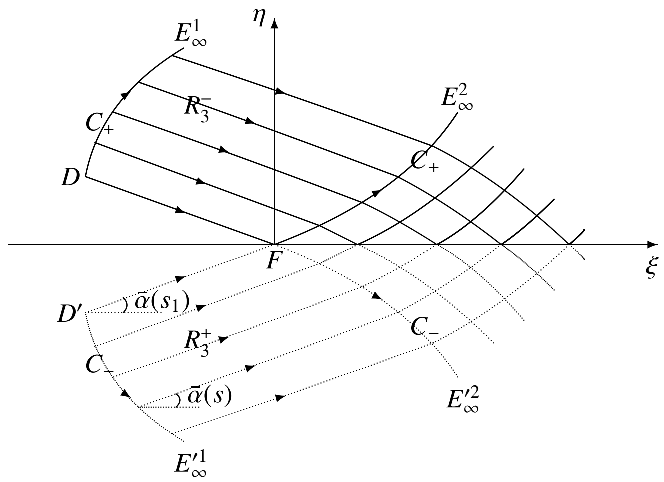


**Figure 10.** Interaction of two symmetrical simple waves.

Assume that the parametric form of $D'E_\infty'^1$ is

$$\xi = g(s), \quad \eta = h(s), \quad s_1 \le s \le s_2,$$

and the values of $u$, $v$, $c$ and $\alpha$ on $D'E_\infty'^1$ are

$$u = \bar{u}(s), \quad v = \bar{v}(s), \quad c = \bar{c}(s), \quad \alpha(s) = \bar{\alpha}(s) \quad (s_1 \le s \le s_2\ ),$$

where $\bar{\alpha}_1 \le \bar{\alpha}(s) \le \bar{\alpha}_2$, and $\bar{\alpha}(s)$ is the $C_+$ characteristic inclination angle, $\bar{\alpha}_1 = \min\limits_{s_1 \le s \le s_2} \bar{\alpha}(s)$ and

$\bar{\alpha}_2 = \max\limits_{s_1 \le s \le s_2} \bar{\alpha}(s)$. By the symmetry, the parametric form of $DE^1_\infty$ is

$$\xi = g(s), \quad \eta = -h(s), \quad s_1 \le s \le s_2,$$

and the values of $u$, $v$, $c$ and $\beta$ on $DE^1_\infty$ are

$$u = \bar{u}(s), \quad v = -\bar{v}(s), \quad c = \bar{c}(s), \quad \beta(s) = -\bar{\alpha}(s) \quad (s_1 \le s \le s_2\ ),$$

where $-\bar{\alpha}(s)$ is the $C_-$ characteristic inclination angle. From Figure 2, the parametric expression of the $C_-$ cross characteristic $FE'^2_\infty$ is

$$\begin{cases} \xi = \xi(s) = \bar{u}(s) - \bar{c}(s)\sin\bar{\alpha}(s) - \bar{c}(s)\cot\dfrac{\bar{\alpha}(s) - \beta(s)}{2}\cos(\bar{\alpha}(s)), \\ \eta = \eta(s) = \bar{v}(s) + \bar{c}(s)\cos\bar{\alpha}(s) - \bar{c}(s)\cot\dfrac{\bar{\alpha}(s) - \beta(s)}{2}\sin(\bar{\alpha}(s)), \end{cases} \tag{5.1}$$

from the last equation of (2.15), the $C_-$ characteristic angle $\beta(s)$ along $FE'^2_\infty$ is determined by

$$\begin{cases} \beta'(s) = -\dfrac{\sin(\bar{\alpha}(s) - \beta(s))}{2\mu^2\bar{c}(s)}\left(m - \tan\dfrac{\bar{\alpha}(s) - \beta(s)}{2}\right)\bar{c}'(s), \\ \beta(s_1) = -\bar{\alpha}(s_1). \end{cases} \tag{5.2}$$

By the symmetry, we can also obtain the parametric expression of the $C_+$ cross characteristic $FE^2_\infty$.

In this section, we study the symmetric Goursat problem (2.1) with boundary data

$$\begin{cases} (u, v, c)|_{FE'^2_\infty} = (u, v, c)\,(\xi(s), \eta(s)) = (\bar{u}, \bar{v}, \bar{c})\,(s), \\ (u, v, c)|_{FE^2_\infty} = (u, v, c)\,(\xi(s), -\eta(s)) = (\bar{u}, -\bar{v}, \bar{c})\,(s), \end{cases} \quad s_1 \le s \le s_2. \tag{5.3}$$

where $(\bar{u}, \bar{v}, \bar{c}) \in C^1$.

5.1.1. Local existence

The problem (2.1) with (5.3) is a standard Goursat problem. Then we have the following local existence theorem.

**Lemma 9.** (Local existence) *There is a $h' > c_1$ and $h'$ is sufficiently close to $c_1$, the Goursat problem* (2.1) *with* (5.3) *admits a unique $C^1$ solution on the angular domain $\Sigma_{h'}$ bounded by $FE'_{h'}$, $FE_{h'}$ and level curve $c(\xi, \eta) = h'$, which intersects with $FE'$ at $E'_{h'}$ and $FE$ at $E_{h'}$.*

*Proof.* The proof is similar to that of Lemma 3, we omit the detail. □

5.1.2. Boundary data estimates

**Lemma 10.** *On the boundary $FE'^2_\infty$,*
*1) when $\bar{\alpha}_1 + \bar{\alpha}_2 \le 2\hat{\theta}$, $(\alpha, \beta)(\bar{\alpha}) \in [\bar{\alpha}_1, \bar{\alpha}_2] \times [\bar{\alpha}_1 - 2\hat{\theta}, -\bar{\alpha}_1]$;*
*2) when $\bar{\alpha}_1 + \bar{\alpha}_2 > 2\hat{\theta}$, $(\alpha, \beta)(\bar{\alpha}) \in [\bar{\alpha}_1, \bar{\alpha}_2] \times [-\bar{\alpha}_2, \bar{\alpha}_2 - 2\hat{\theta})$.*

*Proof.* When $\bar{\alpha}_1 + \bar{\alpha}_2 \le 2\hat{\theta}$, we have $\beta(s_1) = -\bar{\alpha}(s_1) \in [\bar{\alpha}_1 - 2\hat{\theta}, -\bar{\alpha}_1]$. Suppose that there exists an $s' \in (s_1, s_2)$, satisfying $\beta(s') = -\bar{\alpha}_1$, while $\beta(s) < -\bar{\alpha}_1$, $s \in (s_1, s')$. Then we have

$$\frac{\alpha(s') - \beta(s')}{2} = \frac{\bar{\alpha}(s') - (-\bar{\alpha}_1)}{2} = \frac{\bar{\alpha}(s') + \bar{\alpha}_1}{2} < \hat{\theta},$$

by the last equation of (2.15), we get $\bar{\partial}_{-}\beta(s') < 0$. This is a contradiction. Thus $\beta(s) < -\bar{\alpha}_1$, for all $s \in (s_1, s_2)$.

On the other hand, suppose that there exists an $s'' \in (s_1, s_2)$, satisfying $\beta(s'') = \bar{\alpha}_1 - 2\hat{\theta}$, while $\beta(s) > \bar{\alpha}_1 - 2\hat{\theta}$, $s \in (s_1, s'')$. Then, we have

$$\frac{\alpha(s'') - \beta(s'')}{2} = \frac{\bar{\alpha}(s'') - (\bar{\alpha}_1 - 2\hat{\theta})}{2} = \frac{\bar{\alpha}(s'') - \bar{\alpha}_1}{2} + \hat{\theta} > \hat{\theta},$$

by the last equation of (2.15), we get $\bar{\partial}_{-}\beta(s') > 0$. This is a contradiction. Thus $\beta(s) > \bar{\alpha}_1 - 2\hat{\theta}$, for all $s \in (s_1, s_2)$.

The other case can be proved similarly, we omit the details. □

Similarly, On the boundary $FE^2_\infty$, we can also obtain the boundary estimates.

**Lemma 11.** *On the boundary $FE'^2_\infty$,*
*1) when $\bar{\alpha}_1 + \bar{\alpha}_2 \le 2\hat{\theta}$, $(\alpha,\beta)(\bar{\alpha}) \in [\bar{\alpha}_1, -\bar{\alpha}_1 + 2\hat{\theta}] \times [-\bar{\alpha}_2, -\bar{\alpha}_1]$;*
*2) when $\bar{\alpha}_1 + \bar{\alpha}_2 > 2\hat{\theta}$, $(\alpha,\beta)(\bar{\alpha}) \in (-\bar{\alpha}_2 + 2\hat{\theta}, \bar{\alpha}_2] \times [-\bar{\alpha}_2, -\bar{\alpha}_1]$.*

5.1.3. Hyperbolicity and priori $C^0$ estimates

**Lemma 12.** (Invariant region) *In the case of $\bar{\alpha}_1 + \bar{\alpha}_2 \le 2\hat{\theta}$, suppose that the Goursat problem* (2.1) *with* (5.3) *admits a $C^1$ solution on $\Sigma_{h'}$ ($h' > c_1$), and $\bar{\alpha}_1 > 0$, there exists a positive constant $\varepsilon'_0$ such that for any $\varepsilon \in (0, \varepsilon'_0)$,*

$$(\alpha, \beta)(\xi, \eta) \in \Lambda'_\varepsilon, \quad \forall (\xi, \eta) \in \Sigma_{h'}, \tag{5.4}$$

*where $\Lambda'_\varepsilon = \left(\bar{\alpha}_1 - \varepsilon, -\bar{\alpha}_1 + 2\hat{\theta} + \varepsilon\right) \times \left(\bar{\alpha}_1 - 2\hat{\theta} - \varepsilon, -\bar{\alpha}_1 + \varepsilon\right)$.*

*Proof.* Because of $\bar{\alpha}_1 > 0$, we have a closed square domain $\Lambda' := [\bar{\alpha}_1, -\bar{\alpha}_1 + 2\hat{\theta}] \times [\bar{\alpha}_1 - 2\hat{\theta}, -\bar{\alpha}_1] \subset \Gamma$, where $\Gamma$ denotes the open strip region between the lines $\alpha - \beta = 0$ and $\alpha - \beta = \pi$. By computations, for sufficient small constant $\varepsilon'_0 > 0$, the open square domain $\Lambda'_\varepsilon \subset \Gamma$ for any $\varepsilon \in (0, \varepsilon'_0)$. The images of the two boundaries $FE^2_\infty$ and $FE'^2_\infty$ are both in $\Lambda'_\varepsilon$. Refer to Figure 11, the four vertexes of $\Lambda'_\varepsilon$ are $P_1(\bar{\alpha}_1 - \varepsilon, -\bar{\alpha}_1 + \varepsilon)$, $P_2(-\bar{\alpha}_1 + 2\hat{\theta} + \varepsilon, -\bar{\alpha}_1 + \varepsilon)$, $P_3(-\bar{\alpha}_1 + 2\hat{\theta} + \varepsilon, \bar{\alpha}_1 - 2\hat{\theta} - \varepsilon)$ and $P_4(\bar{\alpha}_1 - \varepsilon, \bar{\alpha}_1 - 2\hat{\theta} - \varepsilon)$. We denote that $l_1 = \overline{P_1P_2}$, $l_2 = \overline{P_2P_3}$, $l_3 = \overline{P_3P_4}$ and $l_4 = \overline{P_4P_1}$.

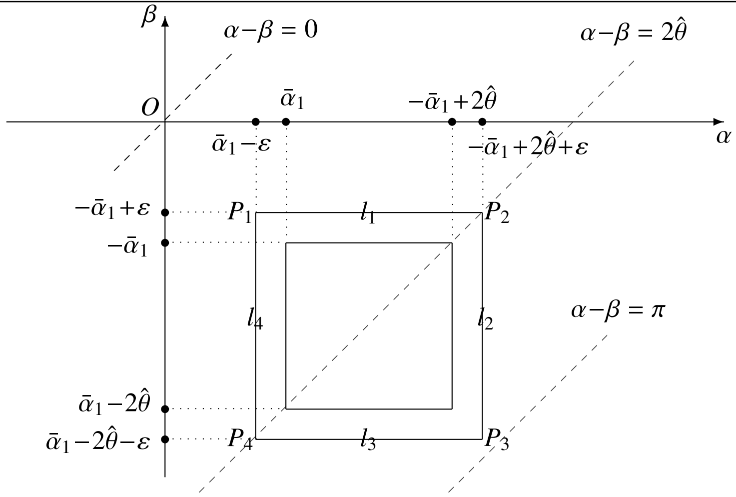


**Figure 11.** Invariant region of $(\alpha,\beta)$.

Then, due to $\bar{\partial}_-c|_{FE'^2_\infty} > 0$, $\bar{\partial}_+c|_{FE^2_\infty} > 0$ and the characteristic decompositions (2.18), we have that

$$\bar{\partial}_\pm c > 0 \quad \text{in} \quad \Sigma_{h'}. \tag{5.5}$$

If the conclusion of Lemma 12 is invalid, by the method of continuity, there must exist a point $H' \in \Sigma_{H'}$, such that $(\alpha,\beta)(H') \in \bigcup\limits_{i=4}^{4} l_i$ and $(\alpha,\beta)(\xi,\eta) \in \Lambda_\varepsilon$ for all $(\xi,\eta) \in \Sigma_{h'} \backslash \{H'\}$, where $\Sigma_{H'}$ is the closed domain bounded by characteristic curves $FH'_+$, $FH'_-$, $H'_+H'$ and $H'_-H'$, and $H'_+$ ($H'_-$) is the intersection point of the $C_+$ ($C_-$) characteristic curve passing through $H'$ with $FE'^2_\infty$ ($FE^2_\infty$), (see Figure 12).

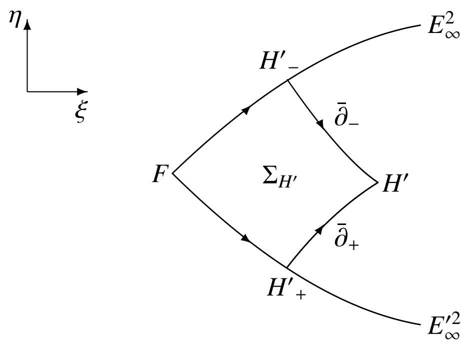


**Figure 12.** Domain $\Sigma_{h'}$.

1) If $(\alpha,\beta)(H') \in l_1 \setminus \{P_2\}$, we have

$$c\bar{\partial}_-\beta(H') = -\frac{\sin 2\delta}{2\mu^2}\left(\tan^2\hat{\theta} - \tan^2\delta\right)\bar{\partial}_-c\Big|_{H'} < 0.$$

which contradicts to $\bar{\partial}_-\beta(H') \geq 0$.
2) If $(\alpha,\beta)(H') \in l_2 \setminus \{P_2\}$, we have

$$c\bar{\partial}_+\alpha(H') = \frac{\sin 2\delta}{2\mu^2}\left(\tan^2\hat{\theta} - \tan^2\delta\right)\bar{\partial}_+c\Big|_{H'} < 0,$$

which contradicts to $\bar{\partial}_+\alpha(H') \geq 0$.
3) If $(\alpha,\beta)(H') \in l_3 \setminus \{P_4\}$ or $(\alpha,\beta)(H') \in l_4 \setminus \{P_4\}$, we get the conclusion similarly.
4) If $(\alpha,\beta)(H') = P_2$, we define $\tilde{\alpha}$ on $H'_+H'$ such that

$$\begin{cases} c\bar{\partial}_+\tilde{\alpha} = \dfrac{\sin 2\delta}{2\mu^2}\left(\tan^2\hat{\theta} - \tan^2\dfrac{\tilde{\alpha}-(-\bar{\alpha}_1+\varepsilon)}{2}\right)\bar{\partial}_+c, \quad \text{along } H'_+H', \\ \tilde{\alpha}(H'_+) = \alpha(H'_+). \end{cases} \tag{5.6}$$

Then by

$$\bar{\alpha}_1 - \varepsilon < \bar{\alpha}_1 < \tilde{\alpha}(H'_+) < \bar{\alpha}_2 < -\bar{\alpha}_1 + 2\hat{\theta} < -\bar{\alpha}_1 + 2\hat{\theta} + \varepsilon,$$

we have

$$\tilde{\alpha}(H'_+) < -\bar{\alpha}_1 + 2\hat{\theta} + \varepsilon.$$

Combining (5.6) with the first equation of (2.15), we get

$$\begin{cases} c\bar{\partial}_+(\alpha-\tilde{\alpha}) = \dfrac{\sin 2\delta}{2\mu^2}\left(\tan^2\dfrac{\tilde{\alpha}-(-\bar{\alpha}_1+\varepsilon)}{2} - \tan^2\dfrac{\alpha-\beta}{2}\right)\bar{\partial}_+c, \quad \text{along } H'_+H', \\ (\alpha-\tilde{\alpha})(H'_+) = 0. \end{cases} \tag{5.7}$$

Substituting $\alpha(H'_+) = \tilde{\alpha}(H'_+)$ into the first equation of (5.7), we get $\bar{\partial}_+(\alpha-\tilde{\alpha})(H'_+) < 0$. We assert that $(\alpha-\tilde{\alpha})(\tilde{H}') < 0$, $\tilde{H}' \in H'_+H'$. If not, there exists a point $H'_1 \in H'_+H' \setminus \{H'_+, H'\}$ such that $(\alpha-\tilde{\alpha})(H'_1) = 0$ and $(\alpha-\tilde{\alpha})(\xi,\eta) < 0$, $(\xi,\eta) \in H'_+H'_1 \setminus \{H'_+, H'_1\}$. Therefore, $\bar{\partial}_+(\alpha-\tilde{\alpha})(H'_1) \geq 0$. But, according to hypothesis $\bar{\alpha}_1 - \varepsilon < \alpha(H'_1) < -\bar{\alpha}_1 + 2\hat{\theta} + \varepsilon$, $\bar{\alpha}_1 - 2\hat{\theta} - \varepsilon < \beta(H'_1) < -\bar{\alpha}_1 + \varepsilon$ and the Eq (5.7), we get $\bar{\partial}_+(\alpha-\tilde{\alpha})(H'_1) < 0$, which leads to a contradiction. Hence, we get

$$\alpha(H') < \tilde{\alpha}(H') < \pi - \bar{\alpha}_1 + 2\hat{\theta} + \varepsilon,$$

which contradicts to the hypothesis $(\alpha,\beta)(H') = P_2$. Similarly, we have that $(\alpha,\beta)(H') = P_4$ is also impossible. □

Similarly, we can obtain the invariant region to the other case.

**Lemma 13.** (Invariant region) *In the case of $\bar{\alpha}_1 + \bar{\alpha}_2 > 2\hat{\theta}$, suppose that the Goursat problem* (2.1) *with* (5.3) *admits a $C^1$ solution on $\Sigma_{h'}$ where $h' > c_1$, and $\bar{\alpha}_2 < 2\hat{\theta}$, there exists a positive constant $\varepsilon_0$ such that for any $\varepsilon \in (0, \varepsilon_0')$,*

$$(\alpha,\beta)(\xi,\eta) \in \Lambda'_\varepsilon, \quad \forall (\xi,\eta) \in \Sigma_{h'}, \tag{5.8}$$

*where $\Lambda'_\varepsilon = \left(-\bar{\alpha}_2 + 2\hat{\theta} - \varepsilon, \bar{\alpha}_2 + \varepsilon\right) \times \left(-\bar{\alpha}_2 - \varepsilon, \bar{\alpha}_2 - 2\hat{\theta} + \varepsilon\right)$.*

Let $\varepsilon \to 0$, immediately, we have that if the Goursat problem (2.1) with (5.3) admits a $C^1$ solution on $\Sigma_{h'}$ under the conditions $0 < \bar{\alpha}_1 < \bar{\alpha}_2 < 2\hat{\theta}$, the solution satisfies that

$$0 < \bar{\alpha}_1 \leq \delta \leq 2\hat{\theta} - \bar{\alpha}_1 < \frac{\pi}{2}, \tag{5.9}$$

or

$$0 < 2\hat{\theta} - \bar{\alpha}_2 \leq \delta \leq \bar{\alpha}_2 < \frac{\pi}{2}. \tag{5.10}$$

**Lemma 14.** ($C^0$ estimates) *Assume that the Goursat problem* (2.1) *with* (5.3) *admits a $C^1$ solution on $\Sigma'_h$ $(h' > c_1)$. Then there exists a function $\mathcal{M}'(h') > 0$, such that*

$$\|(u,v,c)\|_{C^0(\Sigma_{h'})} \leq \mathcal{M}'(h'). \tag{5.11}$$

*Proof.* The proof is similar to that of Lemma 6, we omit the detail. □

5.1.4. Gradient estimates

**Lemma 15.** *Assume that the Goursat problem* (2.1) *with* (5.3) *admits a* $C^1$ *solution on* $\Sigma_{h'}$ ($h' > c_1$). *Then* $\frac{\bar{\partial}_\pm c}{c^3}$ *are uniformly bounded in* $\Sigma_{h'}$, *that is*

$$\left(\frac{\bar{\partial}_+ c}{c^3}, \frac{\bar{\partial}_- c}{c^3}\right) \in (0, \mathcal{N}) \times (0, \mathcal{N}) \tag{5.12}$$

*where* $\mathcal{N} = 2\mu^2$.

*Proof.* From Lemma 7, we know $\frac{\bar{\partial}_+ c}{c^3} \in (0, \mathcal{N})$ on $DE^1_\infty$. Due to $FE^2_\infty$ is the $C_+$ cross characteristic curve of the simple wave $R_3^-$, we get $\frac{\bar{\partial}_+ c}{c^3} \in (0, \mathcal{N})$ on $FE^2_\infty$. By symmetry, we also have $\bar{\partial}_- c \in (0, \mathcal{N})$ on $FE'^2_\infty$. We will prove that the results are also correct in the interior of $\Sigma_{h'}$.

Let $T'$ be an arbitrary point in $\Sigma_{h'}$, and $\Sigma'_T$ is a closed domain bounded by $FT'_+$, $FT'_-$, $T'_+T'$ and $T'_-T'$, $T'_+$ ($T'_-$) is the intersection point of $C_+$ ($C_-$) characteristic curve passing through $T'$ with $FE'^2_\infty$ ($FE^2_\infty$). Without loss of generality, assume $\frac{\bar{\partial}_- c}{c^3}(T') = \mathcal{N}$, by the first equation of (2.18) we have

$$c\bar{\partial}_+\left(\frac{\bar{\partial}_- c}{c^3}\right)(T') = \mathcal{N}\left(\sin 2\delta - \frac{c^3}{2\mu^2\cos^2\delta}\mathcal{N} - c^3\left(2 + \frac{\Omega\cos 2\delta}{2\mu^2}\right)\frac{\bar{\partial}_+ c}{c^3}\right) < 0.$$

It is a contradiction. Therefore, by the method of continuity we prove the uniform boundedness of $\frac{\bar{\partial}_\pm c}{c^3}$. □

**Lemma 16.** (Gradient estimates) *Assume that the Goursat problem* (2.1) *with* (5.3) *admits a* $C^1$ *solution on* $\Sigma_{h'}$ ($h' > c_1$). *Then there exists a positive constant* $\mathcal{P}'$ *which is independent of* $h'$, *such that*

$$\| (Du, Dv, Dc) \|_{C^0(\Sigma_{h'})} \leq \frac{\mathcal{P}' h'^4}{\varepsilon'(h')}.$$

*Proof.* The proof is similar to that of Lemma 8, we omit the detail. □

### *5.2. Global solution*

**Theorem 8.** *Assume that* $0 < \bar{\alpha}_1 < \bar{\alpha}_2 < 2\hat{\theta}$. *The Goursat problem* (2.1) *with* (5.3) *has a solution on a triangle domain bounded by* $FE'^2_\infty$, $FE2_\infty$ (see Figure 13).

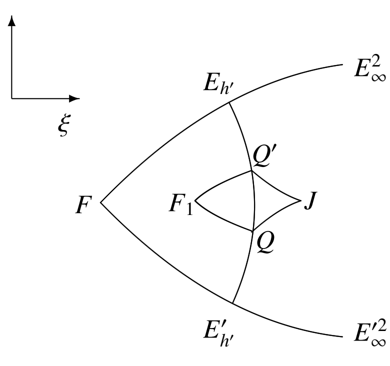


**Figure 13.** 'Small' Goursat problem in $(\xi, \eta)$ plane.

## 6. Global solution of the gas expansion problem around the sharp corner

*Proof.* By the invariant region for $(\alpha,\beta)$ in section 4, we obtain the range of the $C_-$ characteristic inclination angle $\beta$ in $R_3^-$, which is

$$\beta \in \left(-\frac{\pi}{2}, \alpha_1 - 2\hat{\theta}\right).$$

Then the range of angles between the characteristic inclination angles in $R_3^-$ and the rigid wall $OB$ is

$$\left(-\alpha_1 + 2\hat{\theta} - \theta, \frac{\pi}{2} - \theta\right).$$

So we get that $\bar{\alpha}_1 = -\alpha_1 + 2\hat{\theta} - \theta$, $\bar{\alpha}_2 = \frac{\pi}{2} - \theta$. Then, if

$$0 < -\alpha_1 + 2\hat{\theta} - \theta < \frac{\pi}{2} - \theta < 2\hat{\theta}.$$

we obtain the Theorem 8. □

*Proof.* Combining Theorems 7 and 8, we get Theorem 1. □

## 7. Conclusions

In this paper, the self-similar solutions for the 2D pseudo-steady isentropic irrotational supersonic flow of the generalized Chaplygin gas around the convex corner are constructed. The supersonic flow turns the convex corner near the cusp of the corner locally by an incomplete centered expansion wave. Using the methods of characteristic decompositions and invariant regions, we get the global existence of the solution up to infinity of the gas expansion problem.

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## Conflict of interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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