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Research article

An investigation on boundary controllability for Sobolev-type neutral evolution equations of fractional order in Banach space

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Abstract: The main focus of this paper is on the boundary controllability of fractional order Sobolev-type neutral evolution equations in Banach space. We show our key results using facts from fractional calculus, semigroup theory, and the fixed point method. Finally, we give an example to illustrate the theory we have established.

Keywords: fractional integrodifferential system; boundary controllability; fixed point theorem; nonlocal conditions
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1. Introduction

In recent decades, fractional calculus has played a significant role in mathematics. Some physical problems cannot be addressed using integer-order differential equations, while fractional-order differential equations can. Fractional differential equations have received a lot of attention and are utilized widely in engineering, physics, chemistry, biology, and a variety of other subjects. Fractional calculus notions have lately been effectively applied to a wide range of domains, and scientists are increasingly realizing that the fractional system may well correspond to many occurrences in regular sciences and engineering. Rheology, liquid stream, scattering, microscopic structures, viscoelasticity, and optics are just a few of the significant fractional calculus issues that are now being studied. Although diagnostic structures are often difficult to come by, the efficacy of mathematical evaluation methodologies for fractional systems in these disciplines has impressed some academics. Readers can check [1–3, 6, 7, 14–18, 22, 25–31, 33, 35, 37–41, 44, 52, 54–59].

The use of controllability notation in the research and design of control systems is beneficial. Fractional derivatives of various significations can be used to address these types of difficulties. It may be used in a range of sectors, including economics, chemical outgrowth control, biology, power systems, space technology, engineering, electronics, physics, robotics, transportation, chemistry, and so on. The topic of controllability is particularly important in control theory. If the control system is controllable, it can manage a variety of issues such as stability, pole assignment, and optimum control. Boundary controllability plays an important role in the analysis and design of control systems. The researchers in the recent years derived results on controllability for a variety of systems like neutral systems, integrodifferential equations, impulsive systems, fixed delay systems, and time-varying delay systems, etc. Solving these types of seeds has become a significant work for young scholars, one can refer to [1–18, 20, 21, 23, 27–29, 31–33, 38–53].

Fattorini proved the controllability condition on the first and second-order boundary control systems by replacing boundary controls with distributed controls [13]. By assuming exact controllability of the linear system and approximate controllability of linearization, the authors [21] studied global controllability for the abstract semilinear system. [2] obtained results for approximate boundary controllability of stochastic control systems of fractional order with Poisson jump and fractional Brownian motion are cited by the authors. Zhou et al. [59] derived various conditions for the existence of mild solutions with the help of fixed point theorems and fractional power of operators for neutral fractional-order evolution equations having nonlocal conditions.

Also, authors in [22] established results for the neutral integrodifferential fractional-order system having nonlocal conditions and finite delays in abstract space with the help of the measure of noncompactness. In [4] authors established some sufficient conditions for boundary controllability of integrodifferential system of Sobolev-type with the help of Banach contraction principle and theory of strongly continuous operators. With the help of Schauder's fixed point theorem, Ahmed [3] established sufficient conditions for boundary controllability of integrodifferential fractional-order non-linear system in abstract space. Inspired by the above and recent work, to the best of our knowledge there is no article dealing with boundary controllability for Sobolev-type neutral evolution equations of fractional order using this technique. We obtained sufficient conditions for boundary controllability. The results are advanced and weighed as an improvement to the control theory for fractional-order control systems.

The paper is structured in the following manner: In segment 2, we propose a few elementary definitions. In segment 3, we obtained results for boundary controllability. In segment 4, we discussed an example to understand theoretical results.

2. Preliminaries

Assume that Y and Z be two real Banach spaces with $\|\cdot\|$ and $|\cdot|$. Assume that σ be a closed linear and densely defined operator with domain $D(\sigma) \subseteq Y$ and $R(\sigma) \subseteq Z$. Consider Q be a linear operator with $D(Q) \subseteq Y$ and $R(Q) \subseteq X$, a Banach space together $\|\cdot\|_X$.

Assume that the boundary control of neutral evolution equations of Sobolev-type with fractional order of the form

$$
\begin{aligned}
&{}^cD^\alpha[\mathcal{S}x(\varpi) + F(\varpi, x(\varpi), x(b_1(\varpi)), \cdots, x(b_m(\varpi)))] \\
&= \sigma x(\varpi) + G(\varpi, x(\varpi), x(a_1(\varpi)), \cdots, x(a_n(\varpi))), \varpi \in J = [0, b], \qquad (2.1)
\end{aligned}
$$

$$
Qx(\varpi) = B_1 u(\varpi), \quad x(0) = x_0. \qquad (2.2)
$$

In the above, $\mathcal{S} : D(\mathcal{S}) \subset Y \to R(\mathcal{S}) \subset Z$ is a linear operator, the control function $u \in L^2(J, U)$, a Banach space of admissible control function with U as a Banach space, $B_1 : U \to X$ is a linear continuous operator and $a_i, b_j \in C(J, J), i = 1, 2, \cdots, n, \ j = 1, 2, \cdots, m$ where $C(J, J)$ is a set of all continuous function defined from J to J. G and F are the appropriate functions to be specified later and ${}^cD^\alpha, \ 0 < \alpha < 1$ is in the Caputo sense. Let $y(\varpi) = \mathcal{S}x(\varpi)$ for $x \in Y$, then (2.1) and (2.2) can be written as

$$
\begin{aligned}
&{}^cD^\alpha[y(\varpi) + F(\varpi, \mathcal{S}^{-1}y(\varpi), \mathcal{S}^{-1}y(b_1(\varpi)), \cdots, \mathcal{S}^{-1}y(b_m(\varpi)))] \\
&= \sigma\mathcal{S}^{-1}y(\varpi) + G(\varpi, \mathcal{S}^{-1}y(\varpi), \mathcal{S}^{-1}y(a_1(\varpi)), \cdots, \mathcal{S}^{-1}y(a_n(\varpi))), \quad \varpi \in J = [0, b], \qquad (2.3)
\end{aligned}
$$

$$
\widetilde{Q}y(\varpi) = B_1 u(\varpi), \quad y(0) = y_0, \qquad (2.4)
$$

where $\widetilde{Q} = Q\mathcal{S}^{-1} : Z \to X$ is a linear operator. The operator $A : Y \to Z$ given by

$$
D(A\mathcal{S}^{-1}) = \{w \in D(\sigma\mathcal{S}^{-1}) : \widetilde{Q}w = 0\},
$$

$A\mathcal{S}^{-1}w = \sigma\mathcal{S}^{-1}w$ for $w \in D(A\mathcal{S}^{-1})$, (see [4, 23]).

Definition 2.1. *[4, 23] The operators $A : D(A) \subset Y \to Z$ and $\mathcal{S} : D(\mathcal{S}) \subset Y \to Z$ satisfying the following hypotheses:*

$(H1)$ *A and $\mathcal{S}$ are closed linear operators.*

$(H2)$ *$D(\mathcal{S}) \subset D(A)$ and $\mathcal{S}$ is bijective.*

$(H3)$ *$\mathcal{S}^{-1} : Z \to D(\mathcal{S})$ is continuous.*

The hypothesis (H1)–(H3) and the closed graph theorem imply the boundedness of the linear operator $A\mathcal{S}^{-1} : Z \to Z$ and $A\mathcal{S}^{-1}$ generates an analytic compact semigroup of uniformly bounded linear operators $\{T(\varpi) : \varpi \geq 0\}$. This means that there exists a $M \geq 1$ such that $\|T(\varpi)\| \leq M$. Without loss of generality, we assume that $0 \in \lambda(A\mathcal{S}^{-1})$. This allow us to define the fractional power $(-A)^q$, for $0 < q < 1$ as a closed linear operator on its domain $D((-A)^q)$ with inverse $(-A)^{-q}$.

Theorem 2.2. *[34]*

(1) $Y_q = D((-A)^q)$ is a Banach space with the norm $\|x\|_q = \|(-A)^q x\|, \quad x \in Y$.

(2) $T(\varpi) : Y \to Y_q$ for each $(-A)^q T(\varpi)x = T(\varpi)(-A)^q x$, for all $x \in Y_q$ and $\varpi \geq 0$.

(3) For all $\varpi > 0$, $(-A)^q T(\varpi)$ is bounded on Y and there exists a positive constant C_q such that

$$\|(-A)^q T(\varpi)\| \leq \frac{C_q}{\varpi^q}.$$

(4) If $0 < \beta < q \leq 1$, then $D(-A)^q \hookrightarrow D(-A)^\beta$ and the embedding is compact whenever the resolvent operator of A is compact.

Let us recall the following known definitions.

Definition 2.3. *[35] The fractional integral of order $\alpha > 0$ with the lower limit zero for a function f can be defined as*

$$I^\alpha f(\varpi) = \frac{1}{\Gamma\alpha} \int_0^\varpi \frac{f(\nu)d\nu}{(\varpi - \nu)^{1-\alpha}}, \varpi > 0, \alpha > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the Gamma function.

Definition 2.4. *[35] The Caputo derivative of order α with the lower limit zero for a function f can be written as:*

$${}^cD^\alpha f(\varpi) = \frac{1}{\Gamma(n - \alpha)} \int_0^\varpi \frac{f^{(n)}(\nu)d\nu}{(\varpi - \nu)^{\alpha+1-n}} = I^{n-\alpha} f^{(n)}(\varpi), \ \varpi > 0, \quad 0 \leq n - 1 < \alpha < n.$$

If f is an abstract function with the values in Y, then the integrals in the above definition are taken in Bochner's sense.

Lemma 2.5. *A measurable function $\coprod : [0, b] \to Y$ is Bochner integrable if $|\coprod|$ is Lebesgue integrable.*

We now present the following results on the controllability.

(A1) $D(\sigma) \subset D(Q)$ and the restriction of Q to $D(\sigma)$ is continuous relative to graph norm of $D(\sigma)$.

(A2) There exists a linear continuous operator $\mathcal{S} : U \to Z$ such that $\sigma\mathcal{S}^{-1}B \in L(U, Z)$, $\widetilde{Q}(Bu) = B_1 u$, for all $u \in U$. Also $Bu(\varpi)$ is continuously differentiable and

$$\|(-A)^\beta Bu\| \leq C\|B_1 u\|,$$

for all $u \in U$, where C is a constant.

(A3) For all $\varpi \in (0, b]$ and $u \in U$, $T(\varpi)Bu \in D(A\mathcal{S}^{-1})$. Moreover, there exists a positive function $M_0 > 0$ such that $\|A\mathcal{S}^{-1}T(\varpi)\| \leq M_0$ (see [3, 18]).

Let $y(\varpi)$ be the solution of the systems (2.3) and (2.4). Then we define a function $z(\varpi) = y(\varpi) - Bu(\varpi)$. From the assumptions it follows that $z(\varpi) \in D(A\mathcal{S}^{-1})$. Hence, the systems (2.3) and (2.4) can be written in terms of A and B as

$$\begin{aligned} &{}^cD^\alpha[z(\varpi) + F(\varpi, \mathcal{S}^{-1}y(\varpi), \mathcal{S}^{-1}y(b_1(\varpi)), \cdots, \mathcal{S}^{-1}y(b_m(\varpi)))] \\ =&A\mathcal{S}^{-1}z(\varpi) + \sigma\mathcal{S}^{-1}Bu(\varpi) - B^cD^\alpha u(\varpi) + G(\varpi, \mathcal{S}^{-1}y(\varpi), \mathcal{S}^{-1}y(a_1(\varpi)), \cdots, \mathcal{S}^{-1}y(a_n(\varpi))), \end{aligned} \tag{2.5}$$

$$z(0) = y(0) - Bu(0). \tag{2.6}$$

For more details, see [1, 19]. From the systems (2.5) and (2.6), we present the integral form of the systems (2.3) and (2.4) in the following way:

$$\begin{aligned}y(\varpi) &= y(0) + F(0, \mathcal{S}^{-1}y(0), \mathcal{S}^{-1}y(b_1(0)), \cdots, \mathcal{S}^{-1}y(b_m(0)))] \\ &- F(\varpi, \mathcal{S}^{-1}y(\varpi), \mathcal{S}^{-1}y(b_1(\varpi)), \cdots, \mathcal{S}^{-1}y(b_m(\varpi)))] \\ &+ \frac{1}{\Gamma\alpha}\int_0^{\varpi} \frac{\mathcal{S}^{-1}Ay(\nu)d\nu}{(\varpi-\nu)^{1-\alpha}} - \frac{1}{\Gamma\alpha}\int_0^{\varpi} \frac{\mathcal{S}^{-1}ABu(\nu)d\nu}{(\varpi-\nu)^{1-\alpha}} + \frac{1}{\Gamma\alpha}\int_0^{\varpi} \frac{\mathcal{S}^{-1}\sigma Bu(\nu)d\nu}{(\varpi-\nu)^{1-\alpha}} \\ &+ \frac{1}{\Gamma\alpha}\int_0^{\varpi} \frac{G(\nu, \mathcal{S}^{-1}y(\nu), \mathcal{S}^{-1}y(a_1(\nu)), \cdots, \mathcal{S}^{-1}y(a_n(\nu)))d\nu}{(\varpi-\nu)^{1-\alpha}},\end{aligned} \tag{2.7}$$

(see [4, 14, 22, 59]) and hence, the mild solution of the systems (2.1) and (2.2) is presented in the following way:

$$\begin{aligned}x(\varpi) &= \mathcal{S}^{-1}S_\alpha(\varpi)\mathcal{S}x(0) + \mathcal{S}^{-1}S_\alpha(\varpi)F(0, x(0), x(b_1(0)), \cdots, x(b_m(0))) \\ &- \mathcal{S}^{-1}F(\varpi, x(\varpi), x(b_1(\varpi)), \cdots, x(b_m(\varpi))) \\ &- \int_0^{\varpi} (\varpi-\nu)^{\alpha-1}AT_\alpha(\varpi-\nu)\mathcal{S}^{-1}F(\nu, x(\nu), x(b_1(\nu)), \cdots, x(b_m(\nu)))d\nu \\ &+ \int_0^{\varpi} \mathcal{S}^{-1}(\varpi-\nu)^{\alpha-1}[T_\alpha(\varpi-\nu)\sigma\mathcal{S}^{-1}B - A\mathcal{S}^{-1}T_\alpha(\varpi-\nu)B]u(\nu)d\nu \\ &+ \int_0^{\varpi} \mathcal{S}^{-1}(\varpi-\nu)^{\alpha-1}T_\alpha(\varpi-\nu)G(\nu, x(\nu), x(a_1(\nu)), \cdots, x(a_n(\nu)))d\nu,\end{aligned} \tag{2.8}$$

where $\xi_\alpha(\theta)$ is a probability density function defined on $(0, \infty)$ and

$$S_\alpha(\varpi) = \int_0^\infty \xi_\alpha(\theta)T(\varpi^\alpha\theta)xd\theta,$$

and

$$T_\alpha(\varpi)x = \alpha\int_0^\infty \theta\xi_\alpha(\theta)T(\varpi^\alpha\theta)xd\theta.$$

Remark 2.6. *[59] $\xi_\alpha(\theta) \geq 0, \theta \in (0, \infty)$, $\int_0^\infty \xi_\alpha(\theta)d\theta = 1$ and $\int_0^\infty \theta\xi_\alpha(\theta)d\theta = \frac{1}{\Gamma(1+\alpha)}$.*

Definition 2.7. *[4, 14] The systems (2.1) and (2.2) are said to be controllable on the interval J if for every x_0, $x_1 \in Y$, there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of the systems (2.1) and (2.2) satisfies $x(b) = x_1$.*

Lemma 2.8. *[22] The operators $S_\alpha(\varpi)$ and $T_\alpha(\varpi)$ have the following properties:*

(i) *For any fixed $x \in Y$, $\|S_\alpha(\varpi)x\| \leq M\|x\|$, $\|T_\alpha(\varpi)x\| \leq \frac{\alpha M\|x\|}{\Gamma(\alpha+1)}$.*

(ii) *$\{S_\alpha(\varpi),\ \varpi \geq 0\}$ and $\{T_\alpha(\varpi),\ \varpi \geq 0\}$ are strongly continuous.*

(iii) *For all $\varpi \geq 0$, $S_\alpha(\varpi)$ and $T_\alpha(\varpi)$ are also compact operators.*

(iv) *For any $x \in Y$, $\beta \in (0, 1)$ and $\delta \in (0, 1)$, we have*

$$(-A)T_\alpha(\varpi)x = (-A)^{1-\beta}T_\alpha(\varpi)(-A)^\beta x,$$

and

$$\|(-A)^\delta T_\alpha(\varpi)\| \leq \frac{\alpha C_\delta \Gamma 2 - \delta}{\varpi^{\alpha\delta}\Gamma 1 + \alpha(1-\delta)}, \quad \varpi \in (0, b).$$

Remark 2.9. *For any $x \in Y$, $\beta \in (0,1)$ and $\delta = 1$, we have*

$$\|AT_\alpha(\varpi)x\| = \alpha\|\int_0^\infty \theta\xi_\alpha(\theta)AT(\varpi^\alpha\theta)xd\theta\| = \frac{\alpha\|x\|M_0}{\Gamma(1+\alpha)}.$$

Further, assume the following assumptions:

(A4) The linear operator W from $L^2(J,U)$ into Y is given by

$$Wu = \int_0^b \mathcal{S}^{-1}(b-\nu)^{\alpha-1}[T_\alpha(b-\nu)\sigma\mathcal{S}^{-1}B - A\mathcal{S}^{-1}T_\alpha(b-\nu)B]u(\nu)d\nu,$$

induces an invertible operator $\widetilde{W}$ defined on $L^2(J,U)/KerW$, and there exists K_1, K_2 and $K_3 > 0$ such that $(-A)^\beta \le K_1, 0 < \beta \le 1,\ \ \|B\| \le K_2$ and $\|\widetilde{W}^{-1}\| \le K_3$.

(A5) $F : J \times Y^{m+1} \to Y$ is continuous and there exists $\beta \in (0,1)$ and $M_1, M_2 > 0$ such that $(-A)^\beta F$ fulfills the subsequent condition:

$$\|(-A)^\beta F(\nu_1, x_0, x_1, \cdots, x_m) - (-A)^\beta F(\nu_2, y_0, y_1, \cdots, y_m)\| \le M_1(|\nu_1 - \nu_2| + \max_{i=0,1,\cdots,m}\|x_i - y_i\|),$$

for $0 \le \nu_1, \nu_2 \le b,\ x_i, y_i \in Y,\ \ i = 0,1,\cdots,m$ and

$$\|(-A)^\beta F(\varpi, x_0, x_1, \cdots, x_m)\| \le M_2(\max_{i=0,1,\cdots,m}\|x_i\| + 1), \tag{2.9}$$

holds for $(\varpi, x_0, x_1, \cdots, x_m) \in J \times Y^{m+1}$.

(A6) $G : J \times Y^{n+1} \to Y$ fulfill the subsequent conditions:

- **(i)** For every $\varpi \in J$, $G(\varpi,\cdot) : Y^{n+1} \to Y$ is continuous and for every $(x_0, x_1, \cdots, x_n) \in Y^{n+1}$, $G(\cdot, x_0, x_1, \cdots, x_n) : J \to Y$ is strongly measurable.
- **(ii)** For every $k \in \mathbb{N}$, there exists $h_k(\cdot) : [0,b] \to \mathbb{R}^+$ such that

$$\sup_{\|x_0\|,\cdots,\|x_n\|\le k}\|G(\varpi, x_0, x_1, \cdots, x_n)\| \le h_k(\varpi),$$

$$S \to (\varpi-\nu)^{1-\alpha}h_k(\nu) \in L^1([0,\varpi],\mathbb{R}^+)$$

and there exists $\bigwedge > 0$ such that

$$\liminf_{k\to\infty}\frac{\int_0^\varpi(\varpi-\nu)^{1-\alpha}h_k(\nu)d\nu}{k} = \bigwedge < \infty, \varpi \in [0,b].$$

(A7) We assume

$$L^* = \|\mathcal{S}^{-1}\|M_1[(M+1)K_1 + \frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)} < 1, \tag{2.10}$$

$$\Big[M\|\mathcal{S}^{-1}\|K_1M_2 + \|\mathcal{S}^{-1}\|K_1M_2 + \frac{\|\mathcal{S}^{-1}\|C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}M_2}{\beta\Gamma(1+\alpha\beta)} + \frac{\|\mathcal{S}^{-1}\|\alpha M}{\Gamma(1+\alpha)}\bigwedge\Big]$$
$$(\times)\Big(1 + \Big[\frac{\|\mathcal{S}^{-1}\|^2\|\sigma\|b^\alpha M}{\Gamma(1+\alpha)} + \frac{\|\mathcal{S}^{-1}\|b^\alpha M_0}{\Gamma(1+\alpha)}\Big]K_2K_3\Big) < 1. \tag{2.11}$$

Theorem 2.10. *[36, Sadovskii fixed point theorem] Let ϕ be a condensing operator on a Banach space Y, that is ϕ is continuous and takes bounded sets into bounded sets, and $\mu(\phi(B)) \le \mu(B)$ for every bounded set B of Y with $\mu(B) > 0$. If $\phi(\gamma) \subset \gamma$ for a convex, closed and bounded set γ of Y, then ϕ has a fixed point in Y. (Here $\mu(\cdot)$ denotes Kuratowski's measure of non compactness).*

3. Main results

Theorem 3.1. *If* $(A1)$–$(A7)$ *are fulfilled, then the systems* (2.1) *and* (2.2) *are controllable on* J.

Proof. For our convenience, we use the following

$$(\varpi, x(\varpi), x(b_1(\varpi)), \cdots, x(b_m(\varpi)) = (\varpi, v(\varpi)),$$

and

$$(\varpi, x(\varpi), x(a_1(\varpi)), \cdots, x(a_n(\varpi)) = (\varpi, w(\varpi)).$$

Using the assumption $(A5)$, for $x(\cdot)$, we define

$$\begin{aligned} u(\varpi) = \widetilde{W}^{-1}[x_1 &- \mathcal{S}^{-1}S_\alpha(b)\mathcal{S}x_0 - \mathcal{S}^{-1}S_\alpha(b)F(0, v(0)) + \mathcal{S}^{-1}F(b, v(b)) \\ &+ \int_0^b (b-v)^{\alpha-1}AT_\alpha(b-v)\mathcal{S}^{-1}F(v, v(v))dv \\ &- \int_0^b \mathcal{S}^{-1}(b-v)^{\alpha-1}T_\alpha(b-v)G(v, w(v))dv](\varpi). \end{aligned}$$

We now define P as follows:

$$\begin{aligned} (Px)(\varpi) = \mathcal{S}^{-1}S_\alpha(\varpi)\mathcal{S}x(0) &+ \mathcal{S}^{-1}S_\alpha(\varpi)F(0, v(0)) - \mathcal{S}^{-1}F(\varpi, v(\varpi)) \\ &- \int_0^\varpi (\varpi-v)^{\alpha-1}AT_\alpha(\varpi-v)\mathcal{S}^{-1}F(v, v(v))dv \\ &+ \int_0^\varpi \mathcal{S}^{-1}(\varpi-v)^{\alpha-1}T_\alpha(\varpi-v)\sigma\mathcal{S}^{-1}Bu(v)dv \\ &- \int_0^\varpi \mathcal{S}^{-1}(\varpi-v)^{\alpha-1}A\mathcal{S}^{-1}T_\alpha(\varpi-v)Bu(v)dv \\ &+ \int_0^\varpi \mathcal{S}^{-1}(\varpi-v)^{\alpha-1}T_\alpha(\varpi-v)G(v, w(v))dv, \end{aligned}$$

has a fixed point and this fixed point is then a solution of (2.1) and (2.2). So, we have to prove that P has a fixed point. For every $k > 0$, we set

$$B_k = \{x \in Y : \|x(\varpi)\| \le k, 0 \le \varpi \le b\}.$$

Then for every k, B_k is clearly a bounded closed convex set in Y. From Lemma 2.8 and Eq (2.9) yields

$$\begin{aligned} &\Big\|\int_0^\varpi (\varpi-v)^{\alpha-1}AT_\alpha(\varpi-v)\mathcal{S}^{-1}F(v, v(v))dv\Big\| \\ \le &\int_0^\varpi \|(\varpi-v)^{\alpha-1}\mathcal{S}^{-1}(-A)^{1-\beta}T_\alpha(\varpi-v)(-A)^\beta F(v, v(v))\|dv \\ \le &\frac{\|\mathcal{S}^{-1}\|\alpha C_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\int_0^\varpi (\varpi-v)^{\alpha\beta-1}(-A)^\beta F(v, v(v))dv \\ \le &\frac{C_{1-\beta}\Gamma(1+\beta)\|\mathcal{S}^{-1}\|b^{\alpha\beta}M_2}{\Gamma(1+\alpha\beta)}[\max_{i=1,2,\cdots,m}\|x_i\| + 1] \end{aligned}$$

$$\leq \frac{\|\mathcal{S}^{-1}\|C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)} b^{\alpha\beta}(k+1)M_2,$$

then it follows that $(\varpi-\nu)^{\alpha-1}AT_\alpha(\varpi-\nu)\mathcal{S}^{-1}F(\nu, v(\nu))$ is integrable on J, by Lemma (2.5), P is well defined on B_k. From $(A6)(ii)$, one can get

$$\begin{aligned}
\|\int_0^{\varpi} \mathcal{S}^{-1}(\varpi-\nu)^{\alpha-1}T_\alpha(\varpi-\nu)G(\nu, w(\nu))d\nu\| \leq & \int_0^{\varpi} \|\mathcal{S}^{-1}(\varpi-\nu)^{\alpha-1}T_\alpha(\varpi-\nu)G(\nu, w(\nu))\|d\nu \\
\leq & \|\mathcal{S}^{-1}\|\frac{\alpha M}{\Gamma(\alpha+1)}\int_0^{\varpi}(\varpi-\nu)^{\alpha-1}\|G(\nu, w(\nu))\|d\nu \\
\leq & \|\mathcal{S}^{-1}\|\frac{\alpha M}{\Gamma(\alpha+1)}\int_0^{\varpi}(\varpi-\nu)^{\alpha-1}h_k(\nu)d\nu.
\end{aligned}$$

We conclude that for $k > 0$ such that $PB_k \subseteq B_k$. If it fails, then there exist a function $x_k(\cdot) \in B_k$ but $PB_k \nsubseteq B_k$, and $\|Px_k(\varpi)\| > k$, for some $\varpi(k) \in J$, where $\varpi(k)$ denotes that ϖ is independent of k. Then, one can get

$$\begin{aligned}
k \leq & \|(Px_k)(\varpi)\| \\
\leq & \|\mathcal{S}^{-1}S_\alpha(\varpi)\mathcal{S}x(0) + \mathcal{S}^{-1}S_\alpha(\varpi)F(0, v_k(0)) \\
& - \mathcal{S}^{-1}F(\varpi, v_k(\varpi)) - \int_0^{\varpi}(\varpi-\nu)^{\alpha-1}AT_\alpha(\varpi-\nu)\mathcal{S}^{-1}F(\nu, v_k(\nu))d\nu \\
& + \int_0^{\varpi}\mathcal{S}^{-1}(\varpi-\nu)^{\alpha-1}[T_\alpha(\varpi-\nu)\sigma\mathcal{S}^{-1} - A\mathcal{S}^{-1}T_\alpha(\varpi-\nu)]B\widetilde{W}^{-1} \\
& (\times)\{x_1 - \mathcal{S}^{-1}S_\alpha(b)\mathcal{S}x(0) - \mathcal{S}^{-1}S_\alpha(b)F(0, v_k(0)) + \mathcal{S}^{-1}F(b, v_k(b)) \\
& + \int_0^{b}(b-\tau)^{\alpha-1}AT_\alpha(b-\tau)\mathcal{S}^{-1}F(\tau, v_k(\tau))d\nu \\
& - \int_0^{b}\mathcal{S}^{-1}(b-\tau)^{\alpha-1}T_\alpha(b-\tau)G(\tau, w_k(\tau))d\tau\}(\nu)d\nu \\
& + \int_0^{\varpi}\mathcal{S}^{-1}(\varpi-\nu)^{\alpha-1}T_\alpha(\varpi-\nu)G(\nu, w_k(\nu))d\nu\| \\
\leq & M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\| + M\|\mathcal{S}^{-1}\|\|(-A)^{-\beta}(-A)^{\beta}F(0, v_k(0))\| \\
& + \|\mathcal{S}^{-1}\|\|(-A)^{-\beta}(-A)^{\beta}F(\varpi, v_k(\varpi))\| + \|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}b^{\alpha\beta}(k+1)M_2 \\
& + \int_0^{\varpi}\|\mathcal{S}^{-1}\|(\varpi-\nu)^{\alpha-1}[\|\sigma\|\|\mathcal{S}^{-1}\|\|T_\alpha(\varpi-\nu)\| + \|A\mathcal{S}^{-1}T_\alpha(\varpi-\nu)\|] \\
& (\times)\|B\|\|\widetilde{W}^{-1}\|\{\|x_1\| + M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\| + M\|\mathcal{S}^{-1}\|\|(-A)^{-\beta}(-A)^{\beta}F(0, v_k(0))\| \\
& + \|\mathcal{S}^{-1}\|\|(-A)^{-\beta}(-A)^{\beta}F(b, v_k(b))\| + \|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}b^{\alpha\beta}(k+1)M_2 \\
& + \int_0^{b}\|\mathcal{S}^{-1}\|(b-\tau)^{\alpha-1}\|T_\alpha(b-\tau)\|\|G(\tau, w_k(\tau))\|d\tau\}d\nu \\
& + \int_0^{\varpi}\|\mathcal{S}^{-1}\|(\varpi-\nu)^{\alpha-1}\|T_\alpha(\varpi-\nu)\|\|G(\nu, w_k(\nu))\|d\nu
\end{aligned}$$

$$
\begin{aligned}
&\leq M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\| + M\|\mathcal{S}^{-1}\|K_1M_2(k+1) + \|\mathcal{S}^{-1}\|K_1M_2(k+1)\\
&+ \frac{\|\mathcal{S}^{-1}\|C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)}{\beta\Gamma(1+\alpha\beta)}M_2\\
&+ \|\mathcal{S}^{-1}\|\frac{b^{\alpha}}{\alpha}[\|\sigma\|\|\mathcal{S}^{-1}\|\frac{\alpha M}{\Gamma(\alpha+1)} + \frac{\alpha M_0}{\Gamma(\alpha+1)}]K_2K_3\\
&(\times)\{\|x_1\| + M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\| + M\|\mathcal{S}^{-1}\|K_1M_2(k+1) + \|\mathcal{S}^{-1}\|K_1M_2(k+1)\\
&+ \|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}b^{\alpha\beta}(k+1)M_2 + \frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\int_0^b (b-\tau)^{\alpha-1}h_k(\tau)d\tau\}\\
&+ \frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\int_0^{\varpi}(\varpi-\nu)^{\alpha-1}h_k(\nu)d\nu.
\end{aligned}
$$

Dividing by k on both sides of the above inequality and letting $k \to +\infty$, one can get

$$
\begin{aligned}
1 &\leq M\|\mathcal{S}^{-1}\|K_1M_2 + \|\mathcal{S}^{-1}\|K_1M_2 + \|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}b^{\alpha\beta}M_2\\
&+ \|\mathcal{S}^{-1}\|\frac{b^{\alpha}}{\alpha}\left[\|\sigma\|\|\mathcal{S}^{-1}\|\frac{\alpha M}{\Gamma(\alpha+1)} + \frac{\alpha M_0}{\Gamma(\alpha+1)}\right]K_2K_3\Big(M\|\mathcal{S}^{-1}\|K_1M_2\\
&+ \|\mathcal{S}^{-1}\|K_1M_2 + \|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}b^{\alpha\beta}M_2 + \frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\bigwedge\Big) + \|\mathcal{S}^{-1}\|\frac{\alpha M}{\Gamma(\alpha+1)}\bigwedge.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Big[M\|\mathcal{S}^{-1}\|K_1M_2 + \|\mathcal{S}^{-1}\|K_1M_2 + \|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}b^{\alpha\beta}M_2\\
&+\frac{\|\mathcal{S}^{-1}\|\alpha M}{\Gamma(\alpha+1)}\bigwedge\Big]\Big(1+\Big[\frac{\|\mathcal{S}^{-1}\|^2\|\sigma\|b^{\alpha}M}{\Gamma(1+\alpha)} + \frac{\|\mathcal{S}^{-1}\|b^{\alpha}M_0}{\Gamma(1+\alpha)}\Big]K_2K_3\Big) \geq 1.
\end{aligned}
$$

The above equation contradicts Eq (2.11). Thus, for $k > 0$, $PB_k \subseteq B_k$. Now, we need to verify P has a fixed point on B_k, which implies (2.1) and (2.2) have a mild solution. We decompose P as $P = P_1 + P_2$, where P_1 and P_2 are determined on B_k by

$$
\begin{aligned}
(P_1x)(\varpi) &= \mathcal{S}^{-1}S_{\alpha}(\varpi)F(0, \nu(0)) - \mathcal{S}^{-1}F(\varpi, \nu(\varpi))\\
&\quad - \int_0^{\varpi}(\varpi-\nu)^{\alpha-1}AT_{\alpha}(\varpi-\nu)\mathcal{S}^{-1}F(\nu, \nu(\nu))d\nu,
\end{aligned}
$$

and

$$
\begin{aligned}
&(P_2x)(\varpi)\\
=&\mathcal{S}^{-1}S_{\alpha}(\varpi)\mathcal{S}x_0 + \int_0^{\varpi}\mathcal{S}^{-1}(\varpi-\nu)^{\alpha-1}[T_{\alpha}(\varpi-\nu)\sigma\mathcal{S}^{-1} - A\mathcal{S}^{-1}T_{\alpha}(\varpi-\nu)]Bu(\nu)d\nu\\
+&\int_0^{\varpi}\mathcal{S}^{-1}(\varpi-\nu)^{\alpha-1}T_{\alpha}(\varpi-\nu)G(\nu, w(\nu))d\nu,
\end{aligned}
$$

for $0 \leq \varpi \leq b$. We have to verify P_1 is a contraction mapping if P_2 is compact. For checking P_1 fulfills the contraction condition, we assume $x_1, x_2 \in B_k$. Then, for every $\varpi \in J$ and by hypothesis (A5) and Eq (2.10), one can get

$$
\begin{aligned}
&\|(P_1x_1)(\varpi)-(P_1x_2)(\varpi)\|\\
\leq&\|\mathcal{S}^{-1}S_\alpha(\varpi)[F(0,v_1(0))-F(0,v_2(0))\|\\
&+\|\mathcal{S}^{-1}[F(\varpi,v_1(\varpi))-F(\varpi,v_2(\varpi))]\|\\
&+\|\int_0^\varpi \mathcal{S}^{-1}(\varpi-\nu)^{\alpha-1}AT_\alpha(\varpi-\nu)[F(\nu,v_1(\nu))-F(\nu,v_2(\nu))]d\nu\|\\
\leq&\|\mathcal{S}^{-1}\|MK_1M_1\sup_{0\leq\nu\leq b}\|x_1(\nu)-x_2(\nu)\|+\|\mathcal{S}^{-1}\|K_1M_1\sup_{0\leq\nu\leq b}\|x_1(\nu)-x_2(\nu)\|\\
&+\|\mathcal{S}^{-1}\|b^{\alpha\beta}\frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}M_1\sup_{0\leq\nu\leq b}\|x_1(\nu)-x_2(\nu)\|.
\end{aligned}
$$

Hence

$$
\|(P_1x_1)(\varpi)-(P_1x_2)(\varpi)\|\leq\|\mathcal{S}^{-1}\|M_1[(M+1)K_1+\frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}b^{\alpha\beta}]\sup_{0\leq\nu\leq b}\|x_1(\nu)-x_2(\nu)\|.
$$

Thus,

$$
\|(P_1x_1)(\varpi)-(P_1x_2)(\varpi)\|\leq L^*\sup_{0\leq\nu\leq b}\|x_1(\nu)-x_2(\nu)\|,
$$

and by assumption $0 < L^* < 1$, we see that P_1 is a contraction. To prove that P_2 is compact, firstly we prove that P_2 is continuous on B_k. Assume $\{x_n\}\subseteq B_k$ with $x_n\to x$ in B_k, then for every $\nu\in J$, $w_n(\nu)\to w(\nu)$ and by $(A6)(i)$, one can get $G(\nu,w_n(\nu))\to G(\nu,w(\nu))$, when $n\to\infty$. By the dominated convergence theorem, one can get

$$
\begin{aligned}
&\|P_2x_n-P_2x\|\\
=&\sup_{0\leq\varpi\leq b}\|\int_0^\varpi \mathcal{S}^{-1}(\varpi-\nu)^{\alpha-1}\Big(T_\alpha(\varpi-\nu)\sigma\mathcal{S}^{-1}-A\mathcal{S}^{-1}T_\alpha(\varpi-\nu)\Big)B[w_n(\nu)-w(\nu)]d\nu\\
&+\int_0^\varpi \mathcal{S}^{-1}(\varpi-\nu)^{\alpha-1}T_\alpha(\varpi-\nu)[G(\nu,w_n(\nu))-G(\nu,w(\nu))]d\nu\|\to 0,
\end{aligned}
$$

when $n\to\infty$, i.e., P_2 is continuous. Now, we need to verify $\{P_2x : x\in B_k\}$ is an equicontinuous family of functions. For this, we assume $\epsilon>0$ be small, $0<\varpi_1<\varpi_2$, then

$$
\begin{aligned}
&\|(P_2x)(\varpi_2)-(P_2x)(\varpi_1)\|\\
\leq&\|\mathcal{S}^{-1}\|\|S_\alpha(\varpi_2)-S_\alpha(\varpi_1)\|\|x_0\|\\
&+\int_0^{\varpi_1-\epsilon}\|\mathcal{S}^{-1}\|\|\Big((\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)-(\varpi_1-\nu)^{\alpha-1}T_\alpha(\varpi_1-\nu)\Big)\sigma\mathcal{S}^{-1}Bu(\nu)\|d\nu\\
&+\int_{\varpi_1-\epsilon}^{\varpi_1}\|\mathcal{S}^{-1}\|\|\Big((\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)-(\varpi_1-\nu)^{\alpha-1}T_\alpha(\varpi_1-\nu)\Big)\mathcal{S}^{-1}\sigma Bu(\nu)\|d\nu\\
&+\int_{\varpi_1}^{\varpi_2}\|\mathcal{S}^{-1}\|\|\Big((\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)\Big)\mathcal{S}^{-1}\sigma Bu(\nu)\|d\nu\\
&+\int_0^{\varpi_1-\epsilon}\|\mathcal{S}^{-1}\|\|A\mathcal{S}^{-1}\Big((\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)-(\varpi_1-\nu)^{\alpha-1}T_\alpha(\varpi_1-\nu)\Big)Bu(\nu)\|d\nu
\end{aligned}
$$

$$+\int_{\varpi_1-\epsilon}^{\varpi_1}\|\mathcal{S}^{-1}\|\|A\mathcal{S}^{-1}\Big((\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)-(\varpi_1-\nu)^{\alpha-1}T_\alpha(\varpi_1-\nu)\Big)Bu(\nu)\|d\nu$$
$$+\int_{\varpi_1}^{\varpi_2}\|\mathcal{S}^{-1}\|\|A\mathcal{S}^{-1}\Big((\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)\Big)Bu(\nu)\|d\nu$$
$$+\int_{0}^{\varpi_1-\epsilon}\|\mathcal{S}^{-1}\|\|(\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)-(\varpi_1-\nu)^{\alpha-1}T_\alpha(\varpi_1-\nu)\|\|G(\nu,w(\nu))\|d\nu$$
$$+\int_{\varpi_1-\epsilon}^{\varpi_1}\|\mathcal{S}^{-1}\|\|(\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)-(\varpi_1-\nu)^{\alpha-1}T_\alpha(\varpi_1-\nu)\|\|G(\nu,w(\nu))\|d\nu$$
$$+\int_{\varpi_1}^{\varpi_2}\|\mathcal{S}^{-1}\|\|(\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)\|\|G(\nu,w(\nu))\|d\nu.$$

Observe that

$$\|u(\nu)\|\le K_3\{\|x_1\|+M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\|+M\|\mathcal{S}^{-1}\|K_1M_2(k+1)+\|\mathcal{S}^{-1}\|K_1M_2(k+1)$$
$$+\|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)}{\beta\Gamma(1+\alpha\beta)}M_2+\frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\int_0^b(b-\tau)^{\alpha-1}h_k(\tau)d\tau\}.$$

We see that

$$\|(P_2x)(\varpi_2)-(P_2x)(\varpi_1)\|$$
$$\le\|\mathcal{S}^{-1}\|\|S_\alpha(\varpi_2)-S_\alpha(\varpi_1)\|\|x_0\|$$
$$+\int_{0}^{\varpi_1-\epsilon}\|\mathcal{S}^{-1}\|\|(\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)-(\varpi_1-\nu)^{\alpha-1}T_\alpha(\varpi_1-\nu)\|\|\sigma\|\|\mathcal{S}^{-1}\|K_2$$
$$(\times)K_3\{\|x_1\|+M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\|+M\|\mathcal{S}^{-1}\|K_1M_2(k+1)+\|\mathcal{S}^{-1}\|K_1M_2(k+1)$$
$$+\|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)M_2}{\beta\Gamma(1+\alpha\beta)}+\frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\int_0^b(b-\tau)^{\alpha-1}h_k(\tau)d\tau\}d\nu$$
$$+\int_{\varpi_1-\epsilon}^{\varpi_1}\|\mathcal{S}^{-1}\|\|(\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)-(\varpi_1-\nu)^{\alpha-1}T_\alpha(\varpi_1-\nu)\|\|\sigma\|\|\mathcal{S}^{-1}\|K_2K_3$$
$$(\times)\{\|x_1\|+M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\|+M\|\mathcal{S}^{-1}\|K_1M_2(k+1)+\|\mathcal{S}^{-1}\|K_1M_2(k+1)$$
$$+\|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)M_2}{\beta\Gamma(1+\alpha\beta)}+\frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\int_0^b(b-\tau)^{\alpha-1}h_k(\tau)d\tau\}d\nu$$
$$+\int_{\varpi_1}^{\varpi_2}\|\mathcal{S}^{-1}\|\|(\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)\|\|\sigma\|\|\mathcal{S}^{-1}\|K_2K_3\{\|x_1\|+M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\|$$
$$+M\|\mathcal{S}^{-1}\|K_1M_2(k+1)+\|\mathcal{S}^{-1}\|K_1M_2(k+1)+\|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)M_2}{\beta\Gamma(1+\alpha\beta)}$$
$$+\frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\int_0^b(b-\tau)^{\alpha-1}h_k(\tau)d\tau\}d\nu$$
$$+\int_{0}^{\varpi_1-\epsilon}\|\mathcal{S}^{-1}\|\|A\mathcal{S}^{-1}\Big((\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)-(\varpi_1-\nu)^{\alpha-1}T_\alpha(\varpi_1-\nu)\Big)K_2K_3$$
$$(\times)\{\|x_1\|+M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\|+M\|\mathcal{S}^{-1}\|K_1M_2(k+1)+\|\mathcal{S}^{-1}\|K_1M_2(k+1)$$
$$+\|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)M_2}{\beta\Gamma(1+\alpha\beta)}+\frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\int_0^b(b-\tau)^{\alpha-1}h_k(\tau)d\tau\}d\nu$$

$$+\int_{\varpi_1-\epsilon}^{\varpi_1}\|\mathcal{S}^{-1}\|\|A\mathcal{S}^{-1}\Big((\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)-(\varpi_1-\nu)^{\alpha-1}T_\alpha(\varpi_1-\nu)\Big)K_2K_3$$
$$(\times)\{\|x_1\|+M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\|+M\|\mathcal{S}^{-1}\|K_1M_2(k+1)+\|\mathcal{S}^{-1}\|K_1M_2(k+1)$$
$$+\|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)M_2}{\beta\Gamma(1+\alpha\beta)}+\frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\int_0^b(b-\tau)^{\alpha-1}h_k(\tau)d\tau\}d\nu$$
$$+\int_{\varpi_1}^{\varpi_2}\|\mathcal{S}^{-1}\|\|A\mathcal{S}^{-1}\Big((\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)\Big)K_2K_3$$
$$(\times)\{\|x_1\|+M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\|+M\|\mathcal{S}^{-1}\|K_1M_2(k+1)+\|\mathcal{S}^{-1}\|K_1M_2(k+1)$$
$$+\|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)M_2}{\beta\Gamma(1+\alpha\beta)}+\frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\int_0^b(b-\tau)^{\alpha-1}h_k(\tau)d\tau\}d\nu$$
$$+\int_0^{\varpi_1-\epsilon}\|\mathcal{S}^{-1}\|\|(\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)-(\varpi_1-\nu)^{\alpha-1}T_\alpha(\varpi_1-\nu)\|h_k(\nu)d\nu$$
$$+\int_{\varpi_1-\epsilon}^{\varpi_1}\|\mathcal{S}^{-1}\|\|(\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)-(\varpi_1-\nu)^{\alpha-1}T_\alpha(\varpi_1-\nu)\|h_k(\nu)d\nu$$
$$+\int_{\varpi_1}^{\varpi_2}\|\mathcal{S}^{-1}\|\|(\varpi_2-\nu)^{\alpha-1}T_\alpha(\varpi_2-\nu)\|h_k(\nu)d\nu.$$

We check $\|(P_2x)(\varpi_2)-(P_2x)(\varpi_1)\|$ tends to zero independently of $x\in B_k$ when $\varpi_2\to\varpi_1$, with ϵ sufficiently small because of the compactness of $S_\alpha(\varpi)$, for $\varpi>0$ (see [34]) implies the continuity of $S_\alpha(\varpi)$ for $\varpi>0$ in ϖ in the uniform operator topology. We can verify that $P_2x,\ x\in B_k$ is continuous at $\varpi=0$. Therefore, P_2 maps B_k into a family of equicontinuous functions. We need to verify that $V(\varpi)=\{(P_2x)(\varpi):x\in B_k\}$ is relatively compact in Y. Assume that $0<\varpi\le b$ be fixed, $0<\epsilon<\varpi$, for arbitrary $\delta>0$, for $x\in B_k$, we determine

$$\begin{aligned}(P_2^{\epsilon,\delta}x)(\varpi)&=\int_\delta^\infty\xi_\alpha(\theta)T(\varpi^\alpha\theta)\mathcal{S}^{-1}\mathcal{S}x_0d\theta\\
&+\alpha\int_0^{\varpi-\epsilon}\int_\delta^\infty\theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)\sigma\mathcal{S}^{-1}Bu(\nu)d\theta d\nu\\
&-\alpha\int_0^{\varpi-\epsilon}\int_\delta^\infty\theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)A\mathcal{S}^{-1}Bu(\nu)d\theta d\nu\\
&+\alpha\int_0^{\varpi-\epsilon}\int_\delta^\infty\theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)G(\nu,w(\nu))d\theta d\nu\\
&=T(\epsilon^\alpha\delta)\int_\delta^\infty\xi_\alpha(\theta)T(\varpi^\alpha\theta-\epsilon^\alpha\delta)\mathcal{S}^{-1}\mathcal{S}x_0d\theta\\
&+\alpha T(\epsilon^\alpha\delta)\int_0^{\varpi-\epsilon}\int_\delta^\infty\theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta-\epsilon^\alpha\delta)\sigma\mathcal{S}^{-1}Bu(\nu)d\theta d\nu\\
&-\alpha T(\epsilon^\alpha\delta)\int_0^{\varpi-\epsilon}\int_\delta^\infty\theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta-\epsilon^\alpha\delta)A\mathcal{S}^{-1}Bu(\nu)d\theta d\nu\\
&+\alpha T(\epsilon^\alpha\delta)\int_0^{\varpi-\epsilon}\int_\delta^\infty\theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta-\epsilon^\alpha\delta)G(\nu,w(\nu))d\theta d\nu.\end{aligned}$$

Because $T(\epsilon^\alpha\delta),\epsilon^\alpha\delta>0$ is compact, then $V^{\epsilon,\delta}(\varpi)=\{(P_2^{\epsilon,\delta}x)(\varpi):x\in B_k\}$ is relatively compact in Y for every $\epsilon, 0 < \epsilon < \varpi$ and for all $\delta > 0$. Additionally, for each $x \in B_k$, one can get

$$
\begin{aligned}
&\|(P_2x)(\varpi) - (P_2^{\epsilon,\delta}x)(\varpi)\| \\
&\leq \|\int_0^\delta \xi_\alpha(\theta)T(\varpi^\alpha\theta)\mathcal{S}^{-1}\mathcal{S}x_0 d\theta\| \\
&+ \alpha\|\int_0^\varpi\int_0^\delta \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)\sigma\mathcal{S}^{-1}Bu(\nu)d\theta d\nu\| \\
&+ \alpha\|\int_0^\varpi\int_\delta^\infty \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)\sigma\mathcal{S}^{-1}Bu(\nu)d\theta d\nu \\
&- \int_0^{\varpi-\epsilon}\int_\delta^\infty \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)\sigma\mathcal{S}^{-1}Bu(\nu)d\theta d\nu\| \\
&+ \alpha\|\int_0^\varpi\int_0^\delta \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)A\mathcal{S}^{-1}Bu(\nu)d\theta d\nu\| \\
&+ \alpha\|\int_0^\varpi\int_\delta^\infty \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)A\mathcal{S}^{-1}Bu(\nu)d\theta d\nu \\
&- \int_0^{\varpi-\epsilon}\int_\delta^\infty \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)A\mathcal{S}^{-1}Bu(\nu)d\theta d\nu\| \\
&+ \alpha\|\int_0^\varpi\int_0^\delta \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)G(\nu, w(\nu))d\theta d\nu\| \\
&+ \alpha\|\int_0^\varpi\int_\delta^\infty \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)G(\nu, w(\nu))d\theta d\nu \\
&- \int_0^{\varpi-\epsilon}\int_\delta^\infty \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)G(\nu, w(\nu))d\theta d\nu\| \\
&\leq \|\int_0^\delta \xi_\alpha(\theta)T(\varpi^\alpha\theta)\mathcal{S}^{-1}\mathcal{S}x_0 d\theta\| \\
&+ \alpha\|\int_0^\varpi\int_0^\delta \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)\sigma\mathcal{S}^{-1}Bu(\nu)d\theta d\nu\| \\
&+ \alpha\|\int_{\varpi-\epsilon}^\varpi\int_\delta^\infty \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)\sigma\mathcal{S}^{-1}Bu(\nu)d\theta d\nu\| \\
&+ \alpha\|\int_0^\varpi\int_0^\delta \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)A\mathcal{S}^{-1}Bu(\nu)d\theta d\nu\| \\
&+ \alpha\|\int_{\varpi-\epsilon}^\varpi\int_\delta^\infty \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)A\mathcal{S}^{-1}Bu(\nu)d\theta d\nu\| \\
&+ \alpha\|\int_0^\varpi\int_0^\delta \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)G(\nu, w(\nu))d\theta d\nu\| \\
&+ \alpha\|\int_{\varpi-\epsilon}^\varpi\int_\delta^\infty \theta(\varpi-\nu)^{\alpha-1}\xi_\alpha(\theta)\mathcal{S}^{-1}T((\varpi-\nu)^\alpha\theta)G(\nu, w(\nu))d\theta d\nu\| \\
&\leq M\|x_0\|\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\int_0^\delta \xi_\alpha(\theta)d\theta + \alpha M\Big(\int_0^\varpi (\varpi-\nu)^{\alpha-1}\|\sigma\|\|\mathcal{S}^{-1}\|K_2K_3 \\
&(\times)\{\|x_1\| + M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\| + M\|\mathcal{S}^{-1}\|K_1M_2(k+1) + \|\mathcal{S}^{-1}\|K_1M_2(k+1)
\end{aligned}
$$

$$+\|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)M_2}{\beta\Gamma(1+\alpha\beta)}$$
$$+\frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\int_0^b(b-\tau)^{\alpha-1}h_k(\tau)d\tau\}d\nu\Big)\int_0^\delta\theta\xi_\alpha(\theta)d\theta$$
$$+\alpha M\Big(\int_{\varpi-\epsilon}^{\varpi}(\varpi-\nu)^{\alpha-1}\|\sigma\|\|\mathcal{S}^{-1}\|K_2K_3$$
$$(\times)\{\|x_1\|+M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\|+M\|\mathcal{S}^{-1}\|K_1M_2(k+1)+\|\mathcal{S}^{-1}\|K_1M_2(k+1)$$
$$+\|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)M_2}{\beta\Gamma(1+\alpha\beta)}$$
$$+\frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\int_0^b(b-\tau)^{\alpha-1}h_k(\tau)d\tau\}d\nu\Big)\int_0^\infty\theta\xi_\alpha(\theta)d\theta$$
$$+\alpha M_0\Big(\int_0^{\varpi}(\varpi-\nu)^{\alpha-1}K_2K_3$$
$$(\times)\{\|x_1\|+M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\|+M\|\mathcal{S}^{-1}\|K_1M_2(k+1)+\|\mathcal{S}^{-1}\|K_1M_2(k+1)$$
$$+\|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)M_2}{\beta\Gamma(1+\alpha\beta)}$$
$$+\frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\int_0^b(b-\tau)^{\alpha-1}h_k(\tau)d\tau\}d\nu\Big)\int_0^\delta\theta\xi_\alpha(\theta)d\theta$$
$$+\alpha M_0\Big(\int_{\varpi-\epsilon}^{\varpi}(\varpi-\nu)^{\alpha-1}K_2K_3$$
$$(\times)\{\|x_1\|+M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\|+M\|\mathcal{S}^{-1}\|K_1M_2(k+1)+\|\mathcal{S}^{-1}\|K_1M_2(k+1)$$
$$+\|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)M_2}{\beta\Gamma(1+\alpha\beta)}$$
$$+\frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\int_0^b(b-\tau)^{\alpha-1}h_k(\tau)d\tau\}d\nu\Big)\int_0^\infty\theta\xi_\alpha(\theta)d\theta$$
$$+\|\mathcal{S}^{-1}\|\alpha M\Big(\int_0^{\varpi}(\varpi-\nu)^{\alpha-1}h_k(\nu)d\nu\Big)\int_0^\delta\theta\xi_\alpha(\theta)d\theta$$
$$+\|\mathcal{S}^{-1}\|\alpha M\Big(\int_{\varpi-\epsilon}^{\varpi}(\varpi-\nu)^{\alpha-1}h_k(\nu)d\nu\Big)\int_0^\infty\theta\xi_\alpha(\theta)d\theta$$
$$\leq M\|x_0\|\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\int_0^\delta\xi_\alpha(\theta)d\theta+\alpha M\Big(\int_0^{\varpi}(\varpi-\nu)^{\alpha-1}\|\sigma\|\|\mathcal{S}^{-1}\|K_2K_3$$
$$(\times)\{\|x_1\|+M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\|+M\|\mathcal{S}^{-1}\|K_1M_2(k+1)+\|\mathcal{S}^{-1}\|K_1M_2(k+1)$$
$$+\|\mathcal{S}^{-1}\|\frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)M_2}{\beta\Gamma(1+\alpha\beta)}$$
$$+\frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\|\int_0^b(b-\tau)^{\alpha-1}h_k(\tau)d\tau\}d\nu\Big)\int_0^\delta\theta\xi_\alpha(\theta)d\theta$$
$$+\alpha M\Big(\int_{\varpi-\epsilon}^{\varpi}(\varpi-\nu)^{\alpha-1}\|\sigma\|\|\mathcal{S}^{-1}\|K_2K_3$$
$$(\times)\{\|x_1\|+M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\|+M\|\mathcal{S}^{-1}\|K_1M_2(k+1)+\|\mathcal{S}^{-1}\|K_1M_2(k+1)$$

$$
\begin{aligned}
&+ \|\mathcal{S}^{-1}\| \frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)M_2}{\beta\Gamma(1+\alpha\beta)} \\
&+ \frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\| \int_0^b (b-\tau)^{\alpha-1} h_k(\tau)d\tau\}d\nu\Big) \\
&+ \alpha M_0\Big(\int_0^{\varpi} (\varpi-\nu)^{\alpha-1} K_2K_3 \\
&(\times)\{\|x_1\| + M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\| + M\|\mathcal{S}^{-1}\|K_1M_2(k+1) + \|\mathcal{S}^{-1}\|K_1M_2(k+1) \\
&+ \|\mathcal{S}^{-1}\| \frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)M_2}{\beta\Gamma(1+\alpha\beta)} \\
&+ \frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\| \int_0^b (b-\tau)^{\alpha-1} h_k(\tau)d\tau\}d\nu\Big) \int_0^{\delta} \theta\xi_\alpha(\theta)d\theta \\
&+ \alpha M_0\Big(\int_{\varpi-\epsilon}^{\varpi} (\varpi-\nu)^{\alpha-1} K_2K_3 \\
&(\times)\{\|x_1\| + M\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|x_0\| + M\|\mathcal{S}^{-1}\|K_1M_2(k+1) + \|\mathcal{S}^{-1}\|K_1M_2(k+1) \\
&+ \|\mathcal{S}^{-1}\| \frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}(k+1)M_2}{\beta\Gamma(1+\alpha\beta)} \\
&+ \frac{\alpha M}{\Gamma(\alpha+1)}\|\mathcal{S}^{-1}\| \int_0^b (b-\tau)^{\alpha-1} h_k(\tau)d\tau\}d\nu\Big) \\
&+ \|\mathcal{S}^{-1}\|\alpha M\Big(\int_0^{\varpi} (\varpi-\nu)^{\alpha-1} h_k(\nu)d\nu\Big) \int_0^{\delta} \theta\xi_\alpha(\theta)d\theta \\
&+ \|\mathcal{S}^{-1}\|\alpha M\Big(\int_{\varpi-\epsilon}^{\varpi} (\varpi-\nu)^{\alpha-1} h_k(\nu)d\nu\Big).
\end{aligned}
$$

Hence, there are relative compact sets arbitrary close to $V(\varpi), \varpi > 0$. Therefore, $V(\varpi), \varpi > 0$ is also relatively compact in Y. Consequently, with the help of Arzela-Ascoli theorem it can be say that P_2 is compact. The above evidence demonstrates that $P = P_1 + P_2$ is a condensing mapping on B_k, and by the Theorem 2.10, $x(\cdot)$ exists for P on B_k and the systems (2.1) and (2.2) have a mild solution. □

Remark 3.2. *Many authors have recently investigated the boundary controllability of fractional evolution differential systems utilizing fractional theories, mild solutions, Caputo fractional derivatives, and fixed-point techniques. Very particularly, in [1–5], the authors discussed the existence and boundary controllability outcomes for integer and fractional order systems with and without delay by referring to multivalued functions, various fixed point theorems, fractional calculus, and nonlocal conditions. One can extend our current study to the integro-differential systems, Volterra-Fredholm integro-differential systems with integer and fractional order settings by using well-known fixed point theorems.*

4. Example

Let us assume that Ω be a bounded, open subset of $\mathbb{R}^n$. Consider Γ be a sufficiently smooth boundary of Ω. Assume that the following fractional differential system:

$$
\begin{aligned}
&{}^{c}\partial_{\varpi}^{\alpha}[(\chi(\varpi,\nu)-\Delta\chi(\varpi,\nu))+F(\varpi,\chi(\varpi,\nu),\chi(b_1(\varpi),\nu),\cdots,\chi(b_m(\varpi),\nu))]\\
=&\Delta\chi(\varpi,\nu)+G(\varpi,\chi(\varpi,\nu),\chi(a_1(\varpi),\nu),\cdots,\chi(a_n(\varpi),\nu)) \ \ in \ \ Q=(0,b)\times\Omega,\\
&\chi(\varpi,0)=u(\varpi,0), \ \text{on } \Sigma=(0,b)\times\Gamma, \ \varpi\in[0,b],\\
&\chi(\varpi,\nu)=0, \chi(0,\nu)=\chi_0(\nu), \ \ \text{for } \nu\in\Omega.
\end{aligned}
$$

In the above, $u \in L^2(\Sigma)$, $\chi_0 \in L^2(\Delta)$, $F, G \in L^2(Q)$ and ${}^{c}\partial_{\varpi}^{\alpha}$ is a Caputo fractional partial derivative of order $0 < \alpha < 1$. We can formulate the above problem as the boundary control problem (2.1) and (2.2) by suitably taking the space $Y = Z = L^2(\Omega), X = H^{\frac{-1}{2}}(\Gamma), U = L^2(\Gamma), B_1 = I$. Now $\mathcal{S} : D(\mathcal{S}) \subset Y \to Z$ given by $\mathcal{S}\hbar = \hbar - \Delta\hbar$ with $D(\mathcal{S}) = H^2(\Omega)$ and

$$
D(\sigma)=\Big\{\chi\in L^2(\Omega):\Delta\chi\in L^2(\Omega)\Big\}, \ \ \sigma\chi=\Delta\chi.
$$

The trace operator θ is well defined and expressed as $\theta\chi = \chi|_\Gamma$ and for $\chi \in D(\sigma)$, $\theta\chi \in H^{\frac{-1}{2}}(\Gamma)$ [24]. Define $A : D(A) \subset Y \to Z$ in the following way: $A\mathcal{S}^{-1}\hbar = \Delta\mathcal{S}^{-1}\hbar$ with $D(A\mathcal{S}^{-1}) = H_0^1(\Omega) \cup H^2(\Omega)$. Here $H^k(\Omega), H^\nu(\Omega)$ are the usual Sobolev space on Ω, Γ. Then, we introduce A and $\mathcal{S}$ in the following way:

(i) $A\hbar = \sum_{p=1}^{\infty} p^2(\hbar, \hbar_p)\hbar_p, \ \ \hbar \in D(A)$, and $\mathcal{S}\hbar = \sum_{p=1}^{\infty}(1 + p^2)(\hbar, \hbar_p)\hbar_p, \ \ \hbar \in D(\mathcal{S})$.

(ii) For every $\hbar \in Y$,

$$
A^{\frac{-1}{2}}\hbar=\sum_{p=1}^{\infty}\frac{1}{p}(\hbar,\hbar_p)\hbar_p.
$$

(iii) $A^{\frac{1}{2}}$ is defined as follows:

$$
A^{\frac{1}{2}}=\sum_{p=1}^{\infty}p(\hbar,\hbar_p)\hbar_p.
$$

On $D(A^{\frac{1}{2}}) = \{\hbar(\cdot) \in Y; \sum_{p=1}^{\infty} p(\hbar, \hbar_p)\hbar_p \in Y\}$, $\hbar_p(y) = \sqrt{2}\sin p\nu, \ p = 1, 2, 3, \cdots$ be the orthogonal set of eigen vectors of A. Additionally, for $\hbar \in Y$.

$$
\begin{aligned}
\mathcal{S}^{-1}\hbar&=\sum_{p=1}^{\infty}\frac{1}{1+p^2}(\hbar,\hbar_p)\hbar_p,\\
A\mathcal{S}^{-1}\hbar&=\sum_{p=1}^{\infty}\frac{p^2}{1+p^2}(\hbar,\hbar_p)\hbar_p,\\
T(\varpi)\hbar&=\sum_{p=1}^{\infty}e^{\frac{p^2}{1+p^2}}(\hbar,\hbar_p)\hbar_p.
\end{aligned}
$$

Clearly, $A\mathcal{S}^{-1}$ generates a strongly continuous compact semigroup $T(\varpi)$ with $\|T(\varpi)\| \le M$, for all $\varpi \ge 0$. To verify $(A2)$ and $(A3)$, we introduce $B : L^2(\Gamma) \to L^2(\Gamma)$ by $Bu = v_u$, v_u is the unique solution to $\Delta v_u = 0$ in Ω, $v_u = u$ in Γ. We now define F and G in the following way:

$$
F(\varpi,x(\varpi),x(b_1(\varpi)),\cdots,x(b_m(\varpi)))=F(\varpi,\chi(\varpi,\nu),\chi(b_1(\varpi),\nu),\cdots,\chi(b_m(\varpi),\nu)),
$$

$$G(\varpi, x(\varpi), x(a_1(\varpi)), \cdots, x(a_n(\varpi))) = G(\varpi, \chi(\varpi, \nu), \chi(a_1(\varpi), \nu), \cdots, \chi(a_n(\varpi), \nu)).$$

We conclude now F, G fulfill the hypotheses $(A5)$ and $(A6)$. Additionally, $\widetilde{W}^{-1}$ also exists. Assume b and the remaining constants fulfill the hypotheses $(A5)$–$(A7)$. Therefore, all the requirements of the Theorem (3.1) are fulfilled and (2.1) and (2.2) are controllable.

5. Conclusions

In this article, we mainly focused on the boundary controllability of fractional order Sobolev-type neutral evolution equations in Banach space. We show our key results using facts from fractional calculus, semigroup theory, and the fixed point method. Finally, we give an example to illustrate the theory we have established. In the future, we will focus on the boundary controllability of Hilfer fractional-order neutral evolution equations and integrodifferential equations in Banach space by using the fixed point theorem approach.

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Conflict of interest

This work does not have any conflict of interest.

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