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Research article

Representation and stability of distributed order resolvent families

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Abstract: We consider the resolvent family of the following abstract Cauchy problem (1.1) with distributed order Caputo derivative, where A is a closed operator with dense domain and satisfies some further conditions. We first prove some stability results of distributed order resolvent family through the subordination principle. Next, we investigate the analyticity and decay estimate of the solution to (1.1) with operator $A = \lambda > 0$, then we show that the resolvent family of Eq (1.1) can be written as a contour integral. If A is self-adjoint, then the resolvent family can also be represented by resolution of identity of A. And we give some examples as an application of our result.

Keywords: integral representation; stability; distributed order calculus; functional calculus; resolvent families
Mathematics Subject Classification: 26A33, 45K05, 47A10, 47A60, 45D05

1. Introduction

In this paper, we consider the following abstract Cauchy problem with the distributed order derivative:

$$\begin{aligned} D^{(\mu)}u(t) &= -Au(t), \quad t > 0 \\ u(0) &= x_0 \in X, \end{aligned} \tag{1.1}$$

where X is a Banach space and μ satisfies the same condition as [16], that is

$$\mu \in L^1(0,1), \qquad \int_0^1 \frac{\mu(\alpha)}{\alpha} d\alpha < \infty,$$

the resolvent family of Eq (1.1) is denoted by $f(t,\mu,A)$ (if A is a constant λ, then $f(t,\mu,\lambda)$ is the resolvent family of this equation). We obtain the decay estimate of the solution if A generates an exponentially stable semigroup, we prove that the operator norm of the resolvent family is controlled by $k(t)$, which is at least logarithmic decay. And faster decay of μ near zero will lead to the faster decay of the solution. And if A is an invertible sectorial operator with some spectral conditions, then we obtain another representation of the solution, by use of contour integral or spectral measure instead of subordination principle. To the author's knowledge, this is the first time to consider the analyticity of resolvent family $f(t,\mu,\lambda)$ with respect to parameter λ, rather than parameter t, then by using the contour method and spectral measure together with some basic estimates proved by [12,14,16,17], we obtain the stability and representation of distributed order resolvent family. This method was already used to get the decay estimate and stability of fractional resolvent family, see more details at [7,25] and reference therein.

Distributed order differential equation with constant coefficient:

$$\begin{aligned} D^{(\mu)}u(t) &= -\lambda u(t), \quad t > 0, \lambda > 0, \\ u(0) &= x_0 \in X \end{aligned} \tag{1.2}$$

is studied by [3,4,14,16,17,21–23] and reference therein. And the logarithmic decay of $f(t,\mu,\lambda)$ is obtained by different methods. For example, A. N. Kochubei [14] use the Karamata-Feller Tauberian theorem, while A.Kubica [16] use the Laplace transform techniques. Inspired by these papers, we got our main results which are obtained by contour integral method.

In [21–23], multi-term fractional equations with initial-boundary value has been considered. If we let function μ in (1.2) be the linear combination of the Dirac delta function, that is:

$$\mu(\alpha) = \sum_{i=1}^{n} \lambda_i \delta_{\alpha_i},$$

where $\lambda_i > 0$ and δ_{α_i} is the Dirac delta function with $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq 1$, then Eq (1.2) becomes:

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i D^{\alpha_i} u(t) &= -\lambda u(t), \quad t > 0 \\ u(0) &= x_0 \in X. \end{aligned} \tag{1.3}$$

And the classical solution of this equation is represented by the multinomial Mittag-Leffler function. By exploiting several properties of this function, the polynomial decay of the solution is proved and can not be faster than $t^{-\beta}, \beta > \alpha_1$.

In [14], the author shows that if $\mu(\alpha)$ satisfies some regularity conditions, the distributed order derivative of f can be defined as:

$$D^{(\mu)}f(t) = \frac{d}{dt}(k * (f - f(0)))(t)$$

for every continuous function $f(t)$, where

$$k(t) = \int_0^1 \frac{s^{-\alpha}}{\Gamma(\alpha)} \mu(\alpha) d\alpha$$

is a positive decreasing function and plays an important role in the distributed order calculus. The author also got the asymptotic estimates of $k(t)$ under the condition $\mu(\alpha) \in C^3[0,1]$ and the asymptotic estimates of $\hat{k}(p) = \int_0^\infty e^{-pt}k(t)dt$ under the condition $\mu(\alpha) \in C^2[0,1]$. By using these results, author proved the completely monotonicity and decay estimate of solution. And author also consider the following ultra-slow equation:

$$D^{(\mu)}u(t,x) = \Delta u(t,x), \quad x \in R, t > 0.$$

Many useful estimates of the fundamental solution of this equation were obtained here.

Assuming only $\mu \in L^1, \mu \neq 0$, A. Kubika and K. Ryszewska proved very important results [16] that the operator norm of the resolvent family can be controlled by $k(t)$. Thus, the change of μ near-zero influence the decay estimate of $k(t)$, and then influence the decay estimate of the solution [16, Propositon 2]. Based on these results, extending the constant coefficient to the abstract densely defined, linear coefficient operator, we draw our conclusion.

The subordination principle is a useful tool to investigate the properties of the resolvent family, It was first proved by [2, Theorem 3.1] for fractional differential equations and then generalized to distributed order differential equations with abstract densely defined, linear operator. In [3], the Author proved the subordination identity of the solution of Eq (1.2):

$$u(t,\lambda) = \int_0^\infty \phi(t,\tau)e^{-\lambda\tau}d\tau, \quad t > 0,$$

where $\phi(t,\tau)$ is a probability density function, satisfies

$$\phi(t,\tau) \geq 0, \quad \int_0^\infty \phi(t,\tau)d\tau = 1.$$

And in [4], the constant coefficient λ is replaced by a densely defined, linear operator A, then the author renewal the subordination identity if A generates a bounded semigroup $T(t)$:

$$u(t,A) = \int_0^\infty \phi(t,\tau)T(\tau)d\tau, \quad t > 0.$$

The order of the articles is as follows: Some preliminaries of distributed order calculus, notations, and some important results in scalar type are provided in Section 2. The proof of decay estimates of solution operator in Banach space is given in Section 3. In Section 4, we first prove the analyticity and boundedness of $f(t,\mu,\lambda)$ with respect to the variable λ, then by use of contour integral method we get the integral representation result of resolvent family, and by using representation we have proved, some approximation results are given here. Spectral measure representation of resolvent is given in Section 5.

2. Preliminaries

Throughout this paper, $\Re(\lambda)$ and $\Im(\lambda)$ means the real and imaginary part of a complex number λ. X is a Banach space, H is a Hilbert space. $L(X)$ is the space of all bounded and linear operators on X. We always assume that A is a densely defined, closed, and linear operator on X, with $N(A)$, $D(A)$, $R(A)$ its kernel, domain, and range, respectively. While $\sigma(A), \rho(A)$ denotes the spectrum and resolvent family of A. As usual, $*$ denotes the convolution on $\mathbb{R}_+$:

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds. \quad k \in L^1(\mathbb{R}_+), f \in L^1(\mathbb{R}_+, X)$$

and $\hat{f}(\lambda)$ denote the Laplace-transform of a exponentially bounded function $f \in L^1(\mathbb{R}_+, X)$, defined by

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt,$$

if this integral is convergent.

Next, we give two important definitions. The first one is the definition of sectorial operator on Banach space X.

Definition 2.1. [12, Chapter 2] Let $\omega \in [0, \pi)$. An operator A on X is called sectorial of angle ω, in short, $A \in \mathrm{sect}(\omega)$, if

(1) $\sigma(A) \subset \overline{\Sigma_\omega}$, where $\Sigma_\omega := \{z \in \mathbb{C} : |\arg z| < \omega\}$;

(2) $M(A, \omega') := \sup\left\{\|\lambda R(\lambda, A)\| : \lambda \in \mathbb{C} \setminus \overline{\Sigma_{\omega'}}\right\} < \infty$ for all $\omega' \in (\omega, \pi)$.

And we call

$$\omega(A) := \min\{0 \le \omega < \pi : A \in \mathrm{sect}(\omega)\}$$

the spectral angle of A.

The second one is the definition of fractional integral and fractional derivative in Caputo sense: Let $\alpha > 0$, $m = \lceil\alpha\rceil$ the smallest integer bigger than α and

$$g_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & t > 0, \\ 0 & t \le 0, \end{cases}$$

where $\Gamma(a)$ is the Gamma function. Then the fractional integral of order $\alpha > 0$ is defined by

$$J^\alpha f(t) := (g_\alpha * f)(t), \quad f \in L^1(I), \quad t \in I \subseteq \mathbb{R}_+$$

and the Caputo fractional derivative of order α is defined by

$$D^\alpha f(t) = J^{\lceil\alpha\rceil-\alpha} D^{\lceil\alpha\rceil} f(t).$$

For a non-negative and measurable function $\mu : [0, 1] \longrightarrow \mathbb{R}$ we define the distributed-order derivative with order μ by

$$D^{(\mu)} f(t) = \int_0^1 (D^\alpha f)(t)\mu(\alpha)d\alpha.$$

If function $f(t)$ is continuous, then the preceding definition can be simplified as follows [14]:

$$D^{(\mu)} f(t) = \frac{d}{dt}((k * f)(t) - f(0)k(t)),$$

where

$$k(t) = \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\mu(\alpha)d\alpha.$$

This function is Laplace-transformable, and the Laplace-transform of this function is given by

$$\hat{k}(\lambda) = \int_0^\infty e^{-\lambda t} k(t)dt = \int_0^1 \lambda^{\alpha-1}\mu(\alpha)d\alpha.$$

We always assume that function μ satisfies condition:

$$\int_0^1 \mu(\alpha)d\alpha = c_\mu > 0, \quad \int_0^1 \frac{\mu(\alpha)}{\alpha}d\alpha < \infty, \quad \mu(\alpha) \in (0,1). \tag{2.1}$$

There are many functions satisfies Eq (2.1), for example, $\mu(\alpha) = \alpha^\beta$, $\beta \in (0,\infty)$, and it is worth mentioning that according to [16], there exists a constant $\gamma \in (0, \frac{1}{2})$ such that:

$$\int_\gamma^{1-\gamma} \mu(\alpha)d\alpha = \frac{1-\gamma}{2}c_\mu > 0 \tag{2.2}$$

for example, if we choose $\mu(\alpha) = \alpha$, then Eq (2.2) is valid for every $\gamma \leq \frac{1}{3}$.

Next we lay some results for the following distributed-order differential equation,

$$\begin{aligned} D^{(\mu)}u(t) &= -\lambda u(t), \quad t > 0, \lambda > 0, \\ u(0) &= x_0, \end{aligned} \tag{2.3}$$

which will be useful in this paper.

Lemma 2.2. *[16] Suppose that $\lambda > 0$ and μ satisfies (2.1), then the unique absolutely continuous solution of (2.3) is given by*

$$u(t) = f(t,\mu,\lambda)x_0 := \frac{\lambda x_0}{\pi}\int_0^\infty e^{-rt}\frac{G(r,\lambda)}{r}dr,$$

where

$$G(r,\lambda) = \frac{\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha}{(\lambda + \int_0^1 r^\alpha \cos(\alpha\pi)\mu(\alpha)d\alpha)^2 + (\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha)^2}.$$

To get the decay estimate of $f(t,\mu,\lambda)$, the following important lemma is needed.

Lemma 2.3. *[16] Let $p_\lambda = \min\left\{(\frac{\lambda}{4c_\mu})^{\frac{1}{\delta}}, 1\right\}$, where $\delta \in (0,1)$ is small enough such that $\int_0^\delta \mu(\alpha)d\alpha \leq \frac{\lambda}{4}$ and c_μ is defined in condition (2.1), then for $r \in (0, p_\lambda]$ we have*

$$\left|\int_0^1 r^\alpha \cos(\alpha\pi)\mu(\alpha)d\alpha\right| \leq \frac{\lambda}{2},$$

and for $r \in (p_\lambda, \infty)$ we have

$$\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha \geq c_1 \min\left\{r^{1-\gamma}, r^{-\gamma}\right\}.$$

By using this lemma, some decay estimates of $f(t,\mu,\lambda)$ is obtained in [16, Proposition 2].

Proposition 2.4. *[16, Proposition 2] If μ satisfies condition 2.1. Then for t large enough the following estimate holds*

$$|f(t,\mu,\lambda)| \leq |k(t)| \leq \frac{c}{ln(t)}, \tag{2.4}$$

where c is a constant depends only on μ. Furthermore, when a is some fixed number from the interval $(0,1)$,

(1) *if* $k,b>0$ *and* $\mu(\alpha)\leq b\alpha^k$ *a.e. on* $(0,a)$, *then* $|k(t)|\leq\frac{c}{(lnt)^{k+1}}$;

(2) *if* $k,b,\beta,m>0$ *and* $\mu(\alpha)\leq b\alpha^k e^{-\frac{\beta}{\alpha^m}}$ *a.e. on* $(0,a)$, *then for any* $q\in(0,1)$

$$|k(t)|\leq c\frac{\Gamma(k+1)}{(1-q)^{k+1}(lnt)^{k+1}}exp(-m^{\frac{1}{m+1}}(1+\frac{1}{m})(q^m\beta)^{\frac{1}{m+1}}(lnt)^{\frac{m}{m+1}});$$

(3) *if* $b>0$ *and* $\mu(\alpha)\leq bexp(-e^{\frac{1}{\alpha}})$ *a.e. on* $(0,a)$, *then* $|k(t)|\leq\frac{c}{t^{lnlnt}}$;

(4) *if* $a\in(0,1)$ *and* $supp(\mu)\subseteq[a,1]$, *then* $|k(t)|\leq\frac{c}{t^a}$.

The first estimates in Proposition 2.4 was generalized to the following problem

$$\begin{aligned} D^{(\mu)}u(t) &= Lu(t), \quad \text{in } \Omega\times(0,T)\\ u(0) &= u_0 \quad \text{in } \Omega \\ u|_{\partial\Omega} &= 0 \quad \text{for } t\in(0,T) \end{aligned} \tag{2.5}$$

with L is uniformly elliptic [16]. In this paper, we will consider more general cases. More precisely, we will replace L by a general sectorial operator.

Next we recall the definition of resolvent families [32] with kernel $a(t)$.

Definition 2.5. A family $\{R(t)\}$ is called a resolvent family for sectorial operator A with kernel $a(t)\in L^1_{loc}(\mathbb{R}_+)$, if

(1) $R(t)x:\mathbb{R}_+\to X$ is continuous for every $x\in X$ and $R(t)=I$;

(2) $R(t)D(A)\subset D(A)$ and $AR(t)x=R(t)Ax$ for all $x\in D(A)$ and $t\geq 0$;

(3) the resolvent equation

$$R(t)x = x-(a*R(t))(t)Ax$$

holds for every $x\in D(A)$.

For example, $f(t,\mu,\lambda)$ is a resolvent family for operator $Ax:=\lambda x$ with kernel

$$a(t):=\aleph(t)=\frac{1}{\pi}\int_0^\infty e^{-tr}\frac{\int_0^1 r^\alpha\sin(\alpha\pi)\mu(\alpha)d\alpha}{(\int_0^1 r^\alpha\cos(\alpha\pi)\mu(\alpha)d\alpha)^2+(\int_0^1 r^\alpha\sin(\alpha\pi)\mu(\alpha)d\alpha)^2}dr,$$

where $(k*\aleph)(t)=1$ (the form and some interesting estimate of $\aleph(t)$ is given by [14]).

Finally, we would like to give a brief introduction to a special function, the Mittag-Leffler function [2], which plays an important role in fractional calculus.

The Mittag-Leffler function is defined as follows

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}=\frac{1}{2\pi i}\int_C\frac{\mu^{\alpha-\beta}e^\mu}{\mu^\alpha-z}d\mu,\quad z\in\mathbb{C},$$

where C is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu|\leq|z|^{\frac{1}{\alpha}}$ counter clockwise. For short, $E_\alpha(z):=E_{\alpha,1}(z)$ is an entire function and satisfies

$$D^\alpha E_\alpha(wt^\alpha)=wE_\alpha(wt^\alpha).$$

The most interesting properties of this function is its asymptotic expansion.

Lemma 2.6. *Let* $0 < \alpha < 2$, $\beta > 0$, *then*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} \exp(z^{\frac{1}{\alpha}}) - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-N}), \quad \left|\arg(z)\right| \le \frac{1}{2}\alpha\pi$$

and

$$E_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-N}), \quad \left|\arg(-z)\right| \le (1 - \frac{1}{2}\alpha)\pi$$

as $z \to \infty$.

3. Decay estimate for resolvent family

Now we consider the abstract distributed order equation:

$$\begin{aligned} D^{(\mu)}u(t) &= -Au(t), \quad t > 0 \\ u(0) &= x_0, \end{aligned} \tag{3.1}$$

and an operator family $f(t, \mu, A)$ is called a distributed order resolvent family if it satisfies the Definition 2.5 with $a(t) = \aleph(t)$.

In [3], the subordination identity of the solution of the equation is obtained:

$$f(t, \mu, A) = \int_0^\infty \phi(t, \tau) T(\tau) d\tau, \quad t > 0 \tag{3.2}$$

where $\phi(t, \tau)$ is a probability density function, satisfies

$$\phi(t, \tau) \ge 0, \quad \int_0^\infty \phi(t, \tau) d\tau = 1.$$

So if A generates a bounded semigroup, it is easy to see that $f(t, \mu, A)x$ is Laplace-transformable for every $x \in X$:

$$\hat{k}(\lambda) R(\lambda \hat{k}(\lambda), -A)x = \int_0^\infty e^{-\lambda t} f(t, \mu, A)x dt, \ \lambda > 0 \quad x \in H.$$

So by using of Laplace transform, subordination principle, and imitating the method used in [24, Theorem 5.2] and [26, Propositon 3.3] we can easily get the following propositions.

Proposition 3.1. *Let* $f(t, \mu, A)$ *be a distributed ordre resolvnet family on* X *with* $\mu(\alpha) \in C[0, 1]$ *and* $\mu(0) \ne 0$. *If*

$$\|f(t, \mu, A)\| \le Mt^{-\delta} \quad as \quad t \longrightarrow \infty$$

for some constant M *and* $\delta \in (0, 1)$, *then* $X = \{0\}$.

Proof. If condition is satisfied, then there exist a constant N such that

$$\|f(t, \mu, A)\| \le Nt^{-\delta}$$

for every $t > 0$, so we have the following inequality for every $x \in X$,

$$\left\|\hat{k}(\lambda) R(\lambda \hat{k}(\lambda), -A)x\right\| \ \le \ \left\|\int_0^\infty e^{-\lambda t} f(t, \mu, A) dt\right\| \|x\|$$

$$\leq \int_0^\infty e^{-\lambda t} \|f(t,\mu,A)\| \, dt \|x\|$$
$$\leq N\lambda^{\delta-1}\|x\|.$$

This means that

$$\left\| R(\lambda\hat{k}(\lambda), -A) \right\| \leq N |\frac{1}{\hat{k}(\lambda)}| \lambda^{\delta-1}.$$

Since $\mu(0) \neq 0$, it follows from [14, Proposition 2.2] that

$$\hat{k}(\lambda) \sim \mu(0)\lambda^{-1}(ln\frac{1}{\lambda})^{-1},$$

then

$$\lim_{\lambda\to 0} R(\lambda\hat{k}(\lambda), -A)x = 0,$$

for x is arbitary, consequently, $D(A) = \{0\}$, as $\rho(A) \neq \emptyset$, then $X = \{0\}$. □

Proposition 3.2. *If $f(t,\mu,A)$ is a distributed resolvent operator generated by A, and if A also generates an exponential stable semigroup $T(t)$, that is, there exist two positive constant M, w such that*

$$\|T(t)\| \leq Me^{-wt}, t > 0,$$

then

$$\|f(t,\mu,A)x\| \leq M\|x\|k(t),$$

for every $x \in X$ and M is a constant and $k(t)$ is defined in Proposition 2.4.

Proof. By using subordination principle (3.2), we have:

$$\begin{aligned} \|f(t,\mu,A)\| &= \|\int_0^\infty \phi(t,\tau)T(\tau)d\tau\| \\ &\leq M\int_0^\infty \phi(t,\tau)e^{-w\tau}d\tau \\ &= Mf(t,\mu,w), \end{aligned}$$

then the conclusion comes from Proposition 2.4. □

We end this section with two examples.

Example 3.3. Consider equation:

$$\begin{gathered} D_t^{(\mu)}u(t,x) = \Delta u(t,x), \quad t > 0 \\ u(0,x) = \phi(x) \in L_0^1(\mathbb{R}), \end{gathered} \tag{3.3}$$

with additional condition $\mu(\alpha) \sim a\alpha^v$ as $\alpha \longrightarrow 0$ and $a > 0, v > 1$, and

$$\Delta : W_0^{2,1}(\mathbb{R}) \subseteq W_0^{1,1}(\mathbb{R}) \longrightarrow L_0^1(\mathbb{R}).$$

Then from [14], the classical solution of Eq (3.3) is

$$u(t,x)=\int_{\mathbb{R}} Z(t,x-\xi)\phi(\xi)d\xi.$$

and $Z(t,x)$ is obtained by Laplace transform in x and Fourier transform in t and has the following estimates [14]:

$$|Z(t,x)|\le C\int_0^\infty e^{-qrt}r^{-1}\,|lnr|^{-\frac{1+v}{2}}\,e^{-a|x||lnr|^{-\frac{1+v}{2}}}dr,$$

and

$$\left|\frac{d}{dx}Z(t,x)\right|\le C\int_0^\infty e^{-qrt}r^{-1}\,|lnr|^{-(1+v)}\,e^{-a|x||lnr|^{-\frac{1+v}{2}}}dr,$$

with $q<0$. Then we have,

$$\begin{aligned}
\|u(t,x)\|_{L^1_0(\mathbb{R})} &\le \|\phi(x)\|_{L^1_0(\mathbb{R})}\,|Z(t,x)|_{L^\infty}\\
&\le C\,\|\phi(x)\|_{L^1_0(\mathbb{R})}\int_0^\infty e^{-qrt}r^{-1}\,|lnr|^{-\frac{1+v}{2}}\,dr\\
&= C\,\|\phi(x)\|_{L^1_0(\mathbb{R})}\,(\int_0^{\frac{1}{2}}+\int_{\frac{1}{2}}^\infty)e^{-qrt}r^{-1}\,|lnr|^{-\frac{1+v}{2}}\,dr\\
&\le C\,\|\phi(x)\|_{L^1_0(\mathbb{R})}\,(e^{-\frac{qt}{2}}+t^{-1}(lnt)^{\frac{1-v}{2}}\\
&\quad +(lnt)^{-\frac{1+v}{2}}\int_{\frac{1}{2t}}^\infty e^{-qk}k^{-1}(lnk)^{-\frac{1+v}{2}}dk),
\end{aligned}$$

and similarily

$$\begin{aligned}
\left\|\frac{d}{dx}u(t,x)\right\|_{L^1(\mathbb{R})} &\le C\,\|\phi(x)\|_{L^1_0(R)}\,(e^{-\frac{qt}{2}}+t^{-1}(lnt)^{1-v}\\
&\quad +(lnt)^{-(1-v)}\int_{\frac{1}{2t}}^\infty e^{-qk}k^{-1}(lnk)^{-1-v}dk).
\end{aligned}$$

Thus,

$$\|u(t,x)\|_{W^{1,1}_0(\mathbb{R})}\le C(lnt)^{-1-v}\quad as\quad t\longrightarrow\infty.$$

Or consider other additional condition, $supp(\mu)\subseteq[\delta,1]$, then,

$$\hat{k}(\lambda)=\int_0^1\lambda^\alpha\mu(\alpha)d\alpha<C\lambda^\delta,$$

thus through the same calculation as [14, Theorem 4.3] we have

$$|Z(t,x)|\le C\int_0^\infty e^{-qrt}r^{\frac{\delta}{2}-1}e^{-p|x|r^{\frac{\delta}{2}}}dr,$$

and

$$\left|\frac{d}{dx}Z(t,x)\right|\le C\int_0^\infty e^{-qrt}r^{\delta-1}e^{-p|x|r^{\frac{\delta}{2}}}dr,$$

for some $p,q>0$. Thus, we can get

$$\|u(t,x)\|_{W^{1,1}_0(\mathbb{R})}\le Ct^{-\delta}\quad as\quad t\longrightarrow\infty.$$

Both of these two estimates coincide with our theorem.

4. Contour integral representation of resolvent family

As we all know that a bounded analytic semigroup or a bounded analytic fractional resolvent operator can be seen as the result of functional calculus, that is, if $-A$ is an operator with sectorial angle $\omega(-A) < \frac{\pi}{2}$, then

$$T(t) := e^{-\lambda t}(-A) = e^{tA},$$

or if $\omega(-A) < \frac{\alpha\pi}{2}$ for some $\alpha \in (0, 2)$, then fractional resolvent operator $S_\alpha(t)$ can defined by

$$S_\alpha(t) := E_\alpha(-\lambda t^\alpha)(-A) = E_\alpha(At^\alpha).$$

since $e^{-\lambda t}$ and $E_\alpha(-\lambda t^\alpha)$ are both entire function and satisfies some boundedness conditions (see more details in [12]). If $f(t, \mu, \lambda)$ satisfies the boundedness condition and analytic in a sector, then we may use functional calculus to represent the distributed order resolvent operator

$$f(t, \mu, A)x := f(t, \mu, \lambda)(A)x,$$

but unfortunately, this is not valid since $f(t, \mu, \lambda)$ is not analytic in any sector.

Proposition 4.1. *For every sector $\Sigma_\theta = \{\lambda : |arg(\lambda)| < \theta\}$, there is a $\lambda_0 \in \Sigma_\theta$, such that $f(t, \mu, \lambda)$ is diverges at λ_0 for every $t > 0$.*

Proof. Since $\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha = \int_0^1 r^\alpha \cos(\alpha\pi)\mu(\alpha)d\alpha = 0$ as $r = 0$, and

$$\lim_{r\to\infty} \frac{\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha}{\int_0^1 r^\alpha \cos(\alpha\pi)\mu(\alpha)d\alpha} = 0$$

then for every $\theta > 0$, there must exist at least one $\lambda_0 \in \Sigma_\theta$ and a positive constant r_0, such that

$$\lambda_0 = -\int_0^1 r_0^\alpha \cos(\alpha\pi)\mu(\alpha)d\alpha + i\int_0^1 r_0^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha$$

so we have

$$\begin{aligned}
&|f(t, \mu, \lambda_0)| \\
= \ & \Big|\frac{\lambda_0}{\pi}\int_0^\infty \frac{e^{-rt}}{r}\frac{\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha}{(\lambda_0 + \int_0^1 r^\alpha \cos(\alpha\pi)\mu(\alpha)d\alpha)^2 + (\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha)^2}dr\Big| \\
= \ & \frac{|\lambda_0|}{\pi}\Big|\int_0^\infty \frac{e^{-rt}}{r}\frac{\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha}{(\lambda_0 + \int_0^1 r^\alpha \cos(\alpha\pi)\mu(\alpha)d\alpha)^2 + (\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha)^2}dr\Big| \\
= \ & \frac{|\lambda_0|}{\pi}\int_0^\infty \frac{e^{-rt}}{r}\frac{\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha}{|(\lambda_0 + \int_0^1 r^\alpha \cos(\alpha\pi)\mu(\alpha)d\alpha)^2 + (\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha)^2|}dr \\
= \ & \frac{|\lambda_0|}{\pi}\int_0^\infty \frac{e^{-rt}}{r}G_0(r)\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha dr,
\end{aligned}$$

where

$$
\begin{aligned}
& G_0(r) \\
= \;& |(\lambda_0 + \int_0^1 r^\alpha \cos(\alpha\pi)\mu(\alpha)d\alpha)^2 + (\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha)^2)|^{-1} \\
= \;& \Big|(\int_0^1 (r^\alpha - r_0^\alpha)\cos(\alpha\pi)\mu(\alpha)d\alpha)^2 - (\int_0^1 r_0^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha)^2 \\
& +(\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha)^2 \\
& +2i(\int_0^1 r_0^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha)(\int_0^1 (r^\alpha - r_0^\alpha)\cos(\alpha\pi)\mu(\alpha)d\alpha)\Big|^{-1},
\end{aligned}
$$

we choose $|r - r_0|$ small enough such that

$$|r^\alpha - r_0^\alpha| \leq M|r - r_0|r_0^{\alpha-1},$$

then we have

$$G_0(r) \geq \frac{M(r_0, r)}{|r - r_0|},$$

where $M(r_0, r)$ is not equal to zero, thus we have

$$|f(t, \mu, \lambda_0)| \geq \frac{M(r_0, r)|\lambda_0|}{\pi} \int_0^\infty \frac{e^{-rt}}{r} \frac{\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha}{|r - r_0|} dr.$$

Obviously, $f(t, \mu, \lambda)$ is diverges at λ_0. □

Though $f(t, \mu, \lambda)$ is not analytic in any sector, it may analytic in some areas, and the contour integral method may be used. Now we prove the main result of this section, at first we give the following definition.

Definition 4.2. We say a pair of positive number (ω, ϵ) satisfies condition (I) if it satisfies following three conditions,

(1) $\omega, \epsilon \in (0, \infty)$.

(2) $\epsilon \leq \frac{\omega}{4}$.

(3) There exist two positive constants $\delta < 1$(δ is given in Lemma 2.3) and $r_1 \leq \min\{\omega^\delta, \omega\}$ such that:

$$\left|\omega + \int_0^1 r^\alpha \cos(\alpha\pi)\mu(\alpha)d\alpha\right| > \epsilon, r \in (0, r_1),$$

and

$$\left|\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha\right| > 2,_\varepsilon r \in (r_1, \infty).$$

Remark 4.3. For ϵ small enough, the existence of r_1 is obvious, so for every $\omega > 0$, there always exist a constant ϵ such that (ω, ϵ) satisfies condition (I).

Now we are in position to prove the analyticity of $f(t,\mu,\lambda)$ for $\lambda \in \Sigma(\omega,\epsilon)$, where (ω,ϵ) satisfies condition (I) and

$$\Sigma(\omega,\epsilon) := \{\lambda \in C | \Re(\lambda) > \omega, |\Im(\lambda)| < \epsilon\}.$$

We first prove the boundedness and decay estimate of $f(t,\mu,\lambda)$.

Lemma 4.4. *Assume (ω,ϵ) satisfies condition (I) and μ satisfies conditon (2.1). Then there exist a constant N, such that for every $t > 0$ and $\lambda \in \Sigma(\omega,\epsilon)$,*

$$|f(t,\mu,\lambda)| \le k(t)\frac{N}{|\lambda|+1}.$$

The estimate of function $k(t)$ is given in Proposition 2.4.

Proof. We define $b(r) = \int_0^1 r^\alpha \cos(\alpha\pi)\mu(\alpha)d\alpha$ and $c(r) = \int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha$ for abbreviation.

We first assume $\omega > 1$. By condition (I) we have $\Re(\lambda)^\delta > \omega^\delta > \max\{r_1, 1\}$, for every $0 < r < \Re(\lambda)^\delta$:

$$\int_0^1 r^\alpha \cos(\alpha\pi)\mu(\alpha)d\alpha > \int_{\frac{1}{2}}^1 r^\alpha \cos(\alpha\pi)\mu(\alpha)d\alpha > \frac{-1}{2}\Re(\lambda),$$

and

$$\frac{1}{2}|\lambda| < \Re(\lambda) \le |\lambda|.$$

Then, we divide function $f(t,\mu,\lambda)$ into two parts:

$$\begin{aligned} |f(t,\mu,\lambda)| &\le \frac{|\lambda|}{\pi}\int_0^\infty e^{-rt}\frac{|G(r,\lambda)|}{r}dr \\ &\le \frac{|\lambda|}{\pi}\Big(\int_0^{\Re(\lambda)^\delta} + \int_{\Re(\lambda)^\delta}^\infty\Big)\frac{e^{-rt}}{r}|G(r,\lambda)|\,dr \\ &:= f_1 + f_2, \end{aligned}$$

for every $0 < r < \Re(\lambda)^\delta$:

$$\begin{aligned} \Re\Big(\lambda + \int_0^1 r^\alpha \cos(\alpha\pi)\mu(\alpha)d\alpha\Big)^2 &:= \Re(\lambda + b(r))^2 \\ &= \Re(\Re(\lambda) + b(r) + i\Im(\lambda))^2 \\ &= (\Re(\lambda) + b(r))^2 - (\Im(\lambda))^2 \\ &> \frac{(\Re(\lambda))^2}{4} - (\Im(\lambda))^2 > 0, \end{aligned}$$

and there exist a constant L satisfies:

$$\left|(\lambda + b(r))^2\right| > \frac{|\lambda|^2}{16} - (\Im\lambda)^2 > \frac{|\lambda|^2}{L}.$$

So,

$$f_1 \le \frac{|\lambda|}{\pi}\int_0^{\Re(\lambda)^\delta}\frac{e^{-rt}}{r}\frac{L}{|\lambda^2|}\int_0^1 r^\alpha \sin(\alpha\pi)\mu(\alpha)d\alpha dr$$

$$\begin{aligned} &\leq \frac{L}{\pi|\lambda|}\int_0^\infty \frac{e^{-rt}}{r}c(r)dr \\ &\leq C\frac{1}{|\lambda|}k(t), \end{aligned}$$

because (ω,ϵ) satisfies condition (I), if $r \geq \Re(\lambda)^\delta > r_1$, then

$$\begin{aligned} |(\lambda+b(r))^2+c(r)^2| &\geq (\Re(\lambda)+b(r))^2-(\Im(\lambda))^2+c(r)^2 \\ &> c(r)^2-(\Im(\lambda))^2 \\ &> c(r)^2-\epsilon^2 \\ &> \frac{3}{4}c(r)^2. \end{aligned}$$

So there is a constant C satisfies:

$$|G(r,\lambda)| < C\frac{1}{c(r)}.$$

Assume $n \in \mathbb{N}$ and $n\delta > 2$,

$$\begin{aligned} f_2 &\leq C\frac{|\lambda|}{\pi}\int_{\Re(\lambda)^\delta}^\infty \frac{e^{-rt}}{r}\frac{1}{c(r)}dr \\ &\leq C\frac{|\lambda|}{\pi}\int_{\Re(\lambda)^\delta}^\infty \frac{e^{-rt}}{r}r^{-\gamma}dr \\ &\leq C\frac{|\lambda|}{\pi}\frac{1}{\Re(\lambda)^{\gamma\delta}}\int_1^\infty e^{-\Re(\lambda)^\delta st}s^{-\gamma-1}ds \\ &\leq C\frac{|\lambda|}{\pi}\frac{1}{\Re(\lambda)^{\gamma\delta}}\int_1^\infty e^{-\Re(\lambda)^\delta st}s^{n-1}ds \\ &\leq C\,|\lambda|^{1-(n+\gamma)\delta}\,t^{-n}. \end{aligned}$$

So our conclusion valid when $\omega > 1$, if $\omega < 1$, consider operator $(1+\frac{1}{\omega})A$ instead of A. □

Next we prove the analyticity of $f(t,\mu,\lambda)$.

Proposition 4.5. *If (ω,ϵ) satisfies condition (I), then for every $t > 0$, $f(t,\mu,\lambda)$ is analytic in $\Sigma(\omega,\epsilon)$.*

Proof. We define $b(r) = \int_0^1 r^\alpha\cos(\alpha\pi)\mu(\alpha)d\alpha$ and $c(r) = \int_0^1 r^\alpha\sin(\alpha\pi)\mu(\alpha)d\alpha$ for abbreviation.

For every $\lambda_0 \in \Sigma(\omega,\epsilon)$ and $r \in \mathbb{R}_+$, there exist a constant $M(\lambda_0)$ satisfies:

$$\max_{r\in\mathbb{R}_+}\left\{\frac{1}{(\lambda_0+b(r))^2+c(r)^2},\frac{b(r)}{(\lambda_0+b(r))^2+c(r)^2}\right\} \leq M(\lambda_0). \tag{4.1}$$

We first prove $f(t,\mu,\lambda)$ is differentiable in $\Sigma(\omega,\epsilon)$. Let $\lambda,\lambda_0 \in \Sigma(\omega,\epsilon)$,

$$\begin{aligned} f(t,\mu,\lambda)-f(t,\mu,\lambda_0) &= \frac{\lambda-\lambda_0}{\pi}\int_0^\infty \frac{e^{-rt}}{r}G(r,\lambda_0)dr \\ &\quad +\frac{\lambda}{\pi}\int_0^\infty \frac{e^{-rt}}{r}(G(r,\lambda)-G(r,\lambda_0))dr. \end{aligned}$$

So,

$$\begin{aligned}\frac{f(t,\mu,\lambda)-f(t,\mu,\lambda_0))}{\lambda-\lambda_0} &= \frac{1}{\pi}\int_0^\infty \frac{e^{-rt}}{r}G(r,\lambda_0)dr \\ &\quad +\frac{\lambda}{\pi}\int_0^\infty \frac{e^{-rt}}{r}\frac{G(r,\lambda)-G(r,\lambda_0)}{\lambda-\lambda_0}dr \\ &= \frac{1}{\pi}\int_0^\infty \frac{e^{-rt}}{r}G(r,\lambda_0)dr \\ &\quad +\frac{\lambda}{\pi}\int_0^\infty \frac{e^{-rt}}{r}G(r,\lambda_0)\frac{\lambda_0+\lambda+2b(r)}{(\lambda+b(r))^2+c^2(r)}dr.\end{aligned}$$

So,

$$\begin{aligned}&\lim_{\lambda\to\lambda_0}\frac{f(t,\mu,\lambda)-f(t,\mu,\lambda_0))}{\lambda-\lambda_0} \\ = &\ \frac{1}{\pi}\int_0^\infty \frac{e^{-rt}}{r}G(r,\lambda_0)dr+\frac{\lambda_0}{\pi}\int_0^\infty \frac{e^{-rt}}{r}G(r,\lambda_0)\frac{2\lambda_0+2b(r)}{(\lambda_0+b(r))^2+c^2(r)}dr.\end{aligned}$$

When $t>0$, by inequality (4.1), preceding equality is meaningful, and thus $f(t,\mu,\lambda)$ is differentiable in $\Sigma(\omega,\epsilon)$. In order to prove that $f(t,\mu,\lambda)$ is analytic, we only need to check whether $f(t,\mu,\lambda)$ satisfies Cauchy-Riemann condition, and this can be done directly. □

By Proposition 4.5 and Lemma 4.4, if operator A satisfies some special conditions, then distributed order resolvent family $f(t,\mu,A)$ can be represented by contour integral.

Theorem 4.6. *Let A be an invertible sectorial operator and μ satisfies condition (2.1). Suppose there exist (ω,ϵ), (ω_1,ϵ_1) satisfies condition (I), such that $\sigma(A)\subseteq\Sigma(\omega,\epsilon)\subseteq\Sigma(\omega_1,\epsilon_1)$. Define $f(t,\mu,A)$:*

$$f(t,\mu,A):=\begin{cases}\int_{\Gamma_{\omega_1,\epsilon_1}} f(t,\mu,\lambda)R(\lambda,A)d\lambda, & t>0,\\ I, & t=0,\end{cases}$$

where $\Gamma(\omega_1,\epsilon_1)$ is the positive oriented boundary of $\Sigma(\omega_1,\epsilon_1)$. Then $f(t,\mu,A)x_0$ is a weak solution of Eq (3.1), strongly continuous in $(0,\infty)$, and satisfies:

$$\|f(t,\mu,A)x_0\|\le Ck(t). \tag{4.2}$$

Proof. We have already prove the decay estimate and analyticity of $f(t,\mu,\lambda)$, so inequality (4.2) can be proved directly.

$$\|f(t,\mu,A)\|\le k(t)C\int_{\Gamma_{\omega_1,\epsilon_1}}\frac{1}{|\lambda|(1+|\lambda|)}|d\lambda|\le Ck(t).$$

Now we prove the strongly continuity of $f(t,\mu,A)x$. for every $\epsilon,t>0$, let t_1 satisfies:

$$0<t_1-t<\min\{t^n\epsilon,\epsilon\},$$

let δ here coincident with δ in Lemma 4.4, and let n satisfies $n\delta>2+\delta$, if $t\neq 0$, then assume $t<t_1$, $\omega>1$, by proof of Lemma 4.4 and mean-value theorem,

$$|f(t_1,\mu,\lambda)x_0-f(t,\mu,\lambda)x_0| \le |x_0|\frac{1}{\pi}\int_0^\infty \left|\frac{e^{-rt_1}-e^{-rt}}{r}\right||\lambda||G(r,\lambda)|dr$$

$$
\begin{aligned}
&\leq |x_0|\frac{N}{|\lambda|}\int_0^{Re(\lambda)^\delta}\frac{e^{-rt}-e^{-rt_1}}{r}c(r)dr \\
&\quad +|x_0|\frac{|\lambda|}{\pi}\int_{Re(\lambda)^\delta}^{\infty}\frac{e^{-rt}-e^{-rt_1}}{r}dr \\
&< |x_0|\frac{N}{|\lambda|}\int_0^{Re(\lambda)^\delta}\frac{e^{-rt}-e^{-rt_1}}{r}c(r)dr \\
&\quad +\frac{t_1-t}{t^n}|x_0|\frac{N}{\Re(\lambda)^{n\delta-1+\gamma\delta-\delta}} \\
&\leq N\frac{|x_0|}{|\lambda|}\epsilon.
\end{aligned}
$$

Constant N is independent of λ, t, and depend on the value of μ and γ. So we have:

$$
\|f(t_1,\mu,A)x - f(t,\mu,A)x\| \leq \|x\|\, N\epsilon \int_{\Gamma_{\omega_1,\epsilon_1}} \frac{1}{|\lambda|}\,\|R(\lambda,A)\|\,|d\lambda| < C\epsilon.
$$

Constant C is depend on the value of μ and (ω_1, ϵ_1).

If $t = 0$, because

$$
f(t,\mu,A)x = \int_{\Gamma_{\omega_1,\epsilon_1}} f(t,\mu,\lambda)R(\lambda,A)xd\lambda,
$$

and

$$
\int_{\Gamma_{\omega_1,\epsilon_1}} \frac{f(t,\mu,\lambda)}{\lambda}xd\lambda = 0,
$$

and for every $t > 0$,

$$
\|f(t,\mu,\lambda)R(\lambda,A)x\| \leq \|x\|\,\frac{M}{(1+|\lambda|)^2},
$$

and the right hand side of the preceding inequality is integrable on $\Gamma_{\omega_1,\epsilon_1}$, and

$$
f(t,\mu,\lambda)x \to x \; as \; t \to 0.
$$

Then by dominant convergence theorem, for every $x \in D(A)$,

$$
\begin{aligned}
\lim_{t\to 0} f(t,\mu,A)x &= \lim_{t\to 0}\int_{\Gamma_{\omega_1,\epsilon_1}} (f(t,\mu,\lambda)R(\lambda,A)x - \frac{f(t,\mu,\lambda)}{\lambda}x)d\lambda \\
&= \int_{\Gamma_{\omega_1,\epsilon_1}} \frac{1}{\lambda}R(\lambda,A)Axd\lambda = x.
\end{aligned}
$$

Then we prove that $f(t,\mu,A)$ is the resolvent family of equation. Since $f(t,\mu,\lambda)$ satisfies $D^{(\mu)}f = \lambda f$ and

$$
\begin{aligned}
\int_0^\infty e^{-at}f(t,\mu,A)x_0dt &= \int_0^\infty e^{-at}\int_{\Gamma_{\omega_1,\epsilon_1}} f(t,\mu,\lambda)R(\lambda,A)x_0d\lambda dt \\
&= \int_{\Gamma_{\omega_1,\epsilon_1}}\int_0^\infty e^{-at}f(t,\mu,\lambda)dtR(\lambda,A)x_0d\lambda
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\Gamma_{\omega_1,\epsilon_1}} \frac{\widehat{k}(a)}{a\widehat{k}(a)-\lambda} R(\lambda, A)x_0 d\lambda \\
&= \widehat{k}(a)R(a\widehat{k}(a), -A)x_0.
\end{aligned}
$$

Then by the uniqueness of Laplace transform and the uniqueness of the solution, we have:

$$D^{(\mu)} f(t,\mu,A)x = -Af(t,\mu,A)x,$$

and for every $x \in D(A)$,

$$f(0,\mu,A)x = x.$$

This concludes the proof. □

At the end of this section, we give an application of the integral expression proved by Theorem 4.6, we will use this representation to prove the approximation property of distributed order resolvent family, and these approximation properties also show the uniqueness of the resolvent family.

Example 4.7. By [12, Proposition 3.1.15], if A is a sectorial operator satisfies Theorem 4.6, then for every $\alpha \in (0,1)$, A^α satisfies Theorem 4.6 and for every $x \in D(A)$,

$$\lim_{\alpha \to 1} A^\alpha x = Ax,$$

and

$$
\begin{aligned}
\|A^\alpha x - Ax\| \leq\ & |\frac{\sin(\alpha\pi)}{\alpha\pi}|(L^\alpha \sup_{t\leq L} \|(t+A)^{-1}Ax - x\| + \frac{\alpha}{\alpha+1} L^{\alpha+1}\|x\| \\
& + \frac{\alpha}{1-\alpha} L^{\alpha-1}(M\|Ax\| + \|x\|)),
\end{aligned}
$$

where $L \in \mathbb{R}_+$. Since A^α and A resolvent commute, then for every $x \in D(A)$,

$$\|f(t,\mu,A^\alpha)x - f(t,\mu,A)x\| \leq \int_\Gamma |f(t,\mu,\lambda\|\|R(\lambda, A^\alpha\|\|\|R\lambda, A\|\|d\lambda\|\|A^\alpha x - Ax\|.$$

Because both A^α and A satisfy the sectorial estimate, then we have

$$\|f(t,\mu,A^\alpha)x - f(t,\mu,A)x\| \leq M(\mu)k(t)\|A^\alpha x - Ax\|.$$

The above inequality shows that we not only prove the approximation property of the resolvent family but also give the approximation rate by Theorem 4.6.

Another important kind of approximation is the Yosida approximation, that is,

$$\lim_{n\to\infty} nA(n+A)^{-1}x := \lim_{n\to\infty} A_n x = Ax,$$

for every $x \in D(A)$, moreover, if $-A$ generaes a semigroup e^{-tA}, then for every $x \in X$ we have

$$e^{-tA}x = \lim_{n\to\infty} e^{-tnA(n+A)^{-1}}x.$$

And some results about the approximation rate of Yosida approximation are given in many literatures such as [9, 10] and references therein.

Example 4.8. Let $-A$ be an operator which satisfies the condition of Theorem 4.6, and A_n be the Yosida approximation of A, then for every $x \in D(A^2)$,

$$\|Ax - A_n x\| = \|A^2(n+A)^{-1}x\| \leq \frac{1}{n}\|A^2 x\|.$$

So we have

$$\begin{aligned} \|f(t,\mu,A^\alpha)x - f(t,\mu,A_n)x\| &\leq \int_\Gamma |f(t,\mu,\lambda|\|R(\lambda,A^\alpha\|\|R\lambda,A\||d\lambda|\|A_n x - Ax\| \\ &\leq \frac{M(\mu)k(t)}{n}\|A^2 x\|. \end{aligned}$$

By Theorem 4.6 and preceding inequality, we can get the approximation rate of Yosida approximation of distributed order resolvent family.

5. Spectral measure representation of resolvent family

If A is a densely defined sectorial operator on Hilbert space H, assume A is invertible and self-adjoint, then there exists a constant w, such that $\sigma(A) \subset [w, \infty)$. So there exists a resolution of identity $E(\lambda)$, such that:

$$\langle Ax, y\rangle = \int_w^\infty \lambda dE_{x,y}(\lambda) = \int_w^\infty \lambda d\langle E(\lambda)x, y\rangle, \ x \in D(A), y \in H.$$

More properties about resolution of identity can be found in [33]. In literature [36], the author gives some examples to show how resolution of identity and spectral measure integral works.

Example 5.1. [36] Let A is a densely defined non-negative self-adjoint operator on $L^2(\Omega)$, $E(\lambda)$ is the resolution of identity of operator A.

The heat-diffusion semigroup generated by A can be written as:

$$e^{-tA} = \int_0^\infty e^{-t\lambda} dE(\lambda), t > 0,$$

and this semigroup is a contraction semigroup.

The positive fractional powers of operator A is A^δ, $\delta \in (0,1)$, $D(A^\delta) \subseteq D(A)$,

$$A^\delta = \int_0^\infty \lambda^\delta dE(\lambda) = \frac{1}{\Gamma(-\delta)} \int_0^\infty (e^{-tA} - I)\frac{dt}{t^{1+\delta}}.$$

The negative fractional powers of operator A is $A^{-\delta}$, $\delta > 0$,

$$A^{-\delta} = \int_0^\infty \lambda^{-\delta} dE(\lambda) = \frac{1}{\Gamma(\delta)} \int_0^\infty e^{-tA}\frac{dt}{t^{1-\delta}}.$$

Let function $f(t,\mu,\lambda)$ be the resolvent family of distributed order equation with operator $A = \lambda$, then $f(t,\mu,\lambda)$ is measurable on interval $[w,\infty)$. Define operator $f(t,\mu,A)$:

$$f(t,\mu,A) := \int_w^\infty f(t,\mu,\lambda)dE(\lambda),$$

this definition means

$$\langle f(t,\mu,A)x,y\rangle = \int_w^\infty f(t,\mu,\lambda)dE_{x,y}(\lambda),$$

and

$$D(f(t,\mu,A)) = \left\{x \in H; \int_w^\infty |(f(t,\mu,\lambda))|^2\, dE_{x,x}(\lambda) < \infty\right\}.$$

We first prove the decay estimate of $f(t,\mu,A)$.

Theorem 5.2. *$f(t,\mu,A)$ is well defined and $D(f(t,\mu,A)) = H$. If μ satisfies* (2.1)*, then*

$$\|f(t,\mu,A)\| < N(\frac{1}{t} + k(t)),$$

N is a constant independent of t, and may change line by line.

Proof. We use notation $p_\lambda = \min\left\{(\frac{\lambda}{4M})^{\frac{1}{\delta}}, 1\right\}$ as we used in Lemma 2.3. First we assume

$$0 < \int_0^1 \mu(\alpha)d\alpha = M < 1,$$

and δ satisfies:

$$\int_0^\delta \mu(\alpha)d\alpha < \frac{w}{4},$$

and there exist a constant $\gamma \in (0, \frac{1}{2})$ such that:

$$\int_\gamma^{1-\gamma} \mu(\alpha)d\alpha = \frac{1-\gamma}{\gamma}M > 0,$$

the existence of these constants is proved in [16].

Let $x, y \in H$, we divide $\langle f(t,\mu,A)x,y\rangle$ into two parts,

$$\begin{aligned}\langle f(t,\mu,A)x,y\rangle &= \int_\omega^\infty \frac{\lambda}{\pi}\int_0^\infty e^{-rt}\frac{G(r)}{r}drdE_{x,y}(\lambda)\\ &= (\int_\omega^1 + \int_1^\infty)\frac{\lambda}{\pi}\int_0^\infty e^{-rt}\frac{G(r)}{r}drdE_{x,y}(\lambda)\\ &:= \phi_1 + \phi_2,\end{aligned}$$

and divide ϕ_1 into two parts,

$$\begin{aligned}\phi_1 &= \int_\omega^1 \frac{\lambda}{\pi}(\int_0^{p_\lambda} + \int_{p_\lambda}^\infty)e^{-rt}\frac{G(r)}{r}drdE_{x,y}(\lambda)\\ &:= \phi_{11} + \phi_{12}.\end{aligned}$$

Estimate ϕ_{11} and ϕ_{12} separately, by using estimates (41) and (49) from [16]

$$G(r) \le \frac{4}{\lambda^2}\int_0^1 r^\alpha \sin(\alpha\pi)d\alpha,\ r \in (0, p_\lambda),$$

and

$$\int_0^{p_\lambda} e^{-rt} \int_0^1 r^\alpha \sin(\alpha\pi) d\alpha \frac{dr}{r} < k(t),$$

$$\begin{aligned}
|\phi_{11}| &\leq \int_\omega^1 \int_0^{p_\lambda} \frac{\lambda}{\pi} e^{-rt} \frac{G(r)}{r} dr d\left|E_{x,y}(\lambda)\right| \\
&\leq \int_\omega^1 \frac{\lambda}{\pi} \frac{4}{\lambda^2} k(t) d\left|E_{x,y}(\lambda)\right| \\
&\leq Nk(t) \|x\| \|y\|.
\end{aligned}$$

$$\begin{aligned}
|\phi_{12}| &\leq \int_\omega^1 \frac{\lambda}{\pi} \int_{p_\lambda}^\infty e^{-rt} \frac{M \min\left\{r^{-\gamma}, r^{\gamma-1}\right\}}{r} dr d\left|E_{x,y}(\lambda)\right| \\
&\leq \frac{N}{t} \int_0^1 \lambda p_\lambda^{-\gamma-1} e^{-p_\lambda t} d\left|E_{x,y}(\lambda)\right| \\
&\leq \frac{N}{t} \|x\| \|y\|.
\end{aligned}$$

In order to estimate ϕ_2, we divide ϕ_2 into four parts,

$$\begin{aligned}
\phi_2 &= \int_1^\infty \frac{\lambda}{\pi} \left(\int_0^{p_\lambda} + \int_{p_\lambda}^1 + \int_1^\lambda + \int_\lambda^\infty\right) e^{-rt} \frac{G(r)}{r} dr dE_{x,y}(\lambda) \\
&:= \phi_{21} + \phi_{22} + \phi_{23} + \phi_{24},
\end{aligned}$$

and estimate these four parts separately,

$$\begin{aligned}
|\phi_{21}| &\leq \int_1^\infty \frac{\lambda}{\pi} \int_0^{p_\lambda} e^{-rt} \frac{G(r)}{r} dr d\left|E_{x,y}(\lambda)\right| \\
&\leq \int_1^\infty \frac{4}{\pi\lambda} k(t) d\left|E_{x,y}(\lambda)\right| \\
&\leq Nk(t) \|x\| \|y\|.
\end{aligned}$$

By definition of p_λ,

$$\begin{aligned}
|\phi_{22}| &\leq \int_1^\infty \frac{\lambda}{\pi} \int_{p_\lambda}^1 e^{-rt} \frac{G(r)}{r} dr d\left|E_{x,y}(\lambda)\right| \\
&\leq \int_1^\infty \frac{\lambda}{\pi} \int_{p_\lambda}^1 e^{-rt} r^{\gamma-2} dr d\left|E_{x,y}(\lambda)\right| \\
&= \int_1^{4M} \frac{\lambda}{\pi} \int_{p_\lambda}^1 e^{-rt} r^{\gamma-2} dr d\left|E_{x,y}(\lambda)\right| \\
&\leq \int_1^{4M} \frac{\lambda}{\pi} \int_{p_\lambda}^1 e^{-rt} dr p_\lambda^{\gamma-2} d\left|E_{x,y}(\lambda)\right| \\
&\leq \frac{N}{t} \int_1^{4M} \lambda p_\lambda^{\gamma-2} d\left|E_{x,y}(\lambda)\right|
\end{aligned}$$

$$\leq \frac{N}{t}\|x\|\|y\|.$$

In order to estimate ϕ_{23}, we shall use inequality $\int_0^1 \mu(\alpha)d\alpha = M < 1$ to estimate $G(r)$, bacause

$$-\int_0^1 \cos(\pi\alpha)\mu(\alpha)d\alpha \leq \int_0^1 |\cos(\pi\alpha)|\,\mu(\alpha)d\alpha < M < 1,$$

and since $r \in (1, \lambda)$,

$$\left|\int_0^1 r^\alpha \cos(\pi\alpha)\mu(\alpha)d\alpha\right| \leq M\lambda,$$

then we have

$$\lambda + \int_0^1 r^\alpha \cos(\pi\alpha)\mu(\alpha)d\alpha \geq (1 - M)\lambda \geq 0,$$

then by the representation of $G(r, \lambda)$, we have

$$G(r, \lambda) \leq \frac{\int_0^1 r^\alpha \sin(\pi\alpha)\mu(\alpha)d\alpha}{(1 - M)^2\lambda^2} \leq \frac{Nr}{\lambda^2},$$

so

$$\begin{aligned}
|\phi_{23}| &\leq \int_1^\infty \frac{\lambda}{\pi} \int_1^\lambda e^{-rt}\frac{G(r)}{r}drd\left|E_{x,y}(\lambda)\right| \\
&\leq N\int_1^\infty \frac{1}{\pi\lambda} \int_1^\lambda e^{-rt}drd\left|E_{x,y}(\lambda)\right| \\
&\leq \frac{N}{t}\,\|x\|\,\|y\|\,.
\end{aligned}$$

Finally we estimate ϕ_{24},

$$\begin{aligned}
|\phi_{24}| &\leq \int_1^\infty \frac{\lambda}{\pi} \int_\lambda^\infty e^{-rt}\frac{G(r)}{r}drd\left|E_{x,y}(\lambda)\right| \\
&\leq \int_1^\infty \frac{\lambda}{\pi} \int_\lambda^\infty e^{-rt}r^{-1-\gamma}drd\left|E_{x,y}(\lambda)\right| \\
&\leq \int_1^\infty \lambda^{-\gamma} \int_\lambda^\infty e^{-rt}drd\left|E_{x,y}(\lambda)\right| \\
&= \int_1^\infty \lambda^{-\gamma}\frac{e^{-\lambda t}}{t}d\left|E_{x,y}(\lambda)\right| \\
&\leq \frac{1}{t}\int_1^\infty e^{-\lambda t}d\left|E_{x,y}(\lambda)\right| \\
&\leq \frac{1}{t}\,\|x\|\,\|y\|\,.
\end{aligned}$$

So for every $x, y \in H$,

$$\langle f(t, \mu, A)x, y\rangle \leq N(\frac{1}{t} + k(t))\,\|x\|\,\|y\|\,.$$

This means
$$\|f(t,\mu,A)\| \le N(\frac{1}{t}+k(t)).$$
If $\int_0^1 \mu(\alpha)d\alpha = K > 1$, then we consider operator $\frac{A}{K+1}$ instead of A and this end the proof. □

Next we prove $f(t,\mu,A)$ satisfies following three conditions:
(1) $f(t,\mu,A)$ is strongly continuous and $f(0,\mu,A) = I$.
(2) $f(t,\mu,A)D(A) \subseteq D(A)$, and for every $x \in D(A)$, $f(t,\mu,A)Ax = Af(t,\mu,A)x$.
(3) For every $x \in D(A)$, $D^{(\mu)} f(t,\mu,A)x = -Af(t,\mu,A)x$.
By proof of Theorem 5.2, condition (1) is satisfied.
Set $x \in D(A), y \in H$,
$$\begin{aligned}\langle \frac{e^{-sA}f(t,\mu,A)x - f(t,\mu,A)x}{s}, y\rangle &= \int_0^\infty \frac{e^{-s\lambda}-I}{s} f(t,\mu,\lambda)dE_{x,y}(\lambda)\\ &= \int_0^\infty f(t,\mu,\lambda)\frac{e^{-s\lambda}-I}{s}dE_{x,y}(\lambda)\\ &= \langle f(t,\mu,A)\frac{e^{-sA}x-x}{s}, y\rangle.\end{aligned}$$
Then $\langle f(t,\mu,A)Ax,y\rangle = \langle Af(t,\mu,A)x,y\rangle$, since $y \in H$ is arbitrary, $f(t,\mu,A)D(A) \subseteq D(A)$, and for every $x \in D(A)$, $f(t,\mu,A)Ax = Af(t,\mu,A)x$.

Finally, let $x \in D(A), y \in H$,
$$\begin{aligned}\langle D^{(\mu)} f(t,\mu,A)x, y\rangle &= \int_0^\infty D^{(\mu)} f(t,\mu,\lambda)dE_{x,y}(\lambda)\\ &= \int_0^\infty -\lambda f(t,\mu,\lambda)dE_{x,y}(\lambda)\\ &= \langle -Af(t,\mu,A)x,y\rangle.\end{aligned}$$
Combining these conclusions, we have proved that $f(t,\mu,A)$ is the resolvent family generated by operator A.

So we have the following theorem.

Theorem 5.3. *Let A is a densely defined self-adjoint operator on Hilbert space H, if there exist a constant $w > 0$, such that $\sigma(A) \subset [w,\infty)$, and function μ satisfies condition* (2.1)*. Then for every $x \in D(A)$, distributed order equation*
$$\begin{aligned}D^{(\mu)}u(t) &= -Au(t),\\ u(0) &= x,\end{aligned} \tag{5.1}$$
has an unique solution $f(t,\mu,A)x$, given by
$$f(t,\mu,A)x = \int_0^\infty f(t,\mu,\lambda)dE(\lambda)x.$$
And exists a constant N satisfies:
$$\|f(t,\mu,A)\| \le N(\frac{1}{t}+k(t)).$$

Remark 5.4. Since every non-negative self-adjoint operator is a sectorial operator, contour integral can also be used here, and obtain the same decay estimate. Here we use the spectral measure to illustrate two things. First, the representation of the resolvent family is not unique, although resolvent family is unique. Second, the decay speed of the resolvent family not only depends on μ but also depends on the spectral of A.

6. Appendix

In addition to the references mentioned in this article, the following literature on this topic provides applications for Caputo fractional calculus and distributed order differential equation: In paper [6, 11, 37], arthurs give some qualitative analyses applications about Caputo fractional calculus; distributed order differential equations with different initial value or boundary value were considered by [15, 20]; many applications about distributed order differential equations, such as numerical analysis and control theory, were given by [5, 13, 27–30].

7. Conclusions

In this paper, we proved the analyticity and decay estimate of $f(t, \mu, \lambda)$ with respect to λ and then use this property to prove the contour integral representation of $f(t, \mu, A)$. If A is self-adjoint, then we represent $f(t, \mu, A)$ by resolution of identity of A, and some examples are given.

Conflict of interest

The authors declare no conflict of interest.

References

1. W. Arendt, C. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Birkhäuser, 2010.
2. E. G. Bajlekova, *Fractional evolution equations in Banach spaces*, PhD thesis, Department of mathematics, Eindhoven University of Technology, 2001.
3. E. G. Bajlekova, Completely monotone functions and some classes of fractional evolution equations, *Integr. Transf. Spec. Funct.*, **26** (2015), 737–769. https://doi.org/10.1080/10652469.2015.1039224
4. E. G. Bajlekova, Estimate for a general fractional relaxation equation and application to an inverse source problem, *Math. Method. Appl. Sci.*, **41** (2018), 9018–9026. https://doi.org/10.1002/mma.4868
5. A. H. Bhrawy, M. A. Zaky, Numerical simulation of multi-dimensional distributed-order generalized Schrödinger equations, *Nonlinear Dynam.*, **89** (2017), 1415–1432. https://doi.org/10.1007/s11071-017-3525-y
6. M. Bohner, O. Tunc, C. Tunc, Qualitative analysis of Caputo fractional integro-differential equations with constant delays, *Comput. Appl. Math.*, **6** (2021). https://doi.org/10.1007/s40314-021-01595-3

7. E. Cuesta, Asymptotic behaviour of the solution of fractional integro-differential equations and some time discretizations, *Discrete Contin. Dyn. Syst*., **2007** (2013), 277–285.
8. A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher transcendental functions*, McGraw-Hill, New York, 1955.
9. K. J. Engel, R. Nagel, *One-Parameter semigroups for linear evolution equations*, Springer, New York, **194** (2000).
10. A. Gomilko, Y. Tomilov, On convergence rates in approximation theory for operator semigroups, *J. Funct. Anal*., **266** (2014), 3040–3082. https://doi.org/10.1016/j.jfa.2013.11.012
11. J. R. Graef, C. Tunc, H. Sevli, Razumikhin qualitative analyses of Volterra integro-fractional delay differential equation with Caputo derivatives, *Commun. Nonlinear Sci.,* **103** (2021). https://doi.org/10.1016/j.cnsns.2021.106037
12. M. Haase, *The functional calculus for sectorial operators*, Operator theory: Advances and applications, Birkhäuser Verlag, Basel, **169** (2006).
13. R. M. Hafez, M. A. Zaky, M. A. Abdelkawy, Jacobi spectral Galerkin method for distributed-order fractional Rayleigh-Stokes problem for a generalized second grade fluid, *Front. Phys.-Lausanne,* **240** (2020). https://doi.org/10.3389/fphy.2019.00240
14. A. N. Kochubei, Distributed order calculus and equations of ultraslow diffusion, *J. Math. Anal. Appl*., **340** (2008), 252–281. https://doi.org/10.1016/j.jmaa.2007.08.024
15. A. N. Kochubei, Asymptotic properties of solutions of the fractional diffusion-wave equation, *Fract. Calc. Appl. Anal*., **17** (2014), 881–896. https://doi.org/10.2478/s13540-014-0203-3
16. A. Kubica, K. Ryszewska, Decay of solutions to parabolic-type problem with distributed order Caputo derivative, *J. Math. Anal. Appl*., **465** (2018), 75–99. https://doi.org/10.1016/j.jmaa.2018.04.067
17. A. Kubica, K. Ryszewska, Fractional diffusion equation with the distributed order Caputo derivative, *J. Integral Equ. Appl*., **31** (2019), 195–243. https://doi.org/10.1216/JIE-2019-31-2-195
18. J. Kemppainen, J. Siljander, R. Zacher, Representation of solutions and large-time behavior for fully nonlocal diffusion equations, *J. Differ. Equations,* **263** (2017), 149–201. https://doi.org/10.1016/j.jde.2017.02.030
19. P. C. Kunstmann, L. Weis, *Maximal L_p-regularity for parabolic equations, fourier multiplier theorems and H^∞-functional calculus*, Springer-Verlag, Berlin, Heidelberg, 2004.
20. Z. Y. Li, Y. Luchko, M. Yamamoto, Asymptotic estimates of solutions to initial-boundary-value problems for distributed order time-fractional diffusion equations, *Fract. Calc. Appl. Anal*., **17** (2014), 1114–1136. https://doi.org/10.2478/s13540-014-0217-x
21. Y. Luchko, Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation, *J. Math. Anal. Appl*., **374** (2011), 538–548. https://doi.org/10.1016/j.jmaa.2010.08.048
22. Y. Luchko, R. Gorenflo, An operational method for solving fractional differential equations with the Caputo derivatives, *Acta Math. Vietnam*., **24** (1999).
23. Z. Li, Y. Liu, M. Yamamoto, Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients, *Appl. Math. Comput*., **257** (2015), 381–397. https://doi.org/10.1016/j.amc.2014.11.073

24. C. G. Li, M. Kostić, M. Li, On a class of time-fractional differential equations, *Fract. Calc. Appl. Anal.*, **15** (2012), 639–668. https://doi.org/10.2478/s13540-012-0044-x

25. C. Y. Li, M. Li, Asymptotic stability of fractional resolvent families, *J. Evol. Equ.*, **21** (2021), 2523–2545. https://doi.org/10.1007/s00028-021-00694-2

26. M. Li, J. Pastor, S. Piskarev, Inverses of generators of integrated fractional resolvent functions, *Fract. Calc. Appl. Anal.*, **21** (2018), 1542–1564. https://doi.org/10.1515/fca-2018-0081

27. M. A. Zaky, E. H. Doha, J. A. T. Machado, A spectral numerical method for solving distributed-order fractional initial value problems, *J. Comput. Nonlinear Dyn.*, **13** (2018). https://doi.org/10.1115/1.4041030

28. M. A. Zaky, A Legendre collocation method for distributed-order fractional optimal control problems, *Nonlinear Dynam.*, **91** (2018), 2667–2681. https://doi.org/10.1007/s11071-017-4038-4

29. M. A. Zaky, J. A. T, Machado, Multi-dimensional spectral tau-methods for distributed-order fractional diffusion equations, *Comput. Math. Appl.*, **79** (2020), 476–488. https://doi.org/10.1016/j.camwa.2019.07.008

30. M. A. Zaky, J. A. T. Machado, On the formulation and numerical simulation of distributed-order fractional optimal control problems, *Commun. Nonlinear Sci.*, **52** (2017), 177–189. https://doi.org/10.1016/j.cnsns.2017.04.026

31. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer, New York, 1993.

32. J. Prüss, *Evolutionary integral equations and applications*, Birkhäuser, Basel, 1993.

33. W. Rudin, *Functional analysis*, New York: McGraw-Hall, 1973.

34. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives, theory and applications*, Gordon and Breach Science Publishers, 1992.

35. K. Sakamoto, M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems, *J. Math. Anal. Appl.*, **384** (2011), 426–447. https://doi.org/10.1016/j.jmaa.2011.04.058

36. P. R. Stinga, J. L. Torrea, Extension problem and Harnack's inequality for some fractional operators, *Commun. Part. Diff. Eq.*, **35** (2010), 2092–2122. https://doi.org/10.1080/03605301003735680

37. C. Tunc, O. Tunc, On the stability, integrability and boundedness analyses of systems of integro-differential equations with time-delay retardation, *RACSAM Rev. R Acad. A,* **115** (2021). https://doi.org/10.1007/s13398-021-01058-8

38. S. Umarov, *Introduction to fractional and pseudo-differential equations with singular symbols*, Springer International Publishing, 2015.

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