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*Research article*

# Quasi-cyclic displacement and inversion decomposition of a quasi-Toeplitz matrix

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**Abstract:** We study a class of column upper-minus-lower (CUML) Toeplitz matrices, which are "close" to the Toeplitz matrices in the sense that their $(1,-1)$-cyclic displacements coincide with $\varphi$-cyclic displacement of some Toeplitz matrices. Among others, we derive the inverse formula for CUML Toeplitz matrices in the form of sums of products of factor circulants by constructing the corresponding displacement of the matrices. In addition, by the relationship between CUML Toeplitz matrices and CUML Hankel matrices, the inverse formula for CUML Hankel matrices is also obtained.

**Keywords:** CUML Toeplitz matrix; CUML Hankel matrix; RFMLR circulants; RSFPLR circulants; Cyclic displacement; inverse
**Mathematics Subject Classification:** 47C05, 47C15, 15A09, 15B05

## 1. Introduction

In [1], Lakatos et al. defined a Markov chain, its state corresponded to the waiting time at the moments of arrivals. The transition probability matrix of the Markov chain is as follows

$$\begin{pmatrix} \sum\limits_{j=-\infty}^{0} f_j & f_1 & f_2 & f_3 & \cdots \\ \sum\limits_{j=-\infty}^{-1} f_j & f_0 & f_1 & f_2 & \cdots \\ \sum\limits_{j=-\infty}^{-2} f_j & f_{-1} & f_0 & f_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the probability $f_j = P((j-1)T < Y_n - Z_n \leq jT)$, $Y_n$ represents the service time of the $n$th customer and $Z_n$ represents the time difference between the $(n+1)$th and the $n$th customers' arrival time. Obviously, the $n$th order truncated matrix of the above matrix is a column upper minus lower Toeplitz (CUML-Toeplitz) matrix [2].

Toeplitz operators and matrices appear in many areas of pure and applied mathematics [3–5]. Toeplitz matrices have important applications in various disciplines including the elliptic Dirichlet-periodic boundary value problems [6], sinc discretizations of partial and ordinary differential equations [7–9], coding [10, 11], image and signal processing, numerical analysis, system theory, etc.

Cao and Huang [12] discussed the commutants of two Toeplitz operators. Wang et al. [13] discussed Toeplitz operators on Fock-Sobolev space with positive measure symbols. By Fock- Carleson measure, they obtained the characterizations for boundedness and compactness of Toeplitz operators. Yang and Lu [14] characterized commuting dual Toeplitz operators with bounded harmonic symbols on the harmonic Bergman space of the unit disk. Zhao and Zheng [15] showed that the spectrum of Toeplitz operators on the Bergman space with harmonic symbols of affine functions of $z$ and $\bar{z}$ equals the image of closed unit disk under the symbol. Ji [16] considered Toeplitz operators and the Hilbert transform associated with $\mathfrak{A}$. He proved that the commutant of left analytic Toeplitz algebra on noncommutative Hardy space $H^2(\mathcal{M})$ is just the right analytic Toeplitz algebra.

Ng et al. [17] presented a modification of G. Labahn-T. Shalom theorem with another (shorter) proof. Labahn [18] proposed that formulae for the inverse of layered or striped Toeplitz matrices in terms of solutions of standard equations are observed. The inverse of an invertible Toeplitz matrix was presented in the form of Toeplitz Bezoutian of two columns in [19]. The Toeplitz inversion formulae involving circulant matrices have also been presented in [20–22]. In [23], Jiang and Wang present an innovative patterned matrix, RFPL-Toeplitz matrix. The group inverse of this new patterned matrix can be represented as the sum of products of lower and upper triangular Toeplitz matrices. The explicit inverse of nonsingular conjugate-Toeplitz and conjugate-Hankel matrices are provided in [24, 25].

It is generally known in [26] that any matrix $A \in \mathbf{C}^{n\times n}$ is uniquely determined by its displacement, i.e., $\nabla_0(A) = A - Z_0 A Z_0^T$, where $Z_0$ is the lower shift matrix. Furthermore, Gohberg and Olshevsky [27] provided new formulae for representation of matrices (in particular, the Toeplitz matrices) and their inverses in the form of sums of products of factor circulants based on the analysis of the factor $\varphi$-cyclic displacement of matrices. Here the $\varphi$-cyclic displacement of a matrix $A \in \mathbf{C}^{n\times n}$ is defined as

$$\nabla_\varphi(A) = A - Z_\varphi A Z_{\frac{1}{\varphi}}^T,$$

where $Z_\varphi$ is the $\varphi$-cyclic lower shift matrix [27, 28] (see also [29] [30] [31] for case $\varphi = 1$).

The main purpose of present work is to derive the inverses of the column upper-minus-lower Toeplitz matrices and the column upper-minus-lower Hankel matrices based on the construct of new cyclic displacements of matrices in a more general situation (see (2.1) below for definition). These formulas involvs the factor $(1, -1)$-circulants, instead of the factor $\varphi$-circulants of the Toeplitz matrices are the implications of the corresponding formulas given in [27], and are useful for the analysis of the complexity of the inversion.

Based on the characteristics and applications of Toeplitz matrices, we are able to study a class of new type matrices "close" to Toeplitz matrices. Specifically we deal with a column upper-minus-lower

(CUML) Toeplitz matrix of the form

$$T_{CUML}=\begin{pmatrix} t_0 & t_{-1} & \cdots & t_{2-n} & t_{1-n} \\ t_1 & t_0-t_1 & \ddots & \ddots & t_{2-n} \\ t_2 & t_1-t_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2}-t_{n-1} & \cdots & t_1-t_2 & t_0-t_1 \end{pmatrix}_{n\times n}, \tag{1.1}$$

where $t_0, t_{\pm 1}, \cdots, t_{\pm(n-1)}$ are complex numbers.

Obviously, the entries $t_{ij}$ of the matrix in (1.1) are given by the following formulae:

$$t_{ij}=\begin{cases} t_{i-j}, \ j=1 \text{ or } j>i \\ t_{i-j}-t_{i-j+1}, \ 2\leq j\leq i. \end{cases} \tag{1.2}$$

Specially, if $t_{1-n}=t_1, t_{2-n}=t_2, \ \cdots, \ t_{-1}=t_{n-1}$, then $T_{CUML}$ is a row first-minus-last right circulant matrix, which was first defined in [32].

A column upper-minus-lower (CUML) Hankel matrix is of the form

$$H_{CUML}=\begin{pmatrix} h_0 & h_1 & \cdots & h_{n-2} & h_{n-1} \\ h_1 & \iddots & \iddots & h_{n-1}-h_n & h_n \\ \vdots & \iddots & \iddots & h_n-h_{n+1} & h_{n+1} \\ h_{n-2} & \iddots & \iddots & \vdots & \vdots \\ h_{n-1}-h_n & h_n-h_{n+1} & \cdots & h_{2n-3}-h_{2n-2} & h_{2n-2} \end{pmatrix}_{n\times n}, \tag{1.3}$$

where $h_0, h_1, \cdots, h_{2n-2}$ are complex numbers.

Obviously, the entries $h_{ij}$ of the matrix in (1.3) are given by the following formulas:

$$h_{ij}=\begin{cases} h_{i+j-2}, \ j=n \text{ or } i+j\leq n \\ h_{i+j-2}-h_{i+j-1}, \ i+j>n \ and \ j<n. \end{cases} \tag{1.4}$$

Specially, if $h_0=h_n, \ h_1=h_{n+1}, \ \cdots, h_{n-2}=h_{2n-2}$, then $H_{CUML}$ is called a row last-minus-first left-circulant matrix, which is firstly defined in [32].

It should be noted that $H_{CUML}\hat{I}_n$ is a CUML Toeplitz matrix, where $\hat{I}_n$ is the "$n\times n$ reversal matrix [33]", having ones along the secondary diagonal and zeros elsewhere.

## 2. The $(1,-1)$-cyclic displacement of a matrix

The $(1,-1)$-cyclic displacement of a matrix $A\in\mathbf{C}^{n\times n}$ is defined as

$$\nabla_{1,-1}(A)=A-\Phi_{1,-1}A\Phi_{1,-1}^{-1}, \tag{2.1}$$

where

$$\Phi_{1,-1} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & -1 & \ddots & & & 0 \\ 0 & 1 & -1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -1 \end{pmatrix}. \tag{2.2}$$

Obviously, the matrix $\Phi_{1,-1}$ is a row first-minus-last right circulant matrix with the first row $(0, \cdots, 0, 1)$. We call $\Phi_{1,-1}$ the $(1,-1)$-cyclic lower shift matrix. $(1,-1)$-cyclic displacement rank of the matrix $A$ is the number $\tau = \mathrm{rank}\nabla_{1,-1}(A)$. If $\tau$ is comparatively small, we say that matrix $A$ has $(1,-1)$-cyclic displacement structure (with respect to $\Phi_{1,-1}$).

The linear transformation $\nabla_{1,-1}(\cdot)$ in $\mathbf{C}^{n\times n}$ presented in (2.1), it is clear that for a nonsingular matrix $A \in \mathbf{C}^{n\times n}$, there exists a relation between the $(1,-1)$-cyclic displacements of the inverse matrix $A^{-1}$ and the $(1,-1)$-cyclic displacement of $A$, namely

$$\nabla_{1,-1}(A) = -A \cdot \nabla_{1,-1}(A^{-1}) \cdot \Phi_{1,-1} A \Phi_{1,-1}^{-1}. \tag{2.3}$$

From (2.3), the $(1,-1)$-cyclic displacement rank is inherited under matrix inversion: $\mathrm{rank}\nabla_{1,-1}(A) = \mathrm{rank}\nabla_{1,-1}(A^{-1})$. Using the $(1,-1)$-cyclic displacement technique, the equation (2.3) provides us with a way of constructing the $(1,-1)$-cyclic displacement of the inverse matrix of A. If, in particular, the $(1,-1)$-cyclic displacement of $A \in \mathbf{C}^{n\times n}$ is given as the outer sum

$$\nabla_{1,-1}(A) = \sum_{i=1}^{\tau} \mathbf{q}_i \cdot \mathbf{s}_i^T, \tag{2.4}$$

where $\mathbf{q}_i$, $\mathbf{s}_i \in \mathbf{C}^n$, $i = 1, 2, \cdots, \tau$, $\tau = \mathrm{rank}\nabla_{1,-1}(A)$, then from (2.3), the analogous representation for $\nabla_{1,-1}(A^{-1})$ can be made by solving $2\tau$ matrix equations, involving the matrix $A$ and the vectors of outer sum (2.4):

$$\nabla_{1,-1}(A^{-1}) = -\sum_{i=1}^{\tau} (A^{-1}\mathbf{q}_i) \cdot (\mathbf{s}_i^T \Phi_{1,-1} A^{-1} \Phi_{1,-1}^{-1}). \tag{2.5}$$

According to the above statement, we set $\hat{\mathbf{s}}_i^T = \mathbf{s}_i^T \cdot \Phi_{1,-1}$ $(i = 1, 2, ..., \tau)$ and let the vectors $\mathbf{c}_i$ and $\hat{\mathbf{d}}_i^T$ be the solutions of the following equations

$$A\mathbf{c}_i = \mathbf{q}_i \quad (i = 1, 2, ..., \tau), \tag{2.6}$$

and

$$\hat{\mathbf{d}}_i^T A = \hat{\mathbf{s}}_i^T \quad (i = 1, 2, ..., \tau), \tag{2.7}$$

furthermore, in view of (2.3), it is not hard to verify that

$$\nabla_{1,-1}(A^{-1}) = -\sum_{i=1}^{\tau} \mathbf{c}_i \cdot \mathbf{d}_i^T, \tag{2.8}$$

where

$$\mathbf{d}_i^T = \hat{\mathbf{d}}_i^T \cdot \Phi_{1,-1}^{-1}. \tag{2.9}$$

Solving the equations

$$\mathbf{y}_1^T A = \mathbf{e}_0^T \tag{2.10}$$

with $\mathbf{e}_0^T = (1, 0, \cdots, 0) \in \mathbf{C}^n$, and

$$A\mathbf{y}_2 = \mathbf{e}_0 \tag{2.11}$$

produces the first row and the first column of $A^{-1}$, respectively. Note that in our consideration the matrix $A$ is supposed to be nonsingular from the very beginning.

On the other hand, the solvability of Eqs (2.6) and (2.11) implies invertibility of $A$. Indeed, let $\mathbf{c}_i$ $(i = 1, 2, \cdots, \tau)$ and $\mathbf{y}_2$ be the solutions of (2.6) and (2.11), respectively, and let $\mathbf{w}^T A = 0$ with $\mathbf{w} = (w_0, w_1, \cdots, w_{n-1})^T \in \mathbf{C}^n$. Then

$$\mathbf{w}^T (A - \Phi_{1,-1} A \Phi_{1,-1}^{-1}) \Phi_{1,-1} \overset{(2.4)}{=\!=\!=} \mathbf{w}^T \sum_{i=1}^{\tau} \mathbf{q}_i \cdot \mathbf{s}_i^T \Phi_{1,-1} \overset{(2.6)}{=\!=\!=} \mathbf{w}^T A \sum_{i=1}^{\tau} \mathbf{c}_i \cdot \mathbf{s}_i^T \Phi_{1,-1} = 0,$$

so that $\mathbf{w}^T \Phi_{1,-1} A = 0$. From

$$\mathbf{w}^T \Phi_{1,-1} (A - \Phi_{1,-1} A \Phi_{1,-1}^{-1}) \Phi_{1,-1} = \mathbf{w}^T \Phi_{1,-1} A \sum_{i=1}^{\tau} \mathbf{c}_i \mathbf{s}_i^T \Phi_{1,-1} = 0,$$

it follows that $\mathbf{w}^T \Phi_{1,-1}^2 A = 0$. A simple induction gives

$$\mathbf{w}^T \Phi_{1,-1}^k A = 0, \ k = 0, 1, \cdots, n-1.$$

In particular, in view of (2.11),

$$0 = \mathbf{w}^T \Phi_{1,-1}^k A \mathbf{y}_2 = \mathbf{w}^T \Phi_{1,-1}^k \mathbf{e}_0, \ k = 0, 1, \cdots, n-1,$$

i.e.,

$$\begin{aligned}
&\mathbf{w}_0 = \mathbf{w}^T \mathbf{e}_0 = 0, \quad \mathbf{w}_1 = \mathbf{w}^T \Phi_{1,-1} \mathbf{e}_0 = \mathbf{w}^T \mathbf{e}_1 = 0, \\
&\mathbf{w}_k = \mathbf{w}^T \Phi_{1,-1}^k \mathbf{e}_0 \\
&\quad = \mathbf{w}^T \cdot (0, (-1)^{k-1}, (-1)^{k-2} C_{k-1}^1, (-1)^{k-3} C_{k-1}^2, \cdots, -C_{k-1}^{k-2}, 1, 0, \cdots, 0)^T = 0, \\
&\quad k = 2, 3, \cdots, n-1,
\end{aligned}$$

where $C_n^i$ is binomial coefficient $\begin{pmatrix} n \\ i \end{pmatrix}$. We can conclude that $\mathbf{w} = 0$ and hence $A$ is nonsingular . Analogously, we may show that the solvability of equations (2.7) and (2.10) yields the invertibility of $A$, as well.

We summarize what we have obtained in the following theorem.

**Theorem 1.** *Let $A \in \mathbf{C}^{n\times n}$, and $\nabla_{1,-1}(A)$ is given by (2.4). If equations (2.6) and (2.11) [(2.7) and (2.10), respectively] have solutions $\mathbf{c}_i$ and $\mathbf{y}_2$ [$\mathbf{d}_i^T$ and $\mathbf{y}_1^T$], respectively, then A is nonsingular , and thus (2.7) and (2.10) [(2.6) and (2.11)] are solvable, and $\nabla_{1,-1}(A^{-1})$ is of the form (2.8) with $\mathbf{d}_i^T = \hat{\mathbf{d}}_i^T \cdot \Phi_{1,-1}^{-1}$, $i = 1, 2, \cdots, \tau$.*

## 3. Quasi-circulant decompositions of CUML Toeplitz matrices

The row first-minus-last right circulant matrix with the first row $\mathbf{w}^T = [w_0 \;\; w_1 \;\; \cdots \;\; w_{n-1}]$ is denoted by RFMLRcircfr($\mathbf{w}^T$) [32]. In this paper, we denote the row first-minus-last right circulant with the first column $\mathbf{w} = [w_0 \;\; w_1 \;\; \cdots \;\; w_{n-1}]^T$ by RFMLRcircfc($\mathbf{w}$), i.e., the matrix of the form

$$\mathrm{RFMLRcircfc}(\mathbf{w}) = \begin{pmatrix} w_0 & w_{n-1} & w_{n-2} & \cdots & w_1 \\ w_1 & w_0 - w_1 & \ddots & \ddots & \vdots \\ w_2 & w_1 - w_2 & \ddots & \ddots & w_{n-2} \\ \vdots & \vdots & \ddots & \ddots & w_{n-1} \\ w_{n-1} & w_{n-2} - w_{n-1} & \cdots & w_1 - w_2 & w_0 - w_1 \end{pmatrix}.$$

Whenever necessary, we shall refer such matrices RFMLRcircfr($\mathbf{w}^T$) and RFMLRcircfc($\mathbf{w}$) as factor $(1, -1)$-circulants. It should be noted that if $\mathbf{w}^T = (w_0, w_1, \cdots, w_{n-1})$, then

$$\mathrm{RFMLRcircfr}(\mathbf{w}^T) = \mathrm{RFMLRcircfc}(\tilde{\mathbf{w}}) \tag{3.1}$$

with $\tilde{\mathbf{w}} = (w_0, w_{n-1}, w_{n-2}, \cdots, w_1)^T$, and that the identity

$$\mathrm{RFMLRcircfr}(\mathbf{w}^T)\mathrm{RFMLRcircfr}(\mathbf{a}^T) = \mathrm{RFMLRcircfr}(\mathbf{a}^T)\mathrm{RFMLRcircfr}(\mathbf{w}^T) \tag{3.2}$$

and

$$\mathrm{RFMLRcircfc}(\mathbf{w})\mathrm{RFMLRcircfc}(\mathbf{a}) = \mathrm{RFMLRcircfc}(\mathbf{a})\mathrm{RFMLRcircfc}(\mathbf{w}) \tag{3.3}$$

hold for any column vector $\mathbf{w}, \mathbf{a} \in \mathbf{C}^n$.

The row skew first-plus-last right circulant matrix with the first row $\mathbf{w}^T = [w_0 \;\; w_1 \;\; \cdots \;\; w_{n-1}]$ is denoted by RSFPLRcircfr($\mathbf{w}^T$) [34–36]. In this paper, we denote the row skew first-plus-last right circulant with the first column $\mathbf{w} = [w_0 \;\; w_1 \;\; \cdots \;\; w_{n-1}]^T$ by RSFPLRcircfc($\mathbf{w}$), i.e., the matrix of the form

$$\mathrm{RSFPLRcircfc}(\mathbf{w}) = \begin{pmatrix} w_0 & -w_{n-1} & -w_{n-2} & \cdots & -w_1 \\ w_1 & w_0 - w_1 & \ddots & \ddots & \vdots \\ w_2 & w_1 - w_2 & \ddots & \ddots & -w_{n-2} \\ \vdots & \vdots & \ddots & \ddots & -w_{n-1} \\ w_{n-1} & w_{n-2} - w_{n-1} & \cdots & w_1 - w_2 & w_0 - w_1 \end{pmatrix}.$$

Whenever necessary, we shall refer such matrices RSFPLRcircfr($\mathbf{w}^T$) and RSFPLRcircfc($\mathbf{w}$) as factor $(-1, 1)$-circulants. It should be noted that if $\mathbf{w}^T = (w_0, w_1, \cdots, w_{n-1})$, then

$$\mathrm{RSFPLRcircfr}(\mathbf{w}^T) = \mathrm{RSFPLRcircfc}(\tilde{\mathbf{w}}) \tag{3.4}$$

with $\tilde{\mathbf{w}} = (w_0 \;\; -w_{n-1} \;\; -w_{n-2} \;\; \cdots \;\; -w_1)^T$, and that the identity

$$\mathrm{RSFPLRcircfr}(\mathbf{w}^T)\mathrm{RSFPLRcircfr}(\mathbf{a}^T) = \mathrm{RSFPLRcircfr}(\mathbf{a}^T)\mathrm{RSFPLRcircfr}(\mathbf{w}^T)$$

and

$$\mathrm{RSFPLRcircfc}(\mathbf{w})\mathrm{RSFPLRcircfc}(\mathbf{a}) = \mathrm{RSFPLRcircfc}(\mathbf{a})\mathrm{RSFPLRcircfc}(\mathbf{w})$$

hold for any column vector $\mathbf{w}, \mathbf{a} \in \mathbf{C}^n$.

In particular, let $T_{CUML}$ be an $n \times n$ CUML Toeplitz matrix with $(t_0\ t_{-1}\ \cdots\ t_{1-n})$ and $(t_0\ t_1\ \cdots\ t_{n-1})^T$ as its first row and first column, respectively. Considering the $(1,-1)$-cyclic displacement of $T_{CUML}$, we have

$$\nabla_{1,-1}(T_{CUML}) = T_{CUML} - \Phi_{1,-1} T_{CUML} \Phi_{1,-1}^{-1} = \mathbf{x} \cdot \mathbf{e}_0^T + \mathbf{e}_0 \cdot \mathbf{z}^T, \tag{3.5}$$

where $\mathbf{x} = (\beta\ \ t_1 - t_{1-n}\ \ \cdots\ \ t_{n-1} - t_{-1})^T$, $\mathbf{z}^T = (-\beta\ \ t_{-1} - t_{n-1}\ \ \cdots\ \ t_{1-n} - t_1)$, $\mathbf{e}_0 = (1\ \ 0\ \ \cdots\ \ 0)^T \in \mathbf{C}^n$ and $\beta$ may be an arbitrary complex number.

Clearly, $(1,-1)$-cyclic displacement rank of a CUML Toeplitz matrix is on greater than 2, so that such $T_{CUML}$ has $(1,-1)$-cyclic displacement structure if $n$ is sufficiently large.

Furthermore, in the CUML Toeplitz matrix case, $\nabla_{1,-1}(T_{CUML})$ also has a specific form given by (3.5). Then the Eqs (2.6), (2.7) reduce respectively to

$$T_{CUML}\mathbf{c}_1 = \mathbf{x}, \quad T_{CUML}\mathbf{c}_2 = \mathbf{e}_0, \tag{3.6}$$

and

$$\hat{\mathbf{d}}_1^T T_{CUML} = \mathbf{e}_0^T \Phi_{1,-1}, \quad \hat{\mathbf{d}}_2^T T_{CUML} = \mathbf{z}^T \Phi_{1,-1}. \tag{3.7}$$

Thus, by (2.8), we have

$$\nabla_{1,-1}(T_{CUML}^{-1}) = -\sum_{i=1}^{2} \mathbf{c}_i \cdot \mathbf{d}_i^T, \tag{3.8}$$

where

$$\begin{aligned} \mathbf{c}_1 &= T_{CUML}^{-1}\mathbf{x}, \quad \mathbf{d}_1^T = \mathbf{e}_0^T \Phi_{1,-1} T_{CUML}^{-1} \Phi_{1,-1}^{-1}, \\ \mathbf{c}_2 &= T_{CUML}^{-1}\mathbf{e}_0, \quad \mathbf{d}_2^T = \mathbf{z}^T \Phi_{1,-1} T_{CUML}^{-1} \Phi_{1,-1}^{-1}. \end{aligned}$$

Then from (2.4) and [30, 31], we easily obtain the following theorem.

**Theorem 2.** *If the equality*

$$\nabla_{1,-1}(T_{CUML}) = \sum_{i=1}^{\tau} \mathbf{q}_i \cdot \mathbf{s}_i^T, \qquad (\mathbf{q}_i,\ \mathbf{s}_i \in \mathbf{C}^n) \tag{3.9}$$

*holds, then*

$$T_{CUML} = \mathrm{RFMLRcircfc}(T_{CUML}) + \sum_{i=1}^{\tau} \mathrm{L}(\mathbf{q}_\mathrm{i}) \cdot \mathrm{Circ}(\mathbf{s}_\mathrm{i}^\mathrm{T}), \tag{3.10}$$

*where* $\mathrm{RFMLRcircfc}(T_{CUML})$ *is the row first-minus-last right circulant with the same first column as that of* $T_{CUML}$*, and* $L(\mathbf{q}_i)$ *is the lower triangular Toeplitz matrix with the first column* $\mathbf{q}_i = (q_{i0}\ \ q_{i1}\ \ \cdots\ \ q_{i,n-1})^T$*, and* $\mathrm{Circ}(\mathbf{s}_\mathrm{i}^\mathrm{T})$ *is the circulant with the first row* $\mathbf{s}_i^T = (s_{i0}\ \ s_{i1}\ \ \cdots\ \ s_{i,n-1})$.

*Proof.* Based on the definitions of the row first-minus-last right circulant matrix, lower triangular Toeplitz matrix and circulant matrix, we know

$$\mathrm{RFMLRcircfc}(\mathbf{T_{CUML}}) = \begin{pmatrix} t_0 & t_{n-1} & t_{n-2} & \cdots & t_1 \\ t_1 & t_0 - t_1 & \ddots & \ddots & \vdots \\ t_2 & t_1 - t_2 & \ddots & \ddots & t_{n-2} \\ \vdots & \vdots & \ddots & \ddots & t_{n-1} \\ t_{n-1} & t_{n-2} - t_{n-1} & \cdots & t_1 - t_2 & t_0 - t_1 \end{pmatrix}_{n\times n}, \tag{3.11}$$

$$\mathrm{L}(\mathbf{q_i}) = \begin{pmatrix} q_{i0} & 0 & \cdots & 0 \\ q_{i1} & q_{i0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ q_{i,n-1} & \cdots & q_{i1} & q_{i0} \end{pmatrix}_{n\times n}, \tag{3.12}$$

$$\mathrm{Circ}(\mathbf{s}_\mathrm{i}^\mathrm{T}) = \begin{pmatrix} s_{i0} & s_{i1} & \cdots & s_{i,n-1} \\ s_{i,n-1} & s_{i0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & s_{i1} \\ s_{i1} & \cdots & s_{i,n-1} & s_{i0} \end{pmatrix}_{n\times n}. \tag{3.13}$$

According to the Eqs (3.9), (3.11), (3.12) and (3.13), we obtain

$$\begin{aligned} &\mathrm{RFMLRcircfc}(T_{CUML}) + \sum_{i=1}^{\tau} \mathrm{L}(\mathbf{q}_\mathrm{i}) \cdot \mathrm{Circ}(\mathbf{s}_\mathrm{i}^\mathrm{T}) \\ = &\begin{pmatrix} t_0 & t_{-1} & \cdots & t_{2-n} & t_{1-n} \\ t_1 & t_0 - t_1 & \ddots & \ddots & t_{2-n} \\ t_2 & t_1 - t_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} - t_{n-1} & \cdots & t_1 - t_2 & t_0 - t_1 \end{pmatrix}_{n\times n} = T_{CUML}. \end{aligned}$$

Which completes the proof. □

The main result of this section is as follows.

**Theorem 3.** *Let $\nabla_{1,-1}(\cdot)$ be the linear operator in $\mathbf{C}^{n\times n}$ defined by (2.1). Then the following statements hold:*
*(i) The equality $\nabla_{1,-1}(A) = 0$ holds if and only if $A$ is a row first-minus-last right circulant matrix.*
*(ii) If the equation*

$$\nabla_{1,-1}(X) = \sum_{i=1}^{\tau} \mathbf{q}_i \cdot \mathbf{s}_i^T, \tag{3.14}$$

*where* $\mathbf{q}_i$, $\mathbf{s}_i^T$ $(i = 1, 2, \cdots, \tau)$ *are given vectors, is solvable with respect to* $X \in \mathbf{C}^{n\times n}$, *then*

$$\sum_{i=1}^{\tau} \mathrm{RFMLRcircfc}(\mathbf{q}_i) \cdot \mathrm{RFMLRcircfr}(\mathbf{s}_i^T) = 0. \tag{3.15}$$

*(iii) If* $2\tau$ *vectors* $\mathbf{q}_i$ *and* $\mathbf{s}_i^T$ $(i = 1, 2, \cdots, \tau)$ *satisfy the condition (3.15), then the equation (3.14) has the solution*

$$X = \mathrm{RFMLRcircfc}(X) + \frac{1}{2}\sum_{i=1}^{\tau} \mathrm{RSFPLRcircfc}(\mathbf{q}_i) \cdot \mathrm{RFMLRcircfr}(\mathbf{s}_i^T), \tag{3.16}$$

*where* RFMLRcircfc($X$) *is the row first-minus-last right circulant with the same first column as that of* $X$.
*(iv) Under the conditions of (iii), the solution* $X$ *of the equation (3.14) may also be written as*

$$X = \mathrm{RFMLRcircfr}(X) + \frac{1}{2}\sum_{i=1}^{\tau} \mathrm{RFMLRcircfc}(\mathbf{q}_i) \cdot \mathrm{RSFPLRcircfr}(\mathbf{s}_i^T), \tag{3.17}$$

*where* RFMLRcircfr($X$) *is the row first-minus-last right circulant with the same first row as that of* $X$.

*Proof.* ($i$) Let matrix $A = (a_{ij})_{i,j=0}^{n-1}$ meet the requirement $\nabla_{1,-1}(A) = 0$, i.e., $A = \Phi_{1,-1}A\Phi_{1,-1}^{-1}$. From this equality it follows that

$$\begin{aligned} a_{ij} &= a_{i+1,j+1} && if\ j \neq 0 \\ a_{0j} &= a_{n-j,0} && if\ j \neq 0 \\ a_{i0} &= a_{0,n-i} && if\ i \neq 0 \\ a_{i1} &= a_{i-1,0} + a_{i,0} && if\ i \neq 0. \end{aligned}$$

By definition, these relations say that $A$ is a row first-minus-last right circulant matrix.
($ii$) Let $\nabla_{1,-1}(X) = \sum_{i=1}^{\tau} \mathbf{q}_i \cdot \mathbf{s}_i^T$. Then taking into account (2.1), we have

$$\begin{aligned} 0 &= \sum_{j=0}^{n-1} \Phi_{1,-1}^{j} \cdot (A - \Phi_{1,-1}A\Phi_{1,-1}^{-1}) \cdot (\Phi_{1,-1}^{T}) \\ &= \sum_{j=0}^{n-1}\sum_{i=1}^{\tau} (\Phi_{1,-1}^{j}\mathbf{q}_i) \cdot (\mathbf{s}_i^T(\Phi_{1,-1}^{T})^{j}) \\ &= \sum_{i=1}^{\tau} \mathrm{RFMLRcircfc}(\mathbf{q}_i) \cdot \mathrm{RFMLRcircfr}(\mathbf{s}_i^T). \end{aligned}$$

The last equality follows from the general identity $U \cdot V^T = \sum_{k=0}^{n-1} \mathbf{c}_k \cdot \mathbf{d}_k^T$, where $\mathbf{c}_k$ and $\mathbf{d}_k$ are the $k$-th columns of the matrices $U$ and $V$, respectively. The assertion ($ii$) is proved.

Now we proceed to proving assertion (*iii*). Suppose that vectors $\mathbf{q}_i$, $\mathbf{s}_i$ $(i = 1, 2, \cdots, \tau)$ satisfy the condition (3.15) and we compute the $(1, -1)$-cyclic displacement of the matrix $X$ defined by (3.16), i. e., perform the $(1, -1)$-cyclic displacement transformation on both sides of equation (3.16). It follows:

$$\begin{aligned}\nabla_{1,-1}(X) &= \nabla_{1,-1}(\mathrm{RFMLRcircfc}(X)) + \frac{1}{2}\sum_{i=1}^{\tau} \nabla_{1,-1}(\mathrm{RSFPLRcircfc}(\mathbf{q}_i)) \cdot \mathrm{RFMLRcircfr}(\mathbf{s}_i^T) \\ &+ \frac{1}{2}\sum_{i=1}^{\tau} \mathrm{RFMLRcircfc}(\mathbf{q}_i) \cdot \mathrm{RFMLRcircfr}(\mathbf{s}_i^T) \\ &= \nabla_{1,-1}(\mathrm{RFMLRcircfc}(X)) + \frac{1}{2}\sum_{i=1}^{\tau} \nabla_{1,-1}(\mathrm{RSFPLRcircfc}(\mathbf{q}_i)) \cdot \mathrm{RFMLRcircfr}(\mathbf{s}_i^T). \end{aligned} \tag{3.18}$$

It is easy to see that $(1, -1)$-cyclic displacement for RSFPLRcircfc($\mathbf{r}$) with the first column $\mathbf{r} = [r_0 \ r_1 \ \cdots \ r_{n-1}]^T$, has the following simple form

$$\nabla_{1,-1}(\mathrm{RSFPLRcircfc}(\mathbf{r})) = 2(\mathbf{r} \cdot \mathbf{e}_0^T - \mathbf{e}_0 \cdot \tilde{\mathbf{r}}^T), \tag{3.19}$$

where $\tilde{\mathbf{r}}^T = [r_0 \ r_{n-1} \ r_{n-2} \ \cdots \ r_1]$ is the first row of the RFMLRcircfc($\mathbf{r}$). Calculating in this way the $(1, -1)$-cyclic displacement for each matrix RSFPLRcircfc($\mathbf{q}_i$) on the right hand side of (3.18) and taking into account that $\nabla_{1,-1}(\mathrm{RFMLRcircfc}(X)) = 0$ in view of (*i*), we have

$$\nabla_{1,-1}(X) = \sum_{i=1}^{\tau} \mathbf{q}_i \cdot \mathbf{e}_0^T \cdot \mathrm{RFMLRcircfr}(\mathbf{s}_i^T) - \sum_{i=1}^{\tau} \mathbf{e}_0 \cdot \tilde{\mathbf{q}}_i^T \cdot \mathrm{RFMLRcircfr}(\mathbf{s}_i^T), \tag{3.20}$$

where $\tilde{\mathbf{q}}_i^T$ are the first rows of the matrices RFMLRcircfc($\mathbf{q}_i$) $(i = 1, 2, \cdots, \tau)$. Therefore, in view of (3.15) the sum of the last $\tau$ terms in (3.20) is equal to zero matrix. Furthermore, $\mathbf{e}_0^T \cdot \mathrm{RFMLRcircfr}(\mathbf{s}_i^T) = \mathbf{s}_i^T$ $(i = 1, 2, \cdots, \tau)$, and hence the matrix $X$ defined by (3.16) satisfies the equation (3.14), therefore in view of (3.15) the last row of the matrix $X$ and RFMLRcircfc($X$) coincide. The assertion (*iii*) is now completely proved.

The assertion (*iv*) can be proved with the same arguments. □

The proposition (i) of the Theorem 3 shows every complex matrix $A$ is determined by its $(1, -1)$-cyclic displacement up to a row first-minus-last right circulant matrix. Therefore, an arbitrary complex matrix is uniquely determined by its $(1, -1)$-cyclic displacement and any one of its rows or columns.

## 4. Inversion decomposition

**Theorem 4.** *Suppose that $T_{CUML}$ be an arbitrary CUML Toeplitz matrix. If there exist solutions $\mathbf{c}_i$ and $\hat{\mathbf{d}}_i^T$ $(i = 1, 2)$ for equations (3.6) and (3.7)respectively, then we have*

$$\sum_{i=1}^{2} \mathrm{RFMLRcircfc}(\mathbf{c}_i) \cdot \mathrm{RFMLRcircfr}(\mathbf{d}_i^T) = 0, \tag{4.1}$$

*where* $\mathbf{d}_i^T = \hat{\mathbf{d}}_i^T \cdot \Phi_{1,-1}^{-1}$.

*Proof.* According to the special structure of matrix $T_{CUML}$, it is easy to verify that the $T_{CUML}$ satisfies the following equation

$$T_{CUML}^T = Z\hat{I}_n\Phi_{1,-1} \cdot T_{CUML} \cdot \Phi_{1,-1}^{-1}\hat{I}_n Z^{-1}, \tag{4.2}$$

where $Z = \mathrm{Circ}(0,\ldots,0,1)$, i.e. $Z$ is the cyclic lower shift matrix [27], and $\Phi_{1,-1}$ is given by the equation (2.1). Let $\Sigma = Z\hat{I}_n\Phi_{1,-1}$. Then the equation (4.2) can be changed to the following form

$$T_{CUML}^T = \Sigma T_{CUML}\Sigma^{-1}, \tag{4.3}$$

with

$$\Sigma = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & & \iddots & 1 & -1 \\ \vdots & \iddots & \iddots & \iddots & 0 \\ 0 & 1 & \iddots & \iddots & \vdots \\ 1 & -1 & 0 & \cdots & 0 \end{pmatrix}.$$

For $T_{CUML}$ and the representation (3.5) with a given $\beta \in \mathbf{C}$ of its $(1,-1)$-cyclic displacement, we suppose that there exist solutions $\mathbf{c}_i$ and $\hat{\mathbf{d}}_i^T$ $(i = 1,2)$ for the equations (3.6) and (3.7)respectively, that is,

$$T_{CUML}\mathbf{c}_1 = \mathbf{x}, \;\; T_{CUML}\mathbf{c}_2 = \mathbf{e}_0,$$

and

$$\hat{\mathbf{d}}_1^T T_{CUML} = \mathbf{e}_0^T \cdot \Phi_{1,-1}, \;\; \hat{\mathbf{d}}_2^T T_{CUML} = \mathbf{z}^T \cdot \Phi_{1,-1}.$$

Set, as in equation (2.9),

$$\mathbf{d}_1^T = \hat{\mathbf{d}}_1^T \cdot \Phi_{1,-1}^{-1}, \;\; \mathbf{d}_2^T = \hat{\mathbf{d}}_2^T \cdot \Phi_{1,-1}^{-1}. \tag{4.4}$$

Performing transformations to both equations in (4.4) and taking into account the equation (4.3), and $\mathbf{e}_0^T = \mathbf{e}_0^T Z\hat{I}_n$, $\mathbf{z}^T = -\mathbf{x}^T Z\hat{I}_n$, we can obtain that vectors $\mathbf{d}_1^T$ and $\mathbf{d}_2^T$ are related with the solutions $\mathbf{c}_2 = (c_{2,0} \;\; c_{2,1} \;\; \cdots \;\; c_{2,n-1})^T$ and $\mathbf{c}_1 = (c_{1,0} \;\; c_{1,1} \;\; \cdots \;\; c_{1,n-1})^T$ of the equations $T_{CUML}\mathbf{c}_2 = \mathbf{e}_0$ and $T_{CUML}\mathbf{c}_1 = \mathbf{x}$ in the following form:

$$\mathbf{d}_1^T = \mathbf{e}_0^T \cdot \Phi_{1,-1} \cdot T_{CUML}^{-1}\Phi_{1,-1}^{-1} = \mathbf{c}_2^T Z\hat{I}_n,$$

$$\mathbf{d}_2^T = \mathbf{z}^T \cdot \Phi_{1,-1} \cdot T_{CUML}^{-1}\Phi_{1,-1}^{-1} = -\mathbf{c}_1^T Z\hat{I}_n.$$

These meaning that $\mathbf{d}_1^T = (c_{2,0} \;\; c_{2,n-1} \;\; \cdots \;\; c_{2,1})$ is the first row of the matrix RFMLRcircfc($\mathbf{c}_2$), and $-\mathbf{d}_2^T = (c_{1,0} \;\; c_{1,n-1} \;\; \cdots \;\; c_{1,1})$ is the first row of the matrix RFMLRcircfc($\mathbf{c}_1$).

According to

$$\mathrm{RFMLRcircfr}(\mathbf{c}_1)\mathrm{RFMLRcircfr}(\mathbf{c}_2) = \mathrm{RFMLRcircfr}(\mathbf{c}_2)\mathrm{RFMLRcircfr}(\mathbf{c}_1).$$

We have

$$\mathrm{RFMLRcircfc}(\mathbf{c}_1) = -\mathrm{RFMLRcircfr}(\mathbf{d}_2^T),$$
$$\mathrm{RFMLRcircfc}(\mathbf{c}_2) = \mathrm{RFMLRcircfr}(\mathbf{d}_1^T).$$

□

We will now present the main result of the paper.

**Theorem 5.** *Let $T_{CUML}$ be a CUML Toeplitz matrix with $\nabla_{1,-1}(T_{CUML}) = \mathbf{x}\cdot\mathbf{e}_0^T + \mathbf{e}_0\cdot\mathbf{z}^T$ as in (3.5).*
*(i) If there exist solutions $\mathbf{c}_i$ $(i = 1, 2)$ and $\mathbf{y}_1^T$ for equations (3.6) and (2.10) respectively, then matrix $T_{CUML}$ is nonsingular and $T_{CUML}^{-1}$ can be decomposed as follows*

$$T_{CUML}^{-1} = \mathrm{RFMLRcircfr}(\mathbf{y}_1^T) - \frac{1}{2}\sum_{i=1}^{2}\mathrm{RFMLRcircfc}(\mathbf{c}_i)\cdot\mathrm{RSFPLRcircfr}(\mathbf{d}_i^T), \tag{4.5}$$

*where $\mathbf{d}_i^T = \hat{\mathbf{d}}_i^T\cdot\Phi_{1,-1}^{-1}$ $(i = 1, 2)$, $\hat{\mathbf{d}}_1^T$ and $\hat{\mathbf{d}}_2^T$ are solutions of the equations in (3.7), and* $\mathrm{RFMLRcircfr}(\mathbf{y}_1^T)$ *is a row first-minus-last right circulant matrix with the first row $\mathbf{y}_1^T$.*
*(ii) If there exist solutions $\hat{\mathbf{d}}_i^T$ $(i = 1, 2)$ and $\mathbf{y}_2$ for equations (3.7) and (2.11) respectively, then matrix $T_{CUML}$ is nonsingular and another decomposition form for $T_{CUML}^{-1}$ is as follows*

$$T_{CUML}^{-1} = \mathrm{RFMLRcircfc}(\mathbf{y}_2) - \frac{1}{2}\sum_{i=1}^{2}\mathrm{RSFPLRcircfc}(\mathbf{c}_i)\cdot\mathrm{RFMLRcircfr}(\mathbf{d}_i^T), \tag{4.6}$$

*where* $\mathrm{RFMLRcircfc}(\mathbf{y}_2)$ *is a row first-minus-last right circulant with the first column $\mathbf{y}_2$ , and $\mathbf{c}_1$ and $\mathbf{c}_2$ are the solutions of the equations (3.6).*

*Proof.* By Theorem 1, the solvability of the corresponding equations (3.6) yields the invertibility of $T_{CUML}$ then the equations (3.7) are also solvable.

In the following, let us confirm the Eq (4.5). By Theorem 4, we know that vectors $\mathbf{c}_i$, $\mathbf{d}_i^T = \hat{\mathbf{d}}_i^T\cdot\Phi_{1,-1}^{-1}$ $(i = 1, 2)$ satisfy the condition (4.1), where $\mathbf{c}_i$, $\hat{\mathbf{d}}_i^T$ $(i = 1, 2)$ are the solutions of the equations (3.6) and (3.7) respectively, and computing the $(1, -1)$-cyclic displacement of the matrix on the right hand side of (4.5), denoted by $B$. The matrices $\mathrm{RFMLRcircfc}(\mathbf{c}_i)$ $(i = 1, 2)$ are row first-minus-last right circulants, $\mathrm{RSFPLRcircfr}(\mathbf{d}_i^T)$ are row skew first-plus-last right circulants and therefore are computable. According to the Eq (4.5), we have

$$\begin{aligned}\nabla_{1,-1}(B) &= \nabla_{1,-1}(\mathrm{RFMLRcircfr}(\mathbf{y}_1^T)) - \frac{1}{2}\sum_{i=1}^{2}\nabla_{1,-1}[\mathrm{RFMLRcircfc}(\mathbf{c}_i)\cdot\mathrm{RSFPLRcircfr}(\mathbf{d}_i^T)]\\ &= \nabla_{1,-1}(\mathrm{RFMLRcircfr}(\mathbf{y}_1^T)) - \frac{1}{2}\sum_{i=1}^{2}\mathrm{RFMLRcircfc}(\mathbf{c}_i)\cdot\nabla_{1,-1}[\mathrm{RSFPLRcircfr}(\mathbf{d}_i^T)]. \end{aligned}\tag{4.7}$$

The last identity follows from (2.1) and Theorem 4, as

$$\begin{aligned}&\nabla_{1,-1}[\mathrm{RFMLRcircfc}(\mathbf{c}_i)\cdot\mathrm{RSFPLRcircfr}(\mathbf{d}_i^T)]\\ =&\mathrm{RFMLRcircfc}(\mathbf{c}_i)\cdot\mathrm{RSFPLRcircfr}(\mathbf{d}_i^T)\\ &-\Phi_{1,-1}\mathrm{RFMLRcircfc}(\mathbf{c}_i)\Phi_{1,-1}^{-1}\cdot\Phi_{1,-1}\mathrm{RSFPLRcircfr}(\mathbf{d}_i^T)\Phi_{1,-1}^{-1}\\ =&\mathrm{RFMLRcircfc}(\mathbf{c}_i)[\mathrm{RSFPLRcircfr}(\mathbf{d}_i^T) - \Phi_{1,-1}\mathrm{RSFPLRcircfr}(\mathbf{d}_i^T)\Phi_{1,-1}^{-1}]\\ =&\mathrm{RFMLRcircfc}(\mathbf{c}_i)\cdot\nabla_{1,-1}[\mathrm{RSFPLRcircfr}(\mathbf{d}_i^T)],\quad i = 1, 2.\end{aligned}$$

According to the Eqs (3.19), (4.7) and the fact of $\nabla_{1,-1}(\mathrm{RFMLRcircfr}(\mathbf{y}_1^T)) = 0$ (see (i) of Theorem 3), we can obtain

$$\nabla_{1,-1}(B) = \sum_{i=1}^{2}\mathrm{RFMLRcircfc}(\mathbf{c}_i)\cdot\tilde{\mathbf{d}}_i\cdot\mathbf{e}_0^T - \sum_{i=1}^{2}\mathrm{RFMLRcircfc}(\mathbf{c}_i)\cdot\mathbf{e}_0\cdot\mathbf{d}_i^T, \tag{4.8}$$

where $\tilde{\mathbf{d}}_i$ is the first column of the RFMLRcircfr($\mathbf{d}_i^T$) ($i = 1, 2$). Therefore, in view of Theorem 4 the first two terms on the right of (4.8) are equal to the zero matrix. Furthermore, RFMLRcircfc($\mathbf{c}_i$)$\cdot\mathbf{e}_0 = \mathbf{c}_i$ ($i = 1, 2$), and hence the matrix $B$ satisfies (4.7), $\nabla_{1,-1}(B) = -\sum_{i=1}^{2} \mathbf{c}_i \cdot \mathbf{d}_i^T$, so that by (3.8), $\nabla_{1,-1}(T_{CUML}^{-1}) = \nabla_{1,-1}(B)$, therefore in view of (4.1) the first rows of the matrices $T_{CUML}^{-1}$ and $B$ (or RFMLRcircfr($\mathbf{y}_1^T$)) coincide. Thus $B = T_{CUML}^{-1}$, i.e., we have the desired result of assertion (i).

The proof of assertion (ii) is similar. □

According to (i) and (ii) of Theorem 5 we could further conclude the following.

**Theorem 6.** *Let $T_{CUML}$ be a CUML Toeplitz matrix with $\nabla_{1,-1}(T_{CUML}) = \mathbf{x} \cdot \mathbf{e}_0^T + \mathbf{e}_0 \cdot \mathbf{z}^T$ as in (3.5). If $\beta \in \mathbf{C}$ and there exist solutions $\mathbf{c}_1$ and $\mathbf{c}_2$ for equation (3.6), then matrix $T_{CUML}$ is nonsingular and $T_{CUML}^{-1}$ can be decomposed as follows*

$$\begin{aligned} T_{CUML}^{-1} = &\frac{1}{2}[\mathrm{RSFPLRcircfc}(2\mathbf{e}_0 - \mathbf{c}_1) \cdot \mathrm{RFMLRcircfc}(\mathbf{c}_2) \\ &+ \mathrm{RSFPLRcircfc}(\mathbf{c}_2) \cdot \mathrm{RFMLRcircfc}(\mathbf{c}_1)]. \end{aligned} \tag{4.9}$$

*Proof.* First of all, we note that $T_{CUML}^{-1}$ does exist in view of (i) of Theorem 5, therefore, it suffices to show that the formula (4.9) holds. Let the vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ be the solutions of the equations (3.6). By $T_{CUML}\mathbf{c}_2 = \mathbf{e}_0$, we have RFMLRcircfc($T_{CUML}^{-1}$) = RFMLRcircfc($\mathbf{c}_2$), furthermore, in view of the arguments of Theorem 4, RFMLRcircfc($\mathbf{c}_1$) = −RFMLRcircfr($\mathbf{d}_2^T$), and RFMLRcircfc($\mathbf{c}_2$)= RFMLRcircfr($\mathbf{d}_1^T$). Now, having these expressions and (4.6), we can obtain the desired result. □

Theorem 6 says that if $T_{CUML}$ is a CUML Toeplitz matrix and the equations (3.6) have solutions $\mathbf{c}_1$ and $\mathbf{c}_2$, then these solutions are sufficient for restoring the whole matrix $T_{CUML}^{-1}$.

Another decomposition of the inverse of a CUML Toeplitz matrix is obtained in the following.

**Theorem 7.** *Let $T_{CUML}$ be a CUML Toeplitz matrix of the form* (1.1). *If for some $\gamma \in \mathbf{C}$, the equations*

$$T_{CUML}\mathbf{c}_2 = \mathbf{e}_0 \text{ and } T_{CUML}\mathbf{d} = (\gamma, t_{-n+1}, \cdots, t_{-1})^T \tag{4.10}$$

*are solvable, then $T_{CUML}$ is nonsingular , and $T_{CUML}^{-1}$ can be expressed as*

$$\begin{aligned} T_{CUML}^{-1} = \frac{1}{2}[&\mathrm{RSFPLRcircfc}(\mathbf{e}_0 + \mathbf{d})\mathrm{RFMLRcircfc}(\mathbf{c}_2) + \\ &\mathrm{RSFPLRcircfc}(\mathbf{c}_2)\mathrm{RFMLRcircfc}(\mathbf{e}_0 - \mathbf{d})]. \end{aligned} \tag{4.11}$$

*Proof.* Let $\mathbf{d} \in \mathbf{C}^n$ be a solution of the second equation in (4.10). Then the vector $\mathbf{c}_1 = \mathbf{e}_0 - \mathbf{d}$ solves the first equation in (3.6) with $\beta = t_0 - \gamma$, that is, $T_{CUML}(\mathbf{e}_0 - \mathbf{d}) = (t_0 - \gamma, t_1 - t_{1-n}, \cdots, t_{n-1} - t_{-1})^T$. Thus the assertions of Theorem 7 are straightforward consequences of Theorem 6. □

The special structure of CUML Toeplitz matrices and CUML Hankel matrices show that there is a relationship between them. Thus we also get the inverse decompositions of CUML Hankel matrices.

**Remark 1.** Let $H_{CUML} = (h_{i,j})_{i,j=0}^{n-1}$ be a CUML Hankel matrix defined by the Eq (1.3). Then there exists an $n \times n$ CUML Toeplitz matrix $T_{CUML}$ such that $T_{CUML} = H_{CUML}\hat{I}_n$, and $H_{CUML}$ is nonsingular if and only if $T_{CUML}$ is. In that case the inverse of matrix $H_{CUML}$ is $H_{CUML}^{-1} = \hat{I}_n T_{CUML}^{-1}$, and Theorem 5, 6 and 7 are applicable to describe the formulas on representation of the inverse of $H_{CUML}$.

## 5. Conclusions

In this paper, we mainly obtained the inverse formula for CUML Toeplitz matrices by constructing the corresponding displacement of the matrices. By the relationship between CUML Toeplitz matrices and CUML Hankel matrices, the inverse formula for CUML Hankel matrices is also given. These obtained results can be used to study queuing theory model based on Markov process.

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## Conflict of interest

The authors declare that they have no conflict of interest.

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