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Research article

Properties for fourth order discontinuous differential operators with eigenparameter dependent boundary conditions

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Abstract: In this paper, a class of fourth order differential operators with eigenparameter-dependent boundary conditions and transmission conditions is considered. A new operator associated with the problem is established, and the self-adjointness of this operator in an appropriate Hilbert space H is proved. The fundamental solutions are constructed. Sufficient and necessary conditions of the eigenvalues are investigated. Then asymptotic formulas for the fundamental solutions and the characteristic functions are given. Finally, the completeness of eigenfunctions in H is given and the Green function is also involved.

Keywords: transmission conditions; eigenparameter-dependent boundary conditions; completeness; Green function
Mathematics Subject Classification: 34B05, 34L30, 47E05

1. Introduction

In recent years, more and more researchers are interested in the discontinuous differential operators for its wide application in physics and engineering (see [1–22]). Such problems are connected with discontinuous material properties, such as heat and mass transfer which can be found in [10], vibrating string problems when the string loaded additionally with point masses, the heat transfer problems of the laminated plate of membrane (that is, the plate which is formed by overlap of materials with different characteristics) and diffraction problems, etc. Lots of important results in this field have been obtained for the case when the eigenparameter appears not only in the differential equation but also in the boundary conditions. Particularly, more and more researchers have paid close attention to Sturm-Liouville problems with the boundary condition depending on eigenparameter and its inverse problem, asymptotic of eigenvalues and eigenfunctions, oscillation theory, etc. The various physics applications of this kind of problem are found in many literature, such as Hinton [17], Fulton [16], Binding [22]. While the general theory and methods of such second-order boundary value problems are highly developed (see [6–17], [21–30]), little is known about a general characteristic of the high-order problems, and the case of fourth-order is also very little [31, 32].

Yang and Wang studied a class of fourth order differential operators with transmission conditions and containing eigenparameter in the boundary conditions at one endpoint (see [19]), and obtained Green function, asymptotic formulas of eigenvalues and the completeness of eigenfunctions. In [20], Erdoğan Şen studied spectral properties of a fourth order differential operators with transmission conditions and containing eigenparameter in the boundary conditions at one endpoint, the completeness of eigenfunctions and the asymptotic formulas of eigenvalues and fundamental solutions are discussed.

In this paper, we shall consider the following fourth-order boundary value transmission problems

$$lu := (p(x)u''(x))'' + q(x)u(x) = \lambda\omega(x)u(x), \quad x \in J = [-1,0) \cup (0,1] \tag{1.1}$$

with eigenparameter dependent boundary conditions at endpoints

$$l_1u := \lambda u(-1) - u'''(-1) = 0, \tag{1.2}$$

$$l_2u := \lambda u'(-1) + u''(-1) = 0, \tag{1.3}$$

$$l_3u := \lambda(\gamma_1' u(1) - \gamma_2' u'''(1)) - (\gamma_1 u(1) - \gamma_2 u'''(1)) = 0, \tag{1.4}$$

$$l_4u := \lambda(\gamma_3' u'(1) - \gamma_4' u''(1)) + (\gamma_3 u'(1) - \gamma_4 u''(1)) = 0, \tag{1.5}$$

and transmission conditions at discontinuous point $x = 0$

$$B_u(0+) = B \cdot B_u(0-), \tag{1.6}$$

where $p(x) = p_1^4$ for $x \in [-1,0)$, $p(x) = p_2^4$ for $x \in (0,1]$; $\omega(x) = \omega_1^4$ for $x \in [-1,0)$, $\omega(x) = \omega_2^4$ for $x \in (0,1]$, $p_i > 0$ and $\omega_i > 0$ are given real numbers $(i = 1,2)$. The real-valued function $q(x) \in L^1[J,\mathbb{R}]$, $\lambda \in \mathbb{C}$ is a complex eigenparameter, $\gamma_i, \gamma_i', (i = 1,2,3,4)$ are real numbers,$B_u(x) = (u(x), u'(x), u''(x), u'''(x))^T$.

$$B = \begin{pmatrix} \delta_1 & \alpha_1 & 0 & \alpha_2 \\ \delta_2 & \alpha_3 & \alpha_4 & 0 \\ \delta_3 & \beta_1 & \beta_2 & 0 \\ \delta_4 & \beta_3 & 0 & \beta_4 \end{pmatrix}$$

is a 4×4 real matrix,

We assume that $\rho_0 = \begin{vmatrix} \delta_1 & \alpha_2 \\ \delta_4 & \beta_4 \end{vmatrix} = \begin{vmatrix} \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 \end{vmatrix} > 0$, $\rho_1 = \begin{vmatrix} \gamma_1' & \gamma_1 \\ \gamma_2' & \gamma_2 \end{vmatrix} > 0$, $\rho_2 = \begin{vmatrix} \gamma_3' & \gamma_3 \\ \gamma_4' & \gamma_4 \end{vmatrix} > 0$, $\rho_3 = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_3 & \beta_4 \end{vmatrix} = \begin{vmatrix} \delta_2 & \alpha_4 \\ \delta_3 & \beta_2 \end{vmatrix} = 0$, $\rho_4 = \begin{vmatrix} \delta_1 & \alpha_1 \\ \delta_4 & \beta_3 \end{vmatrix} = \begin{vmatrix} \delta_2 & \alpha_3 \\ \delta_3 & \beta_1 \end{vmatrix} > 0$.

In order to investigate the problems (1.1)–(1.6), we define the inner product in $L^2(J)$ as

$$\langle f, g \rangle_1 = \frac{\rho_0 \omega_1^4}{p_1^4} \int_a^c f\overline{g}dx + \frac{\omega_2^4}{p_2^4} \int_c^b f\overline{g}dx, \ \forall f, g \in L^2(J),$$

where

$$f(x) = \begin{cases} f_1(x), \ x \in [-1,0), \\ f_2(x), \ x \in (0,-1]. \end{cases}$$

It is easy to verify that $H_1 = L^2(J, \langle\cdot,\cdot\rangle_1)$ is a Hilbert space.

Here we consider a class of fourth order differential operators with discontinuous coefficient and containing eigenparameter in the boundary conditions at two endpoints. By using the classical analysis techniques and spectral theory of linear operator, a new linear operator A associated with the problem in an appropriate Hilbert space H is defined such that the eigenvalues of the problem coincide with those of A. The main results of the present paper are to discuss its eigenvalues, obtain asymptotic formulas for fundamental solutions and characteristic function, prove that the eigenfunctions of A are complete in H, and give its Green function, which promote and deepen the previous conclusions.

The rest of this paper is organized as follows: In Section 2, we define a new self-adjoint operator A such that the eigenvalues of such a problem coincide with those of A. In Section 3, we construct its fundamental solutions, discuss some properties of eigenvalues. In Section 4, we get the asymptotic formulas for the fundamental solutions and the characteristic function. The completeness of eigenfunctions are discussed in Section 5. In Section 6, we constructed its Green function.

2. Operator formulation

In this section, we introduce a special inner product in the Hilbert space $H = H_1 \bigoplus \mathbb{C}^4$, where $H_1 = L^2[-1,0) \bigoplus L^2(0,-1]$ (for any interval $I \subset \mathbb{R}$, $L^2(I)$ denotes all the complex valued functions which satisfy $\int_I |f(x)|^2 dx < \infty$), $\mathbb{C}^4$ denotes the Hilbert space of complex numbers, a symmetric operator A defined on the Hilbert space such that (1.1)–(1.6) can be considered as the eigenvalue problem of this operator. Namely, we define an inner product on H by

$$\langle F, G\rangle = \langle f, g\rangle_1 + \rho_0(\langle h_1, k_1\rangle_2 + \langle h_2, k_2\rangle_2) + \frac{1}{\rho_1}\langle h_3, k_3\rangle_2 + \frac{1}{\rho_2}\langle h_4, k_4\rangle_2$$

for $F := (f(x), h_1, h_2, h_3, h_4),\ G := (g(x), k_1, k_2, k_3, k_4) \in H$, where $\langle h, k\rangle_2 = h\overline{k}$ for $h, k \in \mathbb{C}$.

For convenience, we shall use the following notations:

$$M_1(f) = f'''(-1),\ M_1'(f) = f(-1),$$

$$M_2(f) = f''(-1),\ M_2'(f) = f'(-1),$$

$$N_1(f) = \gamma_1 f(1) - \gamma_2 f'''(1),\ N_1'(f) = \gamma_1' f(1) - \gamma_2' f'''(1),$$

$$N_2(f) = \gamma_3 f'(1) - \gamma_4 f''(1),\ N_2'(f) = \gamma_3' f'(1) - \gamma_4' f''(1).$$

In the Hilbert space H, we consider the operator A with domain

$$\begin{aligned} D(A) = \{&F = (f(x), h_1, h_2, h_3, h_4) \in H |\ f_i(x), f_i'(x) f_i''(x), f_i'''(x) (i = 1, 2) \text{ are absolutely continuous on} \\ &[-1, 0) \cup (0, 1],\ lf \in H_1, B_f(0+) = B \cdot B_f(0-),\ h_1 = M_1'(f),\ h_2 = M_2'(f),\ h_3 = N_1'(f),\ h_4 = N_2'(f)\} \end{aligned}$$

and the rule

$$AF = \left(\frac{1}{\omega(x)}((p(x)f'')'' + q(x)f), M_1(f), -M_2(f), N_1(f), -N_2(f)\right) \tag{2.1}$$

with

$$F = (f(x), M_1'(f), M_2'(f), N_1'(f), N_2'(f)) \in D(A).$$

Now we rewrite the problems (1.1)–(1.6) in the operator form

$$AF = \lambda F.$$

Thus the problems (1.1)–(1.6) can be considered as the eigenvalue problem of the operator A.

Lemma 2.1. *The eigenvalues and the eigenfunctions of the problems* (1.1)–(1.6) *are defined as the eigenvalues and the first component of corresponding eigenelements of the operator A respectively.*

Lemma 2.2. *The domain $D(A)$ is dense in H.*

Proof. Let $F = (f(x), h_1, h_2, h_3, h_4) \in H$, $F \perp D(A)$ and $\mathbb{C}_0^\infty$ be a functional set such that

$$\varphi(x) = \begin{cases} \varphi_1(x), x \in [-1, 0), \\ \varphi_2(x), x \in (0, 1], \end{cases}$$

for $\varphi_1(x) \in \mathbb{C}_0^\infty[-1, 0)$, $\varphi_2(x) \in \mathbb{C}_0^\infty(0, 1]$. Since $\mathbb{C}_0^\infty \oplus 0 \oplus 0 \oplus 0 \oplus 0 \subset D(A)$ $(0 \in \mathbb{C})$, any $U = (u(x), 0, 0, 0, 0) \in \mathbb{C}_0^\infty \oplus 0 \oplus 0 \oplus 0 \oplus 0$ is orthogonal to F, namely,

$$\langle F, U\rangle = \langle f, u\rangle_1 = 0.$$

We can learn that $f(x)$ is orthogonal to $\mathbb{C}_0^\infty$ in H_1, this implies $f(x) = 0$. So for all $V = (v(x), M_1'(v), 0, 0, 0) \in D(A)$, $\langle F, V\rangle = \rho_0 h_1 M_1'(\overline{v}) = 0$. Thus $h_1 = 0$ since $M_1'(v)$ can be chosen as an arbitrary function. Similarly, we can prove $h_2 = h_3 = h_4 = 0$. Hence $F = (0, 0, 0, 0, 0,)$ is null element in the Hilbert space H. Thus, the orthogonal complement of $D(A)$ consists of only the null element, and therefore is dense in the Hilbert space H. □

Theorem 2.1. *The operator A is self-adjoint in H.*

Proof. Let $F, G \in D(A)$. Integration by parts yields

$$\begin{aligned} \langle AF, G\rangle - \langle F, AG\rangle =& \rho_0[W(f, \overline{g}; 0-) - W(f, \overline{g}; -1)] + [W(f, \overline{g}; 1) - W(f, \overline{g}; 0+)] \\ &+ \rho_0[M_1(f)M_1'(\overline{g}) - M_1'(f)M_1(\overline{g}) - M_2(f)M_2'(\overline{g}) + M_2'(f)M_2(\overline{g})] \\ &+ \frac{1}{\rho_1}[N_1(f)N_1'(\overline{g}) - N_1'(f)N_1(\overline{g})] - \frac{1}{\rho_2}[N_2(f)N_2'(\overline{g}) - N_2'(f)N_2(\overline{g})], \end{aligned} \tag{2.2}$$

where, as usual, $W(f, g; x)$ denotes the Wronskians of f and g:

$$W(f, g; x) = f'''(x)g(x) - f(x)g'''(x) + f'(x)g''(x) - f''(x)g'(x). \tag{2.3}$$

By the transmission condition (1.6), we get

$$W(f, \overline{g}; 0+) = \rho_0 W(f, \overline{g}; 0-). \tag{2.4}$$

Further, it is easy to verify that

$$M_1(f)M_1'(\overline{g}) - M_1'(f)M_1(\overline{g}) - M_2(f)M_2'(\overline{g}) + M_2'(f)M_2(\overline{g}) = W(f, \overline{g}; -1), \tag{2.5}$$

$$\frac{1}{\rho_1}[N_1(f)N_1'(\overline{g}) - N_1'(f)N_1(\overline{g})] - \frac{1}{\rho_2}[N_2(f)N_2'(\overline{g}) - N_2'(f)N_2(\overline{g})] = -W(f, \overline{g}; 1). \tag{2.6}$$

Now, substituting (2.4)–(2.6) into (2.2) yields that

$$\langle AF, G\rangle = \langle F, AG\rangle \ (F, G \in D(A)).$$

Hence A is symmetric.

It remains to show that if $\langle AF, W\rangle = \langle F, U\rangle$ for all $F = (f(x), M_1'(f), M_1'(f), N_1'(f), N_2'(f)) \in D(A)$, then $W \in D(A)$ and $AW = U$, where $W = (w(x), h_1, h_2, h_3, h_4)$, $U = (u(x), k_1, k_2, k_3, k_4)$, i.e.,
(i) $w_1^{(i)}(x) \in AC_{loc}((-1, 0))$, $w_2^{(i)}(x) \in AC_{loc}((0, 1))$ $(i = 0, 1, 2, 3)$, $lw \in H_1$;
(ii) $h_1 = M_1'(w) = w(-1)$, $h_2 = M_2'(w) = w'(-1)$, $h_3 = N_1'(w) = \gamma_1' w(1) - \gamma_2' w'''(1)$, $h_4 = N_2'(w) = \gamma_3' w'(1) - \gamma_4' w''(1)$;
(iii) $B_w(0+) = B \cdot B_w(0-)$;
(iv) $u(x) = lw$;
(v) $k_1 = M_1(w) = w'''(-1)$, $k_2 = -M_2(w) = -w''(-1)$, $k_3 = N_1(w) = \gamma_1 w(1) - \gamma_2 w'''(1)$, $k_4 = -N_2'(w) = -(\gamma_3 w'(1) - \gamma_4 w''(1))$.

For an arbitrary point $F \in \mathbb{C}_0^\infty \oplus 0^4 \in D(A)$ such that

$$\frac{\rho_0\omega_1^4}{p_1^4}\int_{-1}^{0}(lf)\overline{w}dx + \frac{\omega_2^4}{p_2^4}\int_0^1 (lf)\overline{w}dx = \frac{\rho_0\omega_1^4}{p_1^4}\int_{-1}^{0} f\overline{u}dx + \frac{\omega_2^4}{p_2^4}\int_0^1 f\overline{u}dx,$$

that is, $\langle lf, w\rangle_1 = \langle f, u\rangle_1$. According to classical Sturm-Liouville theory, (i) and (iv) hold. By (iv), equation $\langle AF, W\rangle = \langle F, U\rangle$, $\forall F \in D(A)$, becomes

$$\begin{aligned}\langle lf, w\rangle_1 =& \langle f, lw\rangle_1 + \rho_0(M_1'(f)\overline{k}_1 + M_2'(f)\overline{k}_2 - M_1(f)\overline{h}_1 + M_2(f)\overline{h}_2)\\ &+ \frac{1}{\rho_1}(N_1'(f)\overline{k}_3 - N_1(f)\overline{h}_3 + (\frac{1}{\rho_2}(N_2'(f)\overline{k}_4 - N_2(f)\overline{h}_4).\end{aligned}$$

However,

$$\langle lf, w\rangle_1 = \langle f, lw\rangle_1 + \rho_0[W(f, \overline{g}; 0-) - W(f, \overline{g}; -1)] + [W(f, \overline{g}; 1) - W(f, \overline{g}; 0+)].$$

So

$$\begin{aligned}&\rho_0(M_1'(f)\overline{k}_1 + M_2'(f)\overline{k}_2 - M_1(f)\overline{h}_1 + M_2(f)\overline{h}_2) + \frac{1}{\rho_1}(N_1'(f)\overline{k}_3 - N_1(f)\overline{h}_3 + (\frac{1}{\rho_2}(N_2'(f)\overline{k}_4 - N_2(f)\overline{h}_4)\\ &= \rho_0[W(f, \overline{g}; 0-) - W(f, \overline{g}; -1)] + [W(f, \overline{g}; 1) - W(f, \overline{g}; 0+)].\end{aligned} \tag{2.7}$$

By Naimark Patching Lemma 2, there is an $F \in D(A)$ such that

$$f^{(i-1)}(-1) = f^{(i-1)}(0-) = f^{(i-1)}(0+) = 0, \ i = 1, 2, 3, 4,$$

$$f(1) = \gamma_2', \ f'(1) = f''(1) = 0, \ f'''(1) = \gamma_1'.$$

For such an F,

$$M_1'(f) = M_2'(f) = M_1(f) = M_2(f) = 0.$$

$$W(f, \overline{g}; 0-) = W(f, \overline{g}; -1) = W(f, \overline{g}; 0+) = 0.$$

Then from (2.7) we have

$$\frac{1}{\rho_1}(N_1'(f)\overline{k}_3 - N_1(f)\overline{h}_3 + (\frac{1}{\rho_2}(N_2'(f)\overline{k}_4 - N_2(f)\overline{h}_4) = W(f,\overline{g};1).$$

On the one hand,

$$W(f,\overline{g};1) = f'''(1)\overline{w}(1) - f(1)\overline{w}'''(1) + f'(1)\overline{w}''(1) - f''(1)\overline{w}'(1) = \gamma_1'\overline{w}(1) - \gamma_2'\overline{w}'''(1) = N_1'(\overline{w}).$$

On the other hand,

$$\frac{1}{\rho_1}(N_1'(f)\overline{k}_3 - N_1(f)\overline{h}_3 + (\frac{1}{\rho_2}(N_2'(f)\overline{k}_4 - N_2(f)\overline{h}_4) = -\frac{1}{\rho_1}(\overline{h}_3(\gamma_1\gamma_2' - \gamma_2\gamma_1') = \overline{h}_3.$$

So $h_3 = N_1'(w)$. Similarly, we can prove that $h_4 = N_2'(w),\ k_3 = N_1(w),\ k_4 = -N_2(w)$.

For an arbitrary $F \in D(A)$ such that

$$f^{(i-1)}(1) = f^{(i-1)}(0-) = f^{(i-1)}(0+) = 0,\ i = 1,2,3,4,$$
$$f(-1) = f'(-1) = f'''(-1) = 0,\ f''(-1) = 1.$$

For such an F,

$$N_1'(f) = N_2'(f) = N_1(f) = N_2(f) = 0,$$
$$W(f,\overline{g};0-) = W(f,\overline{g};1) = W(f,\overline{g};0+) = 0.$$

Then from (2.7) we have

$$M_1'(f)\overline{k}_1 + M_2'(f)\overline{k}_2 - M_1(f)\overline{h}_1 + M_2(f)\overline{h}_2 = -W(f,\overline{g};-1).$$

On the one hand,

$$M_1'(f)\overline{k}_1 + M_2'(f)\overline{k}_2 - M_1(f)\overline{h}_1 + M_2(f)\overline{h}_2 = \overline{h}_2.$$

On the other hand,

$$-W(f,\overline{g};-1) = f'''(-1)\overline{w}(-1) - f(-1)\overline{w}'''(-1) + f'(-1)\overline{w}''(-1) - f''(-1)\overline{w}'(-1) = \overline{w}'(-1) = M_2'(\overline{w}).$$

So $h_2 = M_2'(\overline{w})$. Similarly, we can proof $h_1 = M_1'(w),\ k_1 = M_1(w),\ k_2 = -M_2(w)$. So (ii) and (v) hold.

Next choose $F \in D(A)$ such that

$$f^{(i-1)}(1) = f^{(i-1)}(-1) = 0,\ i = 1,2,3,4,$$
$$f(0+) = f'(0+) = f''(0+) = 0,\ f'''(0+) = \rho_0,$$
$$f(0-) = -\alpha_2,\ f'(0-) = 0,\ f''(0-) = -\alpha_1,\ f'''(0-) = \delta_1,$$

thus $M_i(f) = M_i'(f) = N_i(f) = N_i'(f) = 0$ $(i = 1,2)$, $W(f,\overline{g};-1) = W(f,\overline{g};1) = 0$. Then from (3.1) we have $W(f,\overline{g};0+) = \rho_0 W(f,\overline{g};0-)$, that is

$$\rho_0 w(0+) = \rho_0(\delta_1 w(0-) + \alpha_1 w(0-) + \alpha_2 w(0-)),$$

so

$$w(0+) = \delta_1 w(0-) + \alpha_1 w(0-) + \alpha_2 w(0-).$$

However, B is a 4×4 real matrix, then using the same method, we can prove

$$B_w(0+) = B \cdot B_w(0-).$$

So (iii) holds.

From the above discussion, we get that A is a self-adjoint operator. □

Corollary 2.1. *All eigenvalues of the problems* (1.1)–(1.6) *are real.*

Corollary 2.2. *Let λ_1 and λ_2 be two different eigenvalues of the problems (1)–(6). Then the corresponding eigenfunctions f and g is orthogonal in the sense that*

$$\frac{\rho_0\omega_1^4}{p_1^4}\int_{-1}^{0} f\overline{g}dx + \frac{\omega_2^4}{p_2^4}\int_0^1 f\overline{g}dx + \rho_0(M_1'(f)M_1'(g) + M_2'(f)M_2'(g)) + \frac{1}{\rho_1}N_1'(f)N_1'(g) + \frac{1}{\rho_2}N_1'(f)N_1'(g)) = 0.$$

Since all eigenvalues are real, it is necessary to study the real-valued eigenfunctions only. Therefore, we can now assume that all eigenfunctions are real-valued.

3. Fundamental solutions

Lemma 3.1. *Let the real-valued function $q(x)$ be continuous in $[-1, 1]$ and $f_i(\lambda)$ $(i = 1, 2, 3, 4)$ be given entire functions. Then for $\forall\lambda \in \mathbb{C}$, the Eq* (1.1) *has a unique solution $u(x, \lambda)$, satisfying the initial conditions*

$$u(-1) = f_1(\lambda),\ u'(-1) = f_2(\lambda),\ u''(-1) = f_3(\lambda),\ u'''(-1) = f_4(\lambda),$$
$$(\text{ or } u(1) = f_1(\lambda),\ u'(1) = f_2(\lambda),\ u''(1) = f_3(\lambda),\ u'''(1) = f_4(\lambda)).$$

Proof. In terms of existence and uniqueness in ordinary differential equation theory, we can conclude this conclusion. □

Let $\phi_{11}(x, \lambda)$ be the solution of Eq (1.1) on the interval $[-1, 0)$, satisfying the initial conditions

$$\phi_{11}(-1) = 1,\ \phi_{11}'(-1) = \phi_{11}''(-1) = 0,\ \phi_{11}'''(-1) = \lambda.$$

By virtue of Lemma 3.1, after defining this solution we can define the solution $\phi_{12}(x, \lambda)$ of Eq (1.1) on the interval (0,1] by the initial conditions

$$B_{\phi_{12}}(0) = B \cdot B_{\phi_{11}}(0).$$

Again let $\phi_{21}(x, \lambda)$ still be the solution of (1.1) on the interval $[-1, 0)$, satisfying the initial conditions

$$\phi_{21}(-1) = 0,\ \phi_{21}'(-1) = 1,\ \phi_{21}''(-1) = -\lambda,\ \phi_{21}'''(-1) = 0.$$

After defining this solution, we can also define the solution $\phi_{22}(x, \lambda)$ of Eq (1.1) on the interval (0,1] by the initial conditions

$$B_{\phi_{22}}(0) = B \cdot B_{\phi_{21}}(0).$$

Analogously, we shall define the solutions $\chi_{12}(x,\lambda)$ and $\chi_{11}(x,\lambda)$ by the initial conditions

$$\chi_{12}(1)=\lambda\gamma_2'-\gamma_2,\ \chi_{12}'(1)=\chi_{12}''(1)=0,\ \chi_{12}'''(1)=\lambda\gamma_1'-\gamma_1,$$
$$B_{\chi_{11}}(0)=C\cdot B_{\chi_{12}}(0),$$

where

$$C=\begin{pmatrix} \frac{\beta_4}{\rho_0} & 0 & 0 & -\frac{\alpha_2}{\rho_0} \\ 0 & \frac{\beta_2}{\rho_0} & -\frac{\alpha_4}{\rho_0} & 0 \\ -\frac{\delta_2\beta_4}{\alpha_4\rho_0} & -\frac{\beta_1}{\rho_0} & \frac{\alpha_3}{\rho_0} & \frac{\alpha_2\delta_2}{\alpha_4\rho_0} \\ -\frac{\delta_4}{\rho_0} & -\frac{\alpha_1\beta_2}{\alpha_2\rho_0} & \frac{\alpha_1\alpha_4}{\alpha_2\rho_0} & \frac{\delta_1}{\rho_0} \end{pmatrix}$$

is a 4×4 real matrix.

In addition, we shall define the solution $\chi_{22}(x,\lambda)$ and $\chi_{21}(x,\lambda)$, satisfying the initial conditions

$$\chi_{22}(1)=0,\ \chi_{22}'(1)=\lambda\gamma_4'+\gamma_4,\chi_{22}''(1)=\lambda\gamma_3'+\gamma_3,\ \chi_{22}'''(1)=0,$$
$$B_{\chi_{21}}(0)=C\cdot B_{\chi_{22}}(0).$$

Let us consider the Wronskians

$$W_1(\lambda):=\begin{vmatrix} \phi_{11}(x,\lambda) & \phi_{21}(x,\lambda) & \chi_{11}(x,\lambda) & \chi_{21}(x,\lambda) \\ \phi_{11}'(x,\lambda) & \phi_{21}'(x,\lambda) & \chi_{11}'(x,\lambda) & \chi_{21}'(x,\lambda) \\ \phi_{11}''(x,\lambda) & \phi_{21}''(x,\lambda) & \chi_{11}''(x,\lambda) & \chi_{21}''(x,\lambda) \\ \phi_{11}'''(x,\lambda) & \phi_{21}'''(x,\lambda) & \chi_{11}'''(x,\lambda) & \chi_{21}'''(x,\lambda) \end{vmatrix}$$

and

$$W_2(\lambda):=\begin{vmatrix} \phi_{12}(x,\lambda) & \phi_{22}(x,\lambda) & \chi_{12}(x,\lambda) & \chi_{22}(x,\lambda) \\ \phi_{12}'(x,\lambda) & \phi_{22}'(x,\lambda) & \chi_{12}'(x,\lambda) & \chi_{22}'(x,\lambda) \\ \phi_{12}''(x,\lambda) & \phi_{22}''(x,\lambda) & \chi_{12}''(x,\lambda) & \chi_{22}''(x,\lambda) \\ \phi_{12}'''(x,\lambda) & \phi_{22}'''(x,\lambda) & \chi_{12}'''(x,\lambda) & \chi_{22}'''(x,\lambda) \end{vmatrix},$$

which are independent of x and are entire functions. Short calculation gives $W_2(\lambda)=\rho_0^2W_1(\lambda)$. Now we may introduce, in consideration, the characteristic function as $W(\lambda)=W_1(\lambda)$.

Theorem 3.1. *The eigenvalues of the problems* (1.1)–(1.6) *consist of the zeros of the function* $W(\lambda)$*.*

Proof. Assume that $W(\lambda)=0$. Then the functions $\phi_{11}(x,\lambda)$, $\phi_{21}(x,\lambda)$ and $\chi_{11}(x,\lambda),\chi_{21}(x,\lambda)$ are linearly dependent, i.e.,

$$k_1\phi_{11}(x,\lambda)+k_2\phi_{21}(x,\lambda)+k_3\chi_{11}(x,\lambda)+k_4\chi_{21}(x,\lambda)=0$$

for some $k_1\neq 0$ or $k_2\neq 0$ and $k_3\neq 0$ or $k_4\neq 0$. From this, it follows that $k_3\chi_{11}(x,\lambda)+k_4\chi_{21}(x,\lambda)$ satisfies the boundary conditions (1.2) and (1.3). Therefore,

$$u(x)=\begin{cases} k_3\chi_{11}(x,\lambda)+k_4\chi_{21}(x,\lambda), & x\in[-1,0), \\ k_3\chi_{12}(x,\lambda)+k_4\chi_{22}(x,\lambda), & x\in(0,1], \end{cases}$$

is an eigenfunction of the problems (1.1)–(1.6) corresponding to the eigenvalue λ.

Now we let $u(x)$ be any eigenfunction corresponding to eigenvalue λ, but $W(\lambda) \neq 0$. Then the function $u(x)$ may be represented in the form

$$u(x) = \begin{cases} c_1\phi_{11}(x,\lambda) + c_2\phi_{21}(x,\lambda) + c_3\chi_{11}(x,\lambda) + c_4\chi_{21}(x,\lambda), & x \in [-1,0), \\ c_5\phi_{12}(x,\lambda) + c_6\phi_{22}(x,\lambda) + c_7\chi_{12}(x,\lambda) + c_8\chi_{22}(x,\lambda), & x \in (0,1], \end{cases} \tag{3.1}$$

where at least one of the constants c_i $(i = 1, 2, \cdots, 8)$ is not zero. Applying the transmission condition (1.6) and the boundary conditions (1.2)–(1.5) to this representation of $u(x)$, we can get a homogenous system of linear equations of the variables c_i $(i = 1, 2, \cdots, 8)$ and taking into account the initial conditions, it follows that the determinant of this system is

$$\begin{vmatrix} 0 & 0 & l_1\chi_{11} & l_1\chi_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & l_2\chi_{11} & l_2\chi_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_3\phi_{12} & l_3\phi_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & l_4\phi_{12} & l_4\phi_{22} & 0 & 0 \\ -\phi_{12}(0) & -\phi_{22}(0) & -\chi_{12}(0) & -\chi_{22}(0) & \phi_{12}(0) & \phi_{22}(0) & \chi_{12}(0) & \chi_{22}(0) \\ -\phi'_{12}(0) & -\phi'_{22}(0) & -\chi'_{12}(0) & -\chi'_{22}(0) & \phi'_{12}(0) & \phi'_{22}(0) & \chi'_{12}(0) & \chi'_{22}(0) \\ -\phi''_{12}(0) & -\phi''_{22}(0) & -\chi''_{12}(0) & -\chi''_{22}(0) & \phi''_{12}(0) & \phi''_{22}(0) & \chi''_{12}(0) & \chi''_{22}(0) \\ -\phi'''_{12}(0) & -\phi'''_{22}(0) & -\chi'''_{12}(0) & -\chi'''_{22}(0) & \phi'''_{12}(0) & \phi'''_{22}(0) & \chi'''_{12}(0) & \chi'''_{22}(0) \end{vmatrix} = -W_2(\lambda)^3.$$

Therefore, the system has only the trivial solution $c_i = 0 (i = 1, 2, \cdots, 8)$. Thus we get a contradiction, which completes the proof. □

4. Asymptotic formulae for the fundamental solutions

In this section, we start by proving several lemmas.

Lemma 4.1. *Let $\lambda = s^4$, $s = \sigma + it$. Then the following integral equations hold for $k = 0, 1, 2, 3$,*

$$\begin{aligned} \frac{d^k}{dx^k}\phi_{11}(x,\lambda) =& \frac{1}{2}\frac{d^k}{dx^k}\cos\frac{\omega_1 s(x+1)}{p_1} - \frac{1}{2}\frac{p_1^3 s}{\omega_1^3}\frac{d^k}{dx^k}\sin\frac{\omega_1 s(x+1)}{p_1} \\ &+ \left(\frac{1}{4} + \frac{1}{4}\frac{p_1^3 s}{\omega_1^3}\right)\frac{d^k}{dx^k}e^{\frac{\omega_1 s(x+1)}{p_1}} - \left(\frac{1}{4}\frac{p_1^3 s}{\omega_1^3} - \frac{1}{4}\right)\frac{d^k}{dx^k}e^{-\frac{\omega_1 s(x+1)}{p_1}} \\ &+ \frac{1}{2\omega_1^3 p_1 s^3}\int_{-1}^{x}\frac{d^k}{dx^k}\left(\sin\frac{\omega_1 s(x-y)}{p_1} - \frac{1}{2}e^{\frac{\omega_1 s(x-y)}{p_1}} + \frac{1}{2}e^{-\frac{\omega_1 s(x-y)}{p_1}}\right)q(y)\phi_{11}(y)dy, \end{aligned} \tag{4.1}$$

$$\begin{aligned} \frac{d^k}{dx^k}\phi_{12}(x,\lambda) =& \left(\frac{1}{2}\phi_{12}(0) - \frac{1}{2}\frac{p_2^2}{\omega_2^2 s^2}\phi''_{12}(0)\right)\frac{d^k}{dx^k}\cos\frac{\omega_2 sx}{p_2} + \left(\frac{1}{2}\frac{p_2}{\omega_2 s}\phi'_{12}(0) - \frac{1}{2}\frac{p_2^3}{\omega_2^3 s^3}\phi'''_{12}(0)\right) \\ &\times \frac{d^k}{dx^k}\sin\frac{\omega_2 sx}{p_2} + \left(\frac{1}{4}\phi_{12}(0) + \frac{1}{4}\frac{p_2}{\omega_2 s}\phi'_{12}(0) + \frac{1}{4}\frac{p_2^2}{\omega_2^2 s^2}\phi''_{12}(0) + \frac{1}{4}\frac{p_2^3}{\omega_2^3 s^3}\phi'''_{12}(0)\right) \\ &\times \frac{d^k}{dx^k}e^{\frac{\omega_2 sx}{p_2}} - \left(-\frac{1}{4}\phi_{12}(0) + \frac{1}{4}\frac{p_2}{\omega_2 s}\phi'_{12}(0) - \frac{1}{4}\frac{p_2^2}{\omega_2^2 s^2}\phi''_{12}(0) + \frac{1}{4}\frac{p_2^3}{\omega_2^3 s^3}\phi'''_{12}(0)\right)\frac{d^k}{dx^k}e^{-\frac{\omega_2 sx}{p_2}} \\ &+ \frac{1}{2\omega_2^3 p_2 s^3}\int_{0}^{x}\frac{d^k}{dx^k}\left(\sin\frac{\omega_2 s(x-y)}{p_2} - \frac{1}{2}e^{\frac{\omega_2 s(x-y)}{p_2}} + \frac{1}{2}e^{-\frac{\omega_2 s(x-y)}{p_2}}\right)q(y)\phi_{12}(y)dy, \end{aligned} \tag{4.2}$$

$$\begin{aligned}\frac{d^k}{dx^k}\phi_{21}(x,\lambda)=&\frac{1}{2}\frac{p_1^2s^2}{\omega_1^2}\frac{d^k}{dx^k}\cos\frac{\omega_1 s(x+1)}{p_1}+\frac{1}{2}\frac{p_1}{\omega_1 s}\frac{d^k}{dx^k}\sin\frac{\omega_1 s(x+1)}{p_1}\\&+\left(\frac{1}{4}\frac{p_1}{\omega_1 s}-\frac{1}{4}\frac{p_1^2s^2}{\omega_1^2}\right)\frac{d^k}{dx^k}e^{\frac{\omega_1 s(x+1)}{p_1}}-\left(\frac{1}{4}\frac{p_1}{\omega_1 s}+\frac{1}{4}\frac{p_1^2s^2}{\omega_1^2}\right)\frac{d^k}{dx^k}e^{-\frac{\omega_1 s(x+1)}{p_1}}\\&+\frac{1}{2\omega_1^3p_1s^3}\int_{-1}^{x}\frac{d^k}{dx^k}\left(\sin\frac{\omega_1 s(x-y)}{p_1}-\frac{1}{2}e^{\frac{\omega_1 s(x-y)}{p_1}}+\frac{1}{2}e^{-\frac{\omega_1 s(x-y)}{p_1}}\right)q(y)\phi_{21}(y)dy,\end{aligned}\tag{4.3}$$

$$\begin{aligned}\frac{d^k}{dx^k}\phi_{22}(x,\lambda)=&\left(\frac{1}{2}\phi_{22}(0)-\frac{1}{2}\frac{p_2^2}{\omega_2^2s^2}\phi_{22}''(0)\right)\frac{d^k}{dx^k}\cos\frac{\omega_2 sx}{p_2}+\left(\frac{1}{2}\frac{p_2}{\omega_2 s}\phi_{22}'(0)-\frac{1}{2}\frac{p_2^3}{\omega_2^3s^3}\phi_{22}'''(0)\right)\\&\times\frac{d^k}{dx^k}\sin\frac{\omega_2 sx}{p_2}+\left(\frac{1}{4}\phi_{22}(0)+\frac{1}{4}\frac{p_2}{\omega_2 s}\phi_{22}'(0)+\frac{1}{4}\frac{p_2^2}{\omega_2^2s^2}\phi_{22}''(0)+\frac{1}{4}\frac{p_2^3}{\omega_2^3s^3}\phi_{22}'''(0)\right)\\&\times\frac{d^k}{dx^k}e^{\frac{\omega_2 sx}{p_2}}-\left(-\frac{1}{4}\phi_{12}(0)+\frac{1}{4}\frac{p_2}{\omega_2 s}\phi_{12}'(0)-\frac{1}{4}\frac{p_2^2}{\omega_2^2s^2}\phi_{22}''(0)+\frac{1}{4}\frac{p_2^3}{\omega_2^3s^3}\phi_{22}'''(0)\right)\frac{d^k}{dx^k}e^{-\frac{\omega_2 sx}{p_2}}\\&+\frac{1}{2\omega_2^3p_2s^3}\int_{0}^{x}\frac{d^k}{dx^k}\left(\sin\frac{\omega_2 s(x-y)}{p_2}-\frac{1}{2}e^{\frac{\omega_2 s(x-y)}{p_2}}+\frac{1}{2}e^{-\frac{\omega_2 s(x-y)}{p_2}}\right)q(y)\phi_{22}(y)dy.\end{aligned}\tag{4.4}$$

Proof. Regarding $\phi_{11}(x,\lambda)$ as the solution of the following non-homogeneous Cauchy problem:

$$\begin{cases}(p(x)u'')''(x)+q(x)u(x)=\lambda\omega(x)u(x),\\ \phi_{11}(-1)=1,\ \phi_{11}'(-1)=0,\\ \phi_{11}''(-1)=0,\ \phi_{11}'''(-1)=\lambda.\end{cases}$$

Using the method of constant variation, $\phi_{11}(x,\lambda)$ satisfies

$$\begin{aligned}\frac{d^k}{dx^k}\phi_{11}(x,\lambda)=&\frac{1}{2}\frac{d^k}{dx^k}\cos\frac{\omega_1 s(x+1)}{p_1}-\frac{1}{2}\frac{p_1^3s}{\omega_1^3}\frac{d^k}{dx^k}\sin\frac{\omega_1 s(x+1)}{p_1}\\&+\left(\frac{1}{4}+\frac{1}{4}\frac{p_1^3s}{\omega_1^3}\right)\frac{d^k}{dx^k}e^{\frac{\omega_1 s(x+1)}{p_1}}-\left(\frac{1}{4}\frac{p_1^3s}{\omega_1^3}-\frac{1}{4}\right)\frac{d^k}{dx^k}e^{-\frac{\omega_1 s(x+1)}{p_1}}\\&+\frac{1}{2\omega_1^3p_1s^3}\int_{-1}^{x}\frac{d^k}{dx^k}\left(\sin\frac{\omega_1 s(x-y)}{p_1}-\frac{1}{2}e^{\frac{\omega_1 s(x-y)}{p_1}}+\frac{1}{2}e^{-\frac{\omega_1 s(x-y)}{p_1}}\right)q(y)\phi_{11}(y)dy.\end{aligned}$$

Then differentiating it with respect to x, we have (4.1). The proof for (4.2)–(4.4) are similar. □

Lemma 4.2. *Let $\lambda=s^4$, $s=\sigma+it$. Then the following integral equations hold for $k=0,1,2,3$,*

$$\begin{aligned}\frac{d^k}{dx^k}\phi_{11}(x,\lambda)=&-\frac{1}{2}\frac{p_1^3s}{\omega_1^3}\frac{d^k}{dx^k}\sin\frac{\omega_1 s(x+1)}{p_1}+\frac{1}{4}\frac{p_1^3s}{\omega_1^3}\frac{d^k}{dx^k}\left(e^{\frac{\omega_1 s(x+1)}{p_1}}-e^{-\frac{\omega_1 s(x+1)}{p_1}}\right)\\&+O(|s|^ke^{|s|\frac{\omega_1(x+1)}{p_1}}),\end{aligned}\tag{4.5}$$

$$\frac{d^k}{dx^k}\phi_{12}(x,\lambda) = \frac{\alpha_2}{2}\phi_{11}'''(0)\frac{d^k}{dx^k}\cos\frac{\omega_2 sx}{p_2} + \frac{p_2\alpha_4}{2\omega_2 s}\phi_{11}''(0)\frac{d^k}{dx^k}\sin\frac{\omega_2 sx}{p_2} + \frac{\alpha_2}{4}\phi_{11}'''(0)\frac{d^k}{dx^k}\left(e^{\frac{\omega_2 sx}{p_2}} + e^{-\frac{\omega_2 sx}{p_2}}\right)$$
$$+ \frac{p_2\alpha_4}{4\omega_2 s}\phi_{11}''(c)\frac{d^k}{dx^k}\left(e^{\frac{\omega_2 sx}{p_2}} - e^{-\frac{\omega_2 sx}{p_2}}\right) + O(|s|^{k+1}e^{|s|[\frac{\omega_1 p_2+\omega_2 p_1}{p_1 p_2}]}), \tag{4.6}$$

$$\frac{d^k}{dx^k}\phi_{21}(x,\lambda) = \frac{1}{2}\frac{p_1^2 s^2}{\omega_1^2}\frac{d^k}{dx^k}\cos\frac{\omega_1 s(x+1)}{p_1} - \frac{1}{4}\frac{p_1^2 s^2}{\omega_1^2}\frac{d^k}{dx^k}\left(e^{\frac{\omega_1 s(x+1)}{p_1}} + e^{-\frac{\omega_1 s(x+1)}{p_1}}\right)$$
$$+ O(|s|^{k-1}e^{|s|\frac{\omega_1(x+1)}{p_1}}), \tag{4.7}$$

$$\frac{d^k}{dx^k}\phi_{22}(x,\lambda) = \frac{\alpha_2}{2}\phi_{21}'''(0)\frac{d^k}{dx^k}\cos\frac{\omega_2 sx}{p_2} + \frac{p_2\alpha_4}{2\omega_2 s}\phi_{21}''(0)\frac{d^k}{dx^k}\sin\frac{\omega_2 sx}{p_2} + \frac{\alpha_2}{4}\phi_{21}'''(0)\frac{d^k}{dx^k}\left(e^{\frac{\omega_2 sx}{p_2}} + e^{-\frac{\omega_2 sx}{p_2}}\right)$$
$$+ \frac{p_2\alpha_4}{4\omega_2 s}\phi_{21}''(0)\frac{d^k}{dx^k}\left(e^{\frac{\omega_2 sx}{p_2}} - e^{-\frac{\omega_2 sx}{p_2}}\right) + O(|s|^{k+2}e^{|s|[\frac{\omega_1 p_2+\omega_2 p_1}{p_1 p_2}]}). \tag{4.8}$$

Each of these asymptotic equalities hold uniformly for $x \in J$, as $|\lambda| \to \infty$.

Proof. Let $\phi_{11}(x,\lambda) = |s|e^{|s|\frac{\omega_1(x+1)}{p_1}}F(x,\lambda)$. We can easily get that $F(x,\lambda)$ is bounded. So $\phi_{11}(x,\lambda) = O(|s|e^{|s|\frac{\omega_1(x+1)}{p_1}})$. Substituting it into (4.1) and differentiating it with respect to x for $k = 0, 1, 2, 3$, we obtain (4.5). Next according to transmission condition (1.6), we have

$$\phi_{12}(0) \approx \alpha_2\phi_{11}'''(0),\ \phi_{12}'(0) \approx \alpha_4\phi_{11}''(0),$$
$$\phi_{12}''(0) \approx \beta_2\phi_{11}''(0),\ \phi_{12}'''(0) \approx \beta_4\phi_{11}'''(0),$$

as $|\lambda| \to \infty$. Substituting these asymptotic expressions into (4.2) for $k = 0$, we get

$$\phi_{12}(x,\lambda) = \frac{\alpha_2}{2}\phi_{11}'''(0)\frac{d^k}{dx^k}\cos\frac{\omega_2 sx}{p_2} + \frac{\alpha_2}{4}\phi_{11}'''(0)\frac{d^k}{dx^k}\left(e^{\frac{\omega_2 sx}{p_2}} + e^{-\frac{\omega_2 sx}{p_2}}\right) + \frac{p_2\alpha_4}{2\omega_2 s}\phi_{11}''(0)\frac{d^k}{dx^k}\sin\frac{\omega_2 sx}{p_2}$$
$$+ \frac{p_2\alpha_4}{4\omega_2 s}\phi_{11}''(0)\frac{d^k}{dx^k}\left(e^{\frac{\omega_2 sx}{p_2}} - e^{-\frac{\omega_2 sx}{p_2}}\right) + O(|s|e^{|s|[\frac{\omega_1 p_2+\omega_2 p_1 x}{p_1 p_2}]})$$
$$+ \frac{1}{2\omega_2^3 p_2 s^3}\int_0^x \frac{d^k}{dx^k}\left(\sin\frac{\omega_2 s(x-y)}{p_2} - \frac{1}{2}e^{\frac{\omega_2 s(x-y)}{p_2}} + \frac{1}{2}e^{-\frac{\omega_2 s(x-y)}{p_2}}\right)q(y)\phi_{12}(y)dy. \tag{4.9}$$

Multiplying through by $|s|^{-4}e^{|s|[\frac{\omega_1}{p_1}+\frac{\omega_2 x}{p_2}]}$, and denoting

$$F_{12}(x,\lambda) := O(|s|^{-4}e^{-|s|[\frac{\omega_1}{p_1}+\frac{\omega_2 x}{p_2}]})\phi_{12}(x,\lambda).$$

Denoting $M(\lambda) := max_{x\in[-1,0)}|F_{12}(x,\lambda)|$, from the last formula, short calculation yields $M(\lambda) < M_0$ for some $M_0 > 0$. It follows that $M(\lambda) = O(1)$ as $|\lambda| \to \infty$, so

$$\phi_{12}(x,\lambda) = O(|s|^4 e^{|s|[\frac{\omega_1}{p_1}+\frac{\omega_2 x}{p_2}]}).$$

Substituting this back into the integral on (4.9) yields (4.6) for $k = 0$. The other assertions can be proved similarly. □

Theorem 4.1. *Let $\lambda = s^4$, $s = \sigma + it$. Then the characteristic function $W(\lambda)$ has the following asymptotic representations:*
Case 1. $\gamma_2' \neq 0$, $\gamma_4' \neq 0$,

$$W(\lambda) = -\frac{\alpha_2\alpha_4\gamma_2'\gamma_4'\omega_2^4 s^{20}}{4\rho_0^2 p_2^4}[1 + (e^{\frac{\omega_1 s}{p_1}} + e^{-\frac{\omega_1 s}{p_1}})\cos\frac{\omega_1 s}{p_1}][1 - \frac{1}{2}(e^{\frac{\omega_2 s}{p_2}} + e^{-\frac{\omega_2 s}{p_2}})\cos\frac{\omega_2 s}{p_2}] + O(|s|^{19} e^{2|s|[\frac{\omega_1 p_2 + \omega_2 p_1}{p_1 p_2}]});$$

Case 2. $\gamma_2' \neq 0$, $\gamma_4' = 0$,

$$W(\lambda) = -\frac{\alpha_2\alpha_4\gamma_2'\gamma_3'\omega_2^3 s^{19}}{8\rho_0^2 p_2^3}[1 + (e^{\frac{\omega_1 s}{p_1}} + e^{-\frac{\omega_1 s}{p_1}})\cos\frac{\omega_1 s}{p_1}][(e^{\frac{\omega_2 s}{p_2}} + e^{-\frac{\omega_2 s}{p_2}})\sin\frac{\omega_2 s}{p_2} + (e^{\frac{\omega_2 s}{p_2}} - e^{-\frac{\omega_2 s}{p_2}})\cos\frac{\omega_2 s}{p_2}] + O(|s|^{18} e^{2|s|[\frac{\omega_1 p_2 + \omega_2 p_1}{p_1 p_2}]});$$

Case 3. $\gamma_2' = 0$, $\gamma_4' \neq 0$,

$$W(\lambda) = \frac{\alpha_2\alpha_4\gamma_1'\gamma_4'\omega_2 s^{17}}{8\rho_0^2 p_2}[1 + (e^{\frac{\omega_1 s}{p_1}} + e^{-\frac{\omega_1 s}{p_1}})\cos\frac{\omega_1 s}{p_1}][(e^{\frac{\omega_2 s}{p_2}} + e^{-\frac{\omega_2 s}{p_2}})\sin\frac{\omega_2 s}{p_2} - (e^{\frac{\omega_2 s(b-c)}{p_2}} - e^{-\frac{\omega_2 s}{p_2}})\cos\frac{\omega_2 s}{p_2}] + O(|s|^{16} e^{2|s|[\frac{\omega_1 p_2 + \omega_2 p_1}{p_1 p_2}]});$$

Case 4. $\gamma_2' = 0$, $\gamma_4' = 0$,

$$W(\lambda) = \frac{\alpha_2\alpha_4\gamma_1'\gamma_3' s^{16}}{4\rho_0^2}[1 + (e^{\frac{\omega_1 s}{p_1}} + e^{-\frac{\omega_1 s}{p_1}})\cos\frac{\omega_1 s}{p_1}][1 + \frac{1}{2}(e^{\frac{\omega_2 s}{p_2}} + e^{-\frac{\omega_2 s}{p_2}})\cos\frac{\omega_2 s}{p_2}] + O(|s|^{15} e^{2|s|[\frac{\omega_1 p_2 + \omega_2 p_1}{p_1 p_2}]}).$$

Proof. The proof is obtained by substituting asymptotic equalities $\frac{d^k}{dx^k}\phi_{12}(1,\lambda)$ and $\frac{d^k}{dx^k}\phi_{22}(1,\lambda)$ into the representation

$$W(\lambda) = \frac{1}{\rho_0^2}W_2(\lambda) = \begin{vmatrix} \phi_{12}(1,\lambda) & \phi_{22}(1,\lambda) & \lambda\gamma_2' - \gamma_2 & 0 \\ \phi_{12}'(1,\lambda) & \phi_{22}'(1,\lambda) & 0 & \lambda\gamma_4' + \gamma_4 \\ \phi_{12}''(1,\lambda) & \phi_{22}''(1,\lambda) & 0 & \lambda\gamma_3' + \gamma_3 \\ \phi_{12}'''(1,\lambda) & \phi_{22}'''(1,\lambda) & \lambda\gamma_1' - \gamma_1 & 0 \end{vmatrix},$$

short calculation, we can get the above conclusions. □

Corollary 4.1. *The eigenvalues of the problems (1)–(6) are bounded below.*

Proof. Putting $s^2 = it$ ($t > 0$) in the above formulae, it follows that $W(-t^2) \to \infty$ as $t \to \infty$. Hence $W(\lambda) \neq 0$ for λ negative and sufficiently large in modulus. □

5. Completeness of eigenfunction

Theorem 5.1. *The operator A has only point spectrum, i.e., $\sigma(A) = \sigma_p(A)$.*

Proof. It suffices to prove that if λ is not an eigenvalue of A, then $\lambda \in \rho(A)$. Here we investigate the equation $(A-\lambda)Y = F \in H$, where $\lambda \in \mathbb{R}$, $F = (f(x), h_1, h_2, h_3, h_4)$. Consider the initial-value problem

$$\begin{cases} ly - \lambda\omega(x)y(x) = f(x)\omega(x), \\ B_y(0+) = B \cdot B_y(0-) \end{cases}, \tag{5.1}$$

and the system of equations

$$\begin{cases} M_1(y) - \lambda M_1'(y) = h_1, \\ M_2(y) + \lambda M_2'(y) = -h_2, \\ N_1(y) - \lambda N_1'(y) = h_3, \\ N_2(y) + \lambda N_2'(y) = -h_4. \end{cases} \tag{5.2}$$

Let

$$u(x) = \begin{cases} u_1(x), \ x \in [-1, 0), \\ u_2(x), \ x \in (0, 1], \end{cases}$$

be the solution of the equation $ly - \lambda\omega(x)y(x) = 0$ satisfying the transmission condition (1.6). Let

$$v(x) = \begin{cases} v_1(x), \ x \in [-1, 0), \\ v_2(x), \ x \in (0, 1], \end{cases}$$

be a special solution of (5.1). Then (5.1) has general solution in the form

$$y(x) = \begin{cases} du_1(x) + v_1(x), \ x \in [-1, 0), \\ du_2(x) + v_2(x), \ x \in (0, 1], \end{cases} \tag{5.3}$$

where $d \in \mathbb{C}$.

Since γ is not an eigenvalue of (1.1)–(1.6), we have

$$\lambda u_1(-1) - u_1'''(-1) \neq 0, \tag{5.4}$$

or

$$\lambda u_1'(-1) + u_1''(-1) \neq 0, \tag{5.5}$$

or

$$\lambda(\gamma_1' u_2(1) - \gamma_2' u_2'''(1)) - (\gamma_1 u_2(1) - \gamma_2 u_2'''(1)) \neq 0, \tag{5.6}$$

or

$$\lambda(\gamma_3' u_2'(1) - \gamma_4' u_2''(1)) + (\gamma_3 u_2'(1) - \gamma_4 u_2''(1)) \neq 0. \tag{5.7}$$

The second, third, fourth, and fifth components of the equation $(A-\lambda)Y = F$ involves the Eq (5.2), so substituting (5.3) into (5.2), and we get

$$(u_1'''(-1) - \lambda u_1(-1))d = h_1 + \lambda v_1(-1) - v_1'''(-1),$$

$$(u_1''(-1) + \lambda u_1'(-1))d = -h_2 - \lambda v_1'(-1) - v_1''(-1),$$
$$(\lambda(\gamma_2' u_2'''(1) - \gamma_1' u_2(1)) - (\gamma_2 u_2'''(1) - \gamma_1 u_2(1)))d = h_3 - \lambda(\gamma_2' v_2'''(1) - \gamma_1' v_2(1)) + \gamma_2 v_2'''(1) - \gamma_1 v_2(1),$$
$$(\lambda(\gamma_3' u_2'(1) - \gamma_4' u_2''(1)) + (\gamma_3 u_2'(1) - \gamma_4 u_2''(1)))d = -h_4 - \lambda(\gamma_3' v_2'(1) - \gamma_4' v_2''(1)) + \gamma_4 v_2''(1) - \gamma_3 v_2'(1).$$

In view of (5.4)–(5.7), we know that d is a unique solution. Thus if λ is not an eigenvalue of (1.1)–(1.6), d is uniquely solvable. Hence y is uniquely determined.

The above arguments show that $(A - \lambda I)^{-1}$ is defined on all of H. We get that $(A - \lambda I)^{-1}$ is bounded by Theorem 2.1 and the Closed Graph Theorem. Thus $\lambda \in \rho(A)$. Hence $\sigma(A) = \sigma_p(A)$. □

Lemma 5.1. *The eigenvalues of the boundary value problems* (1.1)–(1.6) *are bounded below, and form a finite or infinite sequence without finite accumulation point.*

Proof. By the Corollary 4.1, we know that the eigenvalues of boundary value problems (1)–(6) are bounded below. By Theorem 3.1, we obtain that the zeros of the entire function $W(\lambda)$ are the eigenvalues of A. And all the eigenvalues of A are real by the self-adjointness of A, that is to say, for any $\lambda \in \mathbb{C}$ with its imaginary part not vanishing, then $W(\lambda) \neq 0$. Therefore, by the distribution of zeros of entire functions, the conclusion holds. □

Lemma 5.2. *The operator A has compact resolvents, i.e., for each $\delta \in \mathbb{R}/\sigma_p(A)$, $(A - \delta I)^{-1}$ is compact on H.*

Proof. Let $\{\lambda_1, \lambda_2, \cdots\}$ be the eigenvalues of $(A-\delta I)^{-1}$, and let $\{P_1, P_2, \cdots\}$ be the finite rank orthogonal projection on the corresponding eigensubspace. Since $\{\lambda_1, \lambda_2, \cdots\}$ is a bounded sequence and P_n are mutual orthogonality, $\sum_{n=1}^{\infty} \lambda_n P_n$ is strong convergence to $(A - \delta I)^{-1}$, that is, $\sum_{n=1}^{\infty} \lambda_n P_n = (A - \delta I)^{-1}$. In light of the number of $|\lambda_n| > \alpha$ for any $\alpha > 0$ is finite and P_n are finite rank, we have that $(A - \delta I)^{-1}$ is compact. □

By the above Lemmas and the spectral theorem for compact operator, we obtain the following theorem:

Theorem 5.2. *The eigenfunctions of the problems* (1.1)–(1.6)*, augmented to become eigenfunctions of A, are complete in H, i.e., if we let $\{\Phi_n = (\phi_n(x), M_1'(\phi_n), M_2'(\phi_n), N_1'(\phi_n), N_2'(\phi_n)); n \in \mathbb{N}\}$ be a maximum set of orthonormal eigenfunctions of A, where $\{\phi_n(x); n \in \mathbb{N}\}$ are eigenfunctions of the problems* (1.1)–(1.6)*, then for all $F \in H$, $F = \sum_{n=1}^{\infty} \langle F, \Phi_n \rangle \Phi_n$.*

6. Green function

In this section, we will find the Green function defined by (1.1)–(1.6). For convenience, we assume that $p(x) \equiv 1,\ \omega(x) \equiv 1$. Let λ not be an eigenvalue of A, we consider the operator equation $(\lambda I - A)U = F$, $F = (f, h_1, h_2, h_3, h_4)$. This operator equation is equivalent to the inhomogeneous differential equation

$$-u^{(4)} + qu - \lambda u = f(x) \tag{6.1}$$

for $x \in J$, subject to the inhomogeneous boundary conditions

$$\lambda u(-1) - u'''(-1) = h_1, \tag{6.2}$$

$$\lambda u'(-1) + u''(-1) = h_2, \tag{6.3}$$

$$\lambda(\gamma_1' u(1) - \gamma_2' u'''(1)) - (\gamma_1 u(1) - \gamma_2 u'''(1)) = h_3, \tag{6.4}$$

$$\lambda(\gamma_3' u'(1) - \gamma_4' u''(1)) + (\gamma_3 u'(1) - \gamma_4 u''(1)) = h_4, \tag{6.5}$$

and transmission condition (1.6).

By applying the standard method of variation of constants, we search the general solution of the non-homogeneous differential equation (6.1) in the form

$$y(x,\lambda) = \begin{cases} c_1(x,\lambda)\phi_{11}(x,\lambda) + c_2(x,\lambda)\phi_{21}(x,\lambda) + c_3(x,\lambda)\chi_{11}(x,\lambda) + c_4(x,\lambda)\chi_{21}(x,\lambda), x \in [-1,0), \\ c_5(x,\lambda)\phi_{12}(x,\lambda) + c_6(x,\lambda)\phi_{22}(x,\lambda) + c_7(x,\lambda)\chi_{12}(x,\lambda) + c_8(x,\lambda)\chi_{22}(x,\lambda), x \in (0,1]. \end{cases} \tag{6.6}$$

By using the same techniques as in [2], the general solution of the non-homogeneous differential equation (6.1) are obtained as

$$y(x,\lambda) = \begin{cases} y_1(x,\lambda), & x \in [-1,0), \\ y_2(x,\lambda), & x \in (0,1], \end{cases} \tag{6.7}$$

where

$$\begin{aligned} y_1(x,\lambda) =& \frac{1}{W_1(\lambda)}(\phi_{11}(x,\lambda)\int_{-1}^{x} f(\xi)\Delta_1(\xi,\lambda)d\xi + \phi_{21}(x,\lambda)\int_{-1}^{x} f(\xi)\Delta_2(\xi,\lambda)d\xi \\ & - \chi_{11}(x,\lambda)\int_{-1}^{x} f(\xi)\Delta_3(\xi,\lambda)d\xi + \chi_{21}(x,\lambda)\int_{-1}^{x} f(\xi)\Delta_4(\xi,\lambda)d\xi) \\ & + c_1\phi_{11}(x,\lambda) + c_2\phi_{21}(x,\lambda) + c_3\chi_{11}(x,\lambda) + c_4\chi_{21}(x,\lambda), \quad x \in [-1,0), \end{aligned} \tag{6.8}$$

$$\begin{aligned} y_2(x,\lambda) =& \frac{1}{W_2(\lambda)}(\phi_{12}(x,\lambda)\int_{-1}^{x} f(\xi)\Delta_5(\xi,\lambda)d\xi + \phi_{22}(x,\lambda)\int_{-1}^{x} f(\xi)\Delta_6(\xi,\lambda)d\xi \\ & - \chi_{12}(x,\lambda)\int_{-1}^{x} f(\xi)\Delta_7(\xi,\lambda)d\xi + \chi_{22}(x,\lambda)\int_{-1}^{x} f(\xi)\Delta_8(\xi,\lambda)d\xi) \\ & + c_5\phi_{12}(x,\lambda) + c_6\phi_{22}(x,\lambda) + c_7\chi_{12}(x,\lambda) + c_8\chi_{22}(x,\lambda), \quad x \in (0,1], \end{aligned} \tag{6.9}$$

$$\Delta_1(\xi,\lambda) = \begin{vmatrix} \phi_{21}(\xi,\lambda) & \chi_{11}(\xi,\lambda) & \chi_{21}(\xi,\lambda) \\ \phi_{21}'(\xi,\lambda) & \chi_{11}'(\xi,\lambda) & \chi_{21}'(\xi,\lambda) \\ \phi_{21}''(\xi,\lambda) & \chi_{11}''(\xi,\lambda) & \chi_{21}''(\xi,\lambda) \end{vmatrix}, \quad \Delta_2(\xi,\lambda) = \begin{vmatrix} \phi_{11}(\xi,\lambda) & \chi_{11}(\xi,\lambda) & \chi_{21}(\xi,\lambda) \\ \phi_{11}'(\xi,\lambda) & \chi_{11}'(\xi,\lambda) & \chi_{21}'(\xi,\lambda) \\ \phi_{11}''(\xi,\lambda) & \chi_{11}''(\xi,\lambda) & \chi_{21}''(\xi,\lambda) \end{vmatrix},$$

$$\Delta_3(\xi,\lambda) = \begin{vmatrix} \phi_{11}(\xi,\lambda) & \phi_{21}(\xi,\lambda) & \chi_{21}(\xi,\lambda) \\ \phi_{11}'(\xi,\lambda) & \phi_{21}'(\xi,\lambda) & \chi_{21}'(\xi,\lambda) \\ \phi_{11}''(\xi,\lambda) & \phi_{21}''(\xi,\lambda) & \chi_{21}''(\xi,\lambda) \end{vmatrix}, \quad \Delta_4(\xi,\lambda) = \begin{vmatrix} \phi_{11}(\xi,\lambda) & \phi_{21}(\xi,\lambda) & \chi_{11}(\xi,\lambda) \\ \phi_{11}'(\xi,\lambda) & \phi_{21}'(\xi,\lambda) & \chi_{11}'(\xi,\lambda) \\ \phi_{11}''(\xi,\lambda) & \phi_{21}''(\xi,\lambda) & \chi_{11}''(\xi,\lambda) \end{vmatrix},$$

,

$$\Delta_5(\xi,\lambda) = \begin{vmatrix} \phi_{22}(\xi,\lambda) & \chi_{12}(\xi,\lambda) & \chi_{22}(\xi,\lambda) \\ \phi_{22}'(\xi,\lambda) & \chi_{12}'(\xi,\lambda) & \chi_{22}'(\xi,\lambda) \\ \phi_{22}''(\xi,\lambda) & \chi_{12}''(\xi,\lambda) & \chi_{22}''(\xi,\lambda) \end{vmatrix}, \quad \Delta_6(\xi,\lambda) = \begin{vmatrix} \phi_{12}(\xi,\lambda) & \chi_{12}(\xi,\lambda) & \chi_{22}(\xi,\lambda) \\ \phi_{12}'(\xi,\lambda) & \chi_{12}'(\xi,\lambda) & \chi_{22}'(\xi,\lambda) \\ \phi_{12}''(\xi,\lambda) & \chi_{12}''(\xi,\lambda) & \chi_{22}''(\xi,\lambda) \end{vmatrix},$$

$$\Delta_7(\xi,\lambda) = \begin{vmatrix} \phi_{12}(\xi,\lambda) & \phi_{22}(\xi,\lambda) & \chi_{22}(\xi,\lambda) \\ \phi_{12}'(\xi,\lambda) & \phi_{22}'(\xi,\lambda) & \chi_{22}'(\xi,\lambda) \\ \phi_{12}''(\xi,\lambda) & \phi_{22}''(\xi,\lambda) & \chi_{22}''(\xi,\lambda) \end{vmatrix}, \quad \Delta_8(\xi,\lambda) = \begin{vmatrix} \phi_{12}(\xi,\lambda) & \phi_{22}(\xi,\lambda) & \chi_{12}(\xi,\lambda) \\ \phi_{12}'(\xi,\lambda) & \phi_{22}'(\xi,\lambda) & \chi_{12}'(\xi,\lambda) \\ \phi_{12}''(\xi,\lambda) & \phi_{22}'(\xi,\lambda) & \chi_{12}''(\xi,\lambda) \end{vmatrix}.$$

$c_1, c_2 \cdots c_8$ are arbitrary constants. Substituting Eqs (6.8) and (6.9) into transmission condition (1.6), we obtain

$$c_5 = \frac{1}{W_1(\lambda)} \det\left(B_{y_1(0,\lambda)}, B_{\phi_{21}(0,\lambda)}, B_{\chi_{11}(0,\lambda)}, B_{\chi_{21}(0,\lambda)}\right),$$

$$c_6 = \frac{1}{W_1(\lambda)} \det\left(B_{\phi_{11}(0,\lambda)}, B_{y_1(0,\lambda)}, B_{\chi_{11}(0,\lambda)}, B_{\chi_{21}(0,\lambda)}\right),$$

$$c_7 = \frac{1}{W_1(\lambda)} \det\left(B_{\phi_{11}(0,\lambda)}, B_{\phi_{21}(0,\lambda)}, B_{y_1(0,\lambda)}, B_{\chi_{21}(0,\lambda)}\right),$$

$$c_8 = \frac{1}{W_1(\lambda)} \det\left(B_{\phi_{11}(0,\lambda)}, B_{\phi_{21}(0,\lambda)}, B_{\chi_{11}(0,\lambda)}, B_{y_1(0,\lambda)}\right).$$

Meanwhile,

$$\begin{aligned} &\left(B_{\phi_{12}(0,\lambda)}, B_{\phi_{22}(0,\lambda)}, B_{\chi_{12}(0,\lambda)}, B_{\chi_{22}(0,\lambda)}\right)(c_5, c_6, c_7, c_8)^T \\ &= B \cdot \left(B_{\phi_{11}(0,\lambda)}, B_{\phi_{21}(0,\lambda)}, B_{\chi_{11}(0,\lambda)}, B_{\chi_{21}(0,\lambda)}\right) \\ &(\frac{1}{w_1(\lambda)} \begin{pmatrix} -\int_{-1}^{0} f(\xi)\Delta_1(\xi,\lambda)d\xi \\ -\int_{-1}^{0} f(\xi)\Delta_2(\xi,\lambda)d\xi \\ -\int_{-1}^{0} f(\xi)\Delta_3(\xi,\lambda)d\xi \\ -\int_{-1}^{0} f(\xi)\Delta_4(\xi,\lambda)d\xi \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}). \end{aligned}$$

By the initial conditions, we get

$$\begin{pmatrix} c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}) = \frac{1}{W_1(\lambda)} \begin{pmatrix} -\int_{-1}^{0} f(\xi)\Delta_1(\xi,\lambda)d\xi \\ -\int_{-1}^{0} f(\xi)\Delta_2(\xi,\lambda)d\xi \\ -\int_{-1}^{0} f(\xi)\Delta_3(\xi,\lambda)d\xi \\ -\int_{-1}^{0} f(\xi)\Delta_4(\xi,\lambda)d\xi \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}. \tag{6.10}$$

So we can rewrite $y_1(x,\lambda)$ in the form

$$y_1(x,\lambda) = \int_{-1}^{1} K_1(x,\xi,\lambda) f(\xi)d\xi + c_1\phi_{11}(x,\lambda) + c_2\phi_{21}(x,\lambda) + c_3\chi_{11}(x,\lambda) + c_4\chi_{21}(x,\lambda),\ x \in [-1,0),$$

where

$$K_1(x,\xi,\lambda) = \begin{cases} \frac{Z_1(x,\xi,\lambda)}{W_1(\lambda)}, & -1 \le \xi \le x \le 0, \\ 0, & -1 \le x \le \xi \le 0, \\ 0, & -1 \le x \le 0, 0 \le \xi \le 1. \end{cases}$$

$$Z_1(x,\xi,\lambda) = \begin{vmatrix} \phi_{11}(\xi,\lambda) & \phi_{21}(\xi,\lambda) & \chi_{11}(\xi,\lambda) & \chi_{21}(\xi,\lambda) \\ \phi'_{11}(\xi,\lambda) & \phi'_{21}(\xi,\lambda) & \chi'_{11}(\xi,\lambda) & \chi'_{21}(\xi,\lambda) \\ \phi''_{11}(\xi,\lambda) & \phi''_{21}(\xi,\lambda) & \chi''_{11}(\xi,\lambda) & \chi''_{21}(\xi,\lambda) \\ \phi_{11}(x,\lambda) & \phi_{21}(x,\lambda) & \chi_{11}(x,\lambda) & \chi_{12}(x,\lambda) \end{vmatrix}.$$

Substituting (6.10) into (6.9), we have

$$\begin{aligned}
y_2(x,\lambda) =& \frac{1}{W_1(\lambda)}\Big(-\int_{-1}^{0}\phi_{12}(x,\lambda)f(\xi)\Delta_1(\xi,\lambda)d\xi + \int_{-1}^{0}\phi_{22}(x,\lambda)f(\xi)\Delta_2(\xi,\lambda)d\xi \\
&- \int_{-1}^{0}\chi_{12}(x,\lambda)f(\xi)\Delta_3(\xi,\lambda)d\xi + \int_{-1}^{0}\chi_{22}(x,\lambda)f(\xi)\Delta_4(\xi,\lambda)d\xi\Big) \\
&+ \frac{1}{W_2(\lambda)}\Big(-\int_{0}^{x}\phi_{12}(x,\lambda)f(\xi)\Delta_5(\xi,\lambda)d\xi + \int_{0}^{x}\phi_{22}(x,\lambda)f(\xi)\Delta_6(\xi,\lambda)d\xi \\
&+ \int_{0}^{x}\chi_{12}(x,\lambda)f(\xi)\Delta_7(\xi,\lambda)d\xi - \int_{0}^{x}\chi_{22}(x,\lambda)f(\xi)\Delta_8(\xi,\lambda)d\xi\Big) \\
&+ c_1\phi_{12}(x,\lambda) + c_2\phi_{22}(x,\lambda) + c_3\chi_{12}(x,\lambda) + c_4\chi_{22}(x,\lambda), \quad x \in (0,1].
\end{aligned}$$

We can rewrite $y_2(x,\lambda)$ in the form

$$y_2(x,\lambda) = \int_{-1}^{1}K_2(x,\xi,\lambda)f(\xi)d\xi + c_1\phi_{12}(x,\lambda) + c_2\phi_{22}(x,\lambda) + c_3\chi_{12}(x,\lambda) + c_4\chi_{22}(x,\lambda),\ x \in (0,1],$$

where

$$K_2(x,\xi,\lambda) = \begin{cases} \frac{Z_2(x,\xi,\lambda)}{W_1(\lambda)}, & -1 \le \xi \le 0, 0 \le x \le 1, \\ \frac{Z_3(x,\xi,\lambda)}{W_2(\lambda)}, & 0 \le \xi \le x \le 1, \\ 0, & 0 \le x \le \xi \le 1. \end{cases}$$

$$Z_2(x,\xi,\lambda) = \begin{vmatrix} \phi_{11}(\xi,\lambda) & \phi_{21}(\xi,\lambda) & \chi_{11}(\xi,\lambda) & \chi_{21}(\xi,\lambda) \\ \phi'_{11}(\xi,\lambda) & \phi'_{21}(\xi,\lambda) & \chi'_{11}(\xi,\lambda) & \chi'_{21}(\xi,\lambda) \\ \phi''_{11}(\xi,\lambda) & \phi''_{21}(\xi,\lambda) & \chi''_{11}(\xi,\lambda) & \chi''_{21}(\xi,\lambda) \\ \phi_{12}(x,\lambda) & \phi_{22}(x,\lambda) & \chi_{12}(x,\lambda) & \chi_{22}(x,\lambda) \end{vmatrix},$$

$$Z_3(x,\xi,\lambda) = \begin{vmatrix} \phi_{12}(\xi,\lambda) & \phi_{22}(\xi,\lambda) & \chi_{12}(\xi,\lambda) & \chi_{22}(\xi,\lambda) \\ \phi'_{12}(\xi,\lambda) & \phi'_{22}(\xi,\lambda) & \chi'_{12}(\xi,\lambda) & \chi'_{22}(\xi,\lambda) \\ \phi''_{12}(\xi,\lambda) & \phi''_{22}(\xi,\lambda) & \chi''_{12}(\xi,\lambda) & \chi''_{22}(\xi,\lambda) \\ \phi_{12}(x,\lambda) & \phi_{22}(x,\lambda) & \chi_{12}(x,\lambda) & \chi_{22}(x,\lambda) \end{vmatrix}.$$

Obviously, the solution for Eq (31) can be represented in the form:

$$y(x,\lambda) = \int_{-1}^{1}K(x,\xi,\lambda)f(\xi)d\xi + c_1\phi_1(x,\lambda) + c_2\phi_2(x,\lambda) + c_3\chi_1(x,\lambda) + c_4\chi_2(x,\lambda),\ x \in J,$$

with

$$K(x,\xi,\lambda) = \begin{cases} K_1(x,\xi,\lambda), & x \in [-1,0), \\ K_2(x,\xi,\lambda), & x \in (0,1], \end{cases}$$

$$\phi_1(x,\lambda) = \begin{cases} \phi_{11}(x,\lambda), & x \in [-1,0), \\ \phi_{12}(x,\lambda), & x \in (0,1], \end{cases} \quad \phi_2(x,\lambda) = \begin{cases} \phi_{21}(x,\lambda), & x \in [-1,0), \\ \phi_{22}(x,\lambda), & x \in (0,1], \end{cases}$$

$$\chi_1(x,\lambda) = \begin{cases} \chi_{11}(x,\lambda), & x \in [-1,0), \\ \chi_{12}(x,\lambda), & x \in (0,1], \end{cases} \quad \chi_2(x,\lambda) = \begin{cases} \chi_{21}(x,\lambda), & x \in [-1,0), \\ \chi_{22}(x,\lambda), & x \in (0,1]. \end{cases}$$

Denoting

$$U_1(y) = \lambda y(-1) - y'''(-1) = h_1,$$

$$U_1(y) = \lambda y'(-1) + y''(-1) = h_2,$$

$$U_1(y) = \lambda(\gamma_1' y(1) - \gamma_2' y'''(1)) - (\gamma_1 y(1) - \gamma_2 y'''(1)) = h_3,$$

$$U_1(y) = \lambda(\gamma_3' y'(1) - \gamma_4' y''(1)) + (\gamma_3 y'(1) - \gamma_4 y''(1)) = h_4.$$

Substituting $y(x, \lambda)$ into the above conditions, we have

$$c_1 U_1(\phi_1(x,\lambda)) + c_2 U_1(\phi_2(x,\lambda)) + c_3 U_1(\chi_1(x,\lambda)) + c_4 U_1(\chi_2(x,\lambda)) = -\int_{-1}^{1} U_1(K) f(\xi) d\xi + h_1,$$

$$c_1 U_2(\phi_1(x,\lambda)) + c_2 U_2(\phi_2(x,\lambda)) + c_3 U_2(\chi_1(x,\lambda)) + c_4 U_2(\chi_2(x,\lambda)) = -\int_{-1}^{1} U_2(K) f(\xi) d\xi + h_2,$$

$$c_1 U_3(\phi_1(x,\lambda)) + c_2 U_3(\phi_2(x,\lambda)) + c_3 U_3(\chi_1(x,\lambda)) + c_4 U_3(\chi_2(x,\lambda)) = -\int_{-1}^{1} U_3(K) f(\xi) d\xi + h_3,$$

$$c_1 U_4(\phi_1(x,\lambda)) + c_2 U_4(\phi_2(x,\lambda)) + c_3 U_4(\chi_1(x,\lambda)) + c_4 U_4(\chi_2(x,\lambda)) = -\int_{-1}^{1} U_4(K) f(\xi) d\xi + h_4.$$

As the determinant of this system $W_2(\lambda)$ is not zero, so the variables c_i $(i = 1, 2, 3, 4)$ can be unique solved. Therefore,

$$c_1 = \frac{A_1(\lambda) + H_1(\lambda)}{W_2(\lambda)}, \quad c_2 = \frac{A_2(\lambda) + H_2(\lambda)}{W_2(\lambda)},$$

$$c_3 = \frac{A_3(\lambda) + H_3(\lambda)}{W_2(\lambda)}, \quad c_4 = \frac{A_4(\lambda) + H_4(\lambda)}{W_2(\lambda)},$$

where

$$A_1(\lambda) = \begin{vmatrix} -\int_{-1}^{1} U_1(K) f(\xi) d\xi & U_1(\phi_2(x,\lambda)) & U_1(\chi_1(x,\lambda)) & U_1(\chi_1(x,\lambda)) \\ -\int_{-1}^{1} U_2(K) f(\xi) d\xi & U_2(\phi_2(x,\lambda)) & U_2(\chi_1(x,\lambda)) & U_2(\chi_1(x,\lambda)) \\ -\int_{-1}^{1} U_3(K) f(\xi) d\xi & U_3(\phi_2(x,\lambda)) & U_3(\chi_1(x,\lambda)) & U_3(\chi_1(x,\lambda)) \\ -\int_{-1}^{1} U_4(K) f(\xi) d\xi & U_4(\phi_2(x,\lambda)) & U_4(\chi_1(x,\lambda)) & U_4(\chi_1(x,\lambda)) \end{vmatrix},$$

$$H_1(\lambda) = \begin{vmatrix} h_1 & U_1(\phi_2(x,\lambda)) & U_1(\chi_1(x,\lambda)) & U_1(\chi_1(x,\lambda)) \\ h_2 & U_2(\phi_2(x,\lambda)) & U_2(\chi_1(x,\lambda)) & U_2(\chi_1(x,\lambda)) \\ h_3 & U_3(\phi_2(x,\lambda)) & U_3(\chi_1(x,\lambda)) & U_3(\chi_1(x,\lambda)) \\ h_4 & U_4(\phi_2(x,\lambda)) & U_4(\chi_1(x,\lambda)) & U_4(\chi_1(x,\lambda)) \end{vmatrix}.$$

By the Cramer's Role, we can solve $A_i(\lambda)$ and $H_i(\lambda)$. Substituting c_i($i = 1, 2, 3, 4$) into $y(x, \lambda)$ yields that

$$\begin{aligned} y(x,\lambda) &= \int_{-1}^{1} K(x,\xi,\lambda)f(\xi)d\xi + \frac{1}{W_2(\lambda)}(A_1(\lambda)\phi_1(x,\lambda) + A_2(\lambda)\phi_2(x,\lambda) + A_3(\lambda)\chi_1(x,\lambda) + A_4(\lambda)\chi_2(x,\lambda) \\ &+ \frac{1}{W_2(\lambda)}(H_1(\lambda)\phi_1(x,\lambda) + H_2(\lambda)\phi_2(x,\lambda) + H_3(\lambda)\chi_1(x,\lambda) + H_4(\lambda)\chi_2(x,\lambda) \\ &= \int_{-1}^{1}(K(x,\xi,\lambda) + \frac{1}{W_2(\lambda)}B(x,\xi,\lambda))f(\xi)d\xi - \frac{1}{W_2(\lambda)}H(x,\xi,\lambda), \end{aligned}$$

where

$$B(x,\xi,\lambda) = \begin{vmatrix} U_1(\phi_1(x,\lambda)) & U_1(\phi_2(x,\lambda)) & U_1(\chi_1(x,\lambda)) & U_1(\chi_2(x,\lambda)) & U_1(K) \\ U_2(\phi_1(x,\lambda)) & U_2(\phi_2(x,\lambda)) & U_2(\chi_1(x,\lambda)) & U_2(\chi_2(x,\lambda)) & U_2(K) \\ U_3(\phi_1(x,\lambda)) & U_3(\phi_2(x,\lambda)) & U_3(\chi_1(x,\lambda)) & U_3(\chi_2(x,\lambda)) & U_3(K) \\ U_4(\phi_1(x,\lambda)) & U_4(\phi_2(x,\lambda)) & U_4(\chi_1(x,\lambda)) & U_4(\chi_2(x,\lambda)) & U_4(K) \\ \phi_1(x,\lambda) & \phi_2(x,\lambda) & \chi_1(x,\lambda) & \chi_2(x,\lambda) & 0 \end{vmatrix}$$

$$H(x,\xi,\lambda) = \begin{vmatrix} U_1(\phi_1(x,\lambda)) & U_1(\phi_2(x,\lambda)) & U_1(\chi_1(x,\lambda)) & U_1(\chi_2(x,\lambda)) & h_1 \\ U_2(\phi_1(x,\lambda)) & U_2(\phi_2(x,\lambda)) & U_2(\chi_1(x,\lambda)) & U_2(\chi_2(x,\lambda)) & h_2 \\ U_3(\phi_1(x,\lambda)) & U_3(\phi_2(x,\lambda)) & U_3(\chi_1(x,\lambda)) & U_3(\chi_2(x,\lambda)) & h_3 \\ U_4(\phi_1(x,\lambda)) & U_4(\phi_2(x,\lambda)) & U_4(\chi_1(x,\lambda)) & U_4(\chi_2(x,\lambda)) & h_4 \\ \phi_1(x,\lambda) & \phi_2(x,\lambda) & \chi_1(x,\lambda) & \chi_2(x,\lambda) & 0 \end{vmatrix}.$$

Denoting Green function $G(x, \xi, \lambda) = K(x, \xi, \lambda) + \frac{1}{W_2(\lambda)}B(x, \xi, \lambda)$, then $y(x, \lambda)$ can be represented

$$y(x,\lambda) = \int_{-1}^{1} G(x,\xi,\lambda)f(\xi)d\xi - \frac{1}{W_2(\lambda)}H(x,\xi,\lambda). \tag{6.11}$$

Remark 6.1. *Through above discussion, the case of eigenparameter appeared in the boundary conditions of both endpoints is different from the usual case [3], also different from the case of eigenparameter appeared in the boundary conditions of one endpoint [19],* $y(x, \lambda)$ *is not only determined by* $\int_{-1}^{1} G(x, \xi, \lambda)f(\xi)d\xi$*, but also related with* $\frac{1}{W_2(\lambda)}H(x, \xi, \lambda)$*.*

7. Conclusions

In this paper, a class of fourth order differential operators with eigenparameter-dependent boundary conditions and transmission conditions is considered. Using operator theoretic formulation, we transferred the considered problem to an operator in a modified Hilbert space. We investigated some properties of this operator, such as self-adjointness, sufficient and necessary conditions of the eigenvalues, asymptotic formulas for the fundamental solutions and the characteristic functions, the completeness of eigenfunctions in H and the Green function.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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