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Research article

Generalized (p, q)-analogues of Dragomir-Agarwal's inequalities involving Raina's function and applications

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Abstract: In this paper, we introduce the class of generalized strongly convex functions using Raina's function. We derive two new general auxiliary results involving first and second order (p, q)-differentiable functions and Raina's function. Essentially using these identities and the generalized strongly convexity property of the functions, we also found corresponding new generalized post-quantum analogues of Dragomir-Agarwal's inequalities. We discuss some special cases about generalized convex functions. To support our main results, we offer applications to special means, to hypergeometric functions, to Mittag-Leffler functions and also to (p, q)-differentiable functions of first and second order that are bounded in absolute value.

Keywords: Dragomir-Agarwal's inequality; Hölder inequality; power-mean inequality; generalized strongly convex functions; Raina's function; post-quantum calculus
Mathematics Subject Classification: 05A30, 26A33, 26A51, 34A08, 26D07, 26D10, 26D15

1. Introduction and preliminaries

A set $C \subseteq \mathbb{R}$ is said to be convex, if

$$(1-\tau)\upsilon_1 + \tau\upsilon_2 \in C, \quad \forall\, \upsilon_1, \upsilon_2 \in C, \tau \in [0, 1].$$

Similarly, a function $\Psi : C \to \mathbb{R}$ is said to be convex, if

$$\Psi((1-\tau)\upsilon_1 + \tau\upsilon_2) \leq (1-\tau)\Psi(\upsilon_1) + \tau\Psi(\upsilon_2), \quad \forall\, \upsilon_1, \upsilon_2 \in C, \tau \in [0, 1].$$

In recent years, the classical concepts of convexity has been extended and generalized in different directions using novel and innovative ideas.

Let us recall first Raina's function $\mathcal{R}^{\sigma}_{\rho,\lambda}(z)$ that it's defined as follows:

$$\mathcal{R}^{\sigma}_{\rho,\lambda}(z) = \mathcal{R}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(z) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} z^k, \qquad z \in \mathbb{C}, \tag{1.1}$$

where $\rho, \lambda > 0$, with bounded modulus $|z| < M$, and $\sigma = \{\sigma(0), \sigma(1), \ldots, \sigma(k), \ldots\}$ is a bounded sequence of positive real numbers. For details, see [1].

Cortez et al. [2] presented a new generalization of convexity class as follows:

Definition 1. *[2] Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ be a bounded sequence of positive real numbers. A non-empty set $\mathcal{I} \subseteq \mathbb{R}$ is said to be generalized convex, if*

$$\upsilon_1 + \tau \mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1) \in \mathcal{I}, \quad \forall\, \upsilon_1, \upsilon_2 \in \mathcal{I},\ \tau \in [0, 1].$$

Definition 2. *[2] Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ be a bounded sequence of positive real numbers. A function $\Psi : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be generalized convex, if*

$$\Psi(\upsilon_1 + \tau \mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)) \leq (1 - \tau)\Psi(\upsilon_1) + \tau \Psi(\upsilon_2), \quad \forall\, \upsilon_1, \upsilon_2 \in \mathcal{I},\ \tau \in [0, 1].$$

Quantum calculus is the branch of mathematics (often known as calculus without limits) in which we obtain q-analogues of mathematical objects which can be recaptured by taking $q \to 1^-$. Interested readers may find very useful details on quantum calculus in [3]. Recently, quantum calculus has been extended to post quantum calculus. In quantum calculus we deal with q-number with one base q however post quantum calculus includes p and q-numbers with two independent variables p and q. This was first considered by Chakarabarti and Jagannathan [4]. Tunç and Gov [5] introduced the concepts of (p, q)-derivatives and (p, q)-integrals on finite intervals as:

Definition 3. *[5] Let $\mathcal{K} \subseteq \mathbb{R}$ be a non-empty set such that $\upsilon_1 \in \mathcal{K}$, $0 < q < p \leq 1$ and $\Psi : \mathcal{K} \to \mathbb{R}$ be a continuous function. Then, the (p, q)-derivative ${}_{\upsilon_1}\mathcal{D}_{(p,q)}\Psi(\Theta)$ of Ψ at $\Theta \in \mathcal{K}$ is defined by*

$${}_{\upsilon_1}\mathcal{D}_{(p,q)}\Psi(\Theta) = \frac{\Psi(p\Theta + (1 - p)\upsilon_1) - \Psi(q\Theta + (1 - q)\upsilon_1)}{(p - q)(\Theta - \upsilon_1)}, \quad (\Theta \neq \upsilon_1).$$

Note that, if we take $p = 1$ in Definition 3, then we get the definition of q-derivative introduced and studied by Tariboon et al. [6].

Definition 4. *[5] Let $\mathcal{K} \subseteq \mathbb{R}$ be a non-empty set such that $\upsilon_1 \in \mathcal{K}$, $0 < q < p \leq 1$ and $\Psi : \mathcal{K} \to \mathbb{R}$ be a continuous function. Then, the (p, q)-integral on $\mathcal{K}$ is defined by*

$$\int_{\upsilon_1}^{\Theta} \Psi(\tau)\, {}_{\upsilon_1}\mathrm{d}_{(p,q)}\tau = (p - q)(\Theta - \upsilon_1) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Psi\left(\frac{q^n}{p^{n+1}}\Theta + \left(1 - \frac{q^n}{p^{n+1}}\right)\upsilon_1\right)$$

for all $\Theta \in \mathcal{K}$.

Note that, if we take $p = 1$ in Definition 4, then we get the definition of q-integral on finite interval introduced and studied by Tariboon et al. [6].

Theory of convexity has played very important role in the development of theory of inequalities. A wide class of inequalities can easily be obtained using the convexity property of the functions. In this regard Hermite-Hadamard's inequality is one of the most studied result. It provides us an equivalent property for convexity. This famous result of Hermite and Hadamard reads as: Let $\Psi : [\upsilon_1, \upsilon_2] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, then

$$\Psi\left(\frac{\upsilon_1 + \upsilon_2}{2}\right) \leq \frac{1}{\upsilon_2 - \upsilon_1} \int\limits_{\upsilon_1}^{\upsilon_2} \Psi(\Theta) \mathrm{d}\Theta \leq \frac{\Psi(\upsilon_1) + \Psi(\upsilon_2)}{2}.$$

In recent years, several new extensions and generalizations of this classical result have been obtained in the literature. In [7] Dragomir and Agarwal have obtained a new integral identity using the first order differentiable functions:

Lemma 1. *[7] Let $\Psi : X = [\upsilon_1, \upsilon_2] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on X° (the interior set of X), then*

$$\frac{\Psi(\upsilon_1) + \Psi(\upsilon_2)}{2} - \frac{1}{\upsilon_2 - \upsilon_1} \int_{\upsilon_1}^{\upsilon_2} \Psi(\Theta) \mathrm{d}\Theta = \frac{\upsilon_2 - \upsilon_1}{2} \int_0^1 (1 - 2\tau) \Psi'(\tau \upsilon_1 + (1 - \tau) \upsilon_2) \mathrm{d}\tau.$$

Using this identity authors have obtained some new right estimates for Hermite-Hadamard's inequality essentially using the class of first order differentiable convex functions. This idea of Dragomir and Agarwal has inspired many researchers and consequently a variety of new identities and corresponding inequalities have been obtained in the literature using different techniques. Sudsutad et al. [8] and Noor et al. [9] obtained the quantum counterpart of this result and obtained associated q-analogues of trapezium like inequalities. Liu and Zhuang [10] obtained another quantum version of this identity via twice q-differentiable functions and obtained associated q-integral inequalities. Awan et al. [11] extended the results of Dragomir and Agarwal by obtaining a new post-quantum integral identity involving twice (p, q)-differentiable functions and twice (p, q)-differentiable preinvex functions. Du et al. [12] obtained certain quantum estimates on the parameterized integral inequalities and established some applications. Zhang et al. [13] found different types of quantum integral inequalities via (α, m)-convexity. Cortez et al. [14, 15] derived some inequalities using generalized convex functions in quantum analysis.

The main objective of this paper is to introduce the notion of generalized strongly convex functions using Raina's function. We derive two new general auxiliary results involving first and second order (p, q)-differentiable functions and Raina's function. Essentially using these identities and the generalized strongly convexity property of the functions, we also derive corresponding new generalized post-quantum analogues of Dragomir-Agarwal's inequalities. In order to discuss the relation with other results, we also discuss some special cases about generalized convex functions. To support our main results, we give applications to special means, to hypergeometric functions, to Mittag-Leffler functions and also to (p, q)-differentiable functions of first and second order that are bounded in absolute value. Finally, some conclusions and future research are provided as well. We hope that the ideas and techniques of this paper will inspire interested readers working in this field.

2. Main results

In this section, we discuss our main results. First, we introduce the class of generalized strongly convex function involving Raina's function.

Definition 5. *Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ be a bounded sequence of positive real numbers. A function $\Psi : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is called generalized strongly convex, if*

$$\Psi(\upsilon_1 + \tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)) \leq (1-\tau)\Psi(\upsilon_1) + \tau\Psi(\upsilon_2) - c\tau(1-\tau)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1))^2,$$

$\forall\, c > 0, \tau \in [0,1]$ *and* $\upsilon_1, \upsilon_2 \in \mathcal{I}$.

2.1. Auxiliary results

In this section, we derive two new post-quantum integral identities that will be used in a sequel.

Lemma 2. *Let $\Psi : X = [\upsilon_1, \upsilon_1 + \mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function and $0 < q < p \leq 1$. If ${}_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi$ is integrable function on X°, then*

$$\begin{aligned}
&\frac{1}{p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)}\int_{\upsilon_1}^{\upsilon_1 + p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)} \Psi(\tau)\,{}_{\upsilon_1}\mathrm{d}_{(p,q)}\tau - \frac{p\Psi(\upsilon_1 + \mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)) + q\Psi(\upsilon_1)}{p+q} \\
&= \frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)}{p+q}\int_0^1 (1-(p+q)\tau)\,{}_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1 + \tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1))\,{}_0\mathrm{d}_{(p,q)}\tau. \qquad (2.1)
\end{aligned}$$

Proof. Using the right hand side of (3.3), we have

$$I := \frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)}{p+q} I_1,$$

and from the definitions of ${}_{\upsilon_1}\mathrm{D}_{(p,q)}$, and (p,q)-integral, we get

$$\begin{aligned}
I_1 &:= \int_0^1 (1-(p+q)\tau)\,{}_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1 + \tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1))\,{}_0\mathrm{d}_{(p,q)}\tau \\
&= \int_0^1 (1-(p+q)\tau)\frac{\Psi(\upsilon_1 + \tau p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)) - \Psi(\upsilon_1 + q\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1))}{(p-q)\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)}\,{}_0\mathrm{d}_{(p,q)}\tau \\
&= \frac{1}{\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)}\left[\sum_{n=0}^{\infty}\Psi\left(\upsilon_1 + \frac{q^n}{p^n}\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)\right) - \Psi\left(\upsilon_1 + \frac{q^{n+1}}{p^{n+1}}\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)\right)\right] \\
&- \frac{p+q}{\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)}\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Psi\left(\upsilon_1 + \frac{q^n}{p^n}\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)\right) - \sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Psi\left(\upsilon_1 + \frac{q^{n+1}}{p^{n+1}}\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)\right)\right] \\
&= \frac{\Psi(\upsilon_1 + \mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)) - \Psi(\upsilon_1)}{\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)} - \frac{p+q}{\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)}\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Psi\left(\upsilon_1 + \frac{q^n}{p^n}\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)\right)\right. \\
&\left. -\frac{1}{q}\sum_{n=1}^{\infty}\frac{q^n}{p^n}\Psi\left(\upsilon_1 + \frac{q^n}{p^n}\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Psi(\upsilon_1 + \mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)) - \Psi(\upsilon_1)}{\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)} - \frac{p+q}{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)}\Psi(\upsilon_1 + \mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)) \\
&- \frac{p+q}{\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Psi\left(\upsilon_1 + \frac{q^n}{p^n}\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)\right) \\
&+ \frac{p(p+q)}{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Psi\left(\upsilon_1 + \frac{q^n}{p^n}\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)\right) \\
&= -\frac{p\Psi(\upsilon_1 + \mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)) + q\Psi(\upsilon_1)}{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)} + \frac{p+q}{pq(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1))^2} \int_{\upsilon_1}^{\upsilon_1 + p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)} \Psi(\tau)_0 d_{(p,q)}\tau.
\end{aligned}
$$

This completes the proof. □

The second identity for twice (p,q)-differentiable functions states as follows:

Lemma 3. *Let $\Psi : X = [\upsilon_1, \upsilon_1 + \mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)] \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function and $0 < q < p \le 1$. If ${}_{\upsilon_1}D^2{}_{(p,q)}\Psi$ is integrable function on X°, then*

$$
\begin{aligned}
&\frac{p^2\Psi(\upsilon_1 + \mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)) + q\Psi(\upsilon_1)}{p+q} - \frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)} \int_{\upsilon_1}^{\upsilon_1 + p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)} \Psi(\tau)_{\upsilon_1} d_{(p,q)}\tau \\
&= \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1))^2}{p+q} \int_0^1 \tau(1 - q\tau)_{\upsilon_1} D^2{}_{(p,q)}\Psi(\upsilon_1 + \tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1))_0 d_{(p,q)}\tau. \qquad (2.2)
\end{aligned}
$$

Proof. Firstly, applying the definition of ${}_{\upsilon_1}D^2{}_{(p,q)}$ differentiability, we have

$$
\begin{aligned}
&{}_{\upsilon_1}D^2{}_{(p,q)}\Psi(\upsilon_1 + \tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)) \\
&= {}_{\upsilon_1}D_{(p,q)}({}_{\upsilon_1}D_{(p,q)}\Psi(\upsilon_1 + \tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1))) \\
&= \frac{q\Psi(\upsilon_1 + \tau p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)) - (p+q)\Psi(\upsilon_1 + pq\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)) + p\Psi(\upsilon_1 + \tau q^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1))}{pq(p-q)^2\tau^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1))^2}.
\end{aligned}
$$

Now, using the notion of (p,q)-integration, we get

$$
\begin{aligned}
&\int_0^1 \tau(1 - q\tau) \\
&\times \frac{q\Psi(\upsilon_1 + \tau p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)) - (p+q)\Psi(\upsilon_1 + pq\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)) + p\Psi(\upsilon_1 + \tau q^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1))}{pq(p-q)^2\tau^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1))^2} {}_0d_{(p,q)}\tau \\
&= \frac{1}{pq(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1))^2}\left[\sum_{n=0}^{\infty} q\Psi\left(\upsilon_1 + \frac{q^n}{p^{n+1}}p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)\right)\right. \\
&\left.-(p+q)\sum_{n=0}^{\infty}\Psi\left(\upsilon_1 + \frac{q^{n+1}}{p^{n+1}}p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)\right) + p\sum_{n=0}^{\infty}\Psi\left(\upsilon_1 + \frac{q^{n+1}}{p^{n+2}}\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)\right)\right] \\
&- q\left[\frac{q(p-q)\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Psi\left(\upsilon_1 + \frac{q^n}{p^{n+1}}p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1)\right)}{pq(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2 - \upsilon_1))^3}\right.
\end{aligned}
$$

$$
-\frac{(p+q)(p-q)\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+1}}\Psi\left(\upsilon_1+\frac{q^{n+1}}{p^{n+1}}p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)\right)}{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^3}
$$

$$
\left.+\frac{p(p-q)\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)\sum_{n=0}^{\infty}\frac{q^{n+2}}{p^{n+1}}\Psi\left(\upsilon_1+\frac{q^{n+2}}{p^{n+1}}\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)\right)}{pq^3(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^3}\right]
$$

$$
=\frac{q\left(\Psi(\upsilon_1+p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))-\Psi(\upsilon_1)\right)-p\left(\Psi(\upsilon_1+q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))-\Psi(\upsilon_1)\right)}{pq(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}
$$

$$
-\frac{q^2+pq-p}{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}-\frac{p+q}{p^3q^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^3}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau.
$$

After multiplying both sides by $\frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}$, we obtain our required identity. □

2.2. (p,q)-Dragomir-Agarwal like inequalities

We now derive some (p,q)-analogues of Dragomir-Agarwal like inequalities using first order and second order (p,q)-differentiable functions via generalized strongly convex function with modulus $c>0$. Let us recall the following notion that will be used in the sequel.

$$
[n]_{(p,q)}:=\frac{p^n-q^n}{p-q},\qquad n\in\mathbb{N},\ 0<q<p\leq 1.
$$

Theorem 1. *Suppose that all the assumptions of Lemma 2 are satisfied and $|_{\upsilon_1}D_{(p,q)}\Psi|$ is generalized strongly convex function, then*

$$
\left|\frac{1}{p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau-\frac{p\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}\right|
$$

$$
\leq\frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q}\left[S_1|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|+S_2|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|-cS_3(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2\right],
$$

where

$$
S_1:=\frac{2-p-q}{p+q}+\frac{(p+q)^3-2(p+q)-2+(p+q)^2}{(p+q)^2[2]_{(p,q)}},\tag{2.3}
$$

$$
S_2:=\frac{2-(p+q)^2}{(p+q)^2[2]_{(p,q)}}+\frac{(p+q)^3-2}{(p+q)^2[3]_{(p,q)}},\tag{2.4}
$$

and

$$
S_3:=\frac{2-(p+q)^2}{(p+q)^2[2]_{(p,q)}}+\frac{(p+q)^3+p+q-2-(p+q)^4}{(p+q)^3[3]_{(p,q)}}+\frac{(p+q)^4-1}{(p+q)^4[4]_{(p,q)}}.\tag{2.5}
$$

Proof. From Lemma 2, properties of modulus and using the generalized strongly convexity of $|_{\upsilon_1}D_{(p,q)}\Psi|$, we have

$$
\left|\frac{1}{p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau-\frac{p\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}\right|
$$

$$\leq \frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q}\int_0^1 |(1-(p+q)\tau)|\,|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))|_0\mathrm{d}_{(p,q)}\tau$$

$$\leq \frac{q^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q}\left[\int_0^{\frac{1}{p+q}}(1-(p+q)\tau)|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))|_0\mathrm{d}_{(p,q)}\tau\right.$$

$$\left.+\int_{\frac{1}{p+q}}^1((p+q)\tau-1)|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))|_0\mathrm{d}_{(p,q)}\tau\right]$$

$$\leq \frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q}\left[\int_0^{\frac{1}{p+q}}(1-(p+q)\tau)[(1-\tau)|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|+\tau|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|\right.$$

$$-c\tau(1-\tau)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2]_0\mathrm{d}_{(p,q)}\tau+\int_{\frac{1}{p+q}}^1((p+q)\tau-1)[(1-\tau)|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|+\tau|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|$$

$$\left.-c\tau(1-\tau)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2]_0\mathrm{d}_{(p,q)}\tau\right].$$

After simplification, we obtain our required result. □

Corollary 1. *Letting $c\to 0^+$ in Theorem 1, then*

$$\left|\frac{1}{p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau-\frac{p\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}\right|$$
$$\leq \frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q}\left[S_1|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|+S_2|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|\right].$$

Theorem 2. *Suppose that all the assumptions of Lemma 2 are satisfied and $|_{\upsilon_1}D_{(p,q)}\Psi|^m$ is generalized strongly convex function with $m\geq 1$, then*

$$\left|\frac{1}{p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau-\frac{p\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}\right|$$
$$\leq \frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q}S_4^{1-\frac{1}{m}}\left[S_1|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|^m+S_2|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|^m-cS_3(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}},$$

where S_1, S_2 and S_3 are given by (2.3)–(2.5), respectively, and

$$S_4:=\frac{2-(p+q)}{p+q}+\frac{(p+q)((p+q)^2-2)}{(p+q)^2[2]_{(p,q)}}. \tag{2.6}$$

Proof. From Lemma 2, properties of modulus, power-mean inequality and using the generalized strongly convexity of $|_{\upsilon_1}D_{(p,q)}\Psi|^m$, we have

$$\left|\frac{1}{p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau-\frac{p\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}\right|$$
$$\leq \frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q}\left(\int_0^1|(1-(p+q)\tau)|\,_0\mathrm{d}_{(p,q)}\tau\right)^{1-\frac{1}{m}}$$
$$\times\left(\int_0^1|(1-(p+q)\tau)|\,|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))|^m{}_0\mathrm{d}_{(p,q)}\tau\right)^{\frac{1}{m}}$$

$$= \frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q} S_4^{1-\frac{1}{m}} \left[\int_0^{\frac{1}{p+q}} (1-(p+q)\tau) |_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))|^m {}_0\mathrm{d}_{(p,q)}\tau \right.$$

$$\left. + \int_{\frac{1}{p+q}}^1 ((p+q)\tau-1) |_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))|^m {}_0\mathrm{d}_{(p,q)}\tau \right]^{\frac{1}{m}}$$

$$\leq \frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q} S_4^{1-\frac{1}{m}} \left[\int_0^{\frac{1}{p+q}} (1-(p+q)\tau)[(1-\tau)|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|^m + \tau|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|^m \right.$$

$$-c\tau(1-\tau)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2]_0\mathrm{d}_{(p,q)}\tau + \int_{\frac{1}{p+q}}^1 ((p+q)\tau-1)[(1-\tau)|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|^m + \tau|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|^m$$

$$\left. -c\tau(1-\tau)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2]_0\mathrm{d}_{(p,q)}\tau \right]^{\frac{1}{m}}.$$

After simplification, we obtain our required result. □

Corollary 2. *Letting $c \to 0^+$ in Theorem 2, then*

$$\left| \frac{1}{p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)} \int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)} \Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau - \frac{p\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)) + q\Psi(\upsilon_1)}{p+q} \right|$$
$$\leq \frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q} S_4^{1-\frac{1}{m}} \left[S_1|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|^m + S_2|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|^m \right]^{\frac{1}{m}}.$$

Theorem 3. *Suppose that all the assumptions of Lemma 2 are satisfied and $|_{\upsilon_1}D_{(p,q)}\Psi|^m$ is generalized strongly convex function with $\frac{1}{l}+\frac{1}{m}=1$, and $l,m>0$, then*

$$\left| \frac{1}{p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)} \int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)} \Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau - \frac{p\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)) + q\Psi(\upsilon_1)}{p+q} \right|$$
$$\leq \frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q} S_5^{\frac{1}{l}} \left[\frac{p+q-1}{p+q}|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|^m + \frac{1}{p+q}|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|^m \right.$$
$$\left. - c\left(\frac{1}{p+q} - \frac{1}{p^2+pq+q^2}\right)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2 \right]^{\frac{1}{m}},$$

where

$$S_5 := \frac{(p-q)}{1+q} \sum_{n=0}^{\infty} \left[\left(1-(p+q)\frac{q^n}{p^{n+1}}\right)^l + q\left((p+q)\frac{q^n}{p^{n+1}}-1\right)^l \right]. \tag{2.7}$$

Proof. From Lemma 2, properties of modulus, Hölder's inequality and using the generalized strongly convexity of $|_{\upsilon_1}D_{(p,q)}\Psi|^m$, we have

$$\left| \frac{1}{p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)} \int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)} \Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau - \frac{p\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)) + q\Psi(\upsilon_1)}{p+q} \right|$$
$$\leq \frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q} \left(\int_0^1 |(1-(p+q)\tau)|^l {}_0\mathrm{d}_{(p,q)}\tau \right)^{\frac{1}{l}}$$

$$\times\left(\int_0^1 |_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))|^m{}_0\mathrm{d}_{(p,q)}\tau\right)^{\frac{1}{m}}$$
$$\leq \frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q}S_5^{\frac{1}{l}}\left[\int_0^1 [(1-\tau)|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|^m+\tau|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|^m\right.$$
$$\left.-c\tau(1-\tau)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2]_0\mathrm{d}_{(p,q)}\tau\right]^{\frac{1}{m}}.$$

After simplification, we obtain our required result. □

Corollary 3. *Letting $c\to 0^+$ in Theorem 3, then*

$$\left|\frac{1}{p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau-\frac{p\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}\right|$$
$$\leq \frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q}S_5^{\frac{1}{l}}\left[\frac{p+q-1}{p+q}|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|^m+\frac{1}{p+q}|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|^m\right]^{\frac{1}{m}}.$$

Theorem 4. *Suppose that all the assumptions of Lemma 3 are satisfied and $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|$ is generalized strongly convex function, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}\left[S_6|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|+S_7|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|-cS_8(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2\right],$$

where

$$S_6:=\frac{p^2-p-q}{(p+q)(p^2+pq+q^2)}+\frac{q}{p^3+pq(p+q)+q^3}, \tag{2.8}$$

$$S_7:=\frac{p^3}{(p^2+pq+q^2)(p^3+pq(p+q)+q^3)}, \tag{2.9}$$

and

$$S_8:=\frac{1}{p+q}-\frac{1+q}{p^3+pq(p+q)+q^3}+\frac{q}{[5]_{(p,q)}}. \tag{2.10}$$

Proof. From Lemma 3, properties of modulus and using the generalized strongly convexity of $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|$, we have

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}\int_0^1|\tau(1-q\tau)|\left|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))\right|{}_0\mathrm{d}_{(p,q)}\tau$$

$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}\left[\int_0^1 \tau(1-q\tau)[(1-\tau)|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|+\tau|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|\right.$$
$$\left.-c\tau(1-\tau)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2]_0\mathrm{d}_{(p,q)}\tau\right].$$

This completes the proof. □

Corollary 4. *Letting $c\to 0^+$ in Theorem 4, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}\left[S_6|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|+S_7|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|\right].$$

Theorem 5. *Suppose that all the assumptions of Lemma 3 are satisfied and $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$ is generalized strongly convex function with $m\geq 1$, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}S_9^{1-\frac{1}{m}}\left[S_6|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m+S_7|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m-cS_8(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}},$$

where S_6, S_7 and S_8 are given by (2.8)–(2.10), respectively, and

$$S_9:=\frac{p^2}{(p+q)(p^2+pq+q^2)}. \tag{2.11}$$

Proof. From Lemma 3, properties of modulus, power-mean inequality and using the generalized strongly convexity of $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$, we have

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}\left(\int_0^1|\tau(1-q\tau)|_0\mathrm{d}_{(p,q)}\tau\right)^{1-\frac{1}{m}}$$
$$\times\left(\int_0^1|\tau(1-q\tau)|\left|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))\right|^m{}_0\mathrm{d}_{(p,q)}\tau\right)^{\frac{1}{m}}$$
$$=\frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}S_9^{1-\frac{1}{m}}\left[\int_0^1\tau(1-q\tau)|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))|^m{}_0\mathrm{d}_{(p,q)}\tau\right]^{\frac{1}{m}}$$
$$\leq\frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}S_9^{1-\frac{1}{m}}\left[\int_0^1\tau(1-q\tau)[(1-\tau)|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m+\tau|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m\right.$$
$$\left.-c\tau(1-\tau)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2]_0\mathrm{d}_{(p,q)}\tau\right]^{\frac{1}{m}}.$$

This completes the proof. □

Corollary 5. *Letting $c \to 0^+$ in Theorem 5, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}S_9^{1-\frac{1}{m}}\left[S_6|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m+S_7|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m\right]^{\frac{1}{m}}.$$

Theorem 6. *Suppose that all the assumptions of Lemma 3 are satisfied and $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$ is generalized strongly convex function with $\frac{1}{l}+\frac{1}{m}=1$, and $l,m>0$, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}S_{10}^{\frac{1}{l}}\Bigg[\frac{p+q-1}{p+q}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m+\frac{1}{p+q}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m$$
$$-c\left(\frac{1}{p+q}-\frac{1}{p^2+pq+q^2}\right)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2\Bigg]^{\frac{1}{m}},$$

where

$$S_{10}:=(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}-\frac{q^{2n+1}}{p^{2n+2}}\right)^l. \tag{2.12}$$

Proof. From Lemma 3, properties of modulus, Hölder's inequality and using the generalized strongly convexity of $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$, we have

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}\left(\int_0^1|\tau(1-q\tau)|^l_0\mathrm{d}_{(p,q)}\tau\right)^{\frac{1}{l}}$$
$$\times\left(\int_0^1\left|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))\right|^m{}_0\mathrm{d}_{(p,q)}\tau\right)^{\frac{1}{m}}$$
$$=\frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}S_{10}^{\frac{1}{l}}\left[\int_0^1|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))|^m{}_0\mathrm{d}_{(p,q)}\tau\right]^{\frac{1}{m}}$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}S_{10}^{\frac{1}{l}}\Bigg[\int_0^1[(1-\tau)|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m+\tau|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m$$
$$-c\tau(1-\tau)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2]_0\mathrm{d}_{(p,q)}\tau\Bigg]^{\frac{1}{m}}.$$

This completes the proof. □

Corollary 6. *Letting $c \to 0^+$ in Theorem 22, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}S_{10}^{\frac{1}{l}}\left[\frac{p+q-1}{p+q}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m+\frac{1}{p+q}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m\right]^{\frac{1}{m}}.$$

Theorem 7. *Suppose that all the assumptions of Lemma 3 are satisfied and $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$ is generalized strongly convex function with $\frac{1}{l}+\frac{1}{m}=1$, and $l,m>0$, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}S_{11}^{\frac{1}{l}}\left[\left(\frac{1}{[m+1]_{(p,q)}}-\frac{1}{[m+2]_{(p,q)}}\right)|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m\right.$$
$$\left.+\frac{1}{[m+2]_{(p,q)}}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m-c\left(\frac{1}{[m+2]_{(p,q)}}-\frac{1}{[m+3]_{(p,q)}}\right)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{l}},$$

where

$$S_{11}:=(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^l. \tag{2.13}$$

Proof. From Lemma 3, properties of modulus, Hölder's inequality and using the generalized strongly convexity of $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$, we have

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}\left(\int_0^1|(1-q\tau)|^l{}_0\mathrm{d}_{(p,q)}\tau\right)^{\frac{1}{l}}$$
$$\times\left(\int_0^1\tau^m\left|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))\right|^m{}_0\mathrm{d}_{(p,q)}\tau\right)^{\frac{1}{m}}$$
$$=\frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}S_{11}^{\frac{1}{l}}\left[\int_0^1\tau^m|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))|^m{}_0\mathrm{d}_{(p,q)}\tau\right]^{\frac{1}{m}}$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}S_{11}^{\frac{1}{l}}\left[\int_0^1\tau^m[(1-\tau)|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m+\tau|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m\right.$$
$$\left.-c\tau(1-\tau)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2]_0\mathrm{d}_{(p,q)}\tau\right]^{\frac{1}{m}}.$$

This completes the proof. □

Corollary 7. *Letting $c \to 0^+$ in Theorem 7, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$

$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q} S_{11}^{\frac{1}{l}} \left[\left(\frac{1}{[m+1]_{(p,q)}} - \frac{1}{[m+2]_{(p,q)}} \right) |_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m \right.$$
$$\left. + \frac{1}{[m+2]_{(p,q)}} |_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m \right]^{\frac{1}{l}}.$$

Theorem 8. *Suppose that all the assumptions of Lemma 3 are satisfied and* $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$ *is generalized strongly convex function with* $\frac{1}{l}+\frac{1}{m}=1$*, and* $l,m>0$*, then*

$$\left| \frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q} - \frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)} \int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)} \Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau \right|$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q} \left(\frac{1}{[l+1]_{(p,q)}} \right)^{\frac{1}{l}} \left[S_{12}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m \right.$$
$$\left. + S_{13}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m - cS_{14}(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2 \right]^{\frac{1}{m}},$$

where

$$S_{12} := \frac{p+q-1}{p+q} \sum_{n=0}^{\infty} \left(1 - \frac{q^{n+1}}{p^{n+1}} \right)^m, \tag{2.14}$$

$$S_{13} := (p-q) \sum_{n=0}^{\infty} \frac{q^{2n}}{p^{2n+2}} \left(1 - \frac{q^{n+1}}{p^{n+1}} \right)^m, \tag{2.15}$$

and

$$S_{14} := \frac{p^2+pq+q^2-(p+q)}{(p+q)(p^2+pq+q^2)} \sum_{n=0}^{\infty} \left(1 - \frac{q^{n+1}}{p^{n+1}} \right)^m. \tag{2.16}$$

Proof. From Lemma 3, properties of modulus, Hölder's inequality and using the generalized strongly convexity of $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$, we have

$$\left| \frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q} - \frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)} \int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)} \Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau \right|$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q} \left(\int_0^1 \tau^l{}_0\mathrm{d}_{(p,q)}\tau \right)^{\frac{1}{l}}$$
$$\times \left(\int_0^1 (1-q\tau)^m \left|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))\right|^m {}_0\mathrm{d}_{(p,q)}\tau \right)^{\frac{1}{m}}$$
$$= \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q} \left(\frac{1}{[l+1]_{(p,q)}} \right)^{\frac{1}{l}} \left(\int_0^1 (1-q\tau)^m \left|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))\right|^m {}_0\mathrm{d}_{(p,q)}\tau \right)^{\frac{1}{m}}$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q} \left(\frac{1}{[l+1]_{(p,q)}} \right)^{\frac{1}{l}} \left[\int_0^1 (1-q\tau)^m [(1-\tau)|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m + \tau|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m \right.$$
$$\left. - c\tau(1-\tau)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2]_0\mathrm{d}_{(p,q)}\tau \right]^{\frac{1}{m}}.$$

This completes the proof. □

Corollary 8. *Letting $c \to 0^+$ in Theorem 8, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}\left(\frac{1}{[l+1]_{(p,q)}}\right)^{\frac{1}{l}}\left[S_{12}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m+S_{13}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m\right]^{\frac{1}{m}}.$$

3. Applications

In this section, we discuss some applications of our main results.

3.1. Applications to means and their numerical verifications

First of all, we recall some previously known concepts regarding special means. For different real numbers $\upsilon_1 < \upsilon_2$, we have

(1) The arithmetic mean: $A(\upsilon_1,\upsilon_2)=\frac{\upsilon_1+\upsilon_2}{2}$.

(2) The generalized logarithmic mean: $L_n(\upsilon_1,\upsilon_2)=\left[\frac{{\upsilon_2}^{n+1}-{\upsilon_1}^{n+1}}{(\upsilon_2-\upsilon_1)(n+1)}\right]^{\frac{1}{n}}, \quad n\in\mathbb{Z}\setminus\{-1,0\}$.

Proposition 1. *Assume that all the assumptions of Theorem 1 are satisfied, then the following inequality holds*

$$\left|\frac{2A(p{\upsilon_2}^n,q{\upsilon_1}^n)}{p+q}-\frac{1}{[n]_{(p,q)}}L_n^n(\upsilon_1+p(\upsilon_2-\upsilon_1),\upsilon_1)\right|$$
$$\leq \frac{q(\upsilon_2-\upsilon_1)}{p+q}\left[S_1\left|\frac{(p\upsilon_2+(1-p)\upsilon_1)^n-(q\upsilon_2+(1-q)\upsilon_1)^n}{(p-q)(\upsilon_2-\upsilon_1)}\right|+S_2\left|[n]_{(p,q)}{\upsilon_1}^{n-1}\right|\right],$$

where S_1 and S_2 are given by (2.3) and (2.4), respectively.

Proof. If we choose ${}_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(x)=x^n, \mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)=\upsilon_2-\upsilon_1$ and $c=0$ in Theorem 1, we obtain our required result. □

Example 1. *If we take $n=2,\upsilon_1=2,\upsilon_2=4$, $p=\frac{1}{2}$ and $q=\frac{1}{3}$ in Proposition 1, then we have* $0.4 < 9.42$*, which shows the validity of the result.*

Proposition 2. *Assume that all the assumption of Theorem 2 are satisfied, then the following inequality holds*

$$\left|\frac{2A(p{\upsilon_2}^n,q{\upsilon_1}^n)}{p+q}-\frac{1}{[n]_{(p,q)}}L_n^n(\upsilon_1+p(\upsilon_2-\upsilon_1),\upsilon_1)\right|$$
$$\leq \frac{q(\upsilon_2-\upsilon_1)}{p+q}S_4^{1-\frac{1}{m}}\left[S_1\left|\frac{(p\upsilon_2+(1-p)\upsilon_1)^n-(q\upsilon_2+(1-q)\upsilon_1)^n}{(p-q)(\upsilon_2-\upsilon_1)}\right|^m+S_2\left|[n]_{(p,q)}{\upsilon_1}^{n-1}\right|^m\right]^{\frac{1}{m}},$$

where S_1 and S_2 are given by (2.3), (2.4), and S_4 is given by (2.6), respectively.

Proof. If we choose ${}_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(x)=x^n, \mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)=\upsilon_2-\upsilon_1$ and $c=0$ in Theorem 2, then we obtain our required result. □

Example 2. *If we take $n=2,m=2,\upsilon_1=2,\upsilon_2=4$, $p=\frac{1}{2}$ and $q=\frac{1}{3}$ in Proposition 2, then we have* $0.4 < 11.39$*, which shows the validity of the result.*

3.2. Applications to hypergeometric functions

From relation (1.1), if we set $\rho = 1, \lambda = 0$ and $\sigma(k) = \frac{(\phi)_k(\psi)_k}{(\eta)_k} \neq 0$, where ϕ, ψ and η are parameters may be real or complex values and $(m)_k$ is defined as $(m)_k = \frac{\Gamma(m+k)}{\Gamma(m)}$ and its domain is restricted as $|x| \leq 1$, then we have the following hypergeometric function

$$\mathcal{R}(\phi, \psi; \eta, x) := \sum_{k=0}^{\infty} \frac{(\phi)_k(\psi)_k}{k!(\eta)_k} x^k.$$

So using above notations and all the results obtained in this paper, we have the following forms.

Lemma 4. *Let* $\Psi : X = [\upsilon_1, \upsilon_1 + \mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1)] \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function and* $0 < q < p \leq 1$. *If* ${}_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi$ *is integrable function on* X°, *then*

$$\frac{1}{p\mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1)} \int_{\upsilon_1}^{\upsilon_1 + p\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)} \Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau - \frac{p\Psi(\upsilon_1 + \mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1)) + q\Psi(\upsilon_1)}{p + q}$$
$$= \frac{q\mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1)}{p + q} \int_0^1 (1 - (p + q)\tau)_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1 + \tau\mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1))_0\mathrm{d}_{(p,q)}\tau. \tag{3.1}$$

The second identity for twice (p, q)-differentiable functions states as follows:

Lemma 5. *Let* $\Psi : X = [\upsilon_1, \upsilon_1 + \mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1)] \subseteq \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function and* $0 < q < p \leq 1$. *If* ${}_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi$ *is integrable function on* X°, *then*

$$\frac{p^2\Psi(\upsilon_1 + \mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1)) + q\Psi(\upsilon_1)}{p + q} - \frac{1}{p^2\mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1)} \int_{\upsilon_1}^{\upsilon_1 + p^2\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)} \Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau$$
$$= \frac{pq^2(\mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1))^2}{p + q} \int_0^1 \tau(1 - q\tau)_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1 + \tau\mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1))_0\mathrm{d}_{(p,q)}\tau. \tag{3.2}$$

Theorem 9. *Suppose that all the assumptions of Lemma 4 are satisfied and* $|_{\upsilon_1}D_{(p,q)}\Psi|$ *is generalized strongly convex function, then*

$$\left|\frac{1}{p\mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1)} \int_{\upsilon_1}^{\upsilon_1 + p\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)} \Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau - \frac{p\Psi(\upsilon_1 + \mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1)) + q\Psi(\upsilon_1)}{p + q}\right|$$
$$\leq \frac{q\mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1)}{p + q} \left[S_1|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)| + S_2|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)| - cS_3(\mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1))^2\right],$$

where S_1, S_2 *and* S_3 *are given by (2.3)–(2.5).*

Theorem 10. *Suppose that all the assumptions of Lemma 4 are satisfied and* $|_{\upsilon_1}D_{(p,q)}\Psi|^m$ *is generalized strongly convex function with* $m \geq 1$, *then*

$$\left|\frac{1}{p\mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1)} \int_{\upsilon_1}^{\upsilon_1 + p\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)} \Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau - \frac{p\Psi(\upsilon_1 + \mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1)) + q\Psi(\upsilon_1)}{p + q}\right|$$
$$\leq \frac{q\mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1)}{p + q} S_4^{1-\frac{1}{m}} \left[S_1|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|^m + S_2|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|^m - cS_3(\mathcal{R}(\phi, \psi; \eta; \upsilon_2 - \upsilon_1))^2\right]^{\frac{1}{m}},$$

where S_1, S_2 S_3 *and* S_4 *are given by (2.3)–(2.6), respectively.*

Theorem 11. *Suppose that all the assumptions of Lemma 4 are satisfied and* $|_{\upsilon_1}D_{(p,q)}\Psi|^m$ *is generalized strongly convex function with* $\frac{1}{l}+\frac{1}{m}=1$, *and* $l,m>0$, *then*

$$\left|\frac{1}{p\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau-\frac{p\Psi(\upsilon_1+\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}\right|$$
$$\leq\frac{q\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)}{p+q}S_5^{\frac{1}{l}}\left[\frac{p+q-1}{p+q}|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|^m+\frac{1}{p+q}|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|^m\right.$$
$$\left.-c\left(\frac{1}{p+q}-\frac{1}{p^2+pq+q^2}\right)(\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}},$$

where S_5 *is given by (2.7).*

Theorem 12. *Suppose that all the assumptions of Lemma 5 are satisfied and* $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|$ *is generalized strongly convex function, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq\frac{pq^2(\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))^2}{p+q}\left[S_6|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|+S_7|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|-cS_8(\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))^2\right],$$

where S_6,S_7 *and* S_8 *are given as (2.8)–(2.10).*

Theorem 13. *Suppose that all the assumptions of Lemma 5 are satisfied and* $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$ *is generalized strongly convex function with* $m\geq 1$, *then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq\frac{pq^2(\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))^2}{p+q}S_9^{1-\frac{1}{m}}\left[S_6|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m+S_7|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m-cS_8(\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}},$$

where S_6, S_7,S_8 *and* S_9 *are given by (2.8)–(2.11), respectively.*

Theorem 14. *Suppose that all the assumptions of Lemma 5 are satisfied and* $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$ *is generalized strongly convex function with* $\frac{1}{l}+\frac{1}{m}=1$, *and* $l,m>0$, *then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq\frac{pq^2(\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))^2}{p+q}S_{10}^{\frac{1}{l}}\left[\frac{p+q-1}{p+q}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m+\frac{1}{p+q}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m\right.$$
$$\left.-c\left(\frac{1}{p+q}-\frac{1}{p^2+pq+q^2}\right)(\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}}.$$

Theorem 15. *Suppose that all the assumptions of Lemma 5 are satisfied and* $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$ *is generalized strongly convex function with* $\frac{1}{l}+\frac{1}{m}=1$, *and* $l,m>0$, *then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$

$$\leq \frac{pq^2(\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))^2}{p+q} S_{11}^{\frac{1}{l}} \left[\left(\frac{1}{[m+1]_{(p,q)}} - \frac{1}{[m+2]_{(p,q)}} \right) |_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m \right.$$
$$\left. + \frac{1}{[m+2]_{(p,q)}} |_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m - c\left(\frac{1}{[m+2]_{(p,q)}} - \frac{1}{[m+3]_{(p,q)}} \right)(\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))^2 \right]^{\frac{1}{l}},$$

where S_{11} is given by (2.13).

Theorem 16. *Suppose that all the assumptions of Lemma 5 are satisfied and $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$ is generalized strongly convex function with $\frac{1}{l}+\frac{1}{m}=1$, and $l,m>0$, then*

$$\left| \frac{p^2\Psi(\upsilon_1+\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q} - \frac{1}{p^2\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)} \int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1)} \Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau \right|$$
$$\leq \frac{pq^2(\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))^2}{p+q} \left(\frac{1}{[l+1]_{(p,q)}} \right)^{\frac{1}{l}} \left[S_{12}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m \right.$$
$$\left. + S_{13}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m - cS_{14}(\mathcal{R}(\phi,\psi;\eta;\upsilon_2-\upsilon_1))^2 \right]^{\frac{1}{m}},$$

where S_{12}, S_{13}, S_{14} are given by (2.14)–(2.16).

3.3. Applications to Mittag-Leffler functions

Moreover if we take $\sigma=(1,1,1,\ldots),\lambda=1$ and $\rho=\phi$ with $\mathrm{Re}(\phi)>0$ in (1.1), then we obtain well-known Mittag–Leffler function:

$$\mathcal{R}_\phi(x)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(1+\phi k)}x^k.$$

So using this function and all the results obtained in this paper, we have the following forms.

Lemma 6. *Let $\Psi : X=[\upsilon_1,\upsilon_1+\mathcal{R}_\phi(\upsilon_2-\upsilon_1)]\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function and $0<q<p\leq 1$. If ${}_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi$ is integrable function on X°, then*

$$\frac{1}{p\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau-\frac{p\Psi(\upsilon_1+\mathcal{R}_\phi(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}$$
$$=\frac{q\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}{p+q}\int_0^1(1-(p+q)\tau)_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}_\phi(\upsilon_2-\upsilon_1))_0\mathrm{d}_{(p,q)}\tau. \tag{3.3}$$

The second identity for twice (p,q)-differentiable functions states as follows:

Lemma 7. *Let $\Psi : X=[\upsilon_1,\upsilon_1+\mathcal{R}_\phi(\upsilon_2-\upsilon_1)]\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function and $0<q<p\leq 1$. If ${}_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi$ is integrable function on X°, then*

$$\frac{p^2\Psi(\upsilon_1+\mathcal{R}_\phi(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau$$
$$=\frac{pq^2(\mathcal{R}_\phi(\upsilon_2-\upsilon_1))^2}{p+q}\int_0^1\tau(1-q\tau)_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1+\tau\mathcal{R}_\phi(\upsilon_2-\upsilon_1))_0\mathrm{d}_{(p,q)}\tau. \tag{3.4}$$

Theorem 17. *Suppose that all the assumptions of Lemma 6 are satisfied and $|_{\upsilon_1}D_{(p,q)}\Psi|$ is generalized strongly convex function, then*

$$\left|\frac{1}{p\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau-\frac{p\Psi(\upsilon_1+\mathcal{R}_\phi(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}\right|$$
$$\leq\frac{q\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}{p+q}\left[S_1|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|+S_2|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|-cS_3(\mathcal{R}_\phi(\upsilon_2-\upsilon_1))^2\right],$$

where S_1, S_2 and S_3 are (2.3)–(2.5).

Theorem 18. *Suppose that all the assumptions of Lemma 6 are satisfied and $|_{\upsilon_1}D_{(p,q)}\Psi|^m$ is generalized strongly convex function with $m\geq 1$, then*

$$\left|\frac{1}{p\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau-\frac{p\Psi(\upsilon_1+\mathcal{R}_\phi(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}\right|$$
$$\leq\frac{q\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}{p+q}S_4^{1-\frac{1}{m}}\left[S_1|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|^m+S_2|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|^m-cS_3(\mathcal{R}_\phi(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}},$$

where S_1, S_2 S_3 S_4p are given by (2.3)–(2.6), respectively.

Theorem 19. *Suppose that all the assumptions of Lemma 6 are satisfied and $|_{\upsilon_1}D_{(p,q)}\Psi|^m$ is generalized strongly convex function with $\frac{1}{l}+\frac{1}{m}=1$, and $l,m>0$, then*

$$\left|\frac{1}{p\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau-\frac{p\Psi(\upsilon_1+\mathcal{R}_\phi(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}\right|$$
$$\leq\frac{q\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}{p+q}S_5^{\frac{1}{l}}\left[\frac{p+q-1}{p+q}|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_1)|^m+\frac{1}{p+q}|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi(\upsilon_2)|^m\right.$$
$$\left.-c\left(\frac{1}{p+q}-\frac{1}{p^2+pq+q^2}\right)(\mathcal{R}_\phi(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}},$$

where S_5 is given by (2.7).

Theorem 20. *Suppose that all the assumptions of Lemma 7 are satisfied and $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|$ is generalized strongly convex function, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}_\phi(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq\frac{pq^2(\mathcal{R}_\phi(\upsilon_2-\upsilon_1))^2}{p+q}\left[S_6|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|+S_7|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|-cS_8(\mathcal{R}_\phi(\upsilon_2-\upsilon_1))^2\right],$$

where S_6, S_7 and S_8 are given (2.8)–(2.10).

Theorem 21. *Suppose that all the assumptions of Lemma 7 are satisfied and $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$ is generalized strongly convex function with $m\geq 1$, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}_\phi(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$

$$\leq \frac{pq^2(\mathcal{R}_\phi(\upsilon_2-\upsilon_1))^2}{p+q} S_9^{1-\frac{1}{m}} \left[S_6|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m + S_7|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m - cS_8(\mathcal{R}_\phi(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}},$$

where S_6, S_7, S_8 and S_9 are given by (2.8)–(2.11), respectively.

Theorem 22. *Suppose that all the assumptions of Lemma 7 are satisfied and $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$ is generalized strongly convex function with $\frac{1}{l}+\frac{1}{m}=1$, and $l,m>0$, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}_\phi(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q} - \frac{1}{p^2\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}_\phi(\upsilon_2-\upsilon_1)} \Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}_\phi(\upsilon_2-\upsilon_1))^2}{p+q} S_{10}^{\frac{1}{l}}\left[\frac{p+q-1}{p+q}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m + \frac{1}{p+q}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m\right.$$
$$\left. - c\left(\frac{1}{p+q} - \frac{1}{p^2+pq+q^2}\right)(\mathcal{R}_\phi(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}}.$$

Theorem 23. *Suppose that all the assumptions of Lemma 7 are satisfied and $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$ is generalized strongly convex function with $\frac{1}{l}+\frac{1}{m}=1$, and $l,m>0$, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}_\phi(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q} - \frac{1}{p^2\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}_\phi(\upsilon_2-\upsilon_1)} \Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}_\phi(\upsilon_2-\upsilon_1))^2}{p+q} S_{11}^{\frac{1}{l}}\left[\left(\frac{1}{[m+1]_{(p,q)}} - \frac{1}{[m+2]_{(p,q)}}\right)|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m\right.$$
$$\left.+\frac{1}{[m+2]_{(p,q)}}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m - c\left(\frac{1}{[m+2]_{(p,q)}} - \frac{1}{[m+3]_{(p,q)}}\right)(\mathcal{R}_\phi(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{l}},$$

where S_{11} is given by (2.13).

Theorem 24. *Suppose that all the assumptions of Lemma 7 are satisfied and $|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi|^m$ is generalized strongly convex function with $\frac{1}{l}+\frac{1}{m}=1$, and $l,m>0$, then*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}_\phi(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q} - \frac{1}{p^2\mathcal{R}_\phi(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}_\phi(\upsilon_2-\upsilon_1)} \Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq \frac{pq^2(\mathcal{R}_\phi(\upsilon_2-\upsilon_1))^2}{p+q}\left(\frac{1}{[l+1]_{(p,q)}}\right)^{\frac{1}{l}}\left[S_{12}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_1)|^m\right.$$
$$\left.+S_{13}|_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi(\upsilon_2)|^m - cS_{14}(\mathcal{R}_\phi(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}},$$

where S_{12}, S_{13} and S_{14} are given by (2.14)–(2.16).

3.4. Applications to bounded functions

In this section, we discuss applications regarding bounded functions in absolute value of the results obtained from our main results. We suppose that the following two conditions are satisfied:

$$|_{\upsilon_1}\mathrm{D}_{(p,q)}\Psi| \leq \Delta_1 \quad \text{and} \quad |_{\upsilon_1}\mathrm{D}^2{}_{(p,q)}\Psi| \leq \Delta_2,$$

and $0<q<p\leq 1$.

Applying the above conditions, we have the following results.

Corollary 9. *Under the assumptions of Theorem 1, the following inequality holds*

$$\left|\frac{1}{p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau-\frac{p\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}\right|$$
$$\leq\frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q}\left[\Delta_1(S_1+S_2)-cS_3(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2\right].$$

Corollary 10. *Under the assumptions of Theorem 2, the following inequality holds*

$$\left|\frac{1}{p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau-\frac{p\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}\right|$$
$$\leq\frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q}S_4^{1-\frac{1}{m}}\left[\Delta_1^m(S_1+S_2)-cS_3(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}}.$$

Corollary 11. *Under the assumptions of Theorem 3, the following inequality holds*

$$\left|\frac{1}{p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau-\frac{p\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}\right|$$
$$\leq\frac{q\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}{p+q}S_5^{\frac{1}{l}}\left[\Delta_1^m-c\left(\frac{1}{p+q}-\frac{1}{p^2+pq+q^2}\right)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}}.$$

Corollary 12. *Under the assumptions of Theorem 4, the following inequality holds*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq\frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}\left[\Delta_2(S_6+S_7)-cS_8(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2\right].$$

Corollary 13. *Under the assumptions of Theorem 5, the following inequality holds*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq\frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}S_9^{1-\frac{1}{m}}\left[\Delta_2^m(S_6+S_7)-cS_8(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}}.$$

Corollary 14. *Under the assumptions of Theorem 22, the following inequality holds*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq\frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}S_{10}^{\frac{1}{l}}\left[\Delta_2^m-c\left(\frac{1}{p+q}-\frac{1}{p^2+pq+q^2}\right)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}}.$$

Corollary 15. *Under the assumptions of Theorem 7, the following inequality holds*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq\frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}S_{11}^{\frac{1}{l}}\left[\frac{\Delta_2^m}{[m+1]_{(p,q)}}-c\left(\frac{1}{[m+2]_{(p,q)}}-\frac{1}{[m+3]_{(p,q)}}\right)(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{l}}.$$

Corollary 16. *Under the assumptions of Theorem 8, the following inequality holds*

$$\left|\frac{p^2\Psi(\upsilon_1+\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))+q\Psi(\upsilon_1)}{p+q}-\frac{1}{p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_1+p^2\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1)}\Psi(\tau)_{\upsilon_1}\mathrm{d}_{(p,q)}\tau\right|$$
$$\leq\frac{pq^2(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2}{p+q}\left(\frac{1}{[l+1]_{(p,q)}}\right)^{\frac{1}{l}}\left[\Delta_2^m(S_{12}+S_{13})-cS_{14}(\mathcal{R}^{\sigma}_{\rho,\lambda}(\upsilon_2-\upsilon_1))^2\right]^{\frac{1}{m}}.$$

4. Conclusions

In this paper, we introduced the class of generalized strongly convex functions using Raina's function. We have derived two new general auxiliary results involving first and second order (p,q)-differentiable functions and Raina's function. Essentially using these identities and the generalized strongly convexity property of the functions, we also established corresponding new generalized post-quantum analogues of Dragomir-Agarwal's inequalities. We have discussed in details some special cases about generalized convex functions. The efficiency of our main results is also demonstrated with the help of application. We have offered applications to special means, to hypergeometric functions, to Mittag-Leffler functions and also to (p,q)-differentiable functions of first and second order that are bounded in absolute value. We will derive as future works several new post-quantum interesting inequalities using Chebyshev, Markov, Young and Minkowski inequalities. Since the class of generalized strongly convex functions have large applications in many mathematical areas, they can be applied to obtain several results in convex analysis, special functions, quantum mechanics, related optimization theory, and mathematical inequalities and may stimulate further research in different areas of pure and applied sciences. Studies relating convexity, partial convexity, and preinvex functions (as contractive operators) may have useful applications in complex interdisciplinary studies, such as maximizing the likelihood from multiple linear regressions involving Gauss-Laplace distribution. For more details, please see [16–23].

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Conflict of interest

The authors declare no conflict of interest.

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