AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(6): 11443–11463.
DOI: 10.3934/math.2022638
Received: 07 January 2022
Revised: 26 March 2022
Accepted: 10 April 2022
Published: 13 April 2022

Research article

Representations of modified type 2 degenerate poly-Bernoulli polynomials

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Abstract: Research on the degenerate versions of special polynomials provides a new area, introducing the λ-analogue of special polynomials and numbers, such as λ-Sheffer polynomials. In this paper, we propose a new variant of type 2 Bernoulli polynomials and numbers by modifying a generating function. Then we derive explicit expressions and their representations that provide connections among existing λ-Sheffer polynomials. Also, we provide the explicit representations of the proposed polynomials in terms of the degenerate Lah-Bell polynomials and the higher-order degenerate derangement polynomials to confirm the presented identities.

Keywords: degenerate polyexponential functions; degenerate poly-Bernoulli polynomials; λ-linear functionals; λ-Sheffer sequences
Mathematics Subject Classification: 05A19, 11B73, 11B83

1. Introduction

The degenerate Stirling, Bernoulli and Euler polynomials and numbers were first introduced by Carlitz [5, 6] with intriguing results of arithmetic and combinatorial identities. In recent years, much attention has been paid both to providing degenerate versions of special polynomials and numbers, and finding their relations and connections to known degenerate polynomials, as well as transcendental functions such as degenerate gamma functions and degenerate Laplace transforms [18]. In general, degenerate versions of special functions are provided by modifying generating functions in terms of the degenerate exponential functions and the degenerate logarithm function, (see [1, 7, 11, 12, 14, 16, 20, 21, 25] and the references therein). To investigate their arithmetic and combinatorial properties and relations, the research is usually performed by combinatorial methods, umbral calculus techniques, p-adic analysis, and differential equations. In particular, the relation between two λ-Sheffer sequences can be expressed by the representation formula (3.6) that is recently established [17] replacing linear functionals and differential operators by λ-functionals and λ-differential operators, respectively, in Rota's theory [26] of Sheffer sequences.

The aim of the paper is to introduce a new degenerate version of the type 2 poly-Bernoulli polynomials [21] called modified type 2 degenerate poly-Bernoulli polynomials (2.1) by modifying the generating function and present their properties as well as relations to other degenerate versions of polynomials. Also, we provide the computational examples to obtain explicit representations of the proposed polynomials for various k and λ based on two well known special polynomials: the degenerate Lah-Bell polynomials, and the higher-order degenerate derangement polynomials.

The rest of the paper is organized as follows: Section 2 introduces a new class of degenerate type 2 poly-Bernoulli polynomials and related numbers. Then, we present their properties and expressions in terms of the degenerate Stirling numbers of the second kind and the degenerate type 2 Bernoulli polynomials. Section 3 reviews the λ-linear functionals, related λ-differential operators, and λ-Sheffer polynomials, and presents the connection between the proposed polynomials and the known degenerate polynomials, such as the type 2 degenerate Bernoulli polynomials, the degenerate Hermite polynomials, the degenerate derangement polynomials, the higher-order degenerate Euler polynomials, and the degenerate Lah-Bell polynomials. Section 4 presents several specific examples to confirm the computed results between the proposed polynomials and the existing degenerate type polynomials, the degenerate Lah-Bell polynomials and the degenerate derangement polynomials. Section 5 concludes the paper. Before closing this section, we recall several definitions to introduce our new type of poly-Bernoulli polynomials.

Throughout this paper, we denote $\mathbb{N}$ the set of all natural numbers and $\mathbb{Z}$ be the set of all integers. The Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers B_n for $\mathbb{N} \cup \{0\}$ are respectively given by the generating functions (see [2, 16, 21] for details):

$$\frac{t}{e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \text{and} \quad \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad (|t| \ll 1). \tag{1.1}$$

The degenerate Bernoulli polynomials $B_{n,\lambda}(x)$ and the degenerate Bernoulli numbers $B_{n,\lambda}$ are respectively defined by (see [5, 6] for detail)

$$\frac{t}{e_\lambda(t) - 1} e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!} \quad \text{and} \quad \frac{t}{e_\lambda(t) - 1} = \sum_{n=0}^{\infty} B_{n,\lambda} \frac{t^n}{n!}. \tag{1.2}$$

Here, the degenerate exponential functions $e_\lambda^x(t)$ for $\lambda \in \mathbb{R} \backslash \{0\}$ are given by

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad e_\lambda(t) = e_\lambda^1(t) = \sum_{n=0}^{\infty} (1)_{n,\lambda} \frac{t^n}{n!}, \tag{1.3}$$

where $(x)_{n,\lambda}$ is the λ-falling factorial sequence defined by (see [21])

$$(x)_{n,\lambda} = x(x - \lambda)(x - 2\lambda) \cdots (x - (n-1)\lambda) \text{ for } n \geq 1 \text{ and } (x)_{0,\lambda} = 1. \tag{1.4}$$

The degenerate Bernoulli polynomials $B_{n,\lambda}^{(r)}(x)$ of order $r \geq 1$ are defined by (see [16])

$$\left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} \quad \text{and} \quad \left(\frac{t}{e_\lambda(t)-1}\right)^r = \sum_{n=0}^{\infty} B_{n,\lambda}^{(r)}\frac{t^n}{n!}. \tag{1.5}$$

The degenerate type 2-Bernoulli polynomials $\beta_{n,\lambda}(x)$ in [16] are considered as

$$\frac{t}{e_\lambda(t)-e_\lambda^{-1}(t)}e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}, \quad (|t| \ll 1), \tag{1.6}$$

and the related extension, the type 2 degenerate Bernoulli polynomials $\beta_{n,\lambda}^{(r)}(x)$ of order r is introduced (see [19])

$$\left(\frac{t}{e_\lambda(t)-e_\lambda^{-1}(t)}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \quad (|t| \ll 1). \tag{1.7}$$

The type 2 degenerate Euler polynomials $\mathcal{E}_{n,\lambda}(x)$ are introduced in [12] based on the following generating function:

$$\frac{2}{e_\lambda(t)+e_\lambda^{-1}(t)}e_\lambda^x(t) = \sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}, \quad (|t| \ll 1). \tag{1.8}$$

Next, we briefly recall some basic functions used throughout the paper. The polylogarithm functions $\mathrm{Li}_k(t)$ for $k \in \mathbb{Z}$ are defined by (see [13, 19])

$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty}\frac{t^n}{n^k} \quad (|t| < 1).$$

Note that $\mathrm{Li}_k(t)$ for $k = 1$ satisfy

$$\mathrm{Li}_1(t) = \sum_{n=1}^{\infty}\frac{t^n}{n} = -\log(1-t).$$

The degenerate polylogarithm functions $\mathrm{Li}_{k,\lambda}(t)$ for $k \in \mathbb{Z}$ are given by (see [20])

$$\mathrm{Li}_{k,\lambda}(t) = \sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k}t^n \quad (|t| < 1). \tag{1.9}$$

The degenerate Stirling numbers $S_{2,\lambda}(n,m)$ of the second kind are defined by (see [23, 24])

$$\frac{1}{m!}(e_\lambda(t)-1)^m e_\lambda^x(t) = \sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{t^n}{n!}, \quad (n \geq 0). \tag{1.10}$$

Note that (see [23])

$$(t)_{n,\lambda} = \sum_{m=0}^{n}S_{2,\lambda}(n,m)(t)_m, \quad (n \geq 0),$$

where $(t)_n$ is the falling factorial sequence defined by (see [16])

$$(t)_n = t(t-1)(t-2)\cdots(t-(n-1)) \text{ for } n \geq 1 \text{ and } (t)_0 = 1.$$

2. Modified type 2 degenerate poly-Bernoulli polynomials

In this section, we introduce a new type of degenerate poly-Bernoulli polynomials and numbers and discuss their properties and representations in relation to the degenerate type 2-Bernoulli polynomials.

Definition 2.1. *The modified type 2 degenerate poly-Bernoulli polynomials* $\mathfrak{B}_{n,\lambda}^{(k)}(x)$ *for* $k \in \mathbb{Z}$ *are defined by the generating function*

$$\frac{Li_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}e_\lambda^x(t)=\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}.$$

In the case of $x=0$, *we call* $\mathfrak{B}_{n,\lambda}^{(k)} := \mathfrak{B}_{n,\lambda}^{(k)}(0)$ *the modified type 2 degenerate poly-Bernoulli numbers.*

Remark 2.2. *By noting that* $Li_{1,\lambda}(1-e_\lambda(-t)) = t$ *for* $k=1$, $\mathfrak{B}_{n,\lambda}^{(1)}(x)$ *yield the degenerate type 2-Bernoulli polynomials* $\beta_{n,\lambda}(x)$ *listed in* (1.6), *we can show that* $\lim_{\lambda\to 0}\mathfrak{B}_{n,\lambda}^{(1)}(x)=\beta_n(x)$ *for* $k=1$, *where* $\beta_n(x)$ *are type 2-Bernoulli polynomials [13] that satisfy*

$$\lim_{\lambda\to 0}\frac{t}{e_\lambda(t)-e_\lambda(-t)}e_\lambda^x(t)=\frac{t}{e^t-e^{-t}}e^{xt}=\sum_{n=0}^{\infty}\beta_n(x)\frac{t^n}{n!}.$$

We next state the expression for $\mathfrak{B}_{n,\lambda}^{(k)}(x)$ as the sum of the products of the modified type 2 degenerate poly-Bernoulli numbers and λ-falling factorial sequence.

Theorem 2.3. *Let k be any integer. Then the following identity holds true for all* $n \geq 0$,

$$\mathfrak{B}_{n,\lambda}^{(k)}(x)=\sum_{m=0}^{n}\binom{n}{m}\left(\mathfrak{B}_{m,\lambda}^{(k)}\right)(x)_{n-m,\lambda}, \tag{2.1}$$

where $(x)_{n,\lambda}$ *is the* λ*-falling factorial sequence in* (1.4).

Proof. From Definition 2.1, we can consider the generating series of the polynomials $\mathfrak{B}_{n,\lambda}^{(k)}$ as the indicated product:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &=\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}e_\lambda^x(t)\\ &=\left(\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k)}(0)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!}\right)\\ &=\sum_{m=0}^{n}\binom{n}{m}\left(\mathfrak{B}_{m,\lambda}^{(k)}\right)(x)_{n-m,\lambda}\frac{t^n}{n!}.\end{aligned}$$

By the comparison of coefficients, we have the desired identity. □

Remark 2.4. *Since* $\frac{d}{dt}e_\lambda(-t)=\frac{-1}{1-\lambda t}e_\lambda(-t)$ *and* $1-\lambda t = e_\lambda^\lambda(-t)$, *we have*

$$\begin{aligned}\frac{d}{dt}Li_{k,\lambda}(1-e_\lambda(-t)) &= \sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k}\frac{d}{dt}(1-e_\lambda(-t))^n\\ &=\frac{e_\lambda(-t)}{(1-\lambda t)(1-e_\lambda(-t))}Li_{k-1,\lambda}(1-e_\lambda(-t))\\ &=\frac{e_\lambda^{1-\lambda}(-t)}{1-e_\lambda(-t)}Li_{k-1,\lambda}(1-e_\lambda(-t))\end{aligned}$$

for $k \geq 2$. By the multiple integrals of $\mathrm{Li}_{1,\lambda}(1-e_\lambda(-t))$, the function $\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))$ is computed, and $\mathfrak{B}^{(k)}_{n,\lambda}(x)$ can be expressed as

$$\sum_{n=0}^{\infty}\mathfrak{B}^{(k)}_{n,\lambda}(x)\frac{t^n}{n!}=\frac{e^x_\lambda(t)}{e_\lambda(t)-e^{-1}_\lambda(t)}\int_0^t\frac{e^{1-\lambda}_\lambda(-t_1)}{1-e_\lambda(-t_1)}\int_0^{t_1}\frac{e^{1-\lambda}_\lambda(-t_2)}{1-e_\lambda(-t_2)}\cdots\int_0^{t_{k-2}}\frac{e^{1-\lambda}_\lambda(-t_{k-1})}{1-e_\lambda(-t_{k-1})}t_{k-1}dt_{k-1}\cdots dt_2dt_1,$$

in which $k-1$ times of integrals are performed by the product with $\frac{e^{1-\lambda}_\lambda(-t_{k-1})}{1-e_\lambda(-t_{k-1})}$ each time. For example, when $k=2$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}^{(2)}_{n,\lambda}(x)\frac{t^n}{n!}&=\frac{e^x_\lambda(t)}{e_\lambda(t)-e^{-1}_\lambda(t)}\int_0^t\frac{e^{1-\lambda}_\lambda(-t_1)}{1-e_\lambda(-t_1)}t_1dt_1\\&=\frac{e^x_\lambda(t)}{e_\lambda(t)-e^{-1}_\lambda(t)}\int_0^t\sum_{n=0}^{\infty}\beta_{n,\lambda}(1-\lambda)(-1)^n\frac{t_1^n}{n!}dt_1\\&=\frac{te^x_\lambda(t)}{e_\lambda(t)-e^{-1}_\lambda(t)}\sum_{m=0}^{\infty}\frac{\beta_{m,\lambda}(1-\lambda)}{m+1}\frac{(-1)^mt^m}{m!}\\&=\left(\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\beta_{m,\lambda}(1-\lambda)\frac{(-1)^m}{m+1}\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\beta_{n-m,\lambda}(x)\beta_{m,\lambda}(1-\lambda)\frac{(-1)^m}{m+1}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.2}$$

Thus, by comparing coefficients in both sides of (2.2), we have that the identity holds

$$\mathfrak{B}^{(2)}_{n,\lambda}(x)=\sum_{m=0}^{n}\binom{n}{m}\beta_{n-m,\lambda}(x)\beta_{m,\lambda}(1-\lambda)\frac{(-1)^m}{m+1}.$$

Next, we can express the modified type 2 degenerate poly-Bernoulli polynomials in terms of the degenerate Stirling numbers of the second kind and the degenerate type 2-Bernoulli polynomials (1.6).

Theorem 2.5. *For $n \geq 0$ and $\lambda \in \mathbb{R}\backslash\{0\}$, the following identity holds:*

$$\mathfrak{B}^{(k)}_{n,\lambda}(x)=\sum_{m=0}^{n}\binom{n}{m}\beta_{n-m,\lambda}(x)\sum_{\ell=1}^{m+1}\frac{\lambda^{\ell-1}(1)_{\ell,1/\lambda}(-1)^m}{(m+1)\ell^{k-1}}S_{2,\lambda}(m+1,\ell),$$

where $S_{2,\lambda}(n,m)$ are the degenerate Stirling numbers of the second kind defined in (1.10).

Proof. We first note that the degenerate polylogarithm function (1.9) for $0<\lambda<1$ satisfies

$$\begin{aligned}\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{t}&=\frac{1}{t}\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{(m-1)!m^k}(1-e_\lambda(-t))^m\\&=\frac{1}{t}\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^{k-1}}\frac{1}{m!}(1-e_\lambda(-t))^m\\&=\sum_{m=1}^{\infty}\frac{(-\lambda)^{m-1}(1)_{m,1/\lambda}}{m^{k-1}}\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)(-1)^{n+m}\frac{t^{n-1}}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=1}^{n+1}\frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^n}{(n+1)m^{k-1}}S_{2,\lambda}(n+1,m)\right)\frac{t^n}{n!}.\end{aligned}\tag{2.3}$$

From (2.3) and Definition 2.1, we can see that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}&=\frac{e_\lambda^x(t)}{e_\lambda(t)-e_\lambda^{-1}(t)}\mathrm{Li}_{k,\lambda}\left(1-e_\lambda(-t)\right)\\&=\left(\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\sum_{m=1}^{n+1}\frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^n}{(n+1)m^{k-1}}S_{2,\lambda}(n+1,m)\right)\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\beta_{n-m,\lambda}(x)\sum_{\ell=1}^{m+1}\frac{\lambda^{\ell-1}(1)_{\ell,1/\lambda}(-1)^m}{(m+1)\ell^{k-1}}S_{2,\lambda}(m+1,\ell)\right)\frac{t^n}{n!}.\end{aligned}\tag{2.4}$$

Comparison of the coefficients on both sides concludes the desired result for $\lambda\neq 0$. □

Using Theorem 2.3 and Eq (2.3), we have the following result.

Theorem 2.6. *For $n\geq 0$ and $k\in\mathbb{Z}$, the following relation holds:*

$$\mathfrak{B}_{n,\lambda}^{(k)}(1)-\mathfrak{B}_{n,\lambda}^{(k)}(-1)=\sum_{m=1}^{n}\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{n-1}m^{-k+1}S_{2,\lambda}(n,m).\tag{2.5}$$

Proof. First, notice that, from (2.3), we have

$$\mathrm{Li}_{k,\lambda}\left(1-e_\lambda(-t)\right)=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{\lambda^{m-1}(1)_{m,1/\lambda}(-1)^{n-1}}{m^{k-1}}S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}.\tag{2.6}$$

On the other hand, by the virtue of the result of Theorem 2.3, it follows that

$$\begin{aligned}\mathrm{Li}_{k,\lambda}\left(1-e_\lambda(-t)\right)&=\left(e_\lambda(t)-e_\lambda^{-1}(t)\right)\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k)}\frac{t^n}{n!}\\&=\sum_{m=0}^{\infty}\left((1)_{m,\lambda}-(-1)_{m,\lambda}\right)\frac{t^m}{m!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,\lambda}^{(k)}\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\left(\mathfrak{B}_{m,\lambda}^{(k)}(1)_{n-m,\lambda}-\mathfrak{B}_{m,\lambda}^{(k)}(-1)_{n-m,\lambda}\right)\right)\frac{t^n}{n!}\\&=\sum_{n=1}^{\infty}\left(\mathfrak{B}_{n,\lambda}^{(k)}(1)-\mathfrak{B}_{n,\lambda}^{(k)}(-1)\right)\frac{t^n}{n!},\end{aligned}\tag{2.7}$$

which shows the desired result by comparison of the coefficients of (2.6) and (2.7). □

3. Representations of Modified Type 2 Degenerate poly-Bernoulli Polynomials

In this section, we recall a family of λ-linear functionals on the space of polynomials, λ-differential operators related to the family of λ-linear functionals, and λ-Sheffer sequences. The details on these concepts and definitions can be found in [2–4, 8, 10, 17] and the references therein.

Let $\mathcal{F}$ be the algebra of exponential formal power series in t over the field $\mathbb{C}$ of complex numbers

$$\mathcal{F}=\left\{f(t)=\sum_{n=0}^{\infty}a_n\frac{t^n}{n!}\,\middle|\,a_n\in\mathbb{C}\right\},$$

and let $\mathbb{P}$ be the algebra of polynomials in x over $\mathbb{C}$, i.e.,

$$\mathbb{P} = \mathbb{C}[x] = \left\{ \sum_{k=0}^{\infty} a_k x^k \middle| a_k \in \mathbb{C} \text{ with } a_k = 0 \text{ for all but finite number of } i \right\}.$$

Recall that $((x)_{n,\lambda})_{n\in\mathbb{N}}$ form a basis for the $\mathbb{C}$-vector space $\mathbb{P}$. Then, the λ-linear functional $\langle f(t)|\cdot\rangle_\lambda$ on $\mathbb{P}$ for $f(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \in \mathcal{F}$ is defined by

$$\langle f(t)|(x)_{n,\lambda}\rangle_\lambda = a_n, \tag{3.1}$$

and satisfies

$$\langle t^k|(x)_{n,\lambda}\rangle_\lambda = n!\delta_{n,k}, \tag{3.2}$$

where $\delta_{n,k}$ is the Kronecker delta.

The order $o(f(t))$ of the formal power series for a nontrivial $f(t)$ is the smallest integer k for which a_k does not vanish. In particular, $f(t)$ is called a delta series if $o(f(t)) = 1$, while $f(t)$ is called an invertible series if $o(f(t)) = 0$, (see [17] for details).

In [17], the λ- differential operator $(t^k)_\lambda$ of order k on $\mathbb{P}$ is defined by

$$(t^k)_\lambda(x)_{n,\lambda} = \begin{cases} (n)_k(x)_{n-k,\lambda} & \text{if } 0 \le k \le n, \\ 0, & \text{if } k > n, \end{cases} \tag{3.3}$$

where $(n)_k = n(n-1)(n-2)\cdots(n-k+1)$ for $n \ge 1$ and $(n)_0 = 1$. In general, for $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$, the λ-differential operator $(f(t))_\lambda$ is defined by

$$(f(t))_\lambda (x)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} a_k (x)_{n-k,\lambda}. \tag{3.4}$$

Or equivalently, $(f(t))_\lambda$ can be expressed by

$$(f(t))_\lambda = \sum_{k=0}^{\infty} \frac{a_k}{k!}(t^k)_\lambda.$$

In particular, when $f(t) = e_\lambda^x(t)$ where $e_\lambda^x(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}$, $(f(t))_\lambda$ satisfies

$$(e_\lambda^x(t))_\lambda (y)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} (y)_{k,\lambda}(x)_{n-k,\lambda} = (x+y)_{n,\lambda}.$$

For $f(t)$ a delta series and $g(t)$ an invertible series, i.e., $o(f(t)) = 1$ and $o(g(t)) = 0$, there exists a unique sequence $s_{n,\lambda}(x)$ of polynomials $deg(s_{n,\lambda}(x)) = n$ satisfying the orthogonality condition

$$\left\langle g(t)(f(t))^k \middle| s_{n,\lambda}(x) \right\rangle_\lambda = n!\delta_{n,k}, \quad (n, k \ge 0).$$

Here, $s_{n,\lambda}(x)$ is called the λ-Sheffer sequence for $(g(t), f(t))$ and is denoted by $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$.

We recall that $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ if and only if

$$\frac{1}{g(\bar{f}(t))} e_\lambda^x(\bar{f}(t)) = \sum_{n=0}^{\infty} \frac{s_{n,\lambda}(x)}{n!} t^n, \tag{3.5}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ and satisfies $\bar{f}(f(t)) = f\left(\bar{f}(t)\right) = t$.

For given λ-Sheffer sequences $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ and $r_{n,\lambda}(x) \sim (h(t), \ell(t))_\lambda$, the following relation holds

$$s_{n,\lambda}(x) = \sum_{k=0}^{n} c_{n,k} r_{k,\lambda}(x), \tag{3.6}$$

where $c_{n,k}$ is obtained by

$$c_{n,k} = \frac{1}{k!} \left\langle \frac{h\left(\bar{f}(t)\right)}{g\left(\bar{f}(t)\right)} \left(\ell(\bar{f}(t))\right)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda . \tag{3.7}$$

Therefore, since $(x)_{n,\lambda} \sim (1, t)_\lambda$, any λ-Sheffer sequence $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ satisfies

$$s_{n,\lambda}(x) = \sum_{k=0}^{n} \frac{1}{k!} \left\langle \frac{1}{g(\bar{f}(t))} (\bar{f}(t))^k \middle| (x)_{n,\lambda} \right\rangle_\lambda (x)_{k,\lambda}. \tag{3.8}$$

Remark 3.1. *From* (3.1)*, we have*

$$\left\langle \frac{Li_{k,\lambda}(1 - e_\lambda(-t))}{e_\lambda(t) - e_\lambda^{-1}(t)} t^\ell \middle| (x)_{n,\lambda} \right\rangle_\lambda = \sum_{m=0}^{\infty} \mathfrak{B}_{m,\lambda}^{(k)} \frac{1}{m!} \left\langle t^{m+\ell} \middle| (x)_{n,\lambda} \right\rangle_\lambda = \ell! \binom{n}{\ell} \mathfrak{B}_{n-\ell,\lambda}^{(k)},$$

so that (3.8) *yields*

$$\mathfrak{B}_{n,\lambda}^{(k)}(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} \left(\mathfrak{B}_{n-\ell,\lambda}^{(k)}\right) (x)_{\ell,\lambda}. \tag{3.9}$$

Further, using $(x)_{\ell,\lambda} = \sum_{m=0}^{\ell} \sum_{j=0}^{m} S_{2,\lambda}(\ell, m) S_{1,\lambda}(m, j)(x)_{j,\lambda}$ *in* (3.9)*, the identity holds*

$$\mathfrak{B}_{n,\lambda}^{(k)}(x) = \sum_{m=0}^{n} \sum_{j=0}^{m} \sum_{\ell=0}^{m-j} \binom{\ell + j}{j} S_{2,\lambda}(n, m) S_{1,\lambda}(m, \ell + j) \left(\mathfrak{B}_{j,\lambda}^{(k)}\right) (x)_{\ell,\lambda},$$

where $S_{1,\lambda}(n, m)$ *are the degenerate Stirling numbers of the first kind [24] given by*

$$\frac{1}{m!} \left(\log_\lambda(1 + t)\right)^m = \sum_{n=m}^{\infty} S_{1,\lambda}(n, m) \frac{t^n}{n!} \quad (m \geq 0).$$

Now, we explore the representations of the modified type 2 degenerate poly-Bernoulli polynomials in terms of various known degenerate polynomials by using (3.6). We first note that from (3.5), the corresponding λ-Sheffer sequence for $\mathfrak{B}_{n,\lambda}^{(k)}(x)$ is given by

$$\mathfrak{B}_{n,\lambda}^{(k)}(x) \sim \left(\frac{e_\lambda(t) - e_\lambda^{-1}(t)}{\mathrm{Li}_{k,\lambda}\left(1 - e_\lambda(-t)\right)}, t \right)_\lambda . \tag{3.10}$$

Theorem 3.2. *Let $\beta_{n,\lambda}(x)$ be the degenerate type 2-Bernoulli polynomials* (1.6)*. Then for any $n \geq 0$ and integer $k \in \mathbb{Z}$, we have*

$$\mathfrak{B}^{(k)}_{n,\lambda}(x) = \sum_{j=0}^{n}\sum_{\ell=1}^{n-j+1}\binom{n}{j}\frac{\lambda^{\ell-1}(1)_{\ell,1/\lambda}(-1)^{n-j}}{\ell^{k-1}}\frac{S_{2,\lambda}(n-j+1,\ell)}{(n-j+1)}\beta_{j,\lambda}(x).$$

Proof. From (1.6), (2.3), (3.1), (3.2), and Definition 2.1, the following computations are established.

$$\begin{aligned}\mathfrak{B}^{(k)}_{n,\lambda}(y) &= \left\langle \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}e_\lambda^{y}(t)\right| (x)_{n,\lambda}\Big\rangle_\lambda \\ &= \left\langle \frac{1}{t}\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))\right|\left(\frac{t}{e_\lambda(t)-e_\lambda^{-1}(t)}e_\lambda^{y}(t)\right)_\lambda (x)_{n,\lambda}\Big\rangle_\lambda \\ &= \sum_{j=0}^{n}\binom{n}{j}\beta_{j,\lambda}(y)\left\langle \frac{1}{t}\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))\right| (x)_{n-j,\lambda}\Big\rangle_\lambda \\ &= \sum_{j=0}^{n}\binom{n}{j}\beta_{j,\lambda}(y)\sum_{m=0}^{\infty}\left(\sum_{\ell=1}^{m+1}\frac{\lambda^{\ell-1}(1)_{\ell,1/\lambda}(-1)^{m}}{\ell^{k-1}}\frac{S_{2,\lambda}(m+1,\ell)}{(m+1)!}\right)\left\langle t^m \mid (x)_{n-j,\lambda}\right\rangle_\lambda \\ &= \sum_{j=0}^{n}\binom{n}{j}\beta_{j,\lambda}(y)\left(\sum_{\ell=1}^{n-j+1}\frac{\lambda^{\ell-1}(1)_{\ell,1/\lambda}(-1)^{n-j}}{\ell^{k-1}}\frac{S_{2,\lambda}(n-j+1,\ell)}{(n-j+1)!}\right)(n-j)! \\ &= \sum_{j=0}^{n}\sum_{\ell=1}^{n-j+1}\binom{n}{j}\beta_{j,\lambda}(y)\frac{\lambda^{\ell-1}(1)_{\ell,1/\lambda}(-1)^{n-j}}{\ell^{k-1}}\frac{S_{2,\lambda}(n-j+1,\ell)}{(n-j+1)},\end{aligned}$$

which yields the result. □

Next, we consider the type 2 degenerate Bernoulli polynomials of order r defined in (1.7), which satisfy the following λ-Sheffer sequence

$$\beta^{(r)}_{n,\lambda}(x) \sim \left(\frac{(e_\lambda(t)-e_\lambda^{-1}(t))^r}{t^r}, t\right)_\lambda.$$

Theorem 3.3. *For any $n \geq 0$ and integer $k \in \mathbb{Z}$, we have*

$$\mathfrak{B}^{(k)}_{n,\lambda}(x) = \sum_{m=0}^{n}\binom{n}{m}\sum_{\ell=0}^{n-m}\frac{S_{2,\lambda}(\ell+r,r)}{\binom{\ell+r}{r}}\binom{n-m}{\ell}\mathfrak{B}^{(k)}_{n-m-\ell,\lambda}\beta^{(r)}_{m,\lambda}(x).$$

Proof. Let $\mathfrak{B}^{(k)}_{n,\lambda}(x) = \sum_{m=0}^{n} c_{n,m}\beta^{(r)}_{m,\lambda}(x)$, $c_{n,m} \in \mathbb{C}$. Then, by (3.8) with (3.7), we can obtain

$$\begin{aligned}c_{n,m} &= \frac{1}{m!}\left\langle \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}\frac{(e_\lambda(t)-e_\lambda^{-1}(t))^r}{t^r}t^m\right| (x)_{n,\lambda}\Big\rangle_\lambda \\ &= \binom{n}{m}\left\langle \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}\frac{(e_\lambda(t)-e_\lambda^{-1}(t))^r}{t^r}\right| (x)_{n-m,\lambda}\Big\rangle_\lambda \\ &= \binom{n}{m}\sum_{\ell=0}^{n-m}\frac{S_{2,\lambda}(\ell+r,r)}{\binom{\ell+r}{r}\ell!}\left\langle \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}t^\ell\right| (x)_{n-m,\lambda}\Big\rangle_\lambda\end{aligned}$$

$$= \binom{n}{m} \sum_{\ell=0}^{n-m} \frac{S_{2,\lambda}(\ell+r,r)}{\binom{\ell+r}{r}} \binom{n-m}{\ell} \left\langle \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)} \middle| (x)_{n-m-\ell,\lambda} \right\rangle_\lambda ,$$

which yields the conclusion from $\left\langle \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)} \middle| (x)_{n-m-\ell,\lambda} \right\rangle_\lambda = \mathfrak{B}^{(k)}_{n-m-\ell,\lambda}$. □

We also consider the degenerate Hermite polynomials [9], which are defined by

$$e_\lambda^{-1}\left(t^2\right) e_\lambda^x(2t) = \sum_{n=0}^{\infty} H_{n,\lambda}(x) \frac{t^n}{n!},$$

where $H_{n,\lambda}(x)$ can be considered to be the λ-Sheffer sequence for $\left(e_\lambda(\frac{1}{4}t^2), \frac{t}{2}\right)$, or equivalently,

$$H_{n,\lambda} \sim \left(e_\lambda\left(\frac{1}{4}t^2\right), \frac{t}{2}\right)_\lambda .$$

Let $\mathbb{P}_n = \{p(x) \in \mathbb{C}[x] | degp(x) \leq n\}$ be an $(n+1)$ dimensional vector space over $\mathbb{C}$. Since the degenerate Hermite polynomials form a basis of the vector space $\mathbb{P}_n$, for every polynomial $p(x)$ of degree n we have

$$p(x) = \sum_{k=0}^{n} c_{n,k} H_{k,\lambda}(x), \tag{3.11}$$

where $c_{n,k}$ satisfies

$$\begin{aligned} \left\langle e_\lambda\left(\frac{1}{4}t^2\right)\left(\frac{t}{2}\right)^m \middle| p(x) \right\rangle_\lambda &= \sum_{k=0}^{n} c_{n,k} \left\langle e_\lambda\left(\frac{1}{4}t^2\right)\left(\frac{t}{2}\right)^m \middle| H_{k,\lambda}(x) \right\rangle_\lambda \\ &= \sum_{k=0}^{n} c_{n,k} m! \delta_{m,k} = c_{n,m} m!. \end{aligned} \tag{3.12}$$

Thus, we have an explicit form of $c_{n,k}$.

Theorem 3.4. *For any $n \geq 0$ and integer $k \in \mathbb{Z}$, we have*

$$\mathfrak{B}^{(k)}_{n,\lambda}(x) = n! \sum_{\ell=0}^{n} \left(\sum_{\substack{0 \leq m \leq n-\ell \\ m:\ even}} \frac{(1)_{\frac{m}{2},\lambda}}{2^{m+\ell}(m/2)!(n-\ell-m)!\ell!} \mathfrak{B}^{(k)}_{n-\ell-m,\lambda} \right) H_{\ell,\lambda}(x).$$

Proof. Taking $p(x) = \mathfrak{B}^{(k)}_{m,\lambda}(x)$ in (3.11), we have

$$\mathfrak{B}^{(k)}_{m,\lambda}(x) = \sum_{\ell=0}^{n} c_{n,\ell} H_{\ell,\lambda}(x).$$

From (1.3), (3.7) and (3.12), we get

$$c_{n,\ell} = \frac{1}{\ell!} \left\langle e_\lambda\left(\frac{1}{4}t^2\right) \left(\frac{e_\lambda(t)-e_\lambda^{-1}(t)}{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))} \right)^{-1} \left(\frac{t}{2}\right)^\ell \middle| (x)_{n,\lambda} \right\rangle_\lambda$$

$$
\begin{aligned}
&=\frac{1}{2^{\ell}\ell!}\left\langle \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}e_\lambda\left(\frac{1}{4}t^2\right)\middle| (t^\ell)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
&=\frac{1}{2^{\ell}}\binom{n}{\ell}\left\langle \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}\middle| \left(e_\lambda\left(\frac{1}{4}t^2\right)\right)_\lambda (x)_{n-\ell,\lambda}\right\rangle_\lambda \\
&=\frac{1}{2^{\ell}}\binom{n}{\ell}\sum_{m=0}^{\left[\frac{n-\ell}{2}\right]}\frac{(1)_{m,\lambda}}{4^m m!}(n-\ell)_{2m}\left\langle \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}\middle| (x)_{n-\ell-2m,\lambda}\right\rangle_\lambda \\
&=\frac{1}{2^{\ell}}\binom{n}{\ell}\sum_{m=0}^{\left[\frac{n-\ell}{2}\right]}\frac{(1)_{m,\lambda}}{2^{2m} m!}(n-\ell)_{2m}\mathfrak{B}^{(k)}_{n-\ell-2m,\lambda} \\
&=n!\sum_{\substack{0\le m\le n-\ell\\ m:\ \text{even}}}\frac{(1)_{\frac{m}{2},\lambda}}{2^{m+\ell}(m/2)!(n-\ell-m)!\ell!}\mathfrak{B}^{(k)}_{n-\ell-m,\lambda},
\end{aligned}
$$

and yield the conclusive result. □

Next, let $d^{(r)}_{n,\lambda}(x)$ be the degenerate derangement polynomials [22] of order $r\in\mathbb{N}$ given by

$$\frac{1}{(1-t)^r}e_\lambda^{-1}(t)e_\lambda^x(t)=\sum_{n=0}^{\infty}d^{(r)}_{n,\lambda}(x)\frac{t^n}{n!}.$$

These polynomials form a λ-Sheffer sequence such that

$$d^{(r)}_{n,\lambda}(x)\sim\left((1-t)^r e_\lambda(t),t\right)_\lambda.$$

Then, we have the following representation of $\mathfrak{B}^{(k)}_{n,\lambda}(x)$.

Theorem 3.5. *For any $n\geq 0$ and integer $k\in\mathbb{Z}$, we have*

$$\mathfrak{B}^{(k)}_{n,\lambda}(x)=\sum_{\ell=0}^{n}\left(\binom{n}{\ell}\sum_{j=0}^{r}\binom{r}{j}\binom{n-\ell}{j}(-1)^j j!\mathfrak{B}^{(k)}_{n-\ell-j,\lambda}(1)\right)d^{(r)}_{\ell,\lambda}(x).$$

Proof. Let $p(x)=\mathfrak{B}^{(k)}_{n,\lambda}(x)\in\mathbb{P}_n$. Then we can express that

$$\mathfrak{B}^{(k)}_{n,\lambda}(x)=\sum_{\ell=0}^{n}c_{n,\ell}d^{(r)}_{\ell,\lambda}(x),$$

where $c_{n,\ell}$ satisfy

$$
\begin{aligned}
c_{n,\ell}&=\frac{1}{\ell!}\left\langle (1-t)^r e_\lambda(t)t^\ell\middle| \mathfrak{B}^{(k)}_{n,\lambda}(x)\right\rangle_\lambda \\
&=\binom{n}{\ell}\left\langle (1-t)^r\middle| \mathfrak{B}^{(k)}_{n-\ell,\lambda}(x+1)\right\rangle_\lambda \\
&=\binom{n}{\ell}\sum_{j=0}^{r}\binom{r}{j}\binom{n-\ell}{j}(-1)^j j!\left\langle 1\middle| \mathfrak{B}^{(k)}_{n-\ell-j,\lambda}(x+1)\right\rangle_\lambda
\end{aligned}
$$

$$=\binom{n}{\ell}\sum_{j=0}^{r}\binom{r}{j}\binom{n-\ell}{j}(-1)^j j!\mathfrak{B}^{(k)}_{n-\ell-j,\lambda}(1),$$

which provides the conclusion. □

Remark 3.6. *In the case of $r = 1$ in the previous result, the modified type 2 degenerate poly-Bernoulli polynomials have the relation with the degenerate derangement polynomials of order 1 as*

$$\mathfrak{B}^{(k)}_{n,\lambda}(x) = \sum_{\ell=0}^{n}\left(\binom{n}{\ell}\mathfrak{B}^{(k)}_{n-\ell,\lambda}(1) - n\binom{n-1}{\ell}\mathfrak{B}^{(k)}_{n-\ell-1,\lambda}(1)\right)d^{(1)}_{\ell,\lambda}(x).$$

Recalling that the degenerate Bernoulli polynomials [6] of order r are given by

$$\left(\frac{t}{e_\lambda(t)-1}\right)^r e^x_\lambda(t) = \sum_{n=0}^{\infty} B^{(r)}_{n,\lambda}(x)\frac{t^n}{n!},$$

we obtain the following relation.

Theorem 3.7. *For any $n \geq 0$ and integer $k \in \mathbb{Z}$, we have*

$$\mathfrak{B}^{(k)}_{n,\lambda}(x) = \sum_{\ell=0}^{n}\left(r!\binom{n}{\ell}\sum_{m=0}^{n-\ell}\binom{n-\ell}{m}(m+r)_r S_{2,\lambda}(m+r,r)\mathfrak{B}^{(k)}_{n-\ell-r,\lambda}\right)B^{(r)}_{\ell,\lambda}(x).$$

Proof. Let $p(x) = \mathfrak{B}^{(k)}_{n,\lambda}(x) \in \mathbb{P}_n$. Then we can express $p(x)$ in terms of $B^{(r)}_{n,\lambda}(x)$

$$\mathfrak{B}^{(k)}_{n,\lambda}(x) = \sum_{\ell=0}^{n} c_{n,\ell}B^{(r)}_{\ell,\lambda}(x),$$

where $B^{(r)}_{n,\lambda}(x)$ satisfy

$$B^{(r)}_{n,\lambda}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^r, t\right)_\lambda.$$

and $c_{n,\ell}$ is computed by

$$\begin{aligned}
c_{n,\ell} =& \frac{1}{\ell!}\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^r\left(\frac{e_\lambda(t)-e^{-1}_\lambda(t)}{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}\right)^{-1} t^\ell \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
=& \frac{1}{\ell!}\left\langle \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e^{-1}_\lambda(t)}\left(\frac{e_\lambda(t)-1}{t}\right)^r \middle| (t^\ell)_\lambda(x)_{n,\lambda}\right\rangle_\lambda \\
=& \binom{n}{\ell}\left\langle \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e^{-1}_\lambda(t)} \middle| \left(\left(\frac{e_\lambda(t)-1}{t}\right)^r\right)_\lambda (x)_{n-\ell,\lambda}\right\rangle_\lambda \\
=& r!\binom{n}{\ell}\sum_{m=0}^{n-\ell}\binom{n-\ell}{m}(m+r)_r S_{2,\lambda}(m+r,r)\left\langle \frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e^{-1}_\lambda(t)} \middle| (x)_{n-\ell-r,\lambda}\right\rangle_\lambda .
\end{aligned} \tag{3.13}$$

In the above computation, the following relation is applied

$$\left(\frac{e_\lambda(t)-1}{t}\right)^r = \frac{r!}{t^r}\frac{(e_\lambda(t)-1)^r}{r!}$$

$$=r!\sum_{m=r}^{\infty}S_{2,\lambda}(m,r)\frac{t^{m-r}}{m!}$$
$$=r!\sum_{m=0}^{\infty}(m+r)_r S_{2,\lambda}(m+r,r)\frac{t^m}{m!}.$$

Applying the property of (3.1) in the last equation of (3.13) completes the proof. □

Taking into account the degenerate Euler polynomials [6] of order r

$$\left(\frac{2}{e_\lambda(t)+1}\right)^r e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},$$

we have the following result.

Theorem 3.8. *For any $n\geq 0$ and integer $k\in\mathbb{Z}$, we have*

$$\mathfrak{B}_{n,\lambda}^{(k)}(x)=\sum_{\ell=0}^{n}\left(\sum_{m=0}^{r}\sum_{j=0}^{n-\ell}\frac{1}{2^r}\binom{n}{\ell}\binom{r}{m}\binom{n-\ell}{j}(m)_{n-\ell-j,\lambda}\mathfrak{B}_{j,\lambda}^{(k)}\right)E_{\ell,\lambda}^{(r)}(x).$$

Proof. We note that the corresponding λ-Sheffer sequence for $E_{n,\lambda}^{(r)}(x)$ is given by

$$E_{n,\lambda}^{(r)}(x)\sim\left(\left(\frac{e_\lambda(t)+1}{2}\right)^r,t\right)_\lambda.$$

Then, we have

$$\mathfrak{B}_{n,\lambda}^{(k)}(x)=\sum_{\ell=0}^{n}c_{n,\ell}E_{\ell,\lambda}^{(r)}(x),$$

where $c_{n,\ell}$ satisfies

$$c_{n,\ell}=\frac{1}{\ell!}\left\langle\left(\frac{e_\lambda(t)+1}{2}\right)^r\left(\frac{e_\lambda(t)-e_\lambda^{-1}(t)}{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}\right)^{-1}t^\ell\,\middle|\,(x)_{n,\lambda}\right\rangle_\lambda$$
$$=\frac{1}{\ell!}\left\langle\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}\left(\frac{e_\lambda(t)+1}{2}\right)^r\,\middle|\,(t^\ell)_\lambda(x)_{n,\lambda}\right\rangle_\lambda$$
$$=\binom{n}{\ell}\left\langle\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}\,\middle|\,\left(\left(\frac{e_\lambda(t)+1}{2}\right)^r\right)_\lambda(x)_{n-\ell,\lambda}\right\rangle_\lambda$$
$$=\frac{1}{2^r}\binom{n}{\ell}\sum_{m=0}^{r}\binom{r}{m}\left\langle\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}\,\middle|\,(e_\lambda^m(t))_\lambda(x)_{n-\ell,\lambda}\right\rangle_\lambda$$
$$=\frac{1}{2^r}\binom{n}{\ell}\sum_{m=0}^{r}\binom{r}{m}\left\langle\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}\,\middle|\,(x+m)_{n-\ell,\lambda}\right\rangle_\lambda$$
$$=\frac{1}{2^r}\binom{n}{\ell}\sum_{m=0}^{r}\binom{r}{m}\sum_{j=0}^{n-\ell}\binom{n-\ell}{j}(m)_{n-\ell-j,\lambda}\left\langle\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}\,\middle|\,(x)_{j,\lambda}\right\rangle_\lambda.$$

Thus, the desired identity can be obtained by the property of (3.1). □

We finally consider the degenerate Lah-Bell polynomials [15] that are defined by

$$e_\lambda^x\left(\frac{t}{1-t}\right)=\sum_{n=0}^{\infty}B_{n,\lambda}^{L}(x)\frac{t^n}{n!},\quad (|t|<1)$$

and the unsigned Lah numbers $L(n,k)$ defined by

$$L(n,k)=\binom{n-1}{k-1}\frac{n!}{k!}$$

having the generating function

$$\frac{1}{\ell!}\left(\frac{t}{1-t}\right)^{\ell}=\sum_{m=\ell}^{\infty}L(m,\ell)\frac{t^m}{m!}.$$

Theorem 3.9. *For any $n\geq 0$ and integer $k\in\mathbb{Z}$, $\mathfrak{B}_{n,\lambda}^{(k)}(x)$ satisfy*

$$\mathfrak{B}_{n,\lambda}^{(k)}(x)=\sum_{\ell=0}^{n}\left(\sum_{m=\ell}^{n}\binom{n}{m}(-1)^{m-\ell}L(m,\ell)\mathfrak{B}_{n-m,\lambda}^{(k)}\right)B_{\ell,\lambda}^{L}(x).$$

Proof. If we let $p(x)=\mathfrak{B}_{n,\lambda}^{(k)}(x)\in\mathbb{P}_n$, then we have

$$\mathfrak{B}_{n,\lambda}^{(k)}(x)=\sum_{\ell=0}^{n}c_{n,\ell}B_{\ell,\lambda}^{L}(x).$$

By noting that the corresponding λ-Sheffer sequence is given by

$$B_{n,\lambda}^{L}(x)\sim\left(1,\frac{t}{1+t}\right)_{\lambda},$$

the followings are established for $c_{n,\ell}$:

$$\begin{aligned}c_{n,\ell}=&\frac{1}{\ell!}\left\langle\left(\frac{e_\lambda(t)-e_\lambda^{-1}(t)}{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}\right)^{-1}\left(\frac{t}{1+t}\right)^{\ell}\middle|(x)_{n,\lambda}\right\rangle_{\lambda}\\=&\left\langle\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}\middle|\left(\frac{1}{\ell!}\left(\frac{t}{1+t}\right)^{\ell}\right)_{\lambda}(x)_{n,\lambda}\right\rangle_{\lambda}\\=&\sum_{m=\ell}^{n}\binom{n}{m}(-1)^{m-\ell}L(m,\ell)\left\langle\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-e_\lambda^{-1}(t)}\middle|(x)_{n-m,\lambda}\right\rangle_{\lambda}\end{aligned}$$

in which the identity for the unsigned Lah numbers is applied:

$$\frac{1}{\ell!}\left(\frac{t}{1+t}\right)^{\ell}=\sum_{m=\ell}^{\infty}(-1)^{m-\ell}L(m,\ell)\frac{t^m}{m!}.$$

Therefore, $c_{n,\ell}$ is explicitly computed in a the similar way. □

4. Illustrative examples

In this section, we present two examples that show the explicit representations of the modified type 2 degenerate poly-Bernoulli polynomials in terms of the degenerate Lah-Bell polynomials and the degenerate derangement polynomials of order $r \in \mathbb{N}$. We compute the combinatorial results $c_{n,\ell}$ presented in Theorems 3.5 and 3.9 to confirm the connections.

To do this, we first calculate $\mathfrak{B}^{(0)}_{n,\lambda}(x)$ for $k = 0, -1$ and $n = 0, 1, 2, 3, 4$. For $k = 0$, $\mathfrak{B}^{(0)}_{n,\lambda}(x)$ can be obtained using $\mathrm{Li}_{0,\lambda}(t) = t(1-t)^{\lambda-1}$, which are listed below:

$$\begin{aligned}
\mathfrak{B}^{(0)}_{0,\lambda}(x) &= \frac{1}{2},\\
\mathfrak{B}^{(0)}_{1,\lambda}(x) &= \frac{1}{2}x + \frac{1}{4},\\
\mathfrak{B}^{(0)}_{2,\lambda}(x) &= \frac{1}{2}x^2 + \frac{1}{2}(1-\lambda)x - \frac{1}{2}\lambda^2 + \frac{1}{4}\lambda,\\
\mathfrak{B}^{(0)}_{3,\lambda}(x) &= \frac{1}{2}x^3 - \frac{1}{4}(6\lambda - 3)x^2 - \frac{1}{2}\lambda^2 x - \frac{1}{4}\lambda^3 - \frac{1}{4}\lambda^2 + \lambda - \frac{1}{8},\\
\mathfrak{B}^{(0)}_{4,\lambda}(x) &= \frac{1}{2}x^4 + (1-3\lambda)x^3 + \frac{1}{2}(5\lambda^2 - 3\lambda)x^2 - \frac{1}{2}(2\lambda^3 + \lambda^2 - 8\lambda + 1)x - \frac{1}{4}(6\lambda^4 - 2\lambda^3 + 5\lambda).
\end{aligned}$$

For $k = -1$, $\mathfrak{B}^{(-1)}_{n,\lambda}(x)$ are computed as follows:

$$\begin{aligned}
\mathfrak{B}^{(-1)}_{0,\lambda}(x) &= \frac{1}{2},\\
\mathfrak{B}^{(-1)}_{1,\lambda}(x) &= \frac{1}{2}x - \frac{1}{2}\lambda + \frac{3}{2},\\
\mathfrak{B}^{(-1)}_{2,\lambda}(x) &= \frac{1}{2}x^2 - \frac{3}{2}\lambda x + \frac{3}{2}x - \lambda^2 - \frac{1}{4}\lambda + 1,\\
\mathfrak{B}^{(-1)}_{3,\lambda}(x) &= \frac{1}{2}x^3 - \frac{1}{4}(12\lambda - 9)x^2 - (\frac{1}{2}\lambda^2 + 3\lambda - 3)x + \frac{1}{4}\lambda^3 - \frac{15}{4}\lambda^2 + \frac{11}{4}\lambda + \frac{9}{8},\\
\mathfrak{B}^{(-1)}_{4,\lambda}(x) &= \frac{1}{2}x^4 - (5\lambda - 3)x^3 + \frac{1}{2}(11\lambda^2 - 21\lambda + 12)x^2 - \frac{1}{2}(11\lambda^2 - 10\lambda - 9)x\\
&\quad - \frac{1}{4}(2\lambda^3 + 26\lambda^2 + 26\lambda + 94)\lambda + 1.
\end{aligned}$$

Figure 1 illustrates the shapes of $\mathfrak{B}^{(k)}_{n,\lambda}(x)$, $k = -1$ for $n = 0, 1, 2, 3, 4$ when $\lambda = 1/10$ and $\lambda = 1$.

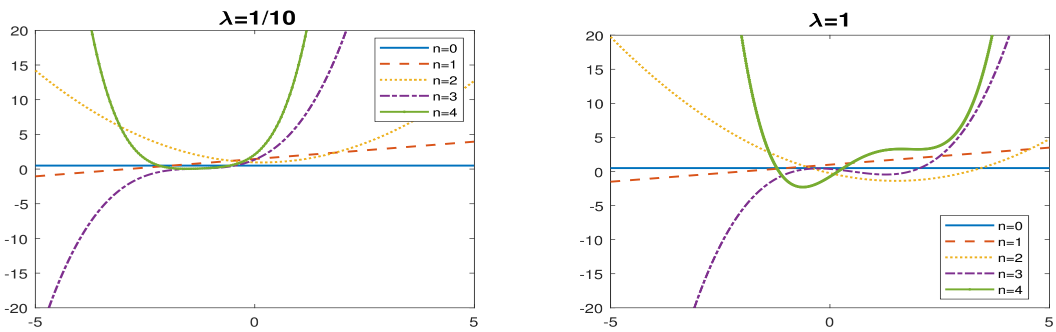

Figure 1. $\mathfrak{B}^{(k)}_{n,\lambda}(x)$, $n = 0, 1, 2, 3, 4$ for $k = -1$.

We express $\mathfrak{B}^{(k)}_{n,\lambda}(x)$, $k = 0, -1,$ $n = 0, 1, 2, 3, 4$ by the degenerate Lah-Bell polynomials $B^L_{n,\lambda}(x)$ for various $\lambda = \frac{1}{100}, \frac{1}{10}, 1$ in the following example.

Example 4.1 (Illustration of Theorem 3.9)**.** *The degenerate Lah-Bell polynomials $B^L_{n,\lambda}(x)$ for $n = 0, 1, 2, 3, 4$ are explicitly given by*

$$\begin{aligned} B^L_{0,\lambda}(x) =& 1,\\ B^L_{1,\lambda}(x) =& x,\\ B^L_{2,\lambda}(x) =& x^2 - (\lambda - 2)x,\\ B^L_{3,\lambda}(x) =& x^3 - 3(\lambda - 2)x^2 + 2(\lambda^2 - 3\lambda + 3)x,\\ B^L_{4,\lambda}(x) =& x^4 - 6(\lambda - 2)x^3 + (11\lambda^2 - 36\lambda + 36)x^2 - 6(\lambda^3 - 4\lambda^2 + 6\lambda - 4)x. \end{aligned}$$

Figure 2 shows the graphs of $B^L_{n,\lambda}(x)$ for $n = 0, 1, 2, 3, 4$ when $\lambda = 1/10, 1$.

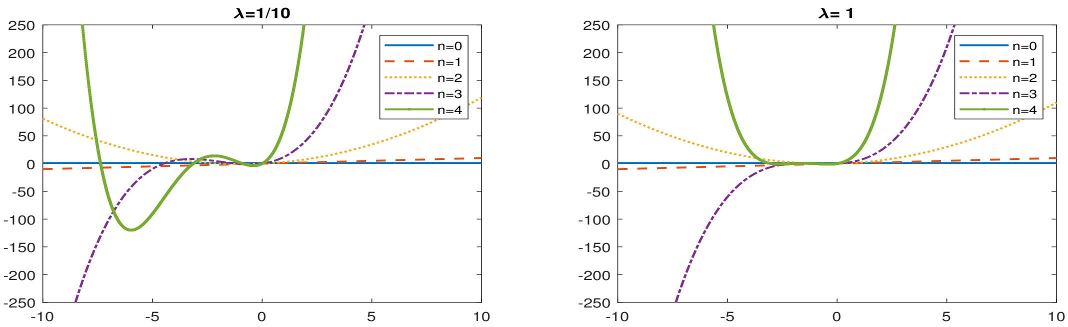

Figure 2. $B^L_{n,\lambda}(x)$ for $n = 0, 1, 2, 3, 4$.

To confirm the established identity in Theorem 3.9, we compute $\sum_{\ell=0}^{n} c_{n,\ell} B^L_{\ell,\lambda}(x)$, where

$$c_{n,\ell} = \sum_{m=\ell}^{n} \binom{n}{m} (-1)^{m-\ell} L(m,\ell)\mathfrak{B}^{(k)}_{n-m,\lambda},\ n = 1, 2, 3, 4.$$

Then, we present that each linear combination of $B^L_{n,\lambda}(x)$ with $c_{n,\ell}$ is identically equal to $\mathfrak{B}^{(k)}_{n,\lambda}(x)$, $n = 1, 2, 3, 4$ for $\lambda = 1/100, 1/10, 1$ and $k = 0, -1$:
Case (I): For $k = 0$ and $\lambda = \frac{1}{100}$,

$$\begin{aligned} \mathfrak{B}^{(0)}_{4,\frac{1}{100}}(x) =& 0.5x^4 + 0.97x^3 - 0.0147x^2 - 0.4601x + 0.0125\\ =& \frac{1}{80}B^L_{0,\frac{1}{100}}(x) - \frac{4017}{619}B^L_{1,\frac{1}{100}}(x) + \frac{817}{68}B^L_{2,\frac{1}{100}}(x) - 5B^L_{3,\frac{1}{100}}(x) + \frac{1}{2}B^L_{4,\frac{1}{100}}(x),\\ \mathfrak{B}^{(0)}_{3,\frac{1}{100}}(x) =& 0.5x^3 + 0.735x^2 - 0.00005x - 0.115\\ =& -\frac{23}{200}B^L_{0,\frac{1}{100}}(x) + \frac{713}{473}B^L_{1,\frac{1}{100}}(x) - \frac{9}{4}B^L_{2,\frac{1}{100}}(x) + \frac{1}{2}B^L_{3,\frac{1}{100}}(x),\\ \mathfrak{B}^{(0)}_{2,\frac{1}{100}}(x) =& 0.5x^2 + 0.495x + 0.0024\\ =& \frac{1}{400}B^L_{0,\frac{1}{100}}(x) - \frac{1}{2}B^L_{1,\frac{1}{100}}(x) + \frac{1}{2}B^L_{2,\frac{1}{100}}(x), \end{aligned}$$

$$\begin{aligned}\mathfrak{B}^{(0)}_{1,\frac{1}{100}}(x) =&0.5x + 0.25\\ =&\frac{1}{4}B^L_{0,\frac{1}{100}}(x) + \frac{1}{2}B^L_{1,\frac{1}{100}}(x).\end{aligned}$$

Case (II): For $k = 0$, $\lambda = 1$,

$$\begin{aligned}\mathfrak{B}^{(0)}_{4,1}(x) =&0.5x^4 - 2x^3 + x^2 + 2x - 0.75\\ =&-\frac{3}{4}B^L_{0,1}(x) - \frac{3}{2}B^L_{1,1}(x) + \frac{21}{2}B^L_{2,1}(x) - 5B^L_{3,1}(x) + \frac{1}{2}B^L_{4,1}(x),\\ \mathfrak{B}^{(0)}_{3,1}(x) =&0.5x^3 - 0.75x^2 - 0.5x + 0.375\\ =&-\frac{3}{8}B^L_{0,1}(x) + \frac{3}{4}B^L_{1,1}(x) - \frac{9}{4}B^L_{2,1}(x) + \frac{1}{2}B^L_{3,1}(x),\\ \mathfrak{B}^{(0)}_{2,1}(x) =&0.5x^2 - 0.25\\ =&-\frac{1}{4}B^L_{0,1}(x) - \frac{1}{2}B^L_{1,1}(x) + \frac{1}{2}B^L_{2,1}(x),\\ \mathfrak{B}^{(0)}_{1,1}(x) =&0.5x + 0.25\\ =&\frac{1}{4}B^L_{0,1}(x) + \frac{1}{2}B^L_{1,1}(x).\end{aligned}$$

Case (III): For $k = -1$, $\lambda = 1/10$,

$$\begin{aligned}\mathfrak{B}^{(0)}_{4,\frac{1}{10}}(x) =&0.5x^4 + 2.5x^3 + 5.005x^2 + 4.925x + 2.1034\\ =&\frac{1485}{706}B^L_{0,\frac{1}{10}}(x) - \frac{1329}{1000}B^L_{1,\frac{1}{10}}(x) + \frac{699}{100}B^L_{2,\frac{1}{10}}(x) - \frac{16}{5}B^L_{3,\frac{1}{10}}(x) + \frac{1}{2}B^L_{4,\frac{1}{10}}(x),\\ \mathfrak{B}^{(0)}_{3,\frac{1}{10}}(x) =&0.5x^3 + 1.95x^2 + 2.695x + 1.3627\\ =&\frac{740}{543}B^L_{0,\frac{1}{10}}(x) + \frac{339}{200}B^L_{1,\frac{1}{10}}(x) - \frac{9}{10}B^L_{2,\frac{1}{10}}(x) + \frac{1}{2}B^L_{3,\frac{1}{10}}(x),\\ \mathfrak{B}^{(0)}_{2,\frac{1}{10}}(x) =&0.5x^2 + 1.35x + 0.965\\ =&\frac{193}{200}B^L_{0,\frac{1}{10}}(x) + \frac{2}{5}B^L_{1,\frac{1}{10}}(x) - \frac{1}{2}B^L_{2,\frac{1}{10}}(x),\\ \mathfrak{B}^{(0)}_{1,\frac{1}{10}}(x) =&0.5x + 0.7\\ =&\frac{7}{10}B^L_{0,\frac{1}{10}}(x) + \frac{1}{2}B^L_{1,\frac{1}{10}}(x).\end{aligned}$$

Case (IV): For $k = -1$, $\lambda = 1$,

$$\begin{aligned}\mathfrak{B}^{(0)}_{4,1}(x) =&0.5x^4 - 2x^3 + x^2 + 2x - 0.75\\ =&-\frac{3}{4}B^L_{0,1}(x) - \frac{3}{2}B^L_{1,1}(x) + \frac{21}{2}B^L_{2,1}(x) - 5B^L_{3,1}(x) + \frac{1}{2}B^L_{4,1}(x),\\ \mathfrak{B}^{(0)}_{3,1}(x) =&0.5x^3 - 0.75x^2 - 0.5x + 0.375\\ =&\frac{3}{8}B^L_{0,1}(x) + \frac{3}{4}B^L_{1,1}(x) - \frac{9}{4}B^L_{2,1}(x) + \frac{1}{2}B^L_{3,1}(x),\\ \mathfrak{B}^{(0)}_{2,1}(x) =&0.5x^2 - 0.25\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{4}B^L_{0,1}(x) - \frac{1}{2}B^L_{1,1}(x) + \frac{1}{2}B^L_{2,1}(x),\\
\mathfrak{B}^{(0)}_{1,1}(x) &= 0.5x + 0.25\\
&= -\frac{1}{4}B^L_{0,1}(x) + \frac{1}{2}B^L_{1,1}(x).
\end{aligned}
$$

In the next example, we express $\mathfrak{B}^{(k)}_{n,\lambda}(x)$, $k = 0, -1,\ n = 0, 1, 2, 3, 4$ by the degenerate derangement polynomials $d^{(r)}_{n,\lambda}(x)$ for $\lambda = \frac{1}{10}, 1, 10$, as selected cases of the results in Theorem 3.5.

Example 4.2 (Illustration of Theorem 3.5)**.** *The degenerate derangement polynomials of order $r = 2$ for $n = 0, 1, 2, 3, 4$ are listed as:*

$$
\begin{aligned}
d^{(2)}_{0,\lambda}(x) &= 1,\\
d^{(2)}_{1,\lambda}(x) &= x + 1,\\
d^{(2)}_{2,\lambda}(x) &= x^2 + (2 - \lambda)x + \lambda + 3,\\
d^{(2)}_{3,\lambda}(x) &= x^3 + (3 - 3\lambda)x^2 + 2\lambda^2 x + 9x - 2\lambda^2 + 3\lambda + 11,\\
d^{(2)}_{4,\lambda}(x) &= x^4 - (6\lambda - 4)x^3 + (11\lambda^2 - 6\lambda + 18)x^2 - (6\lambda^3 + 6\lambda^2 + 6\lambda - 44)x + 6\lambda^3 - 5\lambda^2 + 18\lambda + 53.
\end{aligned}
$$

We plot the graphs of $d^{(r)}_{n,\lambda}(x)$ of order 2 for $n = 0, 1, 2, 3, 4$ when $\lambda = 1/10, 1$ in Figure 3.

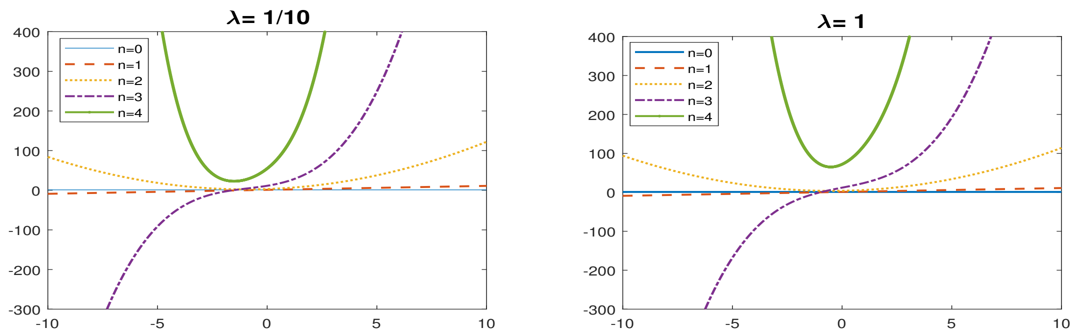

Figure 3. $d^{(2)}_{n,\lambda}(x)$ for $n = 0, 1, 2, 3, 4$.

To confirm the identity presented in Theorem 3.5, we compute $\sum_{\ell=0}^{n} c_{n,\ell} d^{(2)}_{n,\lambda}(x)$, where

$$
c_{n,\ell} = \binom{n}{\ell}\sum_{j=0}^{2}\binom{2}{j}\binom{n-\ell}{j}(-1)^j j!\mathfrak{B}^{(k)}_{n-\ell-j,\lambda}(1),\ n = 1, 2, 3, 4.
$$

Then, the following presentations support the analysis that each linear combination of $d^{(2)}_{n,\lambda}(x)$ with computed $c_{n,\ell}$ is exactly equal to $\mathfrak{B}^{(k)}_{n,\lambda}(x)$, $\quad n = 1, 2, 3, 4$ when $\lambda = 1, 10$ and $k = 0$:
Case (I): For $k = 0,\ \lambda = 10$,

$$
\begin{aligned}
\mathfrak{B}^{(0)}_{4,10}(x) &= 0.5x^4 - 29x^3 + 235x^2 - 1010.5x - 15488\\
&= -14279d^{(2)}_{0,10}(x) - \frac{123}{2}d^{(2)}_{1,10}(x) - 321d^{(2)}_{2,10}(x) - d^{(2)}_{3,10}(x) + \frac{1}{2}d^{(2)}_{4,10}(x),
\end{aligned}
$$

$$\begin{aligned}\mathfrak{B}^{(0)}_{3,10}(x) =&0.5x^3 - 14.25x^2 - 50x - 265.125\\ =&-\frac{123}{8}d^{(2)}_{0,10}(x) - \frac{321}{2}d^{(2)}_{1,10}(x) - \frac{3}{4}d^{(2)}_{2,10}(x) + \frac{1}{2}d^{(2)}_{3,10}(x),\\ \mathfrak{B}^{(0)}_{2,10}(x) =&0.5x^2 - 4.5x - 47.5\\ =&-\frac{107}{2}d^{(2)}_{0,10}(x) - \frac{1}{2}d^{(2)}_{1,10}(x) + \frac{1}{2}d^{(2)}_{2,10}(x),\\ \mathfrak{B}^{(0)}_{1,10}(x) =&0.5x + 0.25\\ =&-\frac{1}{4}d^{(2)}_{0,10}(x) + \frac{1}{2}d^{(2)}_{1,10}(x).\end{aligned}$$

Case (II): For $k = 0$, $\lambda = 1$,

$$\begin{aligned}\mathfrak{B}^{(0)}_{4,1}(x) =&0.5x^4 - 2x^3 + x^2 + 2x - 0.75\\ =&-\frac{27}{4}d^{(2)}_{0,1}(x) + \frac{21}{2}d^{(2)}_{1,1}(x) - \frac{21}{2}d^{(2)}_{2,1}(x) - d^{(2)}_{3,1}(x) + \frac{1}{2}d^{(2)}_{4,1}(x),\\ \mathfrak{B}^{(0)}_{3,1}(x) =&0.5x^3 - 0.75x^2 - 0.5x + 0.375\\ =&-\frac{21}{8}d^{(2)}_{0,1}(x) - \frac{21}{4}d^{(2)}_{1,1}(x) - \frac{3}{4}d^{(2)}_{2,1}(x) + \frac{1}{2}d^{(2)}_{3,1}(x),\\ \mathfrak{B}^{(0)}_{2,1}(x) =&0.5x^2 - 0.25\\ =&-\frac{7}{4}d^{(2)}_{0,1}(x) - \frac{1}{2}d^{(2)}_{1,1}(x) + \frac{1}{2}d^{(2)}_{2,1}(x),\\ \mathfrak{B}^{(0)}_{1,1}(x) =&0.5x + 0.25\\ =&-\frac{1}{4}d^{(2)}_{0,1}(x) + \frac{1}{2}d^{(2)}_{1,1}(x).\end{aligned}$$

5. Conclusions

In this paper, we propose a new variant of type 2 poly-Bernoulli polynomials and numbers using the generating function that is a combination of the degenerate exponential functions with the degenerate polylogarithm function. Then, we derive explicit expressions of the proposed polynomials in terms of the degenerate type 2-Bernoulli polynomials and the degenerate Stirling numbers of the second kind. Furthermore, using λ-linear functionals and λ-Sheffer sequences, we present connections of the modified type 2 degenerate poly-Bernoulli polynomials among existing λ-Sheffer polynomials: the degenerate type 2-Bernoulli polynomials, the higher-order type 2 degenerate Bernoulli polynomials, the degenerate Hermite polynomials, the higher-order degenerate Bernoulli polynomials, the higher-order degenerate derangement polynomials, the degenerate Euler polynomials, and the higher-order degenerate Lah-Bell polynomials. Finally, to confirm the presented combinatorial results, we explicitly compute the connection constants and show combinations with the degenerate Lah-Bell polynomials and the higher-order degenerate derangement polynomials.

Acknowledgments

This work was supported by the Dongguk University Research Fund of 2020 and the first author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government

(No. 2020R1F1A1A0107156412).

Conflict of interest

The authors declare no conflict of interest.

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