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Research article

A numerical study of fractional population growth and nuclear decay model

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Abstract: This paper is devoted to solving the initial value problem (IVP) of the fractional differential equation (FDE) in Caputo sense for arbitrary order $\beta \in (0, 1]$. Based on a few examples and application models, the main motivation is to show that FDE may model more effectively than the ordinary differential equation (ODE). Here, two cubic convergence numerical schemes are developed: the fractional third-order Runge-Kutta (RK3) scheme and fractional strong stability preserving third-order Runge-Kutta (SSRK3) scheme. The approximated solution is derived without taking any assumption of perturbations and linearization. The schemes are presented, and the convergence of the schemes is established. Also, a comparative study has been done of our proposed scheme with fractional Euler method (EM) and fractional improved Euler method (IEM), which has linear and quadratic convergence rates, respectively. Illustrative examples and application examples with the numerical comparison between the proposed scheme, the exact solution, EM, and IEM are given to reveal our scheme's accuracy and efficiency.

Keywords: fractional calculus; FDEs; fractional Euler method; fractional improved Euler method; IVP; Runge-Kutta method; strong stability preserving; population growth model
Mathematics Subject Classification: 26A33, 34A08, 93C10, 93C15, 78A70

1. Introduction

The development of fractional calculus (FC) [1] was traced from a letter dated September 30, 1695, written by L' Hospital to Leibniz regarding the half derivative of the linear function $p(x) = x$. Leibniz response was "An apparent paradox, from which one-day useful consequences will be drawn". Nowadays, FC is identified as the most effective tool for the modelling of physical phenomena not only in mathematics but also in other branches of sciences, engineering, economics, finance, etc. FC has been identified as one of the fastest-growing research areas in the last few decades. Also, FDEs have played a significant role because of their huge range of uses. Several problems in biology, physics, chemistry, applied science and engineering are modelled by FDEs [2–7]. Many research articles were developed to investigate the theory and solutions of FDEs [8, 9]. Also, many researchers [10–12] developed the existence and uniqueness criteria of the IVP if FDEs of fractional order. The most common and oldest derivatives are Riemann Liouville (RL) and Caputo. In this research article, we are taking the derivative in the Caputo frame as it has many advantages for dealing with the IVP of FDEs:

$$ {}_{C}D^{\beta}_{x_0^+} y = p(x,y), \text{ supplementary condition } y(x_0) = y_0, \text{ and } x \in (x_0, x_{\text{end}}], \tag{1.1}$$

where ${}_{C}D^{\beta}_{x_0^+}$ indicates the Caputo fractional order derivative (CFOD) of variable order β such that $\beta \in (0, 1]$.

In the survey, we found that most of the non-linear IVP of FDEs don't have any analytic method for finding the solutions [13], therefore numerical technique must be used for such cases. Some of the analytic and numerical methods for solving FDEs consists of Adomian decomposition method (ADM) [14], variational iteration method (VIM) [15], fractional differential transform scheme [16], fractional finite difference scheme [17], fractional Adams scheme [8], homotopy perturbation scheme [18], spectral collocation scheme [19], extrapolation method [20], homotopy analysis scheme [21], and many others. Out of these methods, the ADM and the VIM are the most well-known methods for solving the FDEs for providing instant and visible symbolic terms of numerical solutions. Moreover, the numerical scheme described in the literature [22–24] have some drawbacks in the RL derivative sense. In this work, we suggest novel schemes to reduce the drawback.

Nowadays, the IVP of FDEs used as a weapon to solve the various mathematical models, dynamical models, and many others. This article establishes two fractional numerical algorithms for the IVP of FDEs of order $\beta \in (0, 1]$. These schemes are fractional third-order RK3 scheme and fractional SSRK3 scheme, which are based on classical third-order RK3 scheme [25–28] and classical SSRK3 scheme, respectively. In [29], Muhammad et al. developed a two-stage generalized Rk2 scheme of second order in CFOD sense. In [30], Kumar et al. established two numerical schemes, which are fractional quadratic midpoint scheme and fractional cubic Heun scheme for an IVP of FDEs (1.1). Also, in the year 2016, Tong et al. [31] suggested numerical schemes, which are fractional EM and fractional IEM, to demonstrate the numerical solution of the IVP of FDEs (1.1). These all referred works motivate us to establish more efficient schemes to solve the IVP of FDE in the CFOD sense. Also, our main aim is to show, based on a few application models and a few concrete examples, that FDEs may model the physical problem more efficiently than the ODEs. In recent decades, many research articles have been devoted for numerical iterative scheme [32–35] of IVP of FDEs. Still, there are few non-linear IVP of FDEs in the Riemann derivative sense where the approximation technique does not work. In our work, we proposed two efficient cubic techniques, which is effectively providing the approximate solution of linear as well as non-linear IVP of FDEs (1.1) of order β where $\beta \in (0, 1]$. Both schemes, fractional RK3, and fractional SSRK3 are cubic convergence in which the SSRK3 scheme is more stable and faster than all other cubic schemes for IVP of FDEs. This paper also provides the comparative and convergence analysis of our suggested technique with fractional EM and fractional IEM, which have linear and quadratic convergence, respectively. We organize this work as follows:

- In Section 2, we introduce mathematical preliminaries, some basic definitions, and the result of FC.
- In Section 3, we proposed our and suggested methodology and their convergence analysis.
- In Section 4, we provide numerical solutions of illustrated examples of IVP of FDEs using suggested schemes.
- In Section 5, we provide the numerical simulation of two real-world application models, which are the fractional World Population Growth (WPG) model and Nuclear Decay (ND) model using suggested schemes.
- In Section 6, we report our conclusion of the proposed scheme with some crucial facts.

2. Preliminaries

This section is devoted to the preliminary concepts of FC that we need in our study. So, we present some definitions and some properties of FC [36–41].

Definition 2.1. For arbitrary $\beta > 0$ and a piecewise integrable function $\kappa : [a, b] \to \mathbb{R}$, the fractional left and right RL integral of order β are defined by

$$ {}_{RL}I^{\beta}_{a+}\kappa(s) = \frac{1}{\Gamma(\beta)} \int_a^s (s-p)^{\beta-1}\kappa(p)dp, \ s > a, $$

and

$$ {}_{RL}I^{\beta}_{b-}\kappa(s) = \frac{1}{\Gamma(\beta)} \int_s^b (p-s)^{\beta-1}\kappa(p)dp, \ s < b, $$

respectively. Here the notation Γ signify the Gamma function.

Definition 2.2. For arbitrary $\beta > 0$ and a piecewise integrable function $\kappa : [a, b] \to \mathbb{R}$, the fractional left and right RL derivative of order β are defined by

$$ {}_{RL}D^{\beta}_{a+}\kappa(s) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{ds^n} \int_a^s (s-p)^{n-\beta-1}\kappa(p)dp, \ s > a, $$

and

$$ {}_{RL}D^{\beta}_{b-}\kappa(s) = \frac{(-1)^n}{\Gamma(n-\beta)} \frac{d^n}{ds^n} \int_s^b (p-s)^{n-\beta-1}\kappa(p)dp, \ s < b, $$

respectively, where $n-1 < \beta < n, n \in \mathbb{N}$. If $0 < \beta < 1$, then the fractional left and right RL derivative are

$$ {}_{RL}D^{\beta}_{a+}\kappa(s) = \frac{1}{\Gamma(1-\beta)} \frac{d}{ds} \int_a^s (s-p)^{-\beta}\kappa(p)dp, \ s > a, $$

and

$$_{RL}D^{\beta}_{b-}\kappa(s) = -\frac{1}{\Gamma(1-\beta)}\frac{d}{ds}\int_{s}^{b}(p-s)^{-\beta}\kappa(p)dp,\ s<b,$$

respectively.

Definition 2.3. For arbitrary $\beta > 0$ and a piecewise integrable function $\kappa : [a, b] \to \mathbb{R}$, the fractional left and right Caputo derivative of order β are defined by

$$_{C}D^{\beta}_{a+}\kappa(s) = \frac{1}{\Gamma(n-\beta)}\int_{a}^{s}(s-p)^{n-\beta-1}\kappa^{n}(p)dp,\ s>a,$$

and

$$_{C}D^{\beta}_{b-}\kappa(s) = \frac{(-1)^{n}}{\Gamma(n-\beta)}\int_{s}^{b}(p-s)^{n-\beta-1}\kappa^{n}(p)dp,\ s<b,$$

respectively, where $n-1 < \beta < n, n \in \mathbb{N}$. If we take $0 < \beta < 1$, then the fractional left and right Caputo derivative are

$$_{C}D^{\beta}_{a+}\kappa(s) = \frac{1}{\Gamma(1-\beta)}\int_{a}^{s}(s-p)^{-\beta}\kappa'(p)dp,\ s>a,$$

and

$$_{C}D^{\beta}_{b-}\kappa(s) = -\frac{1}{\Gamma(1-\beta)}\int_{s}^{b}(p-s)^{-\beta}\kappa'(p)dp,\ s<b,$$

respectively.

The relation between the fractional RL derivative and fractional Caputo derivative is,

$$_{C}D^{\beta}_{a+}\kappa(s) = {}_{RL}D^{\beta}_{a+}\kappa(s) - \sum_{k=0}^{n-1}\kappa^{k}(a)\frac{(s-a)^{k-\beta}}{\Gamma(k-\beta+1)},$$

and

$$_{C}D^{\beta}_{b-}\kappa(s) = {}_{RL}D^{\beta}_{b-}\kappa(s) - \sum_{k=0}^{n-1}\kappa^{k}(b)\frac{(b-s)^{k-\beta}}{\Gamma(k-\beta+1)},$$

where $n-1 < \beta < n, n \in \mathbb{N}$.

Definition 2.4. The one parameter Mittag-Leffler function (E_{β}) which was introduced by Mittag-Leffler, is defined as:

$$E_{\beta}(s) = \sum_{n=0}^{\infty}\frac{s^{n}}{\Gamma(\beta n+1)},\quad s\in\mathbb{C},\ \beta\in\mathbb{C},\ Re(\beta)>0.$$

The two parameter Mittag-Leffler function ($E_{\beta,\gamma}$) which was first appeared in a paper by Wiman [42], is defined as:

$$E_{\beta,\gamma}(s) = \sum_{n=0}^{\infty}\frac{s^{n}}{\Gamma(\beta n+\gamma)},\quad s\in\mathbb{C},\ \beta\in\mathbb{C},\ \gamma\in\mathbb{C},\ Re(\beta)>0,\ Re(\gamma)>0.$$

Lemma 2.1. [36,37] If $Re(\beta) > 0, Re(\gamma) > 0$ and $p(x) \in L_s[a,b]$, where $0 \le s \le \infty$, then the equations

$$ {}_{RL}I^{\beta}_{a^+}{}_{RL}I^{\gamma}_{a^+}p = {}_{RL}I^{\beta+\gamma}_{a^+}p, \text{ and } {}_{RL}I^{\beta}_{b^-}{}_{RL}I^{\gamma}_{b^-}p = {}_{RL}I^{\beta+\gamma}_{b^-}p, \tag{2.1}$$

holds almost everywhere for $x \in [a,b]$. If $\beta + \gamma > 1$, then the expression (2.1) holds at any point on $x \in [a,b]$.

Lemma 2.2. [43] If $p(x) \in C^s[a,b]$, $a < b$ and $s \in \mathbb{N}$. Moreover, If $\beta, \gamma > 0$ be such that, $\exists$ some $m \in \mathbb{N}$ with $m \le s$ and $\beta, \beta + \gamma \in [m-1, m]$. Then,

$$ {}_{C}D^{\beta}_{a^+}{}_{C}D^{\gamma}_{a^+}p = {}_{C}D^{\beta+\gamma}_{a^+}p. $$

Theorem 2.1. (Existence of IVP of FDE) [31] Let $p(x,y)$ be a function that hold the condition $p(x_0, y(x_0)) = 0$ and also the $p(x,y)$ is continuous on the domain $R: 0 \le x - x_0 \le k_1,\ |y - y_0| \le k_2$, then FDEs:

$$ {}_{C}D^{\beta}_{x_0^+}y = p(x,y), \text{ supplementary condition } y(x_0) = y_0 \text{ and } x \in (x_0, x_{\text{end}}], \tag{2.2}$$

has at least one solution in the interval $0 \le x - x_0 \le \lambda$ with $\lambda = \min\left\{k_1, \frac{k_2}{M}\right\}$ and $\max_{(x,y)\in\mathbb{R}} {}_{C}D^{1-\beta}_{a^+}p(x,y) < M$.

Theorem 2.2. (Uniqueness of IVP of FDE) [31] By following the Theorem 2.1, and if $p_x(x,u)$ holds the condition of Lipschitz in the variable u with Lipschitz constant $0 < L$,

$$ |p_x(x,y_1) - p_x(x,y_2)| \le L\,|y_1 - y_2|, $$

then the FDEs (2.2) have unique solution.

3. Proposed methodology

Let us assume, an IVP of FDE (1.1) in CFOD frame of variable order β. By following the Lemma 2.2, we apply a suitable analogous operation so that the FDE (1.1) will become the classical ODE, and we will get the CFOD of order $(1-\beta)$. After that, the revised IVP of FDE:

$$ y' = {}_{C}D^{1-\beta}_{x_0^+}p(x,y), \text{ supplementary condition } y(x_0) = y_0, \text{ and } x \in (x_0, x_{\text{end}}]. \tag{3.1}$$

In order to obtain the efficient and appropriate approximate solution of the IVP of FDEs (3.1), we proposed two effective and fast numerical scheme for IVP of FDEs which are fractional RK3 [25–28] and fractional SSRK3 [44–46]. These two fractional numerical scheme is more accurate and fast compared to all other linear and quadratic convergence scheme for IVP of FDEs (1.1). Below is our fractional RK3 scheme.

3.1. Fractional RK3 scheme of FDE

To establish the numerical solution of the IVP of fractional order differential equation (3.1) in the interval $[a,b]$, we propose the algorithm fractional RK3 scheme [25–28], which is same as classical

RK3 scheme for first order IVP of ODE. For approximating the solution, we consider (x_k, y_k) as our set points, and we consider these points in such a way that the mesh is equally distributed in the interval $[a, b]$ where we set $x_0 = a$ and $x_{end} = b$. This idea will be good by selecting a non-negative integer, say n, and assuming the mesh points. So, the explicit fractional RK3 scheme is given by the Butcher tableau:

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

Here is the algorithm of fractional RK3 scheme of FDE (3.1):

$$\begin{cases} x_k = x_0 + kh \ \text{ for each } \ k = 0, 1, 2, \ldots, n \\ h = x_{k+1} - x_k, \ \text{ (step size)} \\ y_{k+1} = y_k + \frac{h}{6}(l_1 + 4l_2 + l_3), \ \text{ where} \\ l_1 = {}_C D^{1-\beta}_{x_0^+} p(x, y_k)\Big|_{x=x_k}, \\ l_2 = {}_C D^{1-\beta}_{x_0^+} p\left(x + \frac{h}{2}, y_k + \frac{h}{2} l_1\right)\Big|_{x=x_k}, \\ l_3 = {}_C D^{1-\beta}_{x_0^+} p(x + h, y_k - hl_1 + 2hl_2)\Big|_{x=x_k}. \end{cases}$$

This algorithm is a cubic convergence scheme and with the help of Matlab, it is proven to be an efficient and more accurate in the comparison of linear and quadratic convergence scheme.

Before proceeding the convergence of RK3 scheme, first we are going to state some relevant result and lemmas.

Lemma 3.1. [31] Let $p_x(x, u)$ be a function that satisfy the condition of Lipschitz in the unknown variable y, with Lipschitz constant $0 < A$,

$$|p_x(x, y_1) - p_x(x, y_2)| \leq A\,|y_1 - y_2|\,,$$

and also hold the conditions of Theorem 2.1, also consider $P(x, y) = {}_C D^{1-\beta}_{x_0^+} p(x, y)$ holds the condition of Lipschitz in the unknown variable y, with another Lipschitz constant $0 < M$,

$$|P(x, y_1) - P(x, y_2)| \leq M\,|y_1 - y_2|\,.$$

Lemma 3.2. Consider the function $P(x, y)$ satisfy the Lipschitz condition for the unknown variable y and if the conditions of Theorem 2.1 hold, then

$$\begin{aligned} \mu(x, y) = &\frac{1}{6}P(x, y) + \frac{2}{3}P\left(x + \frac{h}{2}, y + \frac{h}{2}P(x, y)\right) \\ &+ \frac{1}{6}P\left(x + h, y - hP(x, y) + 2hP\left(x + \frac{h}{2}, y + \frac{h}{2}P\,(x, y)\right)\right) \end{aligned} \tag{3.2}$$

will always fulfill the condition of Lipschitz in the unknown variable y.

Proof.

$$
\begin{aligned}
|\mu(x,y_1)-\mu(x,y_2)| \leq{}& \frac{1}{6}|P(x,y_1)-P(x,y_2)| + \frac{2}{3}\left|P\left(x+\frac{h}{2}, y_1+\frac{h}{2}P(x,y_1)\right)-\right.\\
&\left.P\left(x+\frac{h}{2}, y_2+\frac{h}{2}P(x,y_2)\right)\right| + \frac{1}{6}\left|P\left(x+h, y_1-hP(x,y_1)+2hP\left(x+\frac{h}{2},\right.\right.\right.\\
&\left.\left.\left.y_1+\frac{h}{2}P(x,y_1)\right)\right)-P\left(x+h, y_2-hP(x,y_2)+2hP\left(x+\frac{h}{2}, y_2+\frac{h}{2}P(x,y_2)\right)\right)\right|\\
&\leq M|y_1-y_2| + \frac{5hM^2}{6}|y_1-y_2| + \frac{h^2M^3}{6}|y_1-y_2| \quad [\text{ Using Lemma 3.1 }]\\
&= M\left(1+\frac{5hM}{6}+\frac{h^2M^2}{6}\right)|y_1-y_2|\\
&= L_\mu |y_1-y_2|.
\end{aligned}
$$

Therefore, $|\mu(x,y_1)-\mu(x,y_2)| \leq L_\mu |y_1-y_2|$, where $L_\mu = M\left(1+\frac{5hM}{6}+\frac{h^2M^2}{6}\right)$.

Theorem 3.1. Let us assume the function $p_x(x,u)$ satisfy the condition of Lipschitz in the unknown variable y, with Lipschitz constant $0 < L$,

$$|p_x(x,y_1)-p_x(x,y_2)| \leq L|y_1-y_2|,$$

and $y(x)$ be the unique solution of IVP of FDEs (3.1). Also let us assume y_k be the approximate solution which is generated by RK3 scheme for non-negative integer n. Then, for every $k = 0,1,2,\ldots n$,

$$y(x_k) - y_k = O(h^3).$$

Proof. First we are taking our fractional RK3 iterative scheme which is based on $y_k = y(x_k)$, then we can write

$$
\begin{aligned}
\bar{y}_{k+1} ={}& y(x_k) + \frac{h}{6}\left[{}_C D_{x_0^+}^{1-\beta} p(x,y_k)\Big|_{x=x_k} + 4\, {}_C D_{x_0^+}^{1-\beta} p\left(x+\frac{h}{2}, y_k + \frac{h}{2}\, {}_C D_{x_0^+}^{1-\beta} p(x,y_k)\Big|_{x=x_k}\right)\Bigg|_{x=x_k}\right.\\
&+ {}_C D_{x_0^+}^{1-\beta} p\left(x+h,\ y_k - h\, {}_C D_{x_0^+}^{1-\beta} p(x,y_k)\Big|_{x=x_k} + 2h\, {}_C D_{x_0^+}^{1-\beta} p\left(x+\frac{h}{2},\right.\right.\\
&\left.\left.\left. y_k + \frac{h}{2}\, {}_C D_{x_0^+}^{1-\beta} p(x,y_k)\Big|_{x=x_k}\right)\Bigg|_{x=x_k}\right)\Bigg|_{x=x_k}\right].
\end{aligned}
$$

Consider, $P(x,y) = {}_C D_{x_0^+}^{1-\beta} p(x,y)$, then the above expression will become,

$$
\begin{aligned}
\bar{y}_{k+1} ={}& y(x_k) + \frac{h}{6}\left[P(x_k,y_k) + 4P\left(x_k+\frac{h}{2},\ y_k+\frac{h}{2}y'(x_k)\right)+\right.\\
&\left.P\left(x_k+h,\ y_k - hy'(x_k) + 2hP\left(x_k+\frac{h}{2}, y_k+\frac{h}{2}y'(x_k)\right)\right)\right]\\
={}& y(x_k) + \frac{h}{6}P(x_k,y_k) + \frac{2h}{3}\left[P(x_k,y_k) + \left(\frac{h}{2}P_x(x_k,y_k) + \frac{h}{2}P_y(x_k,y_k)y'(x_k)\right)+\right.
\end{aligned}
$$

$$\frac{1}{2!}\left[\left(\left(\frac{h}{2}\right)^2 P_{xx}(x_k,y_k)+\frac{h^2}{2}y'(x_k)\,P_{xy}(x_k,y_k)+\left(\frac{h}{2}\right)^2(y'(x_k))^2\,P_{yy}(x_k,y_k)\right)+O(h^3)\right]$$
$$+\frac{h}{6}\Big[\,P(x_k,y_k)+\Big\{hP_x(x_k,y_k)+h\Big(P(x_k,y_k)+hP_x(x_k,y_k)+hy'(x_k)P_y(x_k,y_k)+O(h^2)\Big)$$
$$P_y(x_k,y_k)\Big\}+\frac{h^2}{2!}\Big\{P_{xx}(x_k,y_k)+2P_{xy}(x_k,y_k)\,(y'(x_k)+O(h))+P_{yy}(x_k,y_k)\,(y'(x_k)+O(h))^2\Big\}$$
$$+\frac{h^3}{3!}\Big\{P_{xxx}(\xi,\eta)+3\,(P(x_k,y_k)+O(h))\,P_{xxy}(\xi,\eta)+3\,(P(x_k,y_k)+O(h))^2\,P_{xyy}(\xi,\eta)$$
$$+(P(x_k,y_k)+O(h))^3\,P_{yyy}(\xi,\eta)\Big\}\Big]$$
$$=y(x_k)+hy'(x_k)+\frac{h^2}{2!}\Big[P_x(x_k,y_k)+y'(x_k)P_y(x_k,y_k)\Big]+\frac{h^3}{3!}\Big[P_{xx}(x_k,y_k)+2y'(x_k)P_{xy}(x_k,y_k)$$
$$+P_{yy}(x_k,y_k)\,(y'(x_k))^2+P_x(x_k,y_k)P_y(x_k,y_k)+y'(x_k)\Big(P_y(x_k,y_k)\Big)^2\Big]+O(h^4)$$
$$=y(x_k)+hy'(x_k)+\frac{h^2}{2!}y''(x_k)+\frac{h^3}{3!}y'''(x_k)+O(h^4). \tag{3.3}$$

With the help of Taylor's series, we can write the exact form of the solution,

$$y(x_{k+1})=y(x_k)+hy'(x_k)+\frac{h^2}{2!}y''(x_k)+\frac{h^3}{3!}y'''(x_k)+\frac{h^4}{4!}y''''(x_k)+\ldots \tag{3.4}$$

By Eqs (3.3) and (3.4), we get $|y(x_{k+1})-\bar{y}_{k+1}|=O(h^4)$.

$$\text{So,}\quad |y(x_{k+1})-\bar{y}_{k+1}|\le Ch^4.$$

Taking,

$$\mu(x,y)=\frac{1}{6}P(x,y)+\frac{2}{3}P\left(x+\frac{h}{2},y+\frac{h}{2}P(x,y)\right)$$
$$+\frac{1}{6}P\left(x+h,y-hP(x,y)+2hP\left(x+\frac{h}{2},y+\frac{h}{2}P\,(x,y)\right)\right).$$

From the above Lemmas 3.1 and 3.2, we have

$$|\bar{y}_{k+1}-y_{k+1}|\le|y(x_k)-y_k|+h\,|\mu(x_k,y(x_k))-\mu(x_k,y_k)|$$
$$\le(hL_\mu+1)\,|y(x_k)-y_k|$$

Therefore,

$$|y(x_{k+1})-y_{k+1}|\le|y(x_{k+1})-\bar{y}_{k+1}|+|\bar{y}_{k+1}-y_{k+1}|$$
$$\le Ch^4+(hL_\mu+1)\,|y(x_k)-y_k|\,.$$

So, error estimation will be, $|\epsilon_{k+1}|=(hL_\mu+1)\,|\epsilon_k|+Ch^4$.
Using recursion relation,

$$|\epsilon_k|\le(hL_\mu+1)^k\,|\epsilon_0|+\frac{Ch^3}{L_\mu}\left[\left(hL_\mu+1\right)^k-1\right].$$

Since, $x_k - x_0 = kh$ and $\epsilon_0 = 0$ then, $(hL_\mu + 1)^k \leq e^{khL_\mu} = f_\mu$.
So, we have $|\epsilon_k| \leq \frac{Ch^3}{L_\mu}(f_\mu - 1)$, where $L_\mu = M\left(1 + \frac{5hM}{6} + \frac{h^2M^2}{6}\right)$ [By Lemma 3.2]
Therefore, $|y(x_k) - y_k| = O(h^3)$.
This indicates that our suggested scheme fractional RK3 has a cubic convergence rate. □

3.2. *Fractional SSRK3 scheme of FDE*

Here, we proposed another fractional third-order Runge-Kutta scheme which is more stable and renamed as fractional strong stability preserving third-order Runge Kutta (SSRK3) scheme. So, the explicit fractional SSRK3 scheme is given by the Butcher tableau:

$$
\begin{array}{c|ccc}
0 & & & \\
1 & 1 & & \\
\frac{1}{2} & \frac{1}{6} & \frac{1}{4} & \\
\hline
 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
$$

Here is the algorithm of fractional SSRK3 scheme of IVP of FDE (3.1):

$$
\begin{cases}
x_k = x_0 + kh \ \text{ for each } \ k = 0, 1, 2, \ldots, n \\
h = x_{k+1} - x_k, \ \text{(step size)} \\
y_{k+1} = y_k + \frac{h}{6}(p_1 + p_2 + 4p_3), \ \text{where} \\
p_1 = {}_C D^{1-\beta}_{x_0^+} p(x, y_k)\Big|_{x=x_k}, \\
p_2 = {}_C D^{1-\beta}_{x_0^+} p(x+h, y_k + hp_1)\Big|_{x=x_k}, \\
p_3 = {}_C D^{1-\beta}_{x_0^+} p(x + \frac{h}{2}, y_k + \frac{h}{4}p_1 + \frac{h}{4}p_2)\Big|_{x=x_k}.
\end{cases}
$$

This algorithm is also a cubic convergence scheme that is more stable and accurate than the fractional RK3 scheme and faster in comparing other linear and quadratic convergence schemes for IVP of FDEs.

Lemma 3.3. Consider the function $P(x, y)$ satisfy the Lipschitz condition for the unknown variable y and if the conditions of Theorem 2.1 hold, then

$$
\begin{aligned}
\lambda(x, y) =& \frac{1}{6}P(x, y) + \frac{1}{6}P(x + h, y + hP(x, y)) \\
&+ \frac{2}{3}P\left(x + \frac{h}{2}, y + \frac{h}{4}P(x, y) + \frac{h}{4}P(x + h, y + hP(x, y))\right)
\end{aligned}
\tag{3.5}
$$

will always fulfill the condition of Lipschitz in the unknown variable y.

Proof.

$$
\begin{aligned}
|\lambda(x, y_1) - \lambda(x, y_2)| \leq & \frac{1}{6}|P(x, y_1) - P(x, y_2)| + \frac{1}{6}\Big|P(x + h, y_1 + hP(x, y_1)) - \\
& P(x + h, y_2 + hP(x, y_2))\Big| + \frac{2}{3}\Bigg|P\left(x + \frac{h}{2}, y_1 + \frac{h}{4}P(x, y_1) + \frac{h}{4}P\Big(x + h,\right. \\
& \left. y_1 + P(x, y_1))\Big)\right) - P\left(x + \frac{h}{2}, y_2 + \frac{h}{4}hP(x, y_2) + \frac{h}{4}P(x + h, y_2 + hP(x, y_2))\right)\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\leq M\,|y_1 - y_2| + \frac{hM^2}{3}\,|y_1 - y_2| + \frac{h^2M^3}{6}\,|y_1 - y_2| \quad [\text{ Using Lemma 3.1 }]\\
&= M\left(1 + \frac{hM}{3} + \frac{h^2M^2}{6}\right)|y_1 - y_2|\\
&= L_\lambda\,|y_1 - y_2|,
\end{aligned}
$$

Therefore, $|\lambda(x, y_1) - \lambda(x, y_2)| \leq L_\lambda\,|y_1 - y_2|$, where $L_\lambda = M\left(1 + \frac{hM}{3} + \frac{h^2M^2}{6}\right)$.

Theorem 3.2. Let us assume the function $p_x(x, u)$ satisfy the condition of Lipschitz in the unknown variable y, with Lipschitz constant $0 < L$,

$$
|p_x(x, y_1) - p_x(x, y_2)| \leq L\,|y_1 - y_2|,
$$

and $y(x)$ be the unique solution of IVP of FDEs (3.1). Also, let us assume y_k be the approximate solution which is generated by third order strong stability preserving Runge Kutta scheme for non-negative integer n. Then, for every $k = 0, 1, 2, \ldots n$,

$$
y\,(x_k) - y_k = O(h^3).
$$

Proof. First we are taking our fractional SSRK3 iterative scheme which depend on $y_k = y(x_k)$, then we can write

$$
\begin{aligned}
\bar{y}_{k+1} =& y(x_k) + \frac{h}{6}\Bigg[\,{}_CD^{1-\beta}_{x_0^+}p(x, y_k)\Big|_{x=x_k} + {}_CD^{1-\beta}_{x_0^+}p\left(x + h, y_k + h\,{}_CD^{1-\beta}_{x_0^+}p(x, y_k)\Big|_{x=x_k}\right)\Bigg|_{x=x_k}\\
&+ 4\,{}_CD^{1-\beta}_{x_0^+}p\Bigg(x + \frac{h}{2},\ y_k + \frac{h}{4}\,{}_CD^{1-\beta}_{x_0^+}p(x, y_k)\Big|_{x=x_k} + \frac{h}{4}\,{}_CD^{1-\beta}_{x_0^+}p\Big(x + h,\\
&y_k + h\,{}_CD^{1-\beta}_{x_0^+}p(x, y_k)\Big|_{x=x_k}\Big)\Big|_{x=x_k}\Bigg)\Bigg|_{x=x_k}\Bigg]
\end{aligned}
$$

Consider, $P(x, y) = {}_CD^{1-\beta}_{x_0^+}p(x, y)$, then the above expression will become,

$$
\begin{aligned}
\bar{y}_{k+1} &= y(x_k) + \frac{h}{6}\left[P\left(x_k, y_k\right) + P\left(x_k + h,\ y_k + hy'(x_k)\right) + \right.\\
&\qquad \left. 4P\left(x_k + \frac{h}{2},\ y_k + \frac{h}{4}y'(x_k) + \frac{h}{4}P(x_k + h, y_k + hy'(x_k))\right)\right]\\
&= y(x_k) + \frac{h}{6}P(x_k, y_k) + \frac{h}{6}\Big[P(x_k, y_k) + \{hP_x(x_k, y_k) + hP_y(x_k, y_k)y'(x_k)\} +\\
&\frac{1}{2!}\Big\{h^2P_{xx}(x_k, y_k) + 2h^2y'(x_k)\,P_{xy}(x_k, y_k) + h^2\left(y'(x_k)\right)^2 P_{yy}(x_k, y_k)\Big\} + O(h^3)\Big]\\
&+ \frac{2h}{3}\Bigg[\,P(x_k, y_k) + \Big\{\frac{h}{2}P_x(x_k, y_k) + \frac{h}{4}\Big(2P(x_k, y_k) + hP_x(x_k, y_k) + hy'(x_k)P_y(x_k, y_k) + O(h^2)\Big)\\
&P_y(x_k, y_k)\Big\} + \frac{1}{2!}\Big\{\frac{h^2}{4}P_{xx}(x_k, y_k) + \frac{h^2}{4}P_{xy}(x_k, y_k)(2y'(x_k) + O(h)) + \frac{h^2}{16}P_{yy}(x_k, y_k)\\
&(2y'(x_k) + O(h))^2\Big\} + \frac{1}{3!}\Big\{\frac{h^3}{8}P_{xxx}(\xi, \eta) + \frac{3h^3}{16}(2y'(x_k) + O(h))P_{xxy}(\xi, \eta) +
\end{aligned}
$$

$$
\begin{aligned}
&\frac{3h^3}{32}(2y'(x_k)+O(h))^2P_{xyy}(\xi,\eta)+\frac{h^3}{64}(P(x_k,y_k)+O(h))^3P_{yyy}(\xi,\eta)\Big\}\Big]\\
&=y(x_k)+hy'(x_k)+\frac{h^2}{2!}\left[P_x(x_k,y_k)+y'(x_k)P_y(x_k,y_k)\right]+\frac{h^3}{3!}\Big[P_{xx}(x_k,y_k)+2y'(x_k)P_{xy}(x_k,y_k)\\
&\quad+P_{yy}(x_k,y_k)\left(y'(x_k)\right)^2+P_x(x_k,y_k)P_y(x_k,y_k)+y'(x_k)\left(P_y(x_k,y_k)\right)^2\Big]+O(h^4)\\
&=y(x_k)+hy'(x_k)+\frac{h^2}{2!}y''(x_k)+\frac{h^3}{3!}y'''(x_k)+O(h^4).
\end{aligned}
\tag{3.6}
$$

With the help of Taylor's series, we can write the exact form of the solution,

$$
y(x_{k+1})=y(x_k)+hy'(x_k)+\frac{h^2}{2!}y''(x_k)+\frac{h^3}{3!}y'''(x_k)+\frac{h^4}{4!}y''''(x_k)+\ldots \tag{3.7}
$$

By Eqs (3.6) and (3.7), we get $|y(x_{k+1})-\bar{y}_{k+1}|=O(h^4)$.

$$
\text{So,}\quad |y(x_{k+1})-\bar{y}_{k+1}|\le Ch^4.
$$

Taking,

$$
\begin{aligned}
\lambda(x,y)=&\frac{1}{6}P(x,y)+\frac{1}{6}P(x+h,y+hP(x,y))\\
&+\frac{2}{3}P\left(x+\frac{h}{2},y+\frac{h}{4}P(x,y)+\frac{h}{4}P(x+h,y+hP\,(x,y)\,)\right).
\end{aligned}
$$

From the above Lemmas 3.1 and 3.2, we have

$$
\begin{aligned}
|\bar{y}_{k+1}-y_{k+1}|&\le|y(x_k)-y_k|+h\,|\lambda(x_k,y(x_k))-\lambda(x_k,y_k)|\\
&\le(hL_\lambda+1)\,|y(x_k)-y_k|
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|y(x_{k+1})-y_{k+1}|&\le|y(x_{k+1})-\bar{y}_{k+1}|+|\bar{y}_{k+1}-y_{k+1}|\\
&\le Ch^4+(hL_\lambda+1)\,|y(x_k)-y_k|\,.
\end{aligned}
$$

So, error estimation will be, $|\epsilon_{k+1}|=(hL_\lambda+1)\,|\epsilon_k|+Ch^4$.
Using recursion relation,

$$
|\epsilon_k|\le(hL_\lambda+1)^k\,|\epsilon_0|+\frac{Ch^3}{L_\lambda}\left[(hL_\lambda+1)^k-1\right].
$$

Since, $x_k-x_0=kh$ and $\epsilon_0=0$ then, $(hL_\lambda+1)^k\le e^{khL_\lambda}=f_\lambda$.
So, we have $|\epsilon_k|\le\frac{Ch^3}{L_\lambda}\,(f_\lambda-1)$, where $L_\lambda=M\left(1+\frac{hM}{3}+\frac{h^2M^2}{6}\right)$. [By Lemma 3.3]
Therefore, $|y(x_k)-y_k|=O(h^3)$.
This indicates that our suggested scheme fractional SSRK3 has a cubic convergence rate. □

4. Numerical results and simulation

4.1. Numerical examples

In this section, we present two examples of IVP of FDEs. We provide the numerical solution using our suggested scheme by comparing it with the existing fractional EM and fractional IEM using Matlab. In this illustrated example, the first example is linear IVP of FDE and the second example is non-linear IVP of FDE.

Example 4.1. [47] Consider the following linear IVP of FDE:

$$ {}_{C}D_{0^{+}}^{\beta} y = -y, \quad 0.1 < x \leq 1, \tag{4.1} $$
$$ \text{with suplementary condition, } y(0.1) = E_{\beta}((-0.1)^{\beta}). $$

The analytic solution of FDE (4.1) is,

$$ y(x) = E_{\beta}(-x^{\beta}). $$

For $\beta = \frac{1}{2}$ and with the help of Matlab, by following our proposed scheme with step size $h = \frac{1}{10}$, the analytic and approximate solution of the FDE (4.1) is graphically shown in Figure 1 and tabulated in the Tables 1 and 2. In addition to this, the absolute error graph is shown in the Figure 2. Also, the order of convergence is tabulated in Table 3 and graphically shown in Figure 3.

Table 1. Numerical solution of Example 4.1 when $\beta = \dfrac{1}{2}$, with step length $h = \dfrac{1}{10}$.

		EM		IEM		RK3	
x	y_{exact}	y_{EM}	$\lvert y_{exact} - y_{EM}\rvert$	y_{IEM}	$\lvert y_{exact} - y_{IEM}\rvert$	y_{RK3}	$\lvert y_{exact} - y_{RK3}\rvert$
1/10	0.72358	0.72358	0.00000	0.72358	0.00000	0.72358	0.00000
2/10	0.64379	0.61752	0.02626	0.63966	0.00413	0.64372	0.00007
3/10	0.59202	0.55575	0.03627	0.58687	0.00515	0.59194	0.00008
4/10	0.55361	0.51194	0.04166	0.54805	0.00556	0.55353	0.00008
5/10	0.52316	0.47810	0.04506	0.51739	0.00577	0.52308	0.00008
6/10	0.49802	0.45062	0.04740	0.49213	0.00589	0.49794	0.00008
7/10	0.47670	0.42759	0.04911	0.47073	0.00597	0.47662	0.00008
8/10	0.45825	0.40783	0.05042	0.45223	0.00602	0.45817	0.00008
9/10	0.44202	0.39057	0.05145	0.43596	0.00606	0.44194	0.00008
1	0.42758	0.37530	0.05228	0.42150	0.00608	0.42750	0.00008

Table 2. Numerical solution of Example 4.1 when $\beta = \dfrac{1}{2}$, with step length $h = \dfrac{1}{10}$.

		EM		IEM		SSRK3	
x	y_{exact}	y_{EM}	$\lvert y_{exact} - y_{EM}\rvert$	y_{IEM}	$\lvert y_{exact} - y_{IEM}\rvert$	y_{SSRK3}	$\lvert y_{exact} - y_{SSRK3}\rvert$
1/10	0.72358	0.72358	0.00000	0.72358	0.00000	0.72358	0.00000
2/10	0.64379	0.61752	0.02626	0.63966	0.00413	0.64372	0.00007
3/10	0.59202	0.55575	0.03627	0.58687	0.00515	0.59194	0.00008
4/10	0.55361	0.51194	0.04166	0.54805	0.00556	0.55353	0.00008
5/10	0.52316	0.47810	0.04506	0.51739	0.00577	0.52308	0.00008
6/10	0.49802	0.45062	0.04740	0.49213	0.00589	0.49794	0.00008
7/10	0.47670	0.42759	0.04911	0.47073	0.00597	0.47662	0.00008
8/10	0.45825	0.40783	0.05042	0.45223	0.00602	0.45817	0.00008
9/10	0.44202	0.39057	0.05145	0.43596	0.00606	0.44194	0.00008
1	0.42758	0.37530	0.05228	0.42150	0.00608	0.42750	0.00008

Table 3. Order of convergence table of Example 4.1.

	EM		IEM		RK3		SSRK3	
n	Error	Order	Error	Oredr	Error	Order	Error	Order
10	0.046549	—	0.004972	—	0.000056	—	0.000056	—
20	0.022073	1.0764	0.001285	1.9523	0.000004	2.7265	0.000004	2.7265
40	0.010719	1.0422	0.000324	1.9859	0.000000	2.9098	0.000000	2.9098
80	0.005278	1.0219	0.000081	1.9963	0.000000	2.9750	0.000000	2.9750
160	0.002619	1.0111	0.000020	1.9991	0.000000	2.9935	0.000000	2.9935
320	0.001304	1.0056	0.000005	1.9998	0.000000	2.9984	0.000000	2.9984
640	0.000651	1.0028	0.000001	1.9999	0.000000	2.9993	0.000000	2.9993
1280	0.000325	1.0014	0.000000	2.0000	0.000000	3.0077	0.000000	3.0077
2560	0.000162	1.0007	0.000000	2.0000	0.000000	2.9533	0.000000	2.9533
5120	0.000081	1.0004	0.000000	2.0000	0.000000	3.0559	0.000000	3.0559

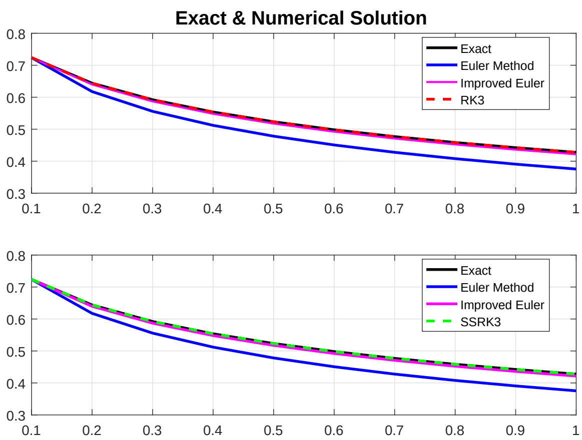

Figure 1. Analytical & numerical solution of Example 4.1.

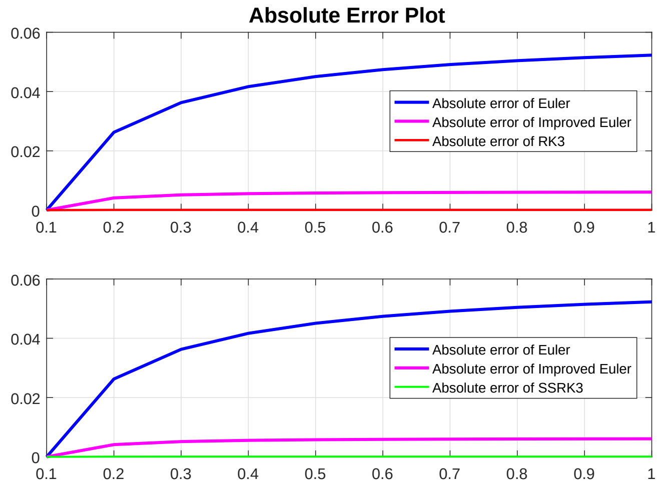

Figure 2. Absolute error visualization of Example 4.1.

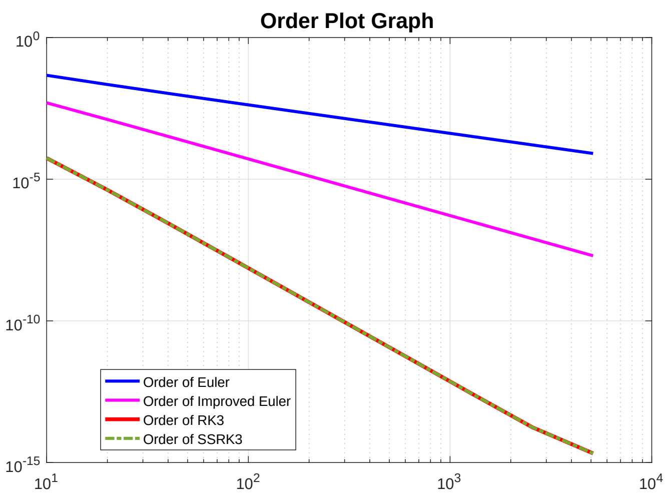

Figure 3. Order of convergence plot for Example 4.1.

Example 4.2. Consider the non-linear IVP of FDE:

$$ {}_{C}D^{\beta}_{1^+}y = \left(\frac{35\sqrt{\pi}}{32}\right)y^{\frac{6}{7}}, \quad 1 < x \leq 2, \tag{4.2} $$

with suplementary condition, $y(1) = 1$.

The analytic solution of FDE (4.2) is,

$$ y(x) = x^{3.5}. $$

For $\beta = \dfrac{1}{2}$ and with the help of Matlab, by following our proposed scheme with step size $h = \dfrac{1}{10}$, the analytic and approximate solution of the FDE (4.2) is graphically shown in Figure 4 and tabulated in the Tables 4 and 5. In addition to this, the absolute error graph is shown in the Figure 5. Also, the order of convergence is tabulated in Table 6 and graphically shown in Figure 6.

Table 4. Numerical solution of Example 4.2 when $\beta = \frac{1}{2}$, with step length $h = \frac{1}{10}$.

		EM		IEM		RK3	
x	y_{exact}	y_{EM}	$\lvert y_{exact} - y_{EM}\rvert$	y_{IEM}	$\lvert y_{exact} - y_{IEM}\rvert$	y_{RK3}	$\lvert y_{exact} - y_{RK3}\rvert$
1	1.00000	1.00000	0.00000	1.00000	0.00000	1.00000	0.00000
11/10	1.39596	1.35000	0.04596	1.39184	0.00413	1.39567	0.00029
12/10	1.89293	1.78367	0.10925	1.88345	0.00948	1.89229	0.00064
13/10	2.50497	2.31282	0.19214	2.48883	0.01614	2.50391	0.00106
14/10	3.24674	2.94987	0.29688	3.22256	0.02418	3.24521	0.00153
15/10	4.13351	3.70782	0.42569	4.09983	0.03369	4.13144	0.00208
16/10	5.18108	4.60027	0.58081	5.13635	0.04473	5.17839	0.00269
17/10	6.40577	5.64135	0.76442	6.34839	0.05738	6.40240	0.00336
18/10	7.82445	6.84574	0.97870	7.75275	0.07169	7.82033	0.00411
19/10	9.45448	8.22866	1.22581	9.36673	0.08775	9.44954	0.00494
2	11.31371	9.80582	1.50789	11.20811	0.10559	11.30787	0.00583

Table 5. Numerical solution of Example 4.2 when $\beta = \frac{1}{2}$, with step length $h = \frac{1}{10}$.

		EM		IEM		SSRK3	
x	y_{exact}	y_{EM}	$\lvert y_{exact} - y_{EM}\rvert$	y_{IEM}	$\lvert y_{exact} - y_{IEM}\rvert$	y_{SSRK3}	$\lvert y_{exact} - y_{SSRK3}\rvert$
1	1.00000	1.00000	0.00000	1.00000	0.00000	1.00000	0.00000
11/10	1.39596	1.35000	0.04596	1.39184	0.00413	1.39575	0.00021
12/10	1.89293	1.78367	0.10925	1.88345	0.00948	1.89247	0.00046
13/10	2.50497	2.31282	0.19214	2.48883	0.01614	2.50420	0.00076
14/10	3.24674	2.94987	0.29688	3.22256	0.02418	3.24564	0.00110
15/10	4.13351	3.70782	0.42569	4.09983	0.03369	4.13202	0.00149
16/10	5.18108	4.60027	0.58081	5.13635	0.04473	5.17914	0.00193
17/10	6.40577	5.64135	0.76442	6.34839	0.05738	6.40335	0.00242
18/10	7.82445	6.84574	0.97870	7.75275	0.07169	7.82149	0.00296
19/10	9.45448	8.22866	1.22581	9.36673	0.08775	9.45093	0.00355
2	11.31371	9.80582	1.50789	11.20811	0.10559	11.30952	0.00419

Table 6. Order of convergence table of Example 4.2.

	EM		IEM		RK3		SSRK3	
n	Error	Order	Error	Order	Error	Order	Error	Order
10	1.507887	—	0.105594	—	0.005834	—	0.004193	—
20	0.803028	0.9090	0.028540	1.8875	0.000787	2.8901	0.000563	2.8971
40	0.414814	0.9530	0.007426	1.9424	0.000102	2.9470	0.000073	2.9488
80	0.210871	0.9761	0.001894	1.9708	0.000013	2.9741	0.000009	2.9745
160	0.106319	0.9880	0.000478	1.9853	0.000002	2.9872	0.000001	2.9873
320	0.053383	0.9940	0.000120	1.9926	0.000000	2.9937	0.000000	2.9936
640	0.026748	0.9970	0.000030	1.9963	0.000000	2.9968	0.000000	2.9968
1280	0.013388	0.9985	0.000008	1.9982	0.000000	2.9984	0.000000	2.9984
2560	0.006697	0.9992	0.000002	1.9991	0.000000	2.9992	0.000000	2.9992

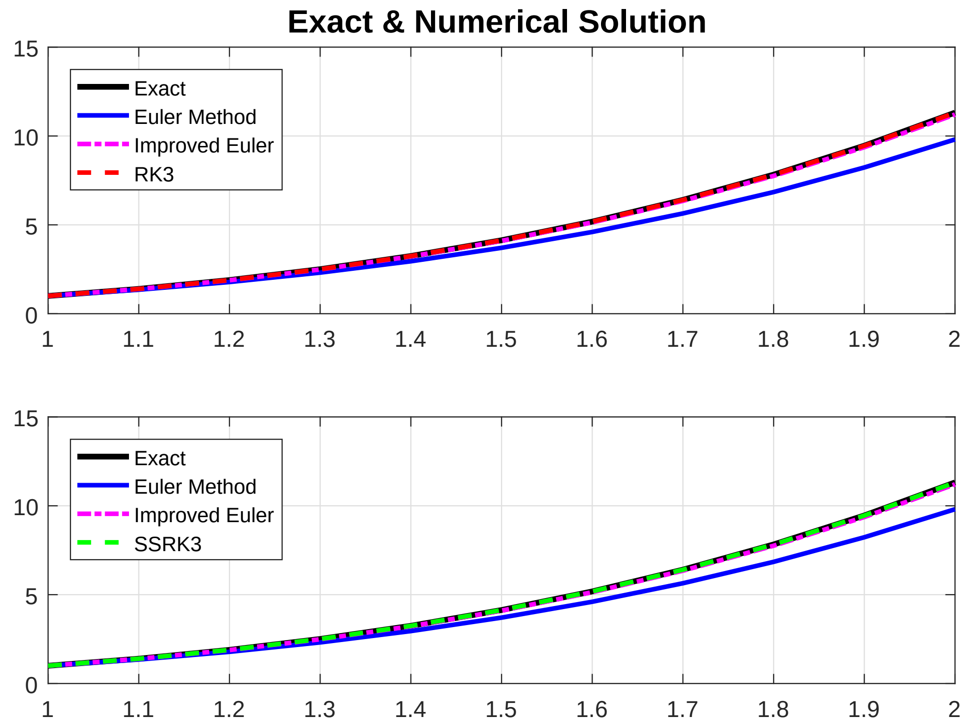

Figure 4. Analytical & numerical solution of Example 4.2.

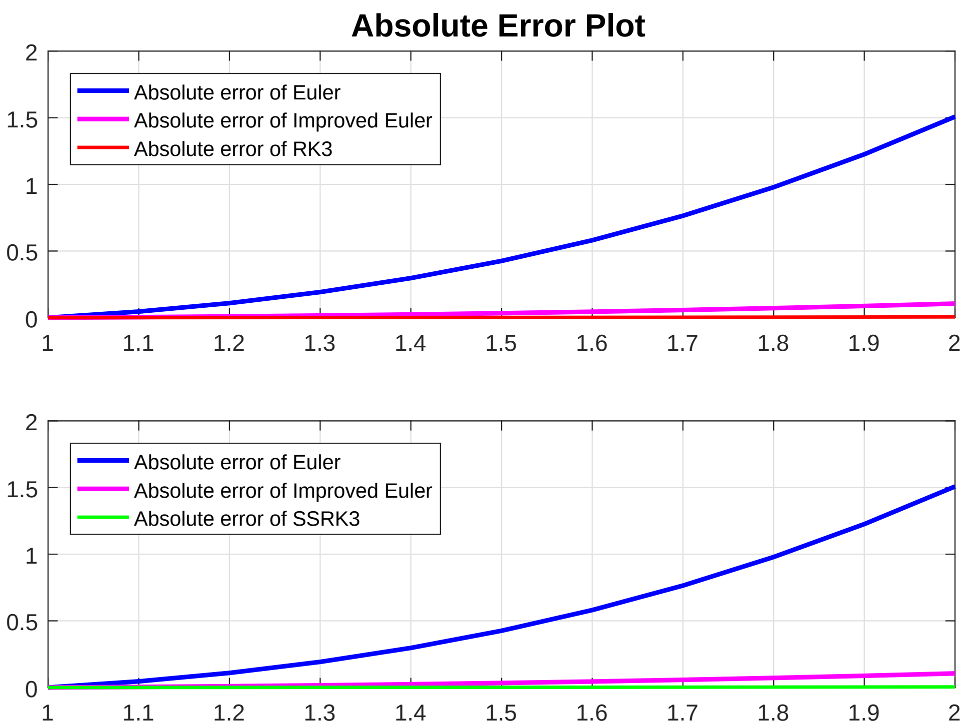

Figure 5. Absolute error visualization of Example 4.2.

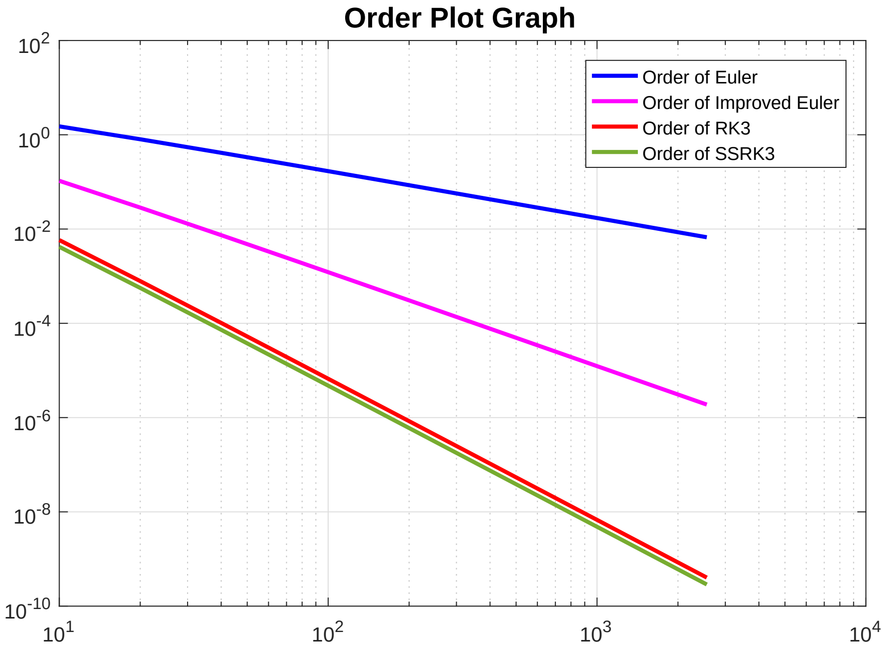

Figure 6. Order plot for Example 4.2.

5. Application examples and simulation

Example 5.1. (Fractional WPG model) [48] Consider the following linear IVP of FDE of WPG model,

$$ {}_{C}D^{\beta}_{t_0^+} n(t) = Pn(t), \quad t > t_0, \tag{5.1} $$
$$ \text{with suplementary condition, } n(t_0) = n_0. $$

Here, $P = B - M$ is the population production rate where B is the birth rate and M is the mortality rate. Also, $n(t)$ is the individuals population at time t and ${}_{C}D^{\beta}$ is the β order rate of change of population. This model (5.1) is known as fractional WPG model of order β. The analytic solution of (5.1) is,

$$ n(t) = n_0 \, E_{\beta}(Pt), \; t \geq 0. $$

Particularly, if we take $\beta = 1$, then that model will be classical WPG such as:

$$ \frac{dn(t)}{dt} = Pn(t), \quad t > t_0, \tag{5.2} $$
$$ \text{with, } n(t_0) = n_0. $$

The analytic solution of (5.2) is,

$$ n(t) = n_0 \, e^{Pt}, \quad t \geq 0. $$

The population at the initial time t_0 is denoted by the symbol n_0. It is found from the survey that the fractional population model (5.1) precisely fitted with world statistical population data with proper order β. In this article, we have taken the world census population data from the year 1920 to 2018 which is collected from the world population data sites SITE-1 or SITE-2 (taken from `https://www.worldbank.org/`) and also one population data from United Nations [49]. Here, this statistical population data is fitted for fractional order $\beta = 1.3932987548432$ [48] and production rate $P \approx 0.0034399$. As per statistical world population data, we found that the initial population in the year 1920 is $n_0 = 1860$ million. Also, we have taken the classical population model for which the production rate is $P \approx 0.013501$.

With the help of Matlab by following our proposed scheme with step size $h = 1$, the statistical world population data from the year 1920 to 2018 is graphically shown in Figure 7. The analytic and approximate solution of the FDE (5.1) graphically shown in Figure 8. In addition to this, the analytic and approximate solution is tabulated in Tables 7 and 8. The absolute error graph is represented in the Figure 9. Thus, from the figure and table, we found that our suggested scheme fractional RK3 and fractional SSRK3 are much more effective and accurate to the statistical population data than EM and IEM.

Table 7. Numerical solution of Example (5.1) when $\beta = 1$, $\beta = 1.3932987548432$, with $h = 1$ year.

	Analytical		EM		IEM		RK3	
Year(t)	n_{Linear}	n_{Frac}	n_{EM}	Error	n_{IEM}	Error	n_{RK3}	Error
1920	1.8600×10^3	1.8600×10^3	1.8600×10^3	0.0000×10^0	1.8600×10^3	0.0000×10^0	1.8600×10^3	0.0000×10^0
1930	2.1289×10^3	1.9909×10^3	1.9799×10^3	1.1060×10^1	1.9892×10^3	1.7315×10^0	1.9906×10^3	3.1405×10^{-1}
1940	2.4366×10^3	2.2169×10^3	2.2020×10^3	1.4921×10^1	2.2152×10^3	1.7413×10^0	2.2166×10^3	3.1405×10^{-1}
1950	2.7888×10^3	2.5173×10^3	2.4987×10^3	1.8625×10^1	2.5156×10^3	1.7390×10^0	2.5170×10^3	3.1405×10^{-1}
1960	3.1919×10^3	2.8943×10^3	2.8717×10^3	2.2615×10^1	2.8926×10^3	1.7320×10^0	2.8940×10^3	3.1405×10^{-1}
1970	3.6533×10^3	3.3561×10^3	3.3290×10^3	2.7117×10^1	3.3544×10^3	1.7219×10^0	3.3558×10^3	3.1405×10^{-1}
1980	4.1814×10^3	3.9147×10^3	3.8824×10^3	3.2308×10^1	3.9130×10^3	1.7090×10^0	3.9144×10^3	3.1405×10^{-1}
1990	4.7858×10^3	4.5858×10^3	4.5474×10^3	3.8363×10^1	4.5841×10^3	1.6930×10^0	4.5855×10^3	3.1405×10^{-1}
2000	5.4775×10^3	5.3885×10^3	5.3431×10^3	4.5471×10^1	5.3869×10^3	1.6737×10^0	5.3882×10^3	3.1405×10^{-1}
2010	6.2693×10^3	6.3462×10^3	6.2923×10^3	5.3847×10^1	6.3445×10^3	1.6505×10^0	6.3459×10^3	3.1405×10^{-1}
2020	7.1755×10^3	7.4865×10^3	7.4228×10^3	6.3739×10^1	7.4849×10^3	1.6229×10^0	7.4862×10^3	3.1405×10^{-1}

Table 8. Numerical solution of Example 5.1 when $\beta = 1$, $\beta = 1.3932987548432$, with $h = 1$ year.

	Analytical		EM		IEM		SSRK3	
Year(t)	n_{Linear}	n_{Frac}	n_{EM}	Error	n_{IEM}	Error	n_{SSRK3}	Error
1920	1.8600×10^3	1.8600×10^3	1.8600×10^3	0.0000×10^0	1.8600×10^3	0.0000×10^0	1.8600×10^3	0.0000×10^0
1930	2.1289×10^3	1.9909×10^3	1.9799×10^3	1.1060×10^1	1.9892×10^3	1.7315×10^0	1.9906×10^3	3.1405×10^{-1}
1940	2.4366×10^3	2.2169×10^3	2.2020×10^3	1.4921×10^1	2.2152×10^3	1.7413×10^0	2.2166×10^3	3.1405×10^{-1}
1950	2.7888×10^3	2.5173×10^3	2.4987×10^3	1.8625×10^1	2.5156×10^3	1.7390×10^0	2.5170×10^3	3.1405×10^{-1}
1960	3.1919×10^3	2.8943×10^3	2.8717×10^3	2.2615×10^1	2.8926×10^3	1.7320×10^0	2.8940×10^3	3.1405×10^{-1}
1970	3.6533×10^3	3.3561×10^3	3.3290×10^3	2.7117×10^1	3.3544×10^3	1.7219×10^0	3.3558×10^3	3.1405×10^{-1}
1980	4.1814×10^3	3.9147×10^3	3.8824×10^3	3.2308×10^1	3.9130×10^3	1.7090×10^0	3.9144×10^3	3.1405×10^{-1}
1990	4.7858×10^3	4.5858×10^3	4.5474×10^3	3.8363×10^1	4.5841×10^3	1.6930×10^0	4.5855×10^3	3.1405×10^{-1}
2000	5.4775×10^3	5.3885×10^3	5.3431×10^3	4.5471×10^1	5.3869×10^3	1.6737×10^0	5.3882×10^3	3.1405×10^{-1}
2010	6.2693×10^3	6.3462×10^3	6.2923×10^3	5.3847×10^1	6.3445×10^3	1.6505×10^0	6.3459×10^3	3.1405×10^{-1}
2020	7.1755×10^3	7.4865×10^3	7.4228×10^3	6.3739×10^1	7.4849×10^3	1.6229×10^0	7.4862×10^3	3.1405×10^{-1}

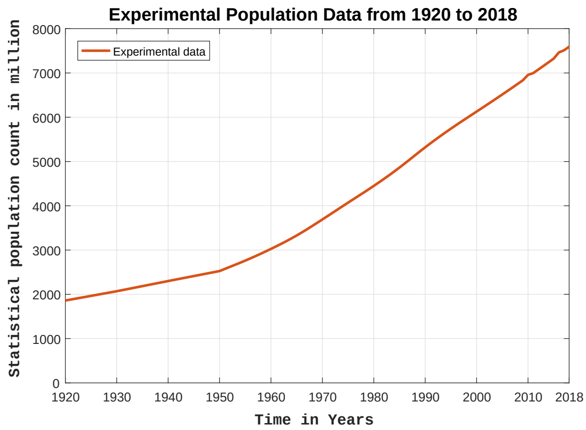

Figure 7. Population date from 1920 to 2018.

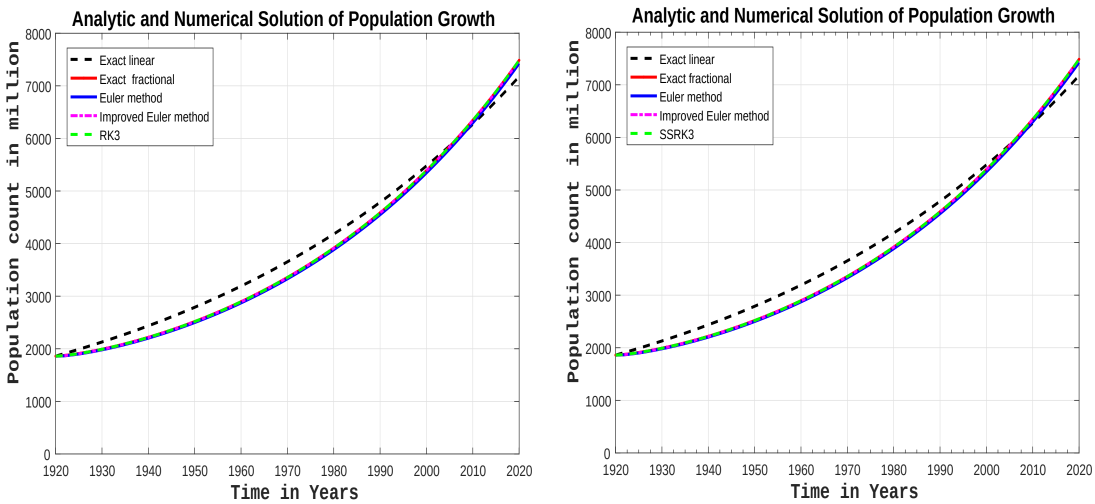

Figure 8. Analytical & numerical solution of Example 5.1.

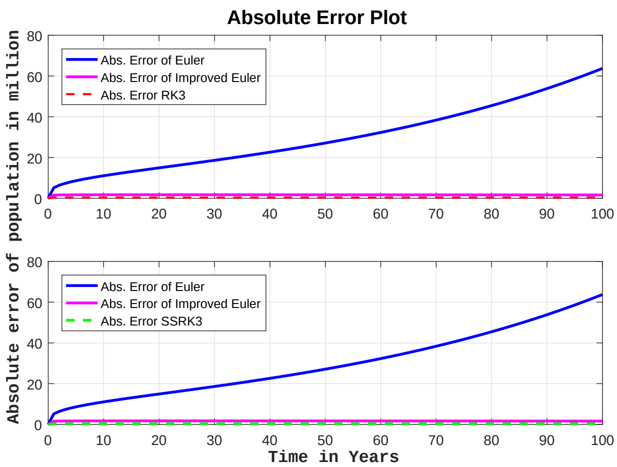

Figure 9. Absolute error plot for Example 5.1.

As per "The Census Bureau's International Data Base", the world population reached around 7.5 billion till June 2018. So, we conclude that our fractional population model for $\beta = 1.3932987548432$ is more accurate and near the statistical population data by following our fractional RK3 scheme and fractional SSRK3 scheme. Also, both our schemes give more accurate and faster results than EM and IEM algorithms.

Example 5.2. (Fractional ND model) [50] Consider the following time fractional radioactive decay equation:

$$
\begin{aligned}
&{}_C D^{\beta}_{t_0^+} N(t) = -\lambda N(t), \quad t > t_0, \\
&\text{supplementary condition, } N(t_0) = N_0.
\end{aligned} \tag{5.3}
$$

Here ${}_C D^{\beta}_{t_0^+}$ denotes the CFOD of arbitrary order β, $\beta \in (0, 1]$ and $N(t)$ represents the number of radioactive particle at any time t. The quantity N_0 is the initial number of particles at $t = 0$ and λ is decay constant, where $\lambda = \frac{1}{\tau}$, and τ is mean life time. The analytic solution for fractional order $0 < \beta < 1$ of FDE (5.3) is

$$N(t) = N_0\, E_\beta(-\lambda t^\beta), \ t \geq 0.$$

Particularly, for $\beta = 1$, the fractional order nuclear decay equation given by FDE (5.3) reduces to the classical one,

$$
\begin{aligned}
&\frac{dN(t)}{dt} = -\lambda N(t), \quad t > t_0, \\
&\text{with } N(t_0) = N_0.
\end{aligned} \tag{5.4}
$$

The analytical solution of (5.4) is,

$$N(t) = N_0 e^{-\lambda t}.$$

In this example, we have taken the experimental data of radioactivity of isotope of Aluminum $\left({}^{28}Al\right)$,which represents a decay model, since particles are emitted over the time and we collect the data from the survey of research article [51] for both the classical nuclear model and fractional nuclear model. It is also found that our fractional model is best fit with our experimental data [51] for fractional order $\beta = 0.8252$ and decay constant $\lambda = 0.0314$. Here, we have also taken the classical model for comparison purpose where the values of $\beta = 1$ and decay constant $\lambda = 0.0121$. For the radioactivity of Aluminum, the initial number of radioactive particle is $N_0 = 1200$ at time $t = 0$ second.

With the help of Matlab by following our proposed scheme with step size $h = 10$ second, the analytic and approximate solution of the fractional model (5.3) as well as classical model (5.4) is graphically shown in Figure 10. In addition to this, the analytic and approximate solution is tabulated in the Tables 9 and 10. The absolute error graph is represented in the Figure 11. Thus from the figure and table, we found that our suggested scheme fractional RK3 and fractional SSRK3 are much more efficiency and accurate to the statistical population data compare to EM and IEM.

Table 9. Numerical solution of Example 5.2 when $\beta = 10sec$, $\beta = 0.8252$, with step size $h = 10$ second.

	Analytical		EM		IEM		RK3	
Time(sec)	N_{Linear}	N_{Frac}	N_{EM}	Error	N_{IEM}	Error	N_{RK3}	Error
0.00	1200.000000	1200.000000	1200.000000	0.000000	1200.000000	0.000000	1200.000000	0.000000
50.00	655.289312	551.999494	560.334119	8.334625	552.321059	0.321565	552.160287	0.160792
100.00	357.836735	326.316340	336.069553	9.753213	326.689759	0.373419	326.503032	0.186692
150.00	195.405490	225.162844	234.125085	8.962241	225.505641	0.342797	225.334224	0.171379
200.00	106.705941	164.801057	172.792163	7.991106	165.105400	0.304344	164.953198	0.152142
250.00	58.269386	127.698306	134.676180	6.977874	127.963163	0.264857	127.830699	0.132393
300.00	31.819421	103.296152	109.401183	6.105030	103.527234	0.231082	103.411656	0.115504
350.00	17.375772	86.385264	91.765128	5.379864	86.588438	0.203174	86.486814	0.101550
400.00	9.488465	74.133931	78.920527	4.786596	74.314379	0.180448	74.224118	0.090188
450.00	5.181408	64.921999	69.222799	4.300800	65.083905	0.161906	65.002917	0.080918
500.00	2.829434	57.777720	61.677770	3.900050	57.924374	0.146654	57.851014	0.073294

Table 10. Numerical solution of Example 5.2 when $\beta = 1, \beta = 0.8252$, with step size $h = 10$ second.

	Analytical		EM		IEM		SSRK3	
Time(sec)	N_{Linear}	N_{Frac}	N_{EM}	Error	N_{IEM}	Error	N_{SSRK3}	Error
0.00	1200.000000	1200.000000	1200.000000	0.000000	1200.000000	0.000000	1200.000000	0.000000
50.00	655.289312	551.999494	560.334119	8.334625	552.321059	0.321565	552.015574	0.016080
100.00	357.836735	326.316340	336.069553	9.753213	326.689759	0.373419	326.335008	0.018668
150.00	195.405490	225.162844	234.125085	8.962241	225.505641	0.342797	225.179981	0.017136
200.00	106.705941	164.801057	172.792163	7.991106	165.105400	0.304344	164.816268	0.015211
250.00	58.269386	127.698306	134.676180	6.977874	127.963163	0.264857	127.711542	0.013236
300.00	31.819421	103.296152	109.401183	6.105030	103.527234	0.231082	103.307699	0.011547
350.00	17.375772	86.385264	91.765128	5.379864	86.588438	0.203174	86.395416	0.010152
400.00	9.488465	74.133931	78.920527	4.786596	74.314379	0.180448	74.142946	0.009015
450.00	5.181408	64.921999	69.222799	4.300800	65.083905	0.161906	64.930088	0.008089
500.00	2.829434	57.777720	61.677770	3.900050	57.924374	0.146654	57.785046	0.007326

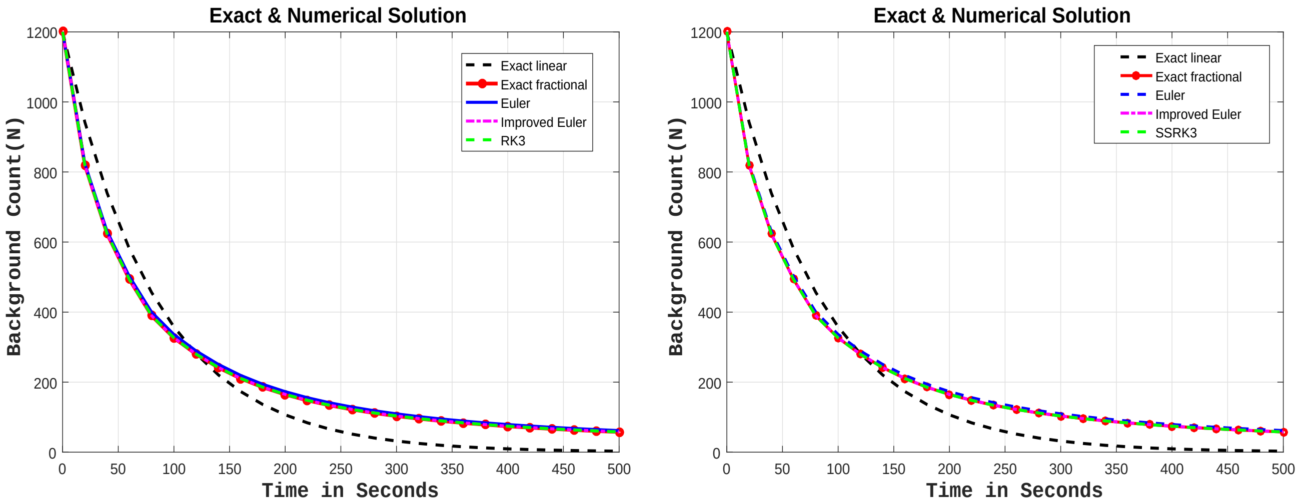

Figure 10. Analytical & numerical solution of Example 5.2.

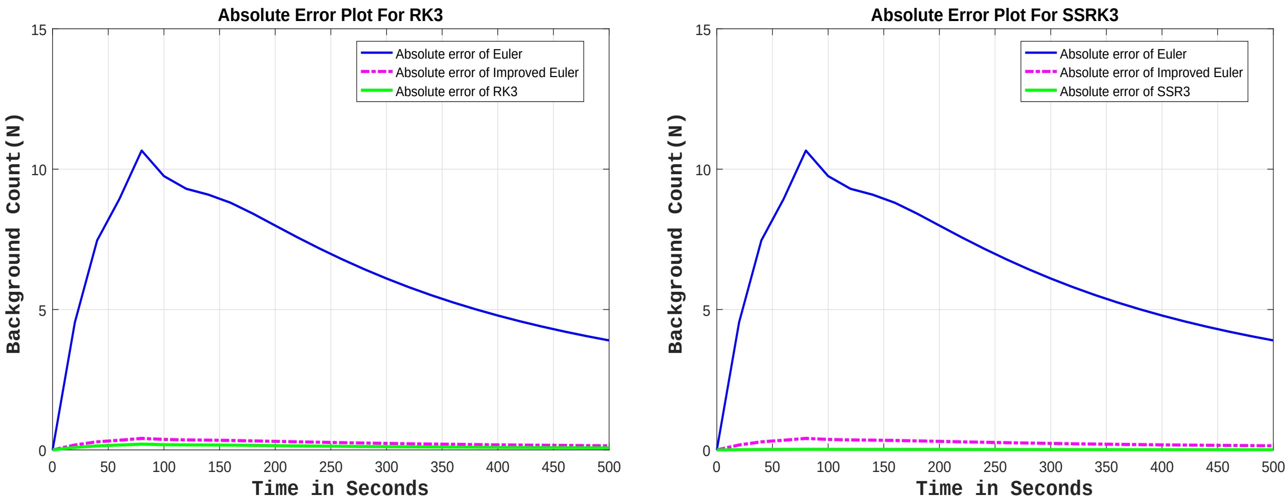

Figure 11. Absolute error plot for Example 5.2.

6. Conclusions

In this research article, a remarkable study has been done for finding the numerical approximation of the IVP of FDEs (1.1) of fractional order β where $\beta \in (0, 1]$. Here, we proposed two cubic convergence schemes: Fractional RK3 scheme and fractional SSRK3 scheme, and these schemes are based on classical RK3 scheme and classical SSRK3 scheme, respectively. We follow the analogous properties of the Caputo derivative to reduce the IVP of FDEs into an IVP of ODE of integer order. From the numerical point of view, both our suggested schemes are efficient and more accurate compared to other cubic convergence schemes as well as all linear and quadratic convergence schemes of IVP of FDEs (1.1). Also, we demonstrate the comparative numerical study of our proposed method with the existing fractional EM and fractional IEM of IVP of FDEs. In addition to this, we provide the numerical solution of two physical application examples: The fractional WPG model and fractional

ND model, using our suggested work to compare with the existing scheme EM and IEM. It is proven that our suggested schemes are faster and more accurate compared with EM and IEM. The significant concluding remark of our scheme are as follows:

- The fractional RK3 scheme is a cubic convergence scheme, faster than the other linear and quadratic convergence schemes for the IVP of FDEs.
- The fractional SSRK3 scheme also has a cubic convergence scheme that is more stable and faster compared to RK3 and as well as all other cubic convergence schemes for the IVP of FDEs.
- Computationally, both schemes are effective, and we get the satisfactory numerical approximation with the help of Matlab by enlargement of step length.

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Conflict of interest

All authors declare no conflicts of interest.

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