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Research article

Multiple solutions of a Sturm-Liouville boundary value problem of nonlinear differential inclusion with nonlocal integral conditions

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Abstract: The existence of solutions for a Sturm-Liouville boundary value problem of a nonlinear differential inclusion with nonlocal integral condition is studied. The maximal and minimal solutions will be studied. The existence of multiple solutions of the nonhomogeneous Sturm-Liouville boundary value problem of differential equation with nonlocal integral condition is considered. The eigenvalues and eigenfunctions are investigated.

Keywords: multiple solutions; Sturm-Liouville boundary value problem; nonlocal integral conditions; eigenvalues; eigenfunctions
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1. Introduction

The Sturm-Liouville problem arises within many areas of science, engineering and applied mathematics. It has been studied for more than two decades. Many physical, biological and chemical processes are described using models based on it (see [1–3], [8], [9] and [11]).

For the homogeneous Sturm-Liouville problem with nonlocal conditions you can see [2], [9] and [11–15]. For the nonhomogeneous equation see [7]. In [7] the authors studied the nonhomogeneous Sturm-Liouville boundary value problem of the differential equation

$$x''(t) + m(t) = -\lambda^2 x(t), \quad t \in (0, \pi),$$

with the conditions

$$x(0) = 0, \qquad x'(\xi) + \lambda x(\xi) = 0, \quad \xi \in (0, \pi].$$

Here, we are concerned, firstly, with the nonlocal problem of the nonlinear differential inclusion

$$-x''(t) \in F(t, \lambda x(t)), \qquad a.e. \quad t \in (0, \pi), \tag{1.1}$$

with the nonlocal conditions ($\eta > \xi$)

$$x'(0) - \lambda x(0) = 0 \quad \text{and} \quad \int_{\xi}^{\eta} x(\tau)d\tau = 0, \quad \xi \in [0, \pi), \quad \eta \in (0, \pi]. \tag{1.2}$$

For

$$h(t, \lambda) \ + \ \lambda^2 x(t) \ = \ f(t, \lambda x(t)) \in F(t, \lambda x(t)),$$

we study the existence of multiple solutions (eignevalues and eignefunctions) of the nonhomogeneous Sturm-Liouville problem of the differential equation

$$x''(t) + h(t, \lambda) = -\lambda^2 x(t), \quad t \in (0, \pi), \tag{1.3}$$

with the conditions (1.2).
The special case of the nonlocal condition (1.2)

$$x'(0) - \lambda x(0) = 0 \quad \text{and} \quad \int_{0}^{\pi} x(\tau)d\tau = 0, \tag{1.4}$$

will be considered.

2. Existence of solutions

Consider the nonlocal boundary value problem of the nonlinear differential inclusion (1.1)-(1.2) under the following assumptions.
(i) The set $F(t, x)$ is nonempty, closed and convex for all $(t, x) \in [0, 1] \times R \times R$.
(ii) $F(t, x)$ is measurable in $t \in [0, 1]$ for every $x, y \in R$.
(iii) $F(t, x)$ is upper semicontinuous in x and y for every $t \in [0, 1]$.
(iv) There exist a bounded measurable function $m : [0, 1] \longrightarrow R$ and a constant λ, such that

$$||F(t, x)|| = \sup\{|f| : f \in F(t, x)\} \ \leq |m(t)| + \lambda^2 |x|.$$

Remark 1. *From the assumptions (i)-(iv) we can deduce that (see [1], [5] and [6]) there exists $f \in F(t, x)$, such that*
(v) $f : I \times R \longrightarrow R$ is measurable in t for every $x, y \in R$ and continuous in x for $t \in [0, 1]$ and there exist a bounded measurable function $m : [0, \pi] \to R$ and a constant λ^2 such that

$$|f(t, x)| \leq |m(t)| + \lambda^2 |x|,$$

and f satisfies the nonlinear differential equation

$$- x''(t) \ = \ f(t, \lambda x(t)), \quad a.e. \quad t \in (0, \pi). \tag{2.1}$$

So, any solution of (2.1) is a solution of (1.1).

(vi) $\lambda\,(\eta - \xi) \neq -2, \ \lambda \in R$.
(vii)

$$\frac{2(1 + |\lambda|\pi)\pi^2 + \pi}{|\mathcal{A}|} \ \lambda^2 \pi < 1.$$

For the integral representation of the solution of (2.1) and (1.2) we have the following lemma.

Lemma 2.1. *If the solution of the problem (2.1) and (1.2) exists, then it can be represented by the integral equation*

$$\begin{aligned} x(t) &= \frac{2(1+\lambda t)}{\mathcal{A}}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds\Big] \\ &\quad - \int_0^t (t-s) f(s,\lambda x(s))ds, \end{aligned} \tag{2.2}$$

where $\mathcal{A} = (\eta-\xi)[2+\lambda(\eta-\xi)] \neq 0.$

Proof. Integrating both sides of Eq (2.1) twice, we obtain

$$x(t) - x(0) - tx'(0) = -\int_0^t (t-s) f(s,\lambda x(s))ds \tag{2.3}$$

and using the assumption $x'(0) - \lambda x(0) = 0$, we obtain

$$x(0) = \frac{1}{\lambda} x'(0). \tag{2.4}$$

The assumption $\int_\xi^\eta x(\tau)d\tau = 0$ implies that

$$\begin{aligned} x(0)\int_\xi^\eta d\tau + x'(0)\int_\xi^\eta \tau d\tau &= \int_\xi^\eta \int_0^\tau (\tau-s) f(s,\lambda x(s))ds\, d\tau, \\ (\eta-\xi)x(0) + \frac{(\eta-\xi)^2}{2}\lambda x(0) &= \int_0^\xi \int_\xi^\eta (\tau-s)d\tau\, f(s,\lambda x(s))ds \\ &\quad + \int_\xi^\eta \int_s^\eta (\tau-s)d\tau\, f(s,\lambda x(s))ds, \\ \frac{(\eta-\xi)[2+\lambda(\eta-\xi)]}{2} x(0) &= \int_0^\xi \Big[\frac{(\eta-s)^2}{2} - \frac{(\xi-s)^2}{2}\Big] f(s,\lambda x(s))ds \\ &\quad + \int_\xi^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds \\ &= \int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds \end{aligned}$$

and we can get

$$\begin{aligned} x(0) &= \frac{2}{(\eta-\xi)[2+\lambda(\eta-\xi)]}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds\Big] \\ &= \frac{2}{\mathcal{A}}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds\Big]. \end{aligned} \tag{2.5}$$

Substituting (2.5) into (2.4), we obtain

$$x'(0) = \frac{2\lambda}{\mathcal{A}}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds\Big]. \tag{2.6}$$

Now from (2.3), (2.5) and (2.6), we obtain

$$\begin{aligned} x(t) &= \frac{2(1+\lambda t)}{\mathcal{A}}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds\Big] \\ &\quad - \int_0^t (t-s) f(s,\lambda x(s))ds. \end{aligned}$$

To complete the proof, differentiate equation (2.2) twice, we obtain

$$\begin{aligned} x'(t) &= \frac{2\lambda}{\mathcal{A}}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds\Big] \\ &\quad - \int_0^t f(s,\lambda x(s))ds, \end{aligned}$$

and

$$x''(t) = -f(t,\lambda x(t)), \quad a.e. t \in (0,T).$$

Now

$$x'(0) = \frac{2\lambda}{\mathcal{A}}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds\Big],$$

and

$$\lambda x(0) = \frac{2\lambda}{\mathcal{A}}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds\Big].$$

From that, we get $x'(0) - \lambda x(0) = 0$.
Now, to ensure that $\int_\xi^\eta x(\tau)d\tau = 0$,
we have

$$\int_\xi^\eta \frac{2(1+\lambda t)}{\mathcal{A}} = \frac{2(\eta-\xi)+\lambda(\eta^2-\xi^2)}{\mathcal{A}} = \frac{(\eta-\xi)[2+\lambda(\eta-\xi)]}{\mathcal{A}} = 1,$$

from that, we obtain as before

$$\begin{aligned} \int_\xi^\eta x(\tau)d\tau &= \int_\xi^\eta \frac{2(1+\lambda\tau)}{\mathcal{A}}\, d\tau \Big[\int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds\Big] \\ &\quad - \int_\xi^\eta \int_0^\tau (\tau-s) f(s,\lambda x(s))ds\, d\tau, \\ &= \int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds \\ &\quad - \int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds + \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds = 0. \end{aligned}$$

This proves the equivalence between the integral equation (2.2) and the nonlocal boundary value problem (1.1)-(1.2).

□

Now, for the existence of at least one continuous solution for the problem of the integral equation (2.2), we have the following theorem.

Theorem 2.1. *Let the assumptions (v)-(vii) be satisfied, then there exists at least one solution $x \in C[0,\pi]$ of the nonlocal boundary value problem (2.1) and (1.2). Moreover, from Remark 1, then there exists at least one solution $x \in C[0,\pi]$ of the nonlocal boundary value problem (1.1)-(1.2).*

Proof. Define the set $Q_r \subset C[0,\pi]$ by

$$Q_r = \{x \in C : \| x \| \leq r\}, \quad r \geq \frac{2(1+|\lambda|\pi)\pi^2 + \pi \, \|m\|_{L^1}}{|\mathcal{A}| - [2(1+|\lambda|\pi)\pi^2 + \pi] \, \lambda^2\pi}.$$

It is clear that the set Q_r is nonempty, closed and convex.
Define the operator T associated with (2.2) by

$$\begin{aligned} Tx(t) &= \frac{2(1+\lambda t)}{\mathcal{A}}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds\Big] \\ &\quad - \int_0^t (t-s) f(s,\lambda x(s))ds. \end{aligned}$$

Let $x \in Q_r$, we have

$$\begin{aligned} |Tx(t)| &\leq \frac{2(1+|\lambda| t)}{|\mathcal{A}|}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} |f(s,\lambda x(s))|ds + \int_0^\xi \frac{(\xi-s)^2}{2} |f(s,\lambda x(s))ds|\Big] \\ &\quad + \int_0^t (t-s)|f(s,\lambda x(s))|ds, \\ &\leq \frac{2(1+|\lambda|\pi)\pi^2}{|\mathcal{A}|} \int_0^\pi \{|m(s)| + \lambda^2|x(s)|\}ds + \pi \int_0^\pi \{|m(s)| + \lambda^2|x(s)|\}ds, \\ &\leq \Big[\frac{2(1+|\lambda|\pi)\pi^2}{|\mathcal{A}|} + \pi\Big] \{\|m\|_{L^1} + \lambda^2\pi\|x\|\}, \\ &\leq \frac{2(1+|\lambda|\pi)\pi^2 + \pi}{|\mathcal{A}|} \{\|m\|_{L^1} + \lambda^2\pi r\} \leq r, \end{aligned}$$

and we have

$$\frac{2(1+|\lambda|\pi)\pi^2 + \pi}{|\mathcal{A}|} \|m\|_{L^1} \leq r\Big(1 - \frac{2(1+|\lambda|\pi)\pi^2 + \pi}{|\mathcal{A}|} \lambda^2\pi\Big).$$

Then $T : Q_r \to Q_r$ and the class $\{Tx\} \subset Q_r$ is uniformly bounded in Q_r.
In what follows we show that the class $\{Tx\}$, $x \in Q_r$ is equicontinuous. For $t_1, t_2 \in [0,\pi]$, $t_1 < t_2$ such that $|t_2 - t_1| < \delta$, we have

$$\begin{aligned} Tx(t_2) - Tx(t_1) &= \frac{2(1+\lambda t_2)}{\mathcal{A}}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds\Big] \\ &\quad - \int_0^{t_2} (t_2-s) f(s,\lambda x(s))ds - \frac{2(1+\lambda t_1)}{\mathcal{A}}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds \\ &\quad - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds\Big] - \int_0^{t_1} (t_2-s) f(s,\lambda x(s))ds, \\ |Tx(t_2) - Tx(t_1)| &= \Big|\frac{2(1+\lambda t_2)}{\mathcal{A}}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x(s))ds\Big] \end{aligned}$$

$$
\begin{aligned}
& -\int_0^{t_2}(t_2-s)f(s,\lambda x(s))ds-\frac{2(1+\lambda t_1)}{\mathcal{A}}\Big[\int_0^{\eta}\frac{(\eta-s)^2}{2}f(s,\lambda x(s))ds \\
& -\int_0^{\xi}\frac{(\xi-s)^2}{2}f(s,\lambda x(s))ds\Big]+\int_0^{t_1}(t_1-s)f(s,\lambda x(s))ds\Big|, \\
\leq\ & \frac{2|\lambda|(t_2-t_1)}{\mathcal{A}}\Big[\int_0^{\eta}\frac{(\eta-s)^2}{2}|f(s,\lambda x(s))|ds+\int_0^{\xi}\frac{(\xi-s)^2}{2}|f(s,\lambda x(s))|ds\Big] \\
& +(t_2-t_1)\int_0^{t_1}|f(s,\lambda x(s))|ds+\pi\int_{t_1}^{t_2}|f(s,\lambda x(s))|ds, \\
\leq\ & \frac{2|\lambda|(t_2-t_1)\pi^2}{\mathcal{A}}\int_0^{\pi}|f(s,\lambda x(s))|ds \\
& +(t_2-t_1)\int_0^{\pi}|f(s,\lambda x(s))|ds+\pi\int_{t_1}^{t_2}|f(s,\lambda x(s))|ds, \\
\leq\ & \frac{2|\lambda|(t_2-t_1)\pi^2}{\mathcal{A}}\{\|m\|_{L^1}+\lambda^2\|x\|\}+(t_2-t_1)\{\|m\|_{L^1}+\lambda^2\|x\|\} \\
& +\pi\int_{t_1}^{t_2}\{|m(s)|+\lambda^2|x(s)|\}ds.
\end{aligned}
$$

Hence the class of function $\{Tx\}$, $x\in Q_r$ is equicontinuous. By Arzela-Ascolis [4] Theorem, we found that the class $\{Tx\}$ is relatively compact.
Now we prove that $T:Q_r\to Q_r$ is continuous.
Let $\{x_n\}\subset Q_r$, such that $x_n\to x_0\in Q_r$, then

$$
\begin{aligned}
Tx_n(t) &= \frac{2(1+\lambda t)}{\mathcal{A}}\Big[\int_0^{\eta}\frac{(\eta-s)^2}{2}f(s,\lambda x_n(s))ds-\int_0^{\xi}\frac{(\xi-s)^2}{2}f(s,\lambda x_n(s))ds\Big] \\
&\quad -\int_0^{t}(t-s)f(s,\lambda x_n(s))ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{n\to\infty}Tx_n(t) &= \lim_{n\to\infty}\Big\{\frac{2(1+\lambda t)}{\mathcal{A}}\Big[\int_0^{\eta}\frac{(\eta-s)^2}{2}f(s,\lambda x_n(s))ds-\int_0^{\xi}\frac{(\xi-s)^2}{2}f(s,\lambda x_n(s))ds\Big] \\
&\quad -\int_0^{t}(t-s)f(s,\lambda x_n(s))ds\Big\}.
\end{aligned}
$$

Now, we have

$$f(s,x_n(s))\to f(s,x_0(s)) \ \text{ as } n\to\infty,$$

and

$$|f(s,\lambda x_n(s))|\leq m(s)+\lambda^2|x_n|\in L^1[0,\pi],$$

then applying Lebesgue Dominated convergence theorem [4], we obtain

$$\lim_{n\to\infty}Tx_n(t) = \frac{2(1+\lambda t)}{\mathcal{A}}\Big[\int_0^{\eta}\frac{(\eta-s)^2}{2}\lim_{n\to\infty}f(s,\lambda x_n(s))ds-\int_0^{\xi}\frac{(\xi-s)^2}{2}\lim_{n\to\infty}f(s,\lambda x_n(s))ds\Big]$$

$$
\begin{aligned}
& - \int_0^t (t-s) \lim_{n\to\infty} f(s, \lambda x_n(s))ds, \\
= & \frac{2(1+\lambda t)}{\mathcal{A}} \Big[\int_0^\eta \frac{(\eta - s)^2}{2} f(s, \lambda x_0(s))ds - \int_0^\xi \frac{(\xi - s)^2}{2} f(s, \lambda x_0(s))ds \Big] \\
& - \int_0^t (t-s) f(s, \lambda x_0(s))ds = F(x_0).
\end{aligned}
$$

Then $Tx_n(t) \to Tx_0(t)$. Which means that the operator T is continuous.
Since all conditions of Schauder theorem [4] are hold, then T has a fixed point in Q_r, then the integral equation (2.2) has at least one solution $x \in C[0, \pi]$.
Consequently the nonlocal boundary value problem (2.1)-(1.2) has at least one solution $x \in C[0, \pi]$. Moreover, from Remark 1, then there exists at least one solution $x \in C[0, \pi]$ of the nonlocal boundary value problem (1.1)-(1.2).
Now, we have the following corollaries

Corollary 1. *Let* $\lambda^2 x(t) = f(t, \lambda x(t)) \in F(t, \lambda x(t))$. *Let the assumptions of Theorem 2.1 be satisfied. Then there exists at lease one solution* $x \in C[0, \pi]$ *of*

$$-x''(t) = \lambda^2 x(t), \quad t \in (0, T).$$

with the nonlocal condition (1.2). Moreover, from Remark 1, there exists at lease one solution $x \in C[0, \pi]$ *of the problem (1.1)-(1.2).*

Corollary 2. *Let the assumptions of Theorem 2.1 be satisfied. Then there exists a solution* $x \in C[0, \pi]$ *of the problem (2.1) and (1.4).*

Proof. Putting $\xi = 0$ and $\eta = \pi$ and applying Theorem 2.1 we get the result. □

3. Maximal and minimal solutions

Taking $J = (0, \pi)$. Here, we study the existence of maximal and minimal solutions of the problem (2.1) and (1.2) which is equivalent to the integral equation (2.2).

Definition 3.1. *[10] Let* $q(t)$ *be a solution* $x(t)$ *of (2.2) Then* $q(t)$ *is said to be a maximal solution of (2.2) if every solution of (2.2) on* J *satisfies the inequality* $x(t) \leq q(t)$, $t \in J$ *. A minimal solution* $s(t)$ *can be defined in a similar way by reversing the above inequality i.e.* $x(t) \geq s(t)$, $t \in J$.

We need the following lemma to prove the existence of maximal and minimal solutions of (2.2).

Lemma 3.2. *Let* $f(t, x)$ *satisfies the assumptions in Theorem 2.1 and let* $x(t)$, $y(t)$ *be continuous functions on* J *satisfying*

$$x(t) \leq \frac{2(1+\lambda t)}{\mathcal{A}} \Big[\int_0^\eta \frac{(\eta - s)^2}{2} f(s, \lambda x(s))ds - \int_0^\xi \frac{(\xi - s)^2}{2} f(s, \lambda x(s))ds \Big]$$

$$
\begin{aligned}
& \quad - \int_0^t (t-s) f(s, \lambda x(s)) ds, \\
y(t) \; \geq \; & \frac{2(1+\lambda t)}{\mathcal{A}} \Big[\int_0^\eta \frac{(\eta - s)^2}{2} f(s, \lambda y(s)) ds - \int_0^\xi \frac{(\xi - s)^2}{2} f(s, \lambda y(s)) ds \Big] \\
& \quad - \int_0^t (t-s) f(s, \lambda y(s)) ds
\end{aligned}
$$

where one of them is strict.
Suppose $f(t, x)$ *is nondecreasing function in* x. *Then*

$$
x(t) \; < \; y(t), \; t \in J. \tag{3.1}
$$

Proof. Let the conclusion (3.1) be false; then there exists t_1 such that

$$
x(t_1) \; = \; y(t_1), \qquad t_1 \; > \; 0
$$

and

$$
x(t) \; < \; y(t), \qquad 0 \; < \; t \; < \; t_1.
$$

From the monotonicity of the function f in x, we get

$$
\begin{aligned}
x(t_1) \; \leq \; & \frac{2(1+\lambda t_1)}{\mathcal{A}} \Big[\int_0^\eta \frac{(\eta - s)^2}{2} f(s, \lambda x(s)) ds - \int_0^\xi \frac{(\xi - s)^2}{2} f(s, \lambda x(s)) ds \Big] \\
& - \int_0^{t_1} (t-s) f(s, \lambda x(s)) ds, \\
< \; & \frac{2(1+\lambda t_1)}{\mathcal{A}} \Big[\int_0^\eta \frac{(\eta - s)^2}{2} f(s, \lambda y(s)) ds - \int_0^\xi \frac{(\xi - s)^2}{2} f(s, \lambda y(s)) ds \Big] \\
& - \int_0^{t_1} (t-s) f(s, \lambda y(s)) ds \\
< \; & y(t_1).
\end{aligned}
$$

This contradicts the fact that $x(t_1) \; = \; y(t_1)$; then

$$
x(t) \; < \; y(t), \; t \in J.
$$

□

Theorem 3.2. *Let the assumptions of Theorem 2.1 be satisfied. Furthermore, if* $f(t, x)$ *is nondecreasing function in* x, *then there exist maximal and minimal solutions of (2.2).*

Proof. Firstly, we shall prove the existence of maximal solution of (2.2). Let $\epsilon > 0$ be given. Now consider the integral equation

$$
x_\epsilon(t) \; = \; \frac{2(1+\lambda t)}{\mathcal{A}} \Big[\int_0^\eta \frac{(\eta - s)^2}{2} f_\epsilon(s, \lambda x_\epsilon(s)) ds - \int_0^\xi \frac{(\xi - s)^2}{2} f_\epsilon(s, \lambda x_\epsilon(s)) ds \Big]
$$

$$- \int_0^t (t-s) f_\epsilon(s, \lambda x_\epsilon(s)) ds, \tag{3.2}$$

where

$$f_\epsilon(t, x_\epsilon(t)) \ = \ f(t, x_\epsilon(t)) \ + \ \epsilon.$$

Clearly the function $f_\epsilon(t, x_\epsilon)$ satisfies assumption (v) and

$$|\ f_\epsilon(t, x_\epsilon)\ | \ \leq \ |m(t)| + \lambda^2 |x| + \ \epsilon \leq |m_1(t)| + \lambda^2 |x|, \ |m_1(t)| = |m(t)| + \epsilon.$$

Therefore, Equation (3.2) has a continuous solution $x_\epsilon(t)$ according to Theorem 2.1.
Let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then

$$\begin{aligned} x_{\epsilon_1}(t) &= \frac{2(1+\lambda t)}{\mathcal{A}} \Big[\int_0^\eta \frac{(\eta - s)^2}{2} f_{\epsilon_1}(s, \lambda x_{\epsilon_1}(s)) ds - \int_0^\xi \frac{(\xi - s)^2}{2} f_{\epsilon_1}(s, \lambda x_{\epsilon_1}(s)) ds \Big] \\ &\quad - \int_0^t (t-s) f_{\epsilon_1}(s, \lambda x_{\epsilon_1}(s)) ds, \\ &= \frac{2(1+\lambda t)}{\mathcal{A}} \Big[\int_0^\eta \frac{(\eta - s)^2}{2} (f(s, \lambda x_{\epsilon_1}(s)) + \epsilon_1) ds - \int_0^\xi \frac{(\xi - s)^2}{2} (f(s, \lambda x_{\epsilon_1}(s)) + \epsilon_1) ds \Big] \\ &\quad - \int_0^t (t-s)(f(s, \lambda x_{\epsilon_1}(s)) + \epsilon_1) ds, \\ &> \frac{2(1+\lambda t)}{\mathcal{A}} \Big[\int_0^\eta \frac{(\eta - s)^2}{2} (f(s, \lambda x_{\epsilon_1}(s)) + \epsilon_2) ds - \int_0^\xi \frac{(\xi - s)^2}{2} (f(s, \lambda x_{\epsilon_1}(s)) + \epsilon_2) ds \Big] \\ &\quad - \int_0^t (t-s)(f(s, \lambda x_{\epsilon_1}(s)) + \epsilon_2) ds, \end{aligned} \tag{3.3}$$

$$\begin{aligned} x_{\epsilon_2}(t) &= \frac{2(1+\lambda t)}{\mathcal{A}} \Big[\int_0^\eta \frac{(\eta - s)^2}{2} (f(s, \lambda x_{\epsilon_2}(s)) + \epsilon_2) ds - \int_0^\xi \frac{(\xi - s)^2}{2} (f(s, \lambda x_{\epsilon_2}(s)) + \epsilon_2) ds \Big] \\ &\quad - \int_0^t (t-s)(f(s, \lambda x_{\epsilon_2}(s)) + \epsilon_2) ds. \end{aligned} \tag{3.4}$$

Applying Lemma 3.2, then (3.3) and (3.4) imply that

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t) \qquad for\ t \in \ J.$$

As shown before in the proof of Theorem 2.1, the family of functions $x_\epsilon(t)$ defined by Eq (3.2) is uniformly bounded and of equi-continuous functions. Hence by the Arzela-Ascoli Theorem, there exists a decreasing sequence ϵ_n such that $\epsilon_n \to 0$ as $n \to \infty$, and $\lim_{n\to\infty} x_{\epsilon_n}(t)$ exists uniformly in I. We denote this limit by $q(t)$. From the continuity of the function f_{ϵ_n} in the second argument, we get

$$\begin{aligned} x(t) &= \lim_{n\to\infty} x_{\epsilon_n}(t) = \frac{2(1+\lambda t)}{\mathcal{A}} \Big[\int_0^\eta \frac{(\eta - s)^2}{2} f(s, \lambda q(s)) ds - \int_0^\xi \frac{(\xi - s)^2}{2} f(s, \lambda q(s)) ds \Big] \\ &\quad - \int_0^t (t-s) f(s, \lambda q(s)) ds, \end{aligned}$$

which proves that $q(t)$ is a solution of (2.2).
Finally, we shall show that $q(t)$ is maximal solution of (2.2). To do this, let $x(t)$ be any solution of (2.2). Then

$$\begin{aligned} x_\epsilon(t) &= \frac{2(1+\lambda t)}{\mathcal{A}}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} f_\epsilon(s,\lambda x_\epsilon(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f_\epsilon(s,\lambda x_\epsilon(s))ds\Big] \\ &\quad - \int_0^t (t-s) f_\epsilon(s,\lambda x_\epsilon(s))ds, \\ &= \frac{2(1+\lambda t)}{\mathcal{A}}\Big[\int_0^\eta \frac{(\eta-s)^2}{2}(f(s,\lambda x_\epsilon(s))+\epsilon)ds - \int_0^\xi \frac{(\xi-s)^2}{2}(f(s,\lambda x_\epsilon(s))+\epsilon)ds\Big] \\ &\quad - \int_0^t (t-s)(f(s,\lambda x_\epsilon(s))+\epsilon)ds, \\ &> \frac{2(1+\lambda t)}{\mathcal{A}}\Big[\int_0^\eta \frac{(\eta-s)^2}{2} f(s,\lambda x_\epsilon(s))ds - \int_0^\xi \frac{(\xi-s)^2}{2} f(s,\lambda x_\epsilon(s))ds\Big] \\ &\quad - \int_0^t (t-s) f(s,\lambda x_\epsilon(s))ds. \end{aligned} \tag{3.5}$$

Applying Lemma 3.2, then (2.2) and (3.5 imply that

$$x_\epsilon(t) > x(t) \quad for\, t \in J.$$

From the uniqueness of the maximal solution (see [10]), it is clear that $x_\epsilon(t)$ tends to $q(t)$ uniformly in $t \in J$ as $\epsilon \to 0$.
In a similar way we can prove that there exists a minimal solution of (2.2). □

4. The homogeneous problem

Here, we study the existence and some general properties of the eigenvalues and eigenfunctions of the problem of the homogeneous equation

$$x''(t) = -\lambda^2 x(t), \quad t \in (0,\pi), \tag{4.1}$$

with the nonlocal condition (1.2).

Lemma 4.3. *The eigenfunctions of the nonlocal boundary value problem (4.1) and (1.2) are in the form of*

$$x_n(t) = c_n\Big(\sin\frac{(-\pi+4\pi n)\,t}{2(\eta+\xi)} + \cos\frac{(-\pi+4\pi n)\,t}{2(\eta+\xi)}\Big), \quad n = 1,2,\cdots. \tag{4.2}$$

Proof. Firstly, we prove that the eigenvalues are

$$\lambda_n = \frac{-\pi+4\pi n}{2(\eta+\xi)}, \quad n = 1,2,\cdots. \tag{4.3}$$

The general solution of the problem (4.1) and (1.2) is given by

$$x(t) = c_1\sin\lambda t + c_2\cos\lambda t. \tag{4.4}$$

Differentiating equation (4.4), we obtain

$$x'(t) = \lambda c_1 \cos \lambda t - \lambda c_2 \sin \lambda t.$$

Using the first condition, when $t = 0$, we obtain

$$c_1 = c_2. \tag{4.5}$$

Integrating both sides of (4.4) from ξ to η, we obtain

$$\frac{c_1}{\lambda} \cos \lambda\xi - \frac{c_1}{\lambda} \cos \lambda\eta + \frac{c_2}{\lambda} \sin \lambda\eta - \frac{c_2}{\lambda} \sin \lambda\xi = 0.$$

Substituting $c_1 = c_2$, we obtain

$$\frac{c_1}{\lambda} \cos \lambda\xi - \frac{c_1}{\lambda} \cos \lambda\eta + \frac{c_1}{\lambda} \sin \lambda\eta - \frac{c_1}{\lambda} \sin \lambda\xi = 0. \tag{4.6}$$

Multiplying (4.6) by $\frac{\lambda}{c_1}$, we obtain

$$\begin{aligned}
&\cos \lambda\xi - \cos \lambda\eta + \sin \lambda\eta - \sin \lambda\xi = 0,\\
&2 \sin \frac{\lambda(\xi+\eta)}{2} \ \sin \frac{\lambda(\eta-\xi)}{2} \ + \ 2 \sin \frac{\lambda(\eta-\xi)}{2} \ \cos \frac{\lambda(\eta+\xi)}{2} = 0,\\
&\sin \frac{\lambda(\xi+\eta)}{2} \ + \ \cos \frac{\lambda(\eta+\xi)}{2} = 0,\\
&\tan \frac{\lambda(\xi+\eta)}{2} = -1,\\
&\frac{\lambda(\xi+\eta)}{2} = -\frac{\pi}{4} + n\pi.
\end{aligned} \tag{4.7}$$

From (4.7), we deduce that

$$\lambda_n = \frac{-\pi + 4\pi n}{2(\eta+\xi)}, \quad n = 1, 2, ...$$

Therefore, from (4.4) we can get

$$x_n(t) = c_n\Big(\sin \frac{(-\pi + 4\pi n)\, t}{2(\eta+\xi)} \ + \ \cos \frac{(-\pi + 4\pi n)\, t}{2(\eta+\xi)}\Big), \quad n = 1, 2, ...$$

Corollary 3. *The eigenfunctions of the nonlocal boundary value problem (4.1) and (1.4) are in the form of*

$$x_n(t) = c_n\Big(\sin \frac{(-1 + 4n)\, t}{2} \ + \ \cos \frac{(-1 + 4n)\, t}{2}\Big), \quad n = 1, 2, ... \tag{4.8}$$

Proof. Putting $\xi = 0$ and $\eta = \pi$ and applying Lemma 4.3 we obtain the result. □

5. The nonhomogeneous problem

Now, we study the existence of multiple solutions of the nonhomogeneous problem (1.3) and (1.2). Let x_1, x_2 be two solutions of the problem (1.3) and (1.2). Let $u(t) = x_1(t) - x_2(t)$, then the function u satisfy the Sturm-Liouville problem

$$u''(t) = -\lambda^2 u(t)$$

with the nonlocal conditions

$$u'(0) - \lambda u(0) = 0 \quad \text{and} \quad \int_{\xi}^{\eta} u(\tau)d\tau = 0, \quad \xi \in [0, \pi), \quad \eta \in (0, \pi].$$

So, the values of (eigenvalues) λ_n for the non zero solution of (4.1) and (1.2) is the same values (eigenvalues) of λ_n for the multiple solutions (eigenfunctions) of (1.3) and (1.2), i.e.

$$\lambda_n = \frac{-\pi + 4\pi n}{2(\eta + \xi)}, \quad n = 1, 2, ...$$

Theorem 5.3. *The multiple solutions (eigenfunctions) $x_n(t)$ of the problem (1.3) and (1.2) are given by*

$$x_n(t) = A_n\left(\sin\frac{(-\pi + 4\pi n)\,t}{2(\eta + \xi)} + \cos\frac{(-\pi + 4\pi n)\,t}{2(\eta + \xi)}\right) - \int_0^t \frac{\sin\frac{(-\pi+4\pi n)(t-s)}{2(\eta+\xi)}}{\frac{-\pi+4\pi n}{2(\eta+\xi)}} h(s, \lambda)ds. \tag{5.1}$$

Proof. Here we use the variation of parameter method to get the solution of (1.3) and (1.2). Assume that the solutions of (1.3) and (1.2) are given by

$$x_n(t) = A_1 \cos\lambda t + A_2 \sin\lambda t + x_p(t). \tag{5.2}$$

So, we have

$$x_1(t) = \cos\lambda t, \quad x_2(t) = \sin\lambda t.$$

Now, we can get $W(x_1, x_2) = \lambda$. Hence

$$x_p(t) = -\cos\lambda t \int_0^t \frac{\sin\lambda s}{\lambda} h(s, \lambda)ds + \sin\lambda t \int_0^t \frac{\cos\lambda s}{\lambda} h(s, \lambda)ds,$$

thus

$$x_p(t) = -\int_0^t \frac{\sin\lambda(t - s)}{\lambda} h(s, \lambda)ds. \tag{5.3}$$

From (5.3) and (5.2), we obtain

$$x_n(t) = A_1 \sin\frac{(-\pi + 4\pi n)\,t}{2(\eta + \xi)} + A_2 \cos\frac{(-\pi + 4\pi n)\,t}{2(\eta + \xi)} - \int_0^t \frac{\sin\frac{(-\pi+4\pi n)(t-s)}{2(\eta+\xi)}}{\frac{-\pi+4\pi n}{2(\eta+\xi)}} h(s, \lambda)ds. \tag{5.4}$$

By using the first condition $x'(0) - \lambda x(0) = 0$, we get

$$A_1 = A_2,$$

therefore the multiple solutions of the nonlocal problem (1.3) and (1.2) are given by

$$x_n(t) = A_n\left(sin\frac{(-\pi + 4\pi n)\,t}{2(\eta + \xi)} + cos\frac{(-\pi + 4\pi n)\,t}{2(\eta + \xi)}\right) - \int_0^t \frac{\sin\frac{(-\pi+4\pi n)\,(t-s)}{2(\eta+\xi)}}{\frac{(-\pi+4\pi n)}{2(\eta+\xi)}} h(s, \lambda)ds, \quad n = 1, 2, ...$$

To complete the proof and to ensure that $x_n(t)$ is the solution of (1.3) and (1.2), we firstly prove that

$$x_n''(t) + h(t, \lambda) = -\lambda^2 x_n(t).$$

Differentiating (5.4) twice, we get

$$\begin{aligned} x_n'(t) &= A_n\frac{-\pi+4\pi n}{2(\eta+\xi)}\Big(\cos\frac{(-\pi+4\pi n)\,t}{2(\eta+\xi)}-\sin\frac{(-\pi+4\pi n)\,t}{2(\eta+\xi)}\Big) \\ &\quad -\int_0^t \cos\frac{(-\pi+4\pi n)\,(t-s)}{2(\eta+\xi)}h(s,\lambda)ds \end{aligned}$$

and

$$\begin{aligned} x_n''(t) &= A_n\Big(\frac{-\pi+4\pi n}{2(\eta+\xi)}\Big)^2\Big(-\sin\frac{(-\pi+4\pi n)\,t}{2(\eta+\xi)}-\cos\frac{(-\pi+4\pi n)\,t}{2(\eta+\xi)}\Big) \\ &\quad -g(t)+\frac{-\pi+4\pi n}{2(\eta+\xi)}\int_0^t \sin\frac{(-\pi+4\pi n)\,(t-s)}{2(\eta+\xi)}h(s,\lambda)ds \end{aligned}$$

and

$$\begin{aligned} x_n''(t)+h(t,\lambda) &= A_n\Big(\frac{-\pi+4\pi n}{2(\eta+\xi)}\Big)^2\Big(-\sin\frac{(-\pi+4\pi n)\,t}{2(\eta+\xi)}-\cos\frac{(-\pi+4\pi n)\,t}{2(\eta+\xi)}\Big) \\ &\quad -h(t,\lambda)+\frac{-\pi+4\pi n}{2(\eta+\xi)}\int_0^t \sin\frac{(-\pi+4\pi n)\,(t-s)}{2(\eta+\xi)}h(s,\lambda)ds+h(t,\lambda) \\ &= -\lambda^2 x_n(t). \end{aligned}$$

Also we have $x'(0)-\lambda x(0)=0$. □

Example 1. *Let* $h(t,\lambda) = \lambda^2$. *Then we find that*

$$x_p(t) = -\int_0^t \frac{\sin\lambda(t-s)}{\lambda}\lambda^2 ds = cos\lambda t-1$$

and the multiple solutions of the nonlocal problem (1.3) and (1.2) are given by

$$x_n(t)=A_1\sin\frac{(-\pi+4\pi n)\,t}{2(\eta+\xi)}+A_2\cos\frac{(-\pi+4\pi n)\,t}{2(\eta+\xi)}+cos\lambda t-1.$$

Now consider the Riemann integral boundary condition (1.4).

Corollary 4. *The multiple solutions (eigenfunctions)* $x_n(t)$ *of the problem (1.3)-(1.4) are given by*

$$x_n(t)=A_n\Big(\sin\frac{(-1+4n)\,t}{2}+\cos\frac{(-1+4n)\,t}{2}\Big)-\int_0^t \frac{\sin\frac{(-1+4n)(t-s)}{2}}{\frac{-1+4n}{2}}h(s,\lambda)ds.$$

Proof. In this special case, we put $\xi=0$ and $\eta=\pi$ and applying Theorem 5.3 we get the result. □

Example 2. *Let* $h(t,\lambda) = \lambda^2$. *Then we find that*

$$x_p(t) = -\int_0^t \frac{\sin\lambda(t-s)}{\lambda}\lambda^2 ds = cos\lambda t-1,$$

and the solution $x_n(t)$ *of the problem (1.3)-(1.4) are given by*

$$x_n(t) = A_n\Big(\sin\frac{(-1+4n)\,t}{2}+\cos\frac{(-1+4n)\,t}{2}\Big)+cos\lambda t-1.$$

6. Conclusions

Here, we proved the existence of solutions $x \in C[0, \pi]$ of the nonlocal boundary value problem of the differential inclusion (1.1) with the nonlocal condition (1.2).
The maximal and minimal solutions of the problem (1.1)-(1.2) have been proved. The eigenvalues and eigenfunctions of the homogeneous and nonhomogeneous equations (4.1) and (1.3) with the nonlocal condition (1.2) have been obtained. Two examples have been studied to illustrate our results.

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Conflict of interest

The authors declare no conflict of interest.

References

1. J. P. Aubin, A. Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory*, vol. **264**, Springer, Berlin, 2012.
2. A. V. Bitsadze, A. A. Samarskii, Some elementary generalizations of linear elliptic boundary value problems, *Dokl. Akad. Nauk*, **185** (1969), 739–740.
3. K. Bingele, A. Bankauskiene, A. Štikonas, Investigation of Spectrum Curves for a Sturm–Liouville problem with Two-Point Nonlocal Boundary Conditions, *Math. Model. Anal.*, **25** (2020), 53–70. https://doi.org/10.3846/mma.2020.10787
4. R. F. Curtain, A. J. Pritchard, *Functional analysis in modern appliedmathematics*, Academic press, 1977.
5. A. M. A. El-Sayed, A. G. Ibrahim, Multivalued fractional differential equations, *Appl. Math. Comput.*, **68** (1995), 15–50. https://doi.org/10.1016/0096-3003(94)00080-N
6. A. M. A. El-Sayed, A. G. Ibrahim, Set-valued integral equation of fractional orders, *Appl. Math. Comput.* **118** (2001), 113–121. https://doi.org/10.1016/S0096-3003(99)00087-9
7. A. M. A. El-Sayed, M. Sh. Mohamed, R. E. M Embia, On the multiple solutions of a nonhomogeneous Sturm-Liouville equation with nonlocal boundary conditions, *International Journal of Applied Mathematics*, **32** (2019), 35–43. https://doi.org/10.12732/ijam.v32i1.3
8. A. M. A. El-Sayed, H. H. G. Hashem, Sh. M. Al-Issa, Qualitative properties of solutions of fractional order boundary value problems, *Int. J. Nonlinear Anal. Appl.*, **13** (2022), 3427–3440.
9. N. S. Imanbaev, Y. Kurmysh, On computation of eigenfunctions of composite type equation with regular boundary conditions, *International Journal of Applied Mathematics*, **34** (2021), 681–692. https://doi.org/10.12732/ijam.v34i4.7

10. V. Lakshmikantham, S. Leela, *Differential and Integral Inequalities*, vol. **1**, Academic press, New York-London, 1969.

11. A. Skucaite, A. Stikonas, Spectrum curves for SturmLiouville problem with integral boundary condition, *Math. Model. Anal.*, **20** (2015), 802818. https://doi.org/10.3846/13926292.2015.1116470

12. A. Skucaite, K. Skucaite-Bingele, S. Peciulyte, A. Stikonas, Investigation of the spectrum for the SturmLiouville problem with one integral boundary Condition, *Nonlinear Anal. Model. Control*, **15** (2010), 501512. https://doi.org/10.15388/NA.15.4.14321

13. A. Skucaite, A. Stikonas, Zeroes and poles of a characteristic function for SturmLiouville problem with nonlocal integral condition, *Liet. matem. rink. Proc. LMS, Ser. A*, **56** (2015), 95100. https://doi.org/10.15388/LMR.A.2015.17

14. A. Skucaite, A. Stikonas, Investigation of the spectrum of the Sturm Liouville problem with a nonlocal integral condition, *Liet. matem. rink. Proc. LMS, Ser. A*, **54** (2013), 67–72. https://doi.org/10.15388/LMR.A.2013.15

15. A. Skucaite, A. Stikonas, Investigation of the Sturm Liouville problems with integral boundary condition, *Liet. matem. rink. Proc. LMS, Ser. A*, **52** (2011), 297–302. https://doi.org/10.15388/LMR.2011.sm03

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