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Research article

Traveling wave solutions to the Boussinesq equation via Sardar sub-equation technique

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Abstract: In present study, the Boussinesq equation is obtained by means of the Sardar Sub-Equation Technique (SSET) to create unique soliton solutions containing parameters. Using this technique, different solutions are obtained, such as the singular soliton, the dark-bright soliton, the bright soliton and the periodic soliton. The graphs of these solutions are plotted for a batter understanding of the model. The results show that the technique is very effective in solving nonlinear partial differential equations (PDEs) arising in mathematical physics.

Keywords: Boussinesq equation; Sardar sub-equation method; traveling wave soliton solutions
Mathematics Subject Classification: 32W50, 35C08, 35C15

1. Introduction

In recent decades, researchers have paid considerable attention to nonlinear waves at the ocean surface. Phenomena of nonlinear waves play a key role in many fields of engineering and science, such as ocean engineering, plasma physics, Control theory, tsunami waves, communications industry, fluid dynamics, and coastal engineering, etc.

Non-linear PDEs have great potential for application in various fields; therefore, researchers pay special attention to their analytical and numerical solutions [1–8]. In the literature, many researchers in mathematics and physics have developed various methods to analyze the nonlinear evolution equations (NLEEs), such as the exp-function approach [9], the modified simple equation technique [10–14], the generalized Kudryashov technique [15,16], the $\frac{G'}{G}$-expansion approach [17–23], the extended rational function expansion approach [23], the Hirota bilinear method [24–30], the extended homoclinic approach [31], the traveling wave scheme [32], the Darboux transformation method [32–35], the sine-cosine approach [36], the semi-inverse variational principle [37], He's variational iteration technique [38], the sine-Gordon method [39–41], A special kind of distributive product [42], the Lie symmetric method [43], the extended homogeneous principal method [44], the power series method [45] and some other methods [46–47].

The Boussinesq equation represents a long-wavelength and weakly nonlinear approximation used in numerical models, water waves, and coastal engineering for simulating water waves in shallow seas and harbors. A Scottish engineer named John Scott Russell closely observed solitary waves (also called translational waves and solitons). Joseph Boussinesq based his approximation on the obervation of John Scott Russell observation. In 1872, the simulation of one-dimensional water waves was determined by Boussinesq which states that the horizontal velocity is constant and the vertical velocity is linear in addition to the water depth, referred to as the Boussinesq equation [48]. Previously, the Boussinesq equation was investigated using various mathematical approaches [49–53].

Consider, the Boussinesq equation in the following form

$$Q_{tt} - Q_{xx} = (Q^2)_{xx} + \lambda Q_{xxxx}, \tag{1.1}$$

where $Q = Q(x,t)$ represents the wave envelope containing x as a spatial variable and t as a temporal variable. Here λ is an arbitrary constant. This is called frequency dispersion phenomenon when water waves of different wavelengths are related, and in the case of an infinitesimal wave amplitude it is also called a linear frequency dispersion. For this reason, this is a valid approximation. The Boussinesq equation allows for waves to propagate in different directions as well, but it is advantageous to consider waves that propagate in the same direction. To form strong and reliable solitons of the Boussinesq equation using (SSET) [54–56], the following traveling wave transformation is used.

$$X(x,t) = X(\beta), \quad \beta = \eta x + \chi t. \tag{1.2}$$

Here η and χ are real constants. Applying Eq (1.2) in Eq (1.1), the following ordinary differential equation (ODE) is constructed.

$$\chi^2 Q'' - \eta^2 Q'' - \eta^2 (Q^2)'' - \lambda \eta^4 Q^{(iv)} = 0. \tag{1.3}$$

Integrating Eq (1.3) twice with respect to β and neglecting the integration constants, we obtain the following equation.

$$(\chi^2 - \eta^2) Q - \eta^2 Q^2 - \lambda \eta^4 Q'' = 0. \tag{1.4}$$

Where χ and η are the velocity and frequency of the propagating wave.

2. Mathematical description

We proceed with these main steps in this work:
Step 1:

$$H(X, X_x, X_t, X_{xx}, X_{tt}, ...) = 0, \tag{2.1}$$

where H is a polynomial of X and $X(\beta) = X(x,t)$ is a unknown function. Consider the wave transformation

$$\beta = \eta x + \chi t,$$

where $\chi \neq 0$ is a constant to be determined later.
Using the transformation, Eq (2.1) is converted to the following ODE.

$$G = (Q, Q', Q'', ...,) = 0. \tag{2.2}$$

Step 2: G is a function of $Q(\beta)$ and prime express the derivatives in regard to β.
Solution of Eq (2.2) can be formulated as

$$Q(\beta) = \sum_{l=0}^{N} c_l M^l(\beta), \quad c_l \neq 0, \tag{2.3}$$

where $c_l (0 \leq l \leq N)$ are real constants and $M(\beta)$ satisfying the ODE in the form

$$M'(\beta) = \sqrt{\xi + uM(\beta)^2 + M(\beta)^4}. \tag{2.4}$$

Here ξ and u are real constants and Eq (2.4) presents the solutions as
Case I: If $u > 0$ and $\xi = 0$, then

$$M_1^{\pm}(\beta) = \pm\sqrt{-pqu}\ \mathrm{sech}_{pq}(\sqrt{u}\beta),$$
$$M_2^{\pm}(\beta) = \pm\sqrt{pqu}\ \mathrm{csch}_{pq}(\sqrt{u}\beta),$$

where

$$\mathrm{sech}_{pq}(\beta) = \frac{2}{pe^{\beta}+qe^{-\beta}}, \quad \mathrm{csch}_{pq}(\beta) = \frac{2}{pe^{\beta}-qe^{-\beta}}.$$

Case II: If $u < 0$ and $\xi = 0$, then

$$M_3^{\pm}(\beta) = \pm\sqrt{-pqu}\ \sec_{pq}(\sqrt{-u}\beta),$$
$$M_4^{\pm}(\beta) = \pm\sqrt{-pqu}\ \csc_{pq}(\sqrt{-u}\beta),$$

where

$$\sec_{pq}(\beta) = \frac{2}{pe^{\iota\beta}+qe^{-\iota\beta}}, \quad \csc_{pq}(\beta) = \frac{2\iota}{pe^{\iota\beta}-qe^{-\iota\beta}}.$$

Case III: If $u < 0$ and $\xi = \frac{u^2}{4}$, then

$$M_5^{\pm}(\beta) = \pm\sqrt{\frac{-u}{2}}\tanh_{pq}(\sqrt{\frac{-u}{2}}\beta),$$
$$M_6^{\pm}(\beta) = \pm\sqrt{\frac{-u}{2}}\coth_{pq}(\sqrt{\frac{-u}{2}}\beta),$$
$$M_7^{\pm}(\beta) = \pm\sqrt{\frac{-u}{2}}(\tanh_{pq}(\sqrt{-2u}\beta) \pm \iota\sqrt{pq}\,\operatorname{sech}_{pq}(\sqrt{-2u}\beta)),$$
$$M_8^{\pm}(\beta) = \pm\sqrt{\frac{-u}{2}}(\coth_{pq}(\sqrt{-2u}\beta) \pm \sqrt{pq}\,\operatorname{csch}_{pq}(\sqrt{-2u}\beta)),$$
$$M_9^{\pm}(\beta) = \pm\sqrt{\frac{-u}{8}}(\tanh_{pq}(\sqrt{\frac{-u}{8}}\beta) + \coth_{pq}(\sqrt{\frac{-u}{8}}\beta)),$$

where

$$\tanh_{pq}(\beta) = \frac{pe^{\beta}-qe^{-\beta}}{pe^{\beta}+qe^{-\beta}}, \quad \coth_{pq}(\beta) = \frac{pe^{\beta}+qe^{-\beta}}{pe^{\beta}-qe^{-\beta}}.$$

Case IV: If $u > 0$ and $\xi = \frac{u^2}{4}$, then

$$M_{10}^{\pm}(\beta) = \pm\sqrt{\frac{u}{2}}\tan_{pq}(\sqrt{\frac{u}{2}}\beta),$$
$$M_{11}^{\pm}(\beta) = \pm\sqrt{\frac{u}{2}}\cot_{pq}(\sqrt{\frac{u}{2}}\beta),$$
$$M_{12}^{\pm}(\beta) = \pm\sqrt{\frac{u}{2}}(\tan_{pq}(\sqrt{2u}\beta) \pm \sqrt{pq}\,\sec_{pq}(\sqrt{2u}\beta)),$$
$$M_{13}^{\pm}(\beta) = \pm\sqrt{\frac{u}{2}}(\cot_{pq}(\sqrt{2u}\beta) \pm \sqrt{pq}\,\csc_{pq}(\sqrt{2u}\beta)),$$
$$M_{14}^{\pm}(\beta) = \pm\sqrt{\frac{u}{8}}(\tan_{pq}(\sqrt{\frac{u}{8}}\beta) + \cot_{pq}(\sqrt{\frac{u}{8}}\beta)),$$

where

$$\tan_{pq}(\beta) = -\iota\frac{pe^{\iota\beta}-qe^{-\iota\beta}}{pe^{\iota\beta}+qe^{-\iota\beta}}, \quad \cot_{pq}(\beta) = \iota\frac{pe^{\iota\beta}+qe^{-\iota\beta}}{pe^{\iota\beta}-qe^{-\iota\beta}}.$$

These are generalized trigonometric and hyperbolic functions with parameters p and q. If we take $p = q = 1$, they become known trigonometric and hyperbolic functions.

Step 3: We calculate the integer N by balancing the capital. Substituting Eq (2.3) into Eq (2.2) we obtain an algebraic equation in the form of $M^l(\beta)$, which we balance by equating the powers of $M^l(\beta)$ l=(0,1,2, ...) to zero thus obtaining a set of algebraic equations.

Step 4: This set of equations leads to the required parameters and the exact solution of the given equation.

3. Execution of Sardar sub-equation technique

In this section, SSET is applied to the Boussinesq equation to construct the traveling wave solution. By the equilibrium rule Eq (2.3) reduces into

$$Q(\beta) = c_0 + c_1 M(\beta) + c_2 M(\beta)^2, \tag{3.1}$$

where c_0, c_1, c_2 are constants. Substitute Eq (3.1), Eq (1.4) into Eq (2.4), we obtain a polynomial in the form of $M^l(\beta)$ and equate powers of $M^l(\beta)$ to zero resulting in algebraic equations in $c_0, c_1, c_2, \chi, \eta$ and λ.

Set of algebraic equations are as:

$$-2c_2\eta^4\lambda\xi - c_0^2\eta^2 - c_0\eta^2 + c_0\chi^2 = 0,$$

$$-2c_0c_1\eta^2 - c_1\eta^2 + c_1\eta^4\lambda(-u) + c_1\chi^2 = 0,$$

$$-2c_0c_2\eta^2 - c_2\eta^2 - c_1^2\eta^2 - 4c_2\eta^4\lambda u + c_2\chi^2 = 0,$$

$$-2c_1\eta^4\lambda - 2c_1c_2\eta^2 = 0,$$

$$-6c_2\eta^4\lambda - c_2^2\eta^2 = 0. \tag{3.2}$$

With the help of Mathematica software, the following parameters are determined.

$$c_0 = \tfrac{1}{3}\left(c_2 u - \sqrt{c_2^2u^2 - 3c_2^2\xi}\right), \quad c_1 = 0, \quad c_2 = c_2,$$

$$\chi = -\frac{\sqrt{3\eta^2 - 2\eta^2\sqrt{c_2^2\left(u^2 - 3\xi\right)}}}{\sqrt{3}}, \quad \lambda = -\frac{c_2}{6\eta^2}. \tag{3.3}$$

Using Eqs (2.4), (3.1) and (3.3) with Eq (1.2), following solutions are constructed.
Case I: If $u > 0$ and $\xi = 0$, then

$$Q_1^{\pm}(x,t) = \frac{1}{3}\left(c_2 u - \sqrt{c_2^2u^2 - 3c_2^2\xi}\right) + c_2\left(\pm\sqrt{-pqu}\right)^2 \operatorname{sech}_{pq}\left(\sqrt{u}(t\chi + \eta x)\right)^2, \tag{3.4}$$

$$Q_2^{\pm}(x,t) = \frac{1}{3}\left(c_2 u - \sqrt{c_2^2u^2 - 3c_2^2\xi}\right) + c_2\left(\pm\sqrt{pqu}\right)^2 \operatorname{csch}_{pq}\left(\sqrt{u}(t\chi + \eta x)\right)^2. \tag{3.5}$$

Case II: If $u < 0$ and $\xi = 0$, then

$$Q_3^{\pm}(x,t) = \frac{1}{3}\left(c_2 u - \sqrt{c_2^2u^2 - 3c_2^2\xi}\right) + c_2\left(\pm\sqrt{-pqu}\right)^2 \sec_{pq}\left(\sqrt{-u}(t\chi + \eta x)\right)^2, \tag{3.6}$$

$$Q_4^{\pm}(x,t) = \frac{1}{3}\left(c_2 u - \sqrt{c_2^2 u^2 - 3c_2^2 \xi}\right) + c_2\left(\pm\sqrt{-pqu}\right)^2 \csc_{pq}\left(\sqrt{-u}(t\chi + \eta x)\right)^2. \tag{3.7}$$

Case III: If $u < 0$ and $\xi = \frac{u^2}{4}$, then

$$Q_5^{\pm}(x,t) = \frac{1}{3}\left(c_2 u - \sqrt{c_2^2 u^2 - 3c_2^2 \xi}\right) + c_2\left(\pm\frac{\sqrt{-u}}{\sqrt{2}}\right)^2 \tanh_{pq}\left(\frac{\sqrt{-u}(t\chi + \eta x)}{\sqrt{2}}\right)^2, \tag{3.8}$$

$$Q_6^{\pm}(x,t) = \frac{1}{3}\left(c_2 u - \sqrt{c_2^2 u^2 - 3c_2^2 \xi}\right) + c_2\left(\pm\frac{\sqrt{-u}}{\sqrt{2}}\right)^2 \coth_{pq}\left(\frac{\sqrt{-u}(t\chi + \eta x)}{\sqrt{2}}\right)^2, \tag{3.9}$$

$$\begin{aligned} Q_7^{\pm}(x,t) = \quad & \frac{1}{3}\left(c_2 u - \sqrt{c_2^2 u^2 - 3c_2^2 \xi}\right) + c_2\left(\pm\frac{\sqrt{-u}}{\sqrt{2}}\left(\tanh_{pq}\left(\sqrt{2}\sqrt{-u}(t\chi + \eta x)\right)\right.\right. \\ & \left.\left.\pm i\sqrt{pq}\operatorname{sech}_{pq}\left(\sqrt{2}\sqrt{-u}(t\chi + \eta x)\right)\right)\right)^2, \end{aligned} \tag{3.10}$$

$$\begin{aligned} Q_8^{\pm}(x,t) = \quad & \frac{1}{3}\left(c_2 u - \sqrt{c_2^2 u^2 - 3c_2^2 \xi}\right) + c_2\left(\pm\frac{\sqrt{-u}}{\sqrt{2}}\left(\coth_{pq}\left(\sqrt{2}\sqrt{-u}(t\chi + \eta x)\right)\right.\right. \\ & \left.\left.\pm\sqrt{pq}\operatorname{csch}_{pq}\left(\sqrt{2}\sqrt{-u}(t\chi + \eta x)\right)\right)\right)^2, \end{aligned} \tag{3.11}$$

$$\begin{aligned} Q_9^{\pm}(x,t) = \quad & \frac{1}{3}\left(c_2 u - \sqrt{c_2^2 u^2 - 3c_2^2 \xi}\right) + c_2\left(\pm\frac{\sqrt{-u}}{2\sqrt{2}}\left(\tanh_{pq}\left(\frac{\sqrt{-u}(t\chi + \eta x)}{2\sqrt{2}}\right) + \right.\right. \\ & \left.\left.\coth_{pq}\left(\frac{\sqrt{-u}(t\chi + \eta x)}{2\sqrt{2}}\right)\right)\right)^2. \end{aligned} \tag{3.12}$$

Case IV: If $u > 0$ and $\xi = \frac{u^2}{4}$, then

$$Q_{10}^{\pm}(x,t) = \frac{1}{3}\left(c_2 u - \sqrt{c_2^2 u^2 - 3c_2^2 \xi}\right) + c_2\left(\pm\frac{\sqrt{u}}{\sqrt{2}}\right)^2 \tan_{pq}\left(\frac{\sqrt{u}(t\chi + \eta x)}{\sqrt{2}}\right)^2, \tag{3.13}$$

$$Q_{11}^{\pm}(x,t) = \frac{1}{3}\left(c_2 u - \sqrt{c_2^2 u^2 - 3c_2^2 \xi}\right) + c_2\left(\pm\frac{\sqrt{u}}{\sqrt{2}}\right)^2 \cot_{pq}\left(\frac{\sqrt{u}(t\chi + \eta x)}{\sqrt{2}}\right)^2, \tag{3.14}$$

$$\begin{aligned} Q_{12}^{\pm}(x,t) = \quad & \frac{1}{3}\left(c_2 u - \sqrt{c_2^2 u^2 - 3c_2^2 \xi}\right) + c_2\left(\pm\frac{\sqrt{u}}{\sqrt{2}}\left(\tan_{pq}\left(\sqrt{2}\sqrt{u}(t\chi + \eta x)\right)\right.\right. \\ & \left.\left.\pm\sqrt{pq}\sec_{pq}\left(\sqrt{2}\sqrt{u}(t\chi + \eta x)\right)\right)\right)^2, \end{aligned} \tag{3.15}$$

$$\begin{aligned} Q_{13}^{\pm}(x,t) = \quad & \frac{1}{3}\left(c_2 u - \sqrt{c_2^2 u^2 - 3c_2^2 \xi}\right) + c_2\left(\pm\frac{\sqrt{u}}{\sqrt{2}}\left(\cot_{pq}\left(\sqrt{2}\sqrt{u}(t\chi + \eta x)\right)\right.\right. \\ & \left.\left.\pm\sqrt{pq}\csc_{pq}\left(\sqrt{2}\sqrt{u}(t\chi + \eta x)\right)\right)\right)^2, \end{aligned} \tag{3.16}$$

$$\begin{aligned} Q_{14}^{\pm}(x,t) = \quad & \frac{1}{3}\left(c_2 u - \sqrt{c_2^2 u^2 - 3c_2^2 \xi}\right) + c_2\left(\pm\frac{\sqrt{u}}{2\sqrt{2}}\left(\tan_{pq}\left(\frac{\sqrt{u}(t\chi + \eta x)}{2\sqrt{2}}\right) + \right.\right. \\ & \left.\left.\cot_{pq}\left(\frac{\sqrt{u}(t\chi + \eta x)}{2\sqrt{2}}\right)\right)\right)^2. \end{aligned} \tag{3.17}$$

3.1. Results and discussion

Mathematical calculations of achieved exact solutions are more proficient and advantageous in analyzing the dynamical behavior of non-linear wave phenomena based upon their graphical depiction. These obtained solutions elaborates different types of soliton solutions. Some of which are presented in 3D, 2D and contour plots with the help of maple. Figure 1 represents bright soliton solution, Figure 2 represents singular soliton solution for Eqs (3.4) and (3.5) respectively. Figures 3 and 5 demonstrate periodic singular soliton solutions for Eqs (3.7) and (3.17). Solution (3.10) represents dark-bright soliton and is plotted in Figure 4.

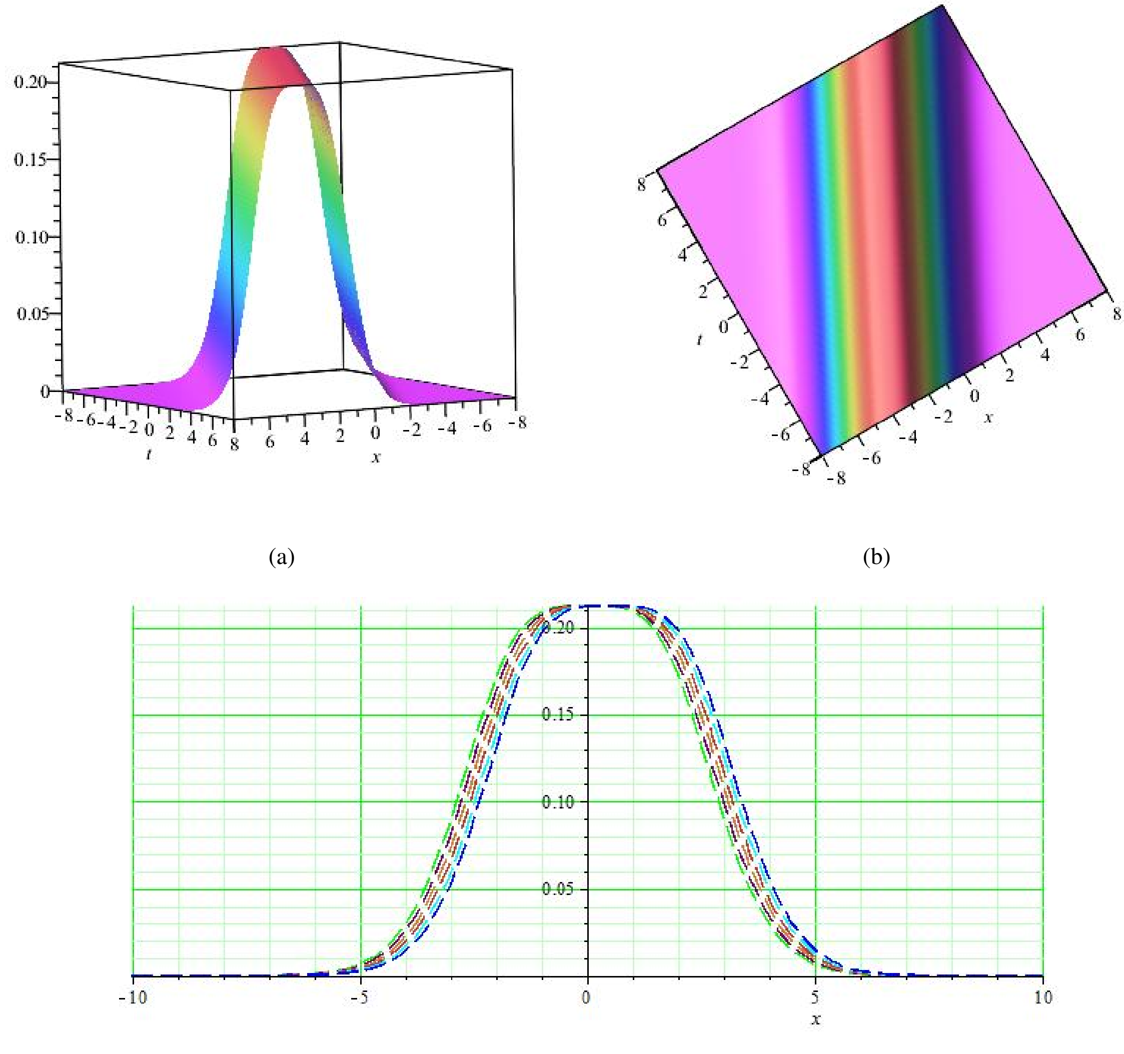
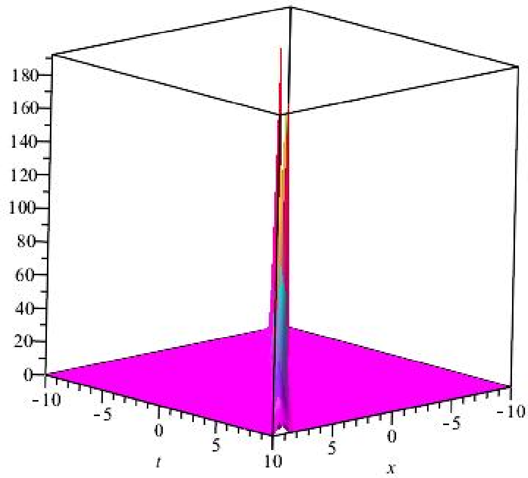
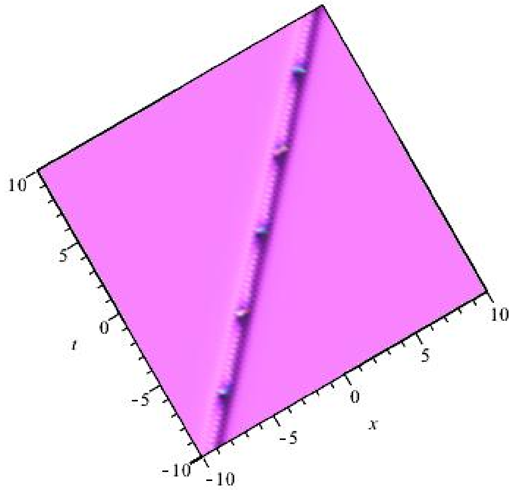


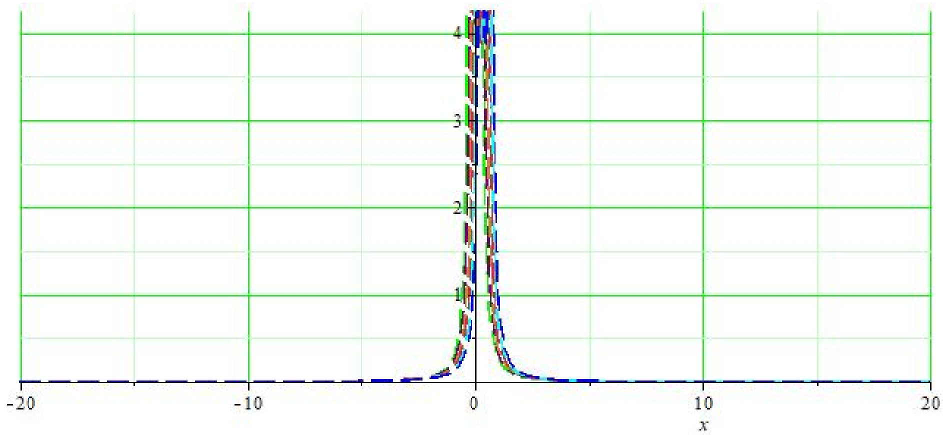
Figure 1. (a) and (b), 3D and contour graphs of $Q_1^{\pm}(x,t)$ are sketched with η=0.91, $\xi = 0$, $p = 0.98$, $q = 0.95$, $u = 0.2$, for $-8 \le x \le 8$, $-8 \le t \le 8$ respectively. (c) 2D graph with η=0.91, $\xi = 0$, $p = 0.98$, $q = 0.95$, $u = 0.2$, and $t = 0, 0.2, 0.4, 0.6, 0.8, 1$ for $-10 \le x \le 10$.



(a)

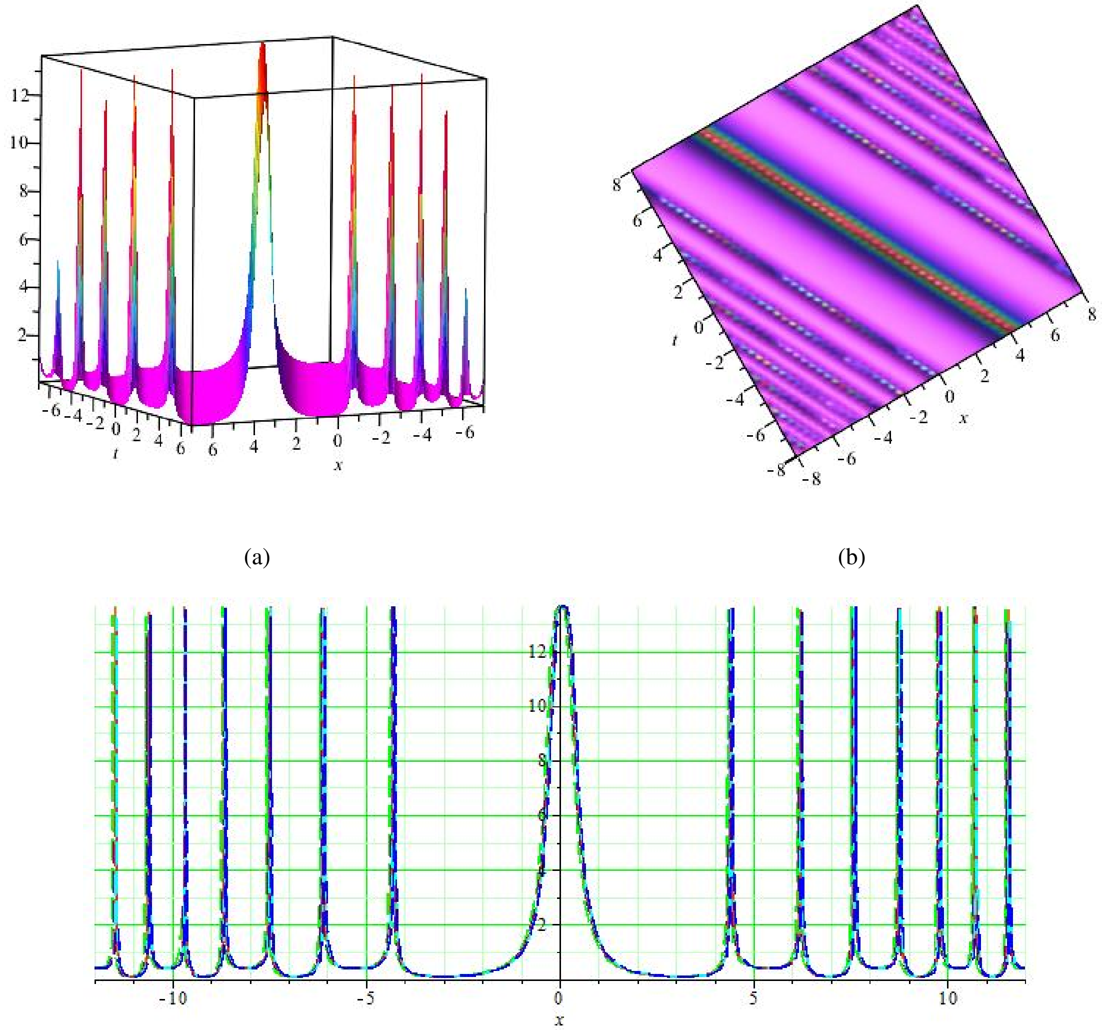


(b)



(c)

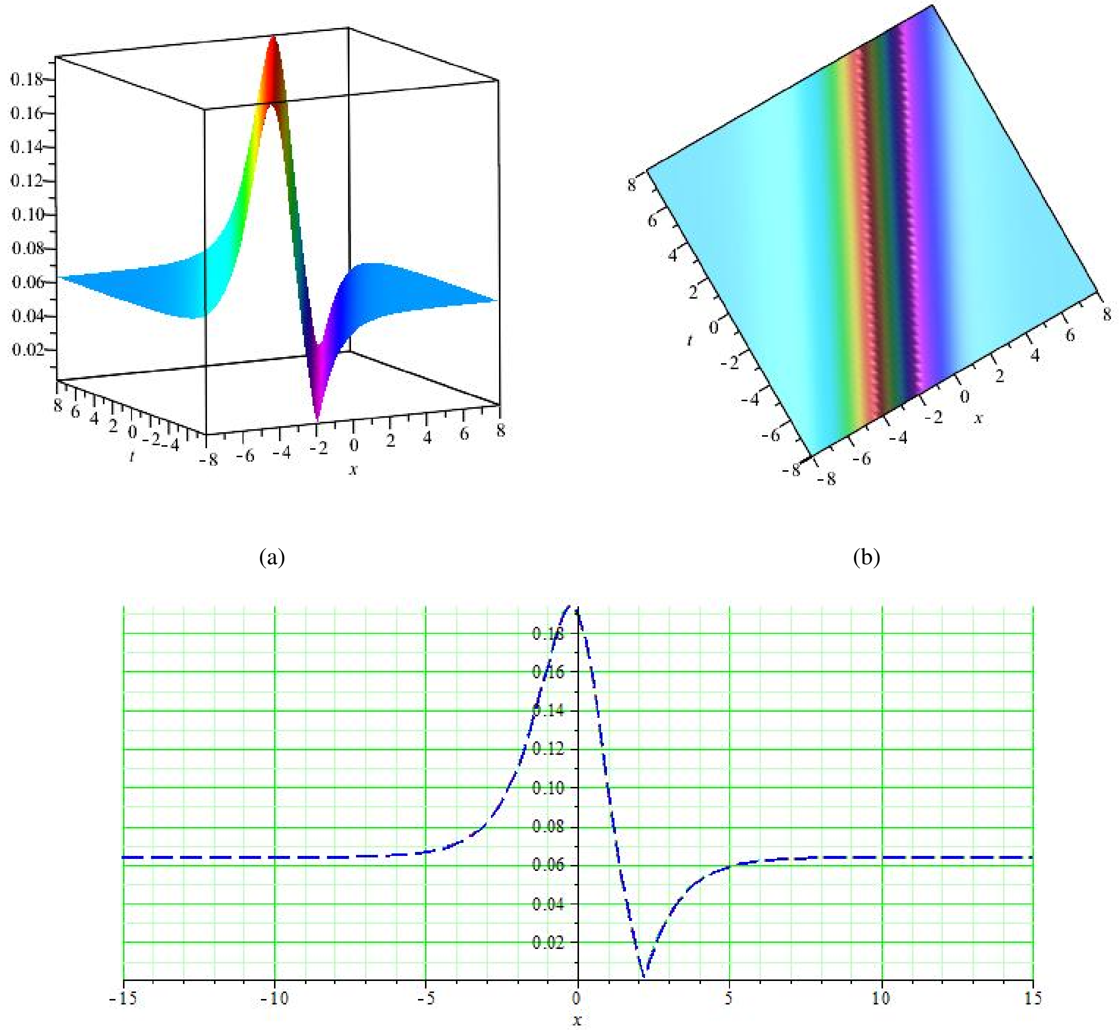
Figure 2. (a) and (b), 3D and contour graphs of $Q_2^{\pm}(x,t)$ are sketched with η=0.9, $\xi = 0$, $p = 0.98, q = 0.99, u = 0.21$ for $-10 \leq x \leq 10$, $-10 \leq t \leq 10$ respectively. (c) 2D graph with η=0.9, $\xi = 0$, $p = 0.98, q = 0.99, u = 0.1$, and $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ for $-20 \leq x \leq 20$.



(a) (b)

(c)

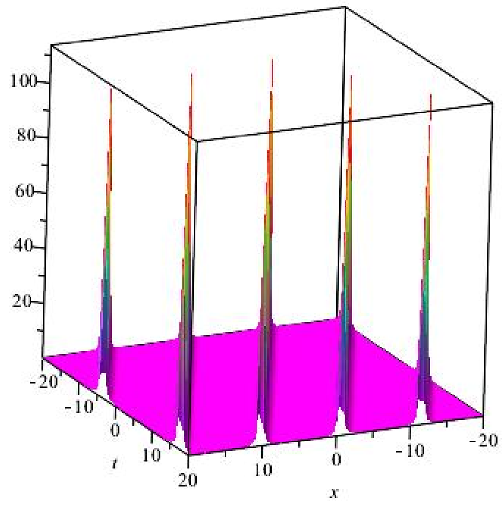
Figure 3. (a) and (b), 3D and contour graphs of $Q_4^{\pm}(x,t)$ are sketched with η=0.91, $\xi = 0$, $p = 0.98$, $q = 0.95$, $u = -0.2$, for $-7 \leq x \leq 7$, $-7 \leq t \leq 7$ respectively. (c) 2D graph with η=0.91, $\xi = 0$, $p = 0.98$, $q = 0.95$, $u = -0.2$, and $t = 0, 0.1, 0.14, 0.16, 0.18, 0.2$ for $-12 \leq x \leq 12$.



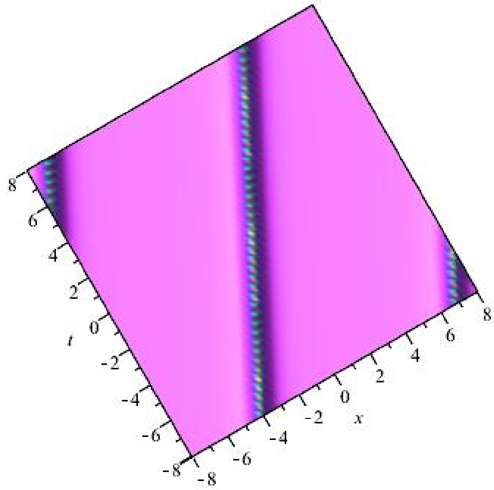
(a) (b)

(c)

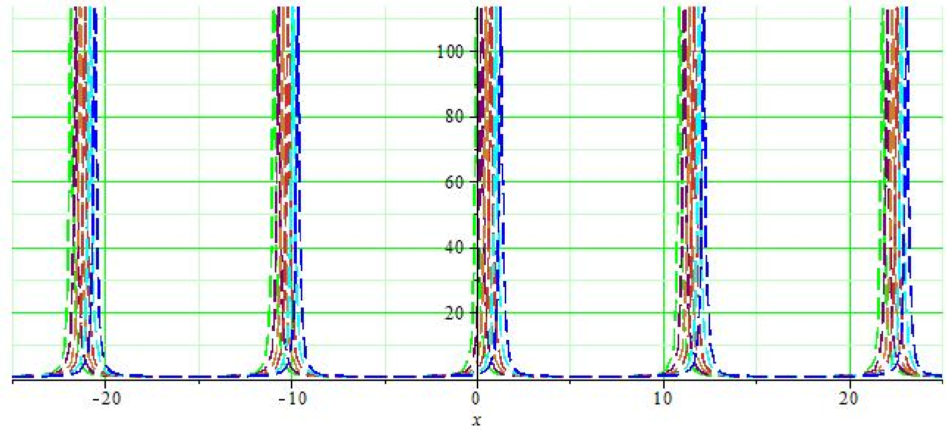
Figure 4. (a) and (b), 3D and contour graphs of $Q_7^{\pm}(x,t)$ are sketched with η=0.91, ξ = 0.0200, $p = 0.98, q = 0.95, u = -0.2$, for $-8 \le x \le 8$, $-8 \le t \le 8$ respectively. (c) 2D graph with η=0.91, $\xi = 0.0200$, $p = 0.98, q = 0.95, u = -0.2$, and $t = 0, 0.01, 0.02, 0.03, 0.04, 0.05$ for $-15 \le x \le 15$.



(a)



(b)



(c)

Figure 5. (a) and (b), 3D and contour graphs of $Q_{14}^{\pm}(x,t)$ are sketched with η=0.91, $\xi = 0.0200, p = 0.98, q = 0.95, u = 0.2$, for $-8 \leq x \leq 8$, $-8 \leq t \leq 8$ respectively. (c) 2D graph with η=0.91, $\xi = 0.0200, p = 0.98, q = 0.95, u = 0.2$, and $t = 0, 0.5, 1, 1.5, 2, 2.5$ for $-25 \leq x \leq 25$.

4. Conclusions

SSET is a realistic, effective, and expressive tool that has been successfully implemented in the Boussinesq equation to extract exact traveling wave solutions that are highly beneficial. The obtained results are in the form of rational, hyperbolic and trigonometric functions. As we can see, this method is powerful, efficient and simple tool for solving various types of nonlinear PDEs found in different models of engineering and natural sciences. The obtained results may be practical, beneficial and can explain the water waves in marine engineering, shallow water with long wavelength, optics, nonlinear grids, coupled circuits, elastic rods and so on. At the end 3D, 2D and contour plots of these solutions are sketched using maple.

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Conflict of interest

The authors declare no conflict of interest.

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