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Research article

Semi-(E, F)-convexity in complex programming problems

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Abstract: Under recent circulars on the notions of convexity for real sets and functions like E-convexity and (E, F)-convexity, we expand the notions of (E, F) and semi-(E, F)-convexity to include domains and functions in complex space. We examine their properties and interrelationships. As a consequence, we apply the associated results on a non-linear semi-(E, F)-convex programming problem with cone-constraints in complex space. We discuss the existence and uniqueness of its optimal solution and establish the necessary and sufficient conditions for a feasible point to be an optimal solution to such a problem. The related results in real space can be deduced as special cases.

Keywords: complex programming; (E, F)-convexity; semi-(E, F)-convexity; optimal solution
Mathematics Subject Classification: 32C15, 90C25, 90C30, 90C46

1. Introduction

Convexity and generalized convexity play important roles in optimization theory and applied mathematics. Various generalizations of convexity have appeared in the literature. A crucial generalization of convexity is E-convexity, introduced by Youness [1] where classes of E-convex sets and E-convex functions have been introduced, based on the effect of an operator E. A class of quasi E-convex functions has been presented in [2]. The initial results have inspired a great deal of subsequent works which have greatly expanded the role of E-convexity in the optimization theory. A class of semi-E-convex functions has been defined, and its properties have been discussed [3]. The classes of semi-(E, F)-convexity have been introduced, and some sufficient conditions of optimality and duality theorem have been established in [4]. The concept of strongly E-convexity has been introduced in [5]. Additional properties about E-convexity have been discussed in [6, 7]. Fulga and Preda [8] have introduced the class of E-preinvex functions as well as the related classes of local E-preinvex and E-prequasiinvex functions. The properties and characterizations of these classes have been given, and some results for the corresponding nonlinear programming have been discussed. Two classes of functions called E-B-vex and E-B-preinvex functions have been introduced in [9]. As a generalization of semi-E-convex functions, a class of semi-E-preinvex functions has been defined in [10]. Some basic properties of E-B-vex functions have been studied in [11]. A class of quasi-(E, F)-convex functions has been introduced in [12]. Some properties of semi-E-preinvex maps in Banach spaces have been studied in [13]. A class of geodesic semi-E-convex functions has been introduced, and their properties have been discussed in [14]. Mirzapour [15] has stated additional properties to semi-E-convex and quasi-semi-E-convex functions. Megahed et al [16,17] have studied the duality and optimality conditions of E-convex programming for E-differentiable functions. In [18], the duality and optimality conditions for more general problem containing (F, α, ρ, d, E)-convex function have been studied. New characterizations of E-convex functions have been discussed in [19]. The concepts of strongly (E, F)-convex sets and functions have been introduced in [20]. Recently, in [21], the notions of $\mathscr{M}$-convex functions, log-$\mathscr{M}$-convex functions, quasi $\mathscr{M}$-convex functions have been introduced where bounds for natural phenomena have been described by integrals. In [22, 23], a problem of E-differentiable vector optimization with the multiple interval-valued objective function has been studied. In [24], a new class of semi strongly (E, F)-convex functions has been presented. In [25], a semi-infinite vector E-convex optimization problem involving support functions has been studied. In [26], the concepts of logarithmic and exponential E-convexity have been introduced for functions defined on a Banach space. However, the concepts of generalized convexity throughout these papers regarding to sets and functions were considered in real space. On the other hand, many concepts of real programming have been generalized to complex programming, see e.g., [27–34].

In this paper, we extend the concept of (E, F)-convexity to complex sets as well as concepts of (E, F)-convexity and semi-(E, F)-convexity to include complex functions, and discuss their properties and interrelations. We apply the associated results on a non-linear semi-(E, F)-convex programming problem with cone-constraints in complex space. We discuss the existence and uniqueness of its optimal solution and establish the necessary and sufficient conditions for a feasible point to be an optimal solution for such a problem. The corresponding results in real space can be deduced from this work as special cases.

The article in the following is prepared as: Section 2 introduces the extended notation and preliminaries of convexity in complex space that will be used throughout the paper. Section 3 discusses some properties and relations of complex (E, F)-convex and semi-(E, F)-convex functions. Section 4 is concerned with results of a non-linear complex semi-(E, F)-convex programming problem. Section 5 is the conclusion of the paper.

2. Notation and preliminaries

In this section, we extend some notation of the generalized convexity to complex space that will be used throughout the paper. The extensions include the concepts of (E, F)-convexity and semi-(E, F)-convexity found in [4].

Definition 1. *A non-empty set $M \subseteq \mathbb{C}^n$ is said to be*

(1) convex if

$$\lambda z_0 + (1 - \lambda)z \in M, \quad \forall z_0, z \in M, \ \forall \lambda \in [0, 1], \tag{2.1}$$

(2) (E, F)*-convex if there are two point-to-set maps* $E, F : M \longrightarrow 2^{\mathbb{C}^n}$ *such that*

$$\lambda E(z_0) + (1 - \lambda)F(z) \subseteq M, \quad \forall z_0, z \in M, \ \forall \lambda \in [0, 1], \tag{2.2}$$

(3) cone if $\lambda M \subseteq M$*, for* $\lambda \geq 0$.

Remark 1. *A convex set* $M \subseteq \mathbb{C}^n$ *is* (E, F)*-convex by taking* $E(z_0) = \{z_0\}$ *and* $F(z) = \{z\}, \ \forall z_0, z \in M$.

It is easy to show, for two point-to-set maps $E, F : M \longrightarrow 2^{\mathbb{C}^n}$, that:

(1) If $M \subseteq \mathbb{C}^n$ is an (E, F)-convex, then $E(M) \bigcup F(M) \subseteq M$.
(2) If $M \subseteq \mathbb{C}^n$ is a convex set and $E(M) \bigcup F(M) \subseteq M$, then M is an (E, F)-convex set.
(3) If $E(M) \bigcup F(M)$ is a convex subset of $M \subseteq \mathbb{C}^n$, then M is an (E, F)-convex set.

Example 1. *Let* $M = \{z \in \mathbb{C} : 0 \leq \arg z \leq \pi/4\}$*, and* $E, F : M \to \mathbb{C}$ *be defined by* $E(z) = \{Re\ z\}$ *and* $F(z) = \{(1 + i)Re\ z\}$*. Since* M *is a convex set in* $\mathbb{C}$ *and* $E(M) \bigcup F(M) \subseteq M$*, then* M *is an* (E, F)*-convex set.*

Example 2. *Let* $S_1 = \{z \in \mathbb{C} : 0 \leq \arg z \leq \pi/4\}$*,* $S_2 = \{z \in \mathbb{C} : 3\pi/4 \leq \arg z \leq \pi\}$ *and* $M = S_1 \cup S_2$*. Let* $E, F : M \to \mathbb{C}$ *be defined by* $E(z) = \{Re\ z\}$ *and* $F(z) = \{-Re\ z\}$*. Since* $E(M) \bigcup F(M)$ *is a convex subset of* M*, then* M *is* (E, F)*-convex in* $\mathbb{C}$.

In Example 1, M is convex and $E(M) \bigcup F(M)$ is not convex. The converse is in Example 2. The following example shows that both M and $E(M) \bigcup F(M)$ may not be convex, but M is (E, F)-convex.

Example 3. *Let* $S_1 = \{z \in \mathbb{C} : 0 \leq \arg z \leq \pi/4\}$*,* $S_2 = \{z \in \mathbb{C} : 3\pi/4 \leq \arg z \leq \pi\}$ *and* $M = S_1 \cup S_2$*. Let* $E, F : M \to \mathbb{C}$ *be defined by* $E(z) = \{|Re\ z|\}$ *and* $F(z) = \{(1 + i)|Re\ z|\}$*. Clearly, the set* M *is* (E, F)*-convex in* $\mathbb{C}$*, although* M *and* $E(M) \bigcup F(M)$ *are not convex sets.*

Definition 2. *For a closed convex cone* T *in* $\mathbb{C}^m$ *and* $M \subseteq \mathbb{C}^n$*, the function* $f : M \to \mathbb{C}^m$ *is said to be*

(1) convex on M *with respect to* T *if* M *is convex and*

$$\lambda f(z_0) + (1 - \lambda)f(z) - f(\lambda z_0 + (1 - \lambda)z) \in T, \quad \forall z_0, z \in M, \ \forall \lambda \in [0, 1], \tag{2.3}$$

(2) (E, F)*-convex on* M *with respect to* T *if there are two point-to-set maps* $E, F : M \longrightarrow 2^{\mathbb{C}^n}$ *such that* M *is* (E, F)*-convex and*

$$\lambda f(w_0) + (1 - \lambda)f(w) - f(\lambda w_0 + (1 - \lambda)w) \in T, \quad \forall z_0, z \in M, \ \forall w_0 \in E(z_0), \ \forall w \in F(z), \ \forall \lambda \in [0, 1], \tag{2.4}$$

(3) semi-(E, F)*-convex on* M *with respect to* T *if there are two point-to-set maps* $E, F : M \longrightarrow 2^{\mathbb{C}^n}$ *such that* M *is* (E, F)*-convex and*

$$\lambda f(z_0) + (1 - \lambda)f(z) - f(\lambda w_0 + (1 - \lambda)w) \in T, \quad \forall z_0, z \in M, \ \forall w_0 \in E(z_0), \ \forall w \in F(z), \ \forall \lambda \in [0, 1]. \tag{2.5}$$

Remark 2. *The convex function* f *on* M *with respect to* T *is* (E, F)*-convex and semi-*(E, F)*-convex on* M *with respect to* T *by taking* $E(z_0) = \{z_0\}$ *and* $F(z) = \{z\}, \ \forall z_0, z \in M$.

In particular, if $T = \mathbb{R}^m_+$, the non-negative orthant of $\mathbb{R}^m$, one obtains

Definition 3. *For $M \subseteq \mathbb{C}^n$, the real part of a function $f : M \to \mathbb{C}^m$ is said to be*

(1) convex on M if M is convex and

$$Re\ f(\lambda z_0 + (1-\lambda)z) \leq Re\ [\lambda f(z_0) + (1-\lambda)f(z)], \quad \forall z_0, z \in M,\ \forall \lambda \in [0,1], \tag{2.6}$$

(2) (E,F)-convex on M if there are two point-to-set maps $E, F : M \longrightarrow 2^{\mathbb{C}^n}$ such that M is (E,F)-convex and

$$Re\ f(\lambda w_0 + (1-\lambda)w) \leq Re\ [\lambda f(w_0) + (1-\lambda)f(w)], \quad \forall z_0, z \in M,\ \forall w_0 \in E(z_0),\ \forall w \in F(z),\ \forall \lambda \in [0,1], \tag{2.7}$$

(3) semi-(E,F)-convex on M if there are two point-to-set maps $E, F : M \longrightarrow 2^{\mathbb{C}^n}$ such that M is (E,F)-convex and

$$Re\ f(\lambda w_0 + (1-\lambda)w) \leq Re\ [\lambda f(z_0) + (1-\lambda)f(z)], \quad \forall z_0, z \in M,\ \forall w_0 \in E(z_0),\ \forall w \in F(z),\ \forall \lambda \in [0,1], \tag{2.8}$$

(4) strictly convex, strictly (E,F)-convex, or strictly semi-(E,F)-convex on M if the inequalities in (2.6), (2.7), or (2.8) holds strictly respectively for $z \neq z_0$ and $\lambda \in (0,1)$.

Moreover, if f is a real function, one obtains the classical definitions.

3. Properties of complex semi-(E,F)-convex functions

In this section, we discuss some properties and relations of complex (E,F)-convex and semi-(E,F)-convex functions. In the the subsequent of the paper, we assume that $E, F : M \longrightarrow 2^{\mathbb{C}^n}$ are two point-to-set maps defined on $M \subseteq \mathbb{C}^n$. We begin with the some properties of (E,F)-convex functions.

Lemma 1. *Let T be a closed convex cone in $\mathbb{C}^m$ and $E(M) \bigcup F(M)$ be a convex subset of M. Then a function $f : M \to \mathbb{C}^m$ is (E,F)-convex on M with respect to T iff f is convex on $E(M) \bigcup F(M)$ with respect to T.*

Corollary 1. *Let $E(M) \bigcup F(M)$ be a convex subset of M. Then a function $Re\ f : M \to \mathbb{R}^m$ is (E,F)-convex on M iff $Re\ f$ is convex on $E(M) \bigcup F(M)$.*

Remark 3. *Intuitively, if $Re\ f$ is a convex function on a convex set M with $E(M) \bigcup F(M) \subseteq M$, then $Re\ f$ is (E,F)-convex on M.*

The following results gives some properties of semi-(E,F)-convex functions.

Theorem 1. *Let T be a closed convex cone in $\mathbb{C}^m$ and M be an (E,F)-convex set. If $f : M \to \mathbb{C}^m$ is semi-(E,F)-convex on M with respect to T, then*

$$f(z_0) - f(w_0) \in T \text{ and } f(z) - f(w) \in T,\ \forall z_0, z \in M,\ \forall w_0 \in E(z_0),\ \forall w \in F(z). \tag{3.1}$$

Proof. If f is semi-(E,F)-convex on M with respect to T, then $\lambda w_0 + (1-\lambda)w \in M$ and we obtain conditions (3.1) by letting $\lambda = 1$ and $\lambda = 0$ in (2.5), separately. □

Corollary 2. *Let M be an (E,F)-convex set and $Re\ f$ is semi-(E,F)-convex on M, then*

$$Re\ f(w_0) \leq Re\ f(z_0) \text{ and } Re\ f(w) \leq Re\ f(z),\ \forall z_0, z \in M,\ \forall w_0 \in E(z_0),\ \forall w \in F(z). \tag{3.2}$$

An (E,F)-convex function on an (E,F)-convex set is not necessary a semi-(E,F)-convex function. The following example provides an (E,F)-convex function, which is not a semi-(E,F)-convex function.

Example 4. *Let $M = \{z \in \mathbb{C} : 0 \le \arg z \le \pi/4\}$ and let $E, F : M \to \mathbb{C}$ be defined by $E(z) = \{z+1\}$ and $F(z) = \{Re\ z\}$. Since $M \subset \mathbb{C}$ is a convex set and $E(M) \bigcup F(M) \subset M$, then M is (E,F)-convex. According to Remark 3, the function $f : M \to \mathbb{C}$ defined by $f(z) = |z|^2$ is so (E,F)-convex on M. For the point $z_0 = 0 \in M$, we have $1 = w_0 \in E(z_0)$, and consequently, $f(w_0) = 1 > 0 = f(z_0)$, then from Corollary 2, it follows that f is not semi-(E,F)-convex on M.*

Theorem 2. *Let T be a closed convex cone in $\mathbb{C}^m$, M be an (E,F)-convex set and $f : M \to \mathbb{C}^m$ be an (E,F)-convex function on M with respect to T. Then f is semi-(E,F)-convex on M with respect to T if conditions (3.1) above are satisfied.*

Proof. Since f is (E,F)-convex on M with respect to T, (2.4) is satisfied.

Suppose that conditions (3.1) are given, then $\forall z_0, z \in M,\ \forall w_0 \in E(z_0),\ \forall w \in F(z),\ \forall \lambda \in [0,1]$, we have

$$\lambda(f(z_0) - f(w_0)) \in T \text{ and } (1-\lambda)(f(z) - f(w)) \in T, \tag{3.3}$$

which implies

$$\lambda f(z_0) + (1-\lambda) f(z) - [\lambda f(w_0) + (1-\lambda) f(w)] \in T. \tag{3.4}$$

By adding the above expression with that in (2.4), then (2.5) follows. Hence f is semi-(E,F)-convex on M with respect to T. □

Corollary 3. *Let M be an (E,F)-convex set and $Re\ f : M \to \mathbb{R}^m$ be an (E,F)-convex function on M. Then $Re\ f$ is semi-(E,F)-convex on M if conditions (3.2) hold.*

By combining Corollaries 2 and 3, we deduce, for an (E,F)-convex function on an (E,F)-convex set M, that a function $Re\ f$ is semi-(E,F)-convex on M if and only if the conditions (3.2) are fulfilled.

Corollary 4. *Let M be an (E,F)-convex set and $Re\ f : M \to \mathbb{R}^m$ be an (E,F)-convex function on M. Then $Re\ f$ is strictly semi-(E,F)-convex on M iff the inequalities in (3.2) hold strictly.*

Corollary 5. *Let M be an (E,F)-convex set and $Re\ f : M \to \mathbb{R}^m$ be a strictly (E,F)-convex function on M. Then $Re\ f$ is strictly semi-(E,F)-convex on M iff conditions (3.2) hold.*

Theorem 3. *Let T be a closed convex cone in $\mathbb{C}^m$, M be an (E,F)-convex set and $f : M \to \mathbb{C}^m$ be a semi-(E,F)-convex function on M with respect to T. Then for any complex vector $\alpha \in \mathbb{C}^m$, the level set*

$$\Gamma_{\alpha,T} := \{z \in M : -f(z) + \alpha \in T\}, \tag{3.5}$$

is an (E,F)-convex set.

Proof. $\forall z_0, z \in \Gamma_{\alpha,S}$ and $\forall \lambda \in [0,1]$, we have

$$\lambda(-f(z_0) + \alpha) \in T, \tag{3.6}$$

and

$$(1-\lambda)(-f(z) + \alpha) \in T. \tag{3.7}$$

Since M is an (E, F)-convex set and f is a semi-(E, F)-convex function on M with respect to T, then $\lambda E(z_0) + (1 - \lambda)F(z) \subseteq M$, and

$$\lambda f(z_0) + (1 - \lambda)f(z) - f(\lambda w_0 + (1 - \lambda)w) \in T, \quad \forall w_0 \in E(z_0),\ \forall w \in F(z). \tag{3.8}$$

Adding (3.6)–(3.8) yields

$$- f(\lambda w_0 + (1 - \lambda)w) + \alpha \in T, \quad \forall w_0 \in E(z_0),\ \forall w \in F(z). \tag{3.9}$$

Hence

$$\lambda w_0 + (1 - \lambda)w \in \Gamma_{\alpha,T}, \tag{3.10}$$

that is

$$\lambda E(z_0) + (1 - \lambda)F(z) \subseteq \Gamma_{\alpha,T}. \tag{3.11}$$

It follows that $\Gamma_{\alpha,T}$ is an (E, F)-convex set. □

Corollary 6. *Let T be a closed convex cone in $\mathbb{C}^m$, M be an (E, F)-convex set, and $f : M \to \mathbb{C}^m$ be a semi-(E, F)-convex function on M with respect to T. Then the set*

$$X := \{z \in M : -f(z) \in T\}, \tag{3.12}$$

is an (E, F)-convex set.

Corollary 7. *Let M be an (E, F)-convex set and $Re\ f : M \to \mathbb{R}^m$ be a semi-(E, F)-convex function on M. Then the set*

$$\Gamma_\alpha := \{z \in M : Re\ f(z) \leq \alpha\}, \tag{3.13}$$

is an (E, F)-convex set for any real vector $\alpha \in \mathbb{R}^m$.

To characterize a semi-(E, F)-convex function, we need to define the extended maps $\hat{E}, \hat{F} : M \times \mathbb{C}^m \longrightarrow 2^{\mathbb{C}^{n+m}}$ by $\hat{E}(z, \alpha) = E \times I(z, \alpha) = (E(z), \alpha)$ and $\hat{F}(z, \alpha) = F \times I(z, \alpha) = (F(z), \alpha)$ for any $(z, \alpha) \in M \times \mathbb{C}^m$.

Theorem 4. *Let T be a closed convex cone in $\mathbb{C}^m$ and M be an (E, F)-convex set. The function $f : M \to \mathbb{C}^m$ is semi-(E, F)-convex on M with respect to T iff its epigaph*

$$epi(f, T) := \{(z, \alpha) : z \in M, \alpha \in \mathbb{C}^m, -f(z) + \alpha \in T\}, \tag{3.14}$$

is an $(\hat{E},\hat{F})$-convex set on $\mathbb{C}^{n+m}$.

Proof. Assume that $epi(f, T)$ is an $(\hat{E}, \hat{F})$-convex set on $\mathbb{C}^{n+m}$, then $\forall z_0, z \in M$, we have $(z_0, f(z_0))$, $(z, f(z)) \in epi(f, T)$. It follows, $\forall \lambda \in [0, 1]$, that

$$\lambda \hat{E}(z_0, f(z_0)) + (1 - \lambda)\hat{F}(z, f(z)) \subseteq epi(f, T), \tag{3.15}$$

i.e.,

$$(\lambda E(z_0) + (1 - \lambda)F(z), \lambda f(z_0) + (1 - \lambda)f(z)) \subseteq epi(f, T). \tag{3.16}$$

Since $\lambda E(z_0) + (1 - \lambda)F(z) \subseteq M$, we have

$$- f(\lambda w_0 + (1 - \lambda)w) + \lambda f(z_0) + (1 - \lambda)f(z) \in T,\ \forall w_0 \in E(z_0),\ \forall w \in F(z) \text{ and } \forall \lambda \in [0, 1]. \tag{3.17}$$

Therefore f is a semi-(E, F)-convex function on M with respect to T.

Conversely, assume that f is a semi-(E, F)-convex on M with respect to T. Then

$$\lambda f(z_0) + (1-\lambda)f(z) - f(\lambda w_0 + (1-\lambda)w) \in T,\ \forall w_0 \in E(z_0),\ \forall w \in F(z) \text{ and } \forall \lambda \in [0,1]. \tag{3.18}$$

To prove $epi(f, T)$ is an $(\hat{E}, \hat{F})$-convex set, let $(z_0, \alpha), (z, \beta) \in epi(f, T)$, so $z_0, z \in M, -f(z_0) + \alpha \in T$ and $-f(z) + \beta \in T$. We get, $\forall \lambda \in [0, 1]$,

$$\lambda(-f(z_0) + \alpha) + (1-\lambda)(-f(z) + \beta) \in T,$$

which, by adding with (3.18), implies

$$-f(\lambda w_0 + (1-\lambda)w) + (\lambda\alpha + (1-\lambda)\beta) \in T\ \forall w_0 \in E(z_0),\ \forall w \in F(z) \text{ and } \forall \lambda \in [0,1].$$

This means that

$$(\lambda w_0 + (1-\lambda)w, \lambda\alpha + (1-\lambda)\beta) \in epi(f, T),$$

that is

$$(\lambda(w_0, \alpha) + (1-\lambda)(w, \beta)) \in epi(f, T),$$

or

$$\lambda(E(z_0), \alpha) + (1-\lambda)(F(z), \beta) \subseteq epi(f, T).$$

It follows that

$$\lambda\hat{E}(z_0, \alpha) + (1-\lambda)\hat{F}(z, \beta) \subseteq epi(f, T),$$

and therefore, $epi(f, T)$ is an $(\hat{E}, \hat{F})$-convex set. □

Corollary 8. *Let T be a closed convex cone in $\mathbb{C}^m$ and M be an (E, F)-convex set. The function $-f : M \to \mathbb{C}^m$ is semi-(E, F)-convex on M with respect to T iff its hypograph*

$$hyp(f, T) := \{(z, \alpha) : z \in M, \alpha \in \mathbb{C}^m, f(z) + \alpha \in T\}, \tag{3.19}$$

is an $(\hat{E}, \hat{F})$-convex set on $\mathbb{C}^{n+m}$.

4. Results of a complex semi-(E, F)-convex programming problem

We consider the non-linear complex programming problem:

$$\begin{aligned} &min\ Re\ f(z) \\ &subject\ to\ z \in M, \end{aligned} \tag{4.1}$$

where $f : M \to \mathbb{C}$. We have the following results.

Theorem 5. *Let M be an (E, F)-convex set and*

$$Re\ f(w_1) \leq Re\ f(z_1) \text{ and } Re\ f(w_2) \leq Re\ f(z_2),\quad \forall z_1, z_2 \in M,\ \forall w_1 \in E(z_1),\ \forall w_2 \in F(z_2). \tag{4.2}$$

If w_0 is a solution of

$$\begin{aligned} &min\ Re\ f(w) \\ &subject\ to\ w \in E(z) \bigcup F(z) \text{ and } z \in M, \end{aligned} \tag{4.3}$$

then w_0 is a solution of problem (4.1).

Proof. Let $w_0 \in E(M)$ be a solution of problem (4.3), i.e., $w_0 \in E(z_0)$ for some $z_0 \in M$. From the (E, F)-convexity of M, $w_0 \in M$. Suppose that w_0 is not a solution of problem (4.1), then there exists $z^* \in M$ such that $Re\ f(z^*) < Re\ f(w_0)$, which implies by (4.2) to $Re\ f(w) \leq Re\ f(z^*) < Re\ f(w_0)$, for any $w \in E(z^*)$, which contradicts the optimality of w_0 of problem (4.3). If $w_0 \in F(M)$, the proof is similar. □

Notice that, under the conditions of the above theorem, a solution of problem (4.1), if exists, should lie in $E(M) \bigcup F(M)$.

Corollary 9. *Let M be an (E, F)-convex set and $Re\ f : M \to \mathbb{R}$ be a semi-(E, F)-convex on M. If w_0 is a solution of problem (4.3), then w_0 is a solution of problem (4.1).*

Proof. The proof follows directly from Corollary 2 above. □

Theorem 6. *Let M be an (E, F)-convex set, and $Re\ f : M \to \mathbb{R}$ be an (E, F)-convex function on M with $Re\ f(w_1) \leq Re\ f(z_1)$, $\forall z_1 \in M$, $\forall w_1 \in F(z_1)$ (or $Re\ f(w_2) \leq Re\ f(z_2)$, $\forall z_2 \in M$, $\forall w_2 \in E(z_2)$). If $w_0 \in E(M) \bigcup F(M))$ is a local solution of problem (4.1), then w_0 is a global solution of problem (4.1).*

Proof. Suppose that $w_0 \in E(z_0) \subseteq E(M)$, for some $z_0 \in M$, is not a global solution of problem (4.1), then there exists another $z^* \in M$ such that $Re\ f(z^*) < Re\ f(w_0)$. Since M is an (E, F)-convex set, then for any $w \in F(z^*)$ and for $\lambda \in (0, 1)$ small enough, $w_0 + \lambda(w - w_0) \in M \cap N_\varepsilon(w_0)$. Then

$$Re\ f(w_0 + \lambda(w - w_0)) = Re\ f(\lambda w + (1 - \lambda)w_0),$$

which implies, by (E, F)-convexity of $Re\ f$, to

$$Re\ f(w_0 + \lambda(w - w_0)) \leq Re\ [\lambda f(w) + (1 - \lambda)f(w_0)].$$

Therefore,

$$Re\ f(w_0 + \lambda(w - w_0)) \leq Re\ [\lambda f(z^*) + (1 - \lambda)f(w_0)] < Re\ [\lambda f(w_0) + (1 - \lambda)f(w_0)] = Re\ f(w_0),$$

which contradicts the fact that w_0 is a local solution of problem (4.1). This proves the theorem. □

Corollary 10. *Let M be an (E, F)-convex set, and $Re\ f : M \to \mathbb{R}$ be a semi-(E, F)-convex function on M. If $w_0 \in E(M) \bigcup F(M))$ is a local solution of problem (4.1), then w_0 is a global solution of problem (4.1).*

Proof. The proof follows directly from Corollary 2 above. □

Corollary 11. *Let M be an (E, F)-convex set, and $Re\ f : M \to \mathbb{R}$ be a semi-(E, F)-convex function on M, and $z_0 \in M$ be a fixed point of E or F, i.e., $z_0 \in E(z_0) \bigcup F(z_0)$. If z_0 is a local solution of problem (4.1), then z_0 is a global solution of problem (4.1).*

Corollary 12. *Let M be an (E, F)-convex set, and $Re\ f : M \to \mathbb{R}$ be a (E, F)-convex function on M satisfying $Re\ f(w_1) \leq Re\ f(z_1)$, $\forall z_1 \in M$, $\forall w_1 \in F(z_1)$ and $Re\ f(w_2) \leq Re\ f(z_2)$, $\forall z_2 \in M$, $\forall w_2 \in E(z_2)$, and $z_0 \in M$ be a fixed point of E or F. If z_0 is a local solution of problem (4.1), then z_0 is a global solution of problem (4.1).*

Theorem 7. *Let M be an (E, F)-convex set, and $Re\ f : M \to \mathbb{R}$ be a strictly semi-(E, F)-convex function on M, then the global solution of problem (4.1) is unique.*

Proof. Let z_1 and z_2 be two different global solutions of problem (4.1), then $Re\ f(z_1) = Re\ f(z_2)$. Since M is an (E, F)-convex set and $Re\ f(z)$ is a strictly semi-(E, F)-convex function on M, then $\forall w_1 \in E(z_1)$, $\forall w_2 \in F(z_2)$ and $\forall \lambda \in (0, 1)$, we have

$$Re\ f(\lambda w_1 + (1 - \lambda)w_2) < Re\ (\lambda f(z_1) + (1 - \lambda)f(z_2) = Re\ f(z_1),$$

which contradicts the global optimality of z_1 of problem (4.1). Hence the global solution is unique. □

Corollary 13. *Let M be an (E, F)-convex set, and $Re\ f : M \to \mathbb{R}$ be an (E, F)-convex function on M satisfying $Re\ f(w_1) < Re\ f(z_1)$, $\forall z_1 \in M$, $\forall w_1 \in E(z_1)$ and $Re\ f(w_2) < Re\ f(z_2)$, $\forall z_2 \in M$, $\forall w_2 \in F(z_2)$, then the global solution of problem (4.1) is unique.*

Corollary 14. *Let M be an (E, F)-convex set, and $Re\ f : M \to \mathbb{R}$ be a strictly (E, F)-convex function on M satisfying $Re\ f(w_1) \leq Re\ f(z_1)$, $\forall z_1 \in M$, $\forall w_1 \in E(z_1)$ and $Re\ f(w_2) \leq Re\ f(z_2)$, $\forall z_2 \in M$, $\forall w_2 \in F(z_2)$, then the global solution of problem (4.1) is unique.*

Theorem 8. *Let M be an (E, F)-convex set, and $Re\ f : M \to \mathbb{R}$ be a semi-(E, F)-convex function on M, then the set of all optimal solutions of problem (4.1) is (E, F)-convex.*

Proof. Let z_0 be an optimal solution of problem (4.1), then the set $Z_0 := \{z \in M : Re\ f(z) \leq Re\ f(z_0)\} = \Gamma_{f(z_0)}$ of all optimal solutions is (E, F)-convex, according to Corollary 7. □

Now, we investigate the necessary and sufficient optimality criteria for a nonlinear complex programming problem (4.1).

Theorem 9. *Let M be an (E, F)-convex set, and $Re\ f : M \to \mathbb{R}$ be a differentiable function on M.*

(1) If $w_0 \in E(M)$ is a solution of problem (4.1), then

$$Re\ [\langle \nabla_z f(w_0), w - w_0 \rangle + \langle \nabla_{\bar{z}} f(w_0), \overline{w} - \overline{w_0} \rangle] \geq 0, \forall z \in M, \forall w \in F(z). \tag{4.4}$$

(2) If $w_0 \in F(M)$ is a solution of problem (4.1), then

$$Re\ [\langle \nabla_z f(w_0), w - w_0 \rangle + \langle \nabla_{\bar{z}} f(w_0), \overline{w} - \overline{w_0} \rangle] \geq 0, \forall z \in M, \forall w \in E(z), \tag{4.5}$$

where k^{th} elements of the vectors $\nabla_z f(w_0)$ and $\nabla_{\bar{z}} f(w_0)$ for $k = 1, ..., n$ are $\frac{\partial f(w_0)}{\partial z_k}$ and $\frac{\partial f(w_0)}{\partial \overline{z_k}}$, respectively.

Proof. (i) Let $w_0 \in E(z_0)$ be an optimal solution of problem (4.1) for some $z_0 \in M$. For any $z \in M$, $w \in F(z)$ and for $\lambda \in (0, 1]$, we note that $\lambda w + (1 - \lambda)w_0 \in M$, and then we have

$$Re\ f(w_0) \leq Re\ f(\lambda w + (1 - \lambda)w_0) = Re\ f(w_0 + \lambda(w - w_0)).$$

Since $Re\ f$ is differentiable on M, it follows for $\lambda \in (0, 1]$ that

$$Re\ f(w_0) \leq Re[\ f(w_0) + \lambda\langle \nabla_z f(w_0), w - w_0 \rangle + \lambda\langle \nabla_{\bar{z}} f(w_0), \overline{w} - \overline{w_0} \rangle + o(\lambda)],$$

where $o(\lambda)/\lambda \to 0$ as $\lambda \to 0^+$. Hence

$$Re[\lambda\langle \nabla_z f(w_0), w - w_0 \rangle + \lambda\langle \nabla_{\bar{z}} f(w_0), \overline{w} - \overline{w_0} \rangle + o(\lambda)] \geq 0.$$

Therefore, by dividing by λ and taking $\lambda \to 0^+$, the condition 4.4 follows.

(ii) The proof is similar to part (i). □

Theorem 10. *Let M be an (E, F)-convex set, and $Re\ f : M \to \mathbb{R}$ be a differentiable semi-(E, F)-convex function on M. Then a fixed point $z_0 \in M$ of E or F is a solution of problem (4.1), if*

$$(i)\ \ Re\ [\langle \nabla_z f(z_0), w - z_0 \rangle + \langle \nabla_{\overline{z}} f(z_0), \overline{w} - \overline{z_0} \rangle] \geq 0, \forall z \in M, \forall w \in F(z), \tag{4.6}$$

or

$$(ii)\ \ Re\ [\langle \nabla_z f(z_0), w - z_0 \rangle + \langle \nabla_{\overline{z}} f(z_0), \overline{w} - \overline{z_0} \rangle] \geq 0, \forall z \in M, \forall w \in E(z). \tag{4.7}$$

Proof. (i) Let $z_0 \in E(z_0)$. Since $Re\ f$ is semi-(E, F)-convex on M, then for any $z \in M$, $w \in F(z)$ and $\lambda \in [0, 1]$, we have

$$Re\ f(z_0 + \lambda(w - z_0)) \leq Re\ [f(z_0) + \lambda(f(z) - f(z_0))],$$

which yields

$$Re\ [(f(z) - f(z_0))] \geq \frac{Re\ f(z_0 + \lambda(w - z_0)) - Re\ f(z_0)}{\lambda},\ \lambda \in (0, 1].$$

As $\lambda \to 0^+$, the differentiability of $Re\ f$ on M yields

$$Re\ [f(z) - f(z_0)] \geq Re\ [\langle \nabla_z f(z_0), w - z_0 \rangle + \langle \nabla_{\overline{z}} f(z_0), \overline{w} - \overline{z_0} \rangle].$$

By using 4.6, we have $Re\ [f(z) - f(z_0)] \geq 0$, proving that z_0 is a solution of (4.1).

(ii) For $z_0 \in F(z_0)$, the proof is similar to part (i). □

Again, the semi-(E, F)-convexity condition of $Re\ f$ can be replaced by (E, F)-convexity with the conditions (3.2).

In the remainder of this section, we reconsider the complex programming problem (4.1) but with cone-constraints as

$$\begin{array}{l} min\ Re\ f(z) \\ subject\ to\ z \in M_g := \{z \in \mathbb{C}^n : g(z) \in S\}, \end{array} \tag{4.8}$$

where $f : \mathbb{C}^n \to \mathbb{C}$ and $g : \mathbb{C}^n \to \mathbb{C}^m$ are two functions, and S is a cone in $\mathbb{C}^m$. We apply the associated results to problem 4.8.

Theorem 11. *Let S be a closed convex cone in $\mathbb{C}^m$ and $-g$ be a semi-(E, F)-convex function on $\mathbb{C}^n$ with respect to S. Then the feasible region $M_g \subseteq \mathbb{C}^n$ of problem 4.8 is an (E, F)-convex set.*

Proof. The proof follows directly according to Corollary 6. □

As a consequent of Theorem 11, if $-g$ is a semi-(E, F)-convex function on $\mathbb{C}^n$ with respect to S, then all results of this section are fulfilled on M_g instead of M.

Example 5. *Let the set $M \subseteq \mathbb{C}$ be defined by $M = \{z \in \mathbb{C} : 0 \leq \arg z \leq \pi/4\}$, the maps $E, F : M \to \mathbb{C}$ be defined by $E(z) = \{Re\ z\}$, $F(z) = \{(1 + i)Im\ z\}$, and the objective $f : M \to \mathbb{C}$ be defined by $f(z) = |z|^2$. It is easy to show that the set M is (E, F)-convex and the function f is semi-(E, F)-convex on M. The point $w_0 = 0$ is the optimal solution of problem (4.3) and according to Corollary 9, it is the optimal solution of problem (4.1). Further, $w_0 = 0$ is the global solution of problem (4.1) according to Corollary 10 or Corollary 11.*

5. Conclusions

In this paper, we have extended the concepts of (E, F)-convexity to include complex sets as well as concepts of (E, F)-convexity and semi-(E, F)-convexity to include complex functions, and we have discussed their properties and relations. We have proved that the solution of a non-linear semi-(E, F)-convex programming problem, if exists, lies in $E(M) \bigcup F(M)$ and that the local solution is global. In the case of $Re\ f$ is strictly semi-(E, F)-convex, the global solution is unique. Finally, we have investigated the necessary and sufficient optimality criteria for a feasible point to be an optimal solution to such a problem. Some illustrative examples have been given. It is worth noting that the related results in real problems can be deduced from this work as special cases.

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Conflict of interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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