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Research article

A quasi-boundary method for solving an inverse diffraction problem

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Abstract: In this paper, we deal with the reconstruction problem of aperture in the plane from their diffraction patterns. The problem is severely ill-posed. The reconstruction solutions of classical Tikhonov method and Fourier truncated method are usually over-smoothing. To overcome this disadvantage of the classical methods, we introduce a quasi-boundary regularization method for stabilizing the problem by adding a-priori assumption on the exact solution. The corresponding error estimate is derived. At the continuation boundary $z = 0$, the error estimate under the a-priori assumption is also proved. In theory without noise, the proposed method has better approximation than the classical Tikhonov method. For illustration, two numerical examples are constructed to demonstrate the feasibility and efficiency of the proposed method.

Keywords: inverse diffraction problem; ill-posed; regularization; quasi-boundary method; Tikhonov method
Mathematics Subject Classification: 35R25, 35R30, 47A52

1. Introduction

The reconstruction problem of aperture in the plane from their diffraction patterns arises in acoustics and optics. Consider the following inverse diffraction problem. Let A be a bounded aperture in an infinite perfectly soft screen which is located in the plane $z = 0$ in $\mathbb{R}^3$. A harmonic plane wave with wave-number k propagates along with the positive z direction. It hits the screen and escapes through the aperture A. The measured data at receiving screen $z = d > 0$ is given. The problem is to reconstruct the shape (domain) of the aperture A. By Kirchhoff approximation, the mathematical modeling can be formulated as follows.

Inverse problem. Let $u(x, y, z)$ be the solution of the following problem:

$$u_{xx} + u_{yy} + u_{zz} + k^2 u(x, y, z) = 0, \quad (x, y) \in \mathbb{R}^2, z > 0, \tag{1.1}$$

$$u(x,y,d) = g(x,y), \qquad (x,y) \in \mathbb{R}^2, \tag{1.2}$$

$$\lim_{r\to\infty} r(\frac{\partial u}{\partial r} - iku) = 0, \text{ where } r = \sqrt{x^2+y^2+z^2}. \tag{1.3}$$

It is explained here that "r" always equals "$\sqrt{x^2+y^2+z^2}$" in this article. Here, we wish to find the $u(x,y,0) = \chi_A(x,y)$ from the data $g(x,y)$, where $\chi_A(x,y)$ is the characteristic function of the domain A.

But we deal with a more general case where the $\chi_A(x,y)$ is replaced by a general function $f(x,y)$ that belongs to $L^2(\mathbb{R}^2)$ (if A is the support of f, this corresponds to assuming an arbitrary incident wave).

Before investigating the inverse problem, we discuss briefly the forward problem. The corresponding forward problem is as follow.

Forward problem. Let $u(x,y,z)$ be the solution of the following problem:

$$u_{xx} + u_{yy} + u_{zz} + k^2 u(x,y,z) = 0, \quad (x,y) \in \mathbb{R}^2, z > 0, \tag{1.4}$$

$$u(x,y,0) = f(x,y), \qquad (x,y) \in \mathbb{R}^2, \tag{1.5}$$

$$\lim_{r\to\infty} r(\frac{\partial u}{\partial r} - iku) = 0. \tag{1.6}$$

We wish to find the $u(x,y,z) = g(x,y) \in L^2(\mathbb{R}^2)$ from the data $f(x,y) \in L^2(\mathbb{R}^2)$ for a fixed $0 < z \le d$.

It is well known that if $f(x,y)$ has bounded support A, then the forward problem admits a unique solution $u(x,y,z)$ which is given by

$$u(x,y,z) = \int_{\mathbb{R}^2} H_z(x-\mu, y-\nu) f(\mu,\nu) d\mu d\nu := F(z) f(x,y), \tag{1.7}$$

where

$$H_z(x,y) = -\frac{1}{2\pi} \frac{\partial \Big(\frac{e^{ik\sqrt{\tau^2+x^2+y^2}}}{\sqrt{\tau^2+x^2+y^2}} \Big)}{\partial \tau} \Big|_{\tau=z},$$

and $F(z) : L^2(A) \to L^2(\mathbb{R}^2)$ is the solution operator.

The problem can be reposed in the Fourier domain. Let

$$\hat{h}(\xi,\eta) = \frac{1}{2\pi} \int_{\mathbb{R}^2} h(x,y) e^{-i(\xi x + \eta y)} dx dy \tag{1.8}$$

be the Fourier transform of the function $h(x,y) \in L^2(\mathbb{R}^2)$. The corresponding inverse Fourier transform of the function $\hat{h}(\xi,\eta)$ is given by

$$h(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{h}(\xi,\eta) e^{i(\xi x + \eta y)} d\xi d\eta. \tag{1.9}$$

By taking Fourier transform of (1.4)–(1.6) with respect to variables x and y, the solution of forward problem in frequency domain is given by

$$\hat{u}(\xi,\eta,z) = e^{-za(\xi,\eta)} \hat{f}(\xi,\eta) = \widehat{F(z)f}, \tag{1.10}$$

where $a(\xi,\eta) = \sqrt{\xi^2+\eta^2-k^2}$ if $\xi^2+\eta^2 \ge k^2$ and $a(\xi,\eta) = i\sqrt{k^2-\xi^2-\eta^2}$ if $\xi^2+\eta^2 < k^2$.

From this formula, formally we have the solution for the inverse problem

$$\hat{f}(\xi, \eta) = e^{da(\xi,\eta)}\hat{g}(\xi, \eta) \tag{1.11}$$

and

$$\hat{u}(\xi, \eta, z) = e^{(d-z)a(\xi,\eta)}\hat{g}(\xi, \eta). \tag{1.12}$$

Following [1], the plane waves with frequencies $\sqrt{\xi^2 + \eta^2} < k$ are called homogeneous waves and the plane waves with frequencies $\sqrt{\xi^2 + \eta^2} > k$ are called evanescent waves. The reconstruction regions are divided two cases:

(1) Near-field region: Corresponds to distances $d < \lambda = \frac{2\pi}{k}$, and in this case the evanescent waves are important.

(2) Far-field region: Corresponds to distances $d > \lambda = \frac{2\pi}{k}$, and in this case the homogeneous waves are important, and one can assume that the $\hat{u}(\xi, \eta, d)$ has a support given by $\mathbb{B} = \{(\xi, \eta) : \xi^2 + \eta^2 \leq k^2\}$.

The problem of inverse diffraction from plane to plane is well-known ill-posed problem in the near-field pattern and in the far-field pattern [2–7]. There are some machine learning methods for solving inverse scattering problems conveniently, such as neural network [8, 9]. In this paper, we discuss the inverse diffraction problem from the aspect of regularization theory for solving ill-posed problems. To stabilize the ill-posed problem and improve the resolution of the solution, a priori information about unknown solution is introduced [10] necessarily.

The problem is similar to the Cauchy problem for the Helmholtz equation. Regularization methods for solving Cauchy problem have been suggested by various authors. For instance, the modified Tikhonov regularisation method [11–13], the truncation method [14, 15], quasi-reversibility method [16,17], mapped regularization methods [18], operator regularization method [19], mollification method [20], fractional Tikhonov method [21], a slow-evolution-from-the-continuation-boundary (SECB) method [22], posteriori regularization [23], Spectral Galerkin method [24], etc.

A priori information about unknown solution has been proved to be essential in the analysis of ill-posed problems in mathematical physics. Otherwise, without the priori information the convergence rate of the constructed regularization method is arbitrarily slow [25–27].

Although this problem has been investigated by Sondhi [28], Bertero [29], Magnanini [10] and Santosa [30], most of the existing results are devoted to the numerical aspects, e.g., a level set method has been proposed for solving the problem in [30]. In this work, we will focus on the regularization method and the error estimate from the viewpoint of regularization for ill-posed problems. In references [22, 31], we studied this problem by slow-evolution-from-the-continuation-boundary (SECB) method and spectral method respectively. Here we propose a quasi-boundary regularization for solving this inverse problem and corresponding error estimates are derived. It is well-known that the classical Tikhonov method has an over-smoothing regularization solution and the truncated Fourier method also has one, therefore we try to give a simple method to overcome this disadvantage. The total variation method [32] is a good alternative for resolve this problem, but it is difficult to treat both numerically and theoretically. In this paper, we find out that the proposed quasi-boundary regularization has better approximation than the classical Tikhonov method in the case of noiseless data theoretically. Although this result doesn't has real meaning in practice, the quasi-boundary method has its advantage. The proposed method could give better approximation by using few cost when we want to reconstruct the non-smooth exact object. The numerical examples show the regularization method is effective and support the results.

The paper is organized as follow. In Section 2, a quasi-boundary regularization method is presented and the error bounds are proved; in Section 3, a comparison between the classical Tikhonov method and the quasi-boundary method is given; in Section 4, some numerical results are reported in order to show that the quasi-boundary method is more effective for the inverse diffraction problem.

2. The quasi-boundary regularization method

Let $g(x,y)$ denotes the exact data, and we would actually have noisy data function $g_\delta(x,y) \in L^2(\mathbb{R}^2)$, for which

$$\|g_\delta(x,y) - g(x,y)\| \le \delta, \tag{2.1}$$

where the constant $\delta > 0$ represents a bound on the measured error, $\|\cdot\|$ denotes the L^2-norm throughout this paper. An a-priori knowledge about the true solution is an essential element in the successful computation of ill-posed inverse problems [25]. We assume there exists a constant $E > 0$, such that the exact solution $u(\cdot,\cdot,0) := f(\cdot,\cdot)$ satisfies

$$\|u(\cdot,\cdot,0)\|_p \le E, \tag{2.2}$$

where $\|\cdot\|_p$ denotes the norm of Sobolev space $H^p(\mathbb{R}^2)$.

In this section, we follow the idea from Ames et al. [33] where they used a quasi-boundary method (or so-called modified boundary method) for solving a classical backward heat equation. Let us consider the problem with noisy data for the current inverse problem:

$$u_{xx} + u_{yy} + u_{zz} + k^2u(x,y,z) = 0, \qquad (x,y) \in \mathbb{R}^2, z > 0, \tag{2.3}$$

$$u(x,y,d) = g_\delta(x,y), \qquad (x,y) \in \mathbb{R}^2, \tag{2.4}$$

$$\lim_{r\to\infty} r(\frac{\partial u}{\partial r} - iku) = 0. \tag{2.5}$$

We try to use a quasi-boundary regularization method to solve the problem (2.3)–(2.5), i.e., let us consider the following modified boundary problem:

$$(v_\delta^\alpha)_{xx} + (v_\delta^\alpha)_{yy} + (v_\delta^\alpha)_{zz} + k^2(v_\delta^\alpha)(x,y,z) = 0, \qquad (x,y) \in \mathbb{R}^2, z > 0, \tag{2.6}$$

$$\alpha(v_\delta^\alpha)(x,y,0) + (v_\delta^\alpha)(x,y,d) = g_\delta(x,y), \qquad (x,y) \in \mathbb{R}^2, \tag{2.7}$$

$$\lim_{r\to\infty} r(\frac{\partial(v_\delta^\alpha)}{\partial r} - ik(v_\delta^\alpha)) = 0, \tag{2.8}$$

where $\alpha > 0$ is a small parameter.

By the technique of Fourier transform, we can get the solution of the modified boundary problem in the frequency domain:

$$(\hat{v_\delta^\alpha})(\xi,\eta,z) = \frac{e^{(d-z)a(\xi,\eta)}}{1 + \alpha e^{da(\xi,\eta)}}\hat{g}_\delta(\xi,\eta), \tag{2.9}$$

where $a(\xi,\eta)$ is the same as in (1.10).

Proposition 2.1. Let $g_\delta \in L^2(\mathbb{R}^2)$ and make the solution of problem (2.3)–(2.5) exist, then the solution of problem (2.6)–(2.8) approximates the solution of problem (2.3)–(2.5) in L^2-norm when $\alpha \to 0$.

Proof. The solution of (2.3)–(2.5) is given by

$$\hat{u_\delta}(\xi,\eta,z) = e^{(d-z)a(\xi,\eta)}\hat{g}_\delta(\xi,\eta).$$

From (2.9) we can easily see that as $\alpha \to 0$, $\hat{v_\delta^\alpha} \to \hat{u_\delta}$ and $\|\hat{v_\delta^\alpha} - \hat{u_\delta}\| \to 0$, uniformly in z.

Lemma 2.1. If $0 \le z < d$, $0 < \alpha < 1$, then

$$\sup_{\xi^2+\eta^2\ge k^2} \Big|\frac{e^{(d-z)\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}\Big| \le \alpha^{\frac{z}{d}-1}. \tag{2.10}$$

$$\sup_{\xi^2+\eta^2< k^2} \Big|\frac{e^{i(d-z)\sqrt{k^2-\xi^2-\eta^2}}}{1+\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}\Big| \le \frac{1}{1-\alpha}. \tag{2.11}$$

Proof. If $\xi^2+\eta^2 \ge k^2$, by inequality (see Lemma 3.1 in [16])

$$\sup_{s>0} \frac{e^{sz}}{1+\alpha e^{sd}} \le \alpha^{-\frac{z}{d}}, \tag{2.12}$$

we have

$$\sup_{\xi^2+\eta^2\ge k^2} \Big|\frac{e^{(d-z)\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}\Big| \le \alpha^{-\frac{d-z}{d}} = \alpha^{\frac{z}{d}-1}.$$

If $\xi^2+\eta^2 < k^2$, we have

$$\sup_{\xi^2+\eta^2< k^2} \Big|\frac{e^{i(d-z)\sqrt{k^2-\xi^2-\eta^2}}}{1+\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}\Big| \le \frac{1}{1-|\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}|} = \frac{1}{1-\alpha}.$$

Proposition 2.2. Let $g_\delta \in L^2(\mathbb{R}^2)$, then the solution of problem (2.6)–(2.8) continuously depends on the data in L^2-norm if the parameter α is selected appropriately.

Proof. For problem (2.6)–(2.8), we have the following conclusion: If any two functions $g_{\delta,1}$ and $g_{\delta,2}$ satisfy $\|g_{\delta,1} - g_{\delta,2}\| \le \varepsilon$, let $v_\delta^{\alpha,1}$ and $v_\delta^{\alpha,2}$ be the corresponding solutions, respectively, setting $\alpha = O(\varepsilon)$, then $\|v_\delta^{\alpha,1} - v_\delta^{\alpha,2}\| \to 0$, as $\varepsilon \to 0$. In fact, by Parseval's identity and Lemma 2.1, we have

$$\begin{aligned}
&\|v_\delta^{\alpha,1} - v_\delta^{\alpha,2}\| \\
=&\|\widehat{v_\delta^{\alpha,1}} - \widehat{v_\delta^{\alpha,2}}\| = \|\frac{e^{(d-z)a(\xi,\eta)}}{1+\alpha e^{da(\xi,\eta)}}(\widehat{g_{\delta,1}} - \widehat{g_{\delta,2}})\| \\
=&\Big(\int_{\xi^2+\eta^2\ge k^2} \Big|\frac{e^{(d-z)\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}(\widehat{g_{\delta,1}} - \widehat{g_{\delta,2}})\Big|^2 d\xi d\eta + \int_{\xi^2+\eta^2<k^2} \Big|\frac{e^{i(d-z)\sqrt{k^2-\xi^2-\eta^2}}}{1+\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}(\widehat{g_{\delta,1}} - \widehat{g_{\delta,2}})\Big|^2 d\xi d\eta\Big)^{1/2} \\
\le& \sup_{\xi^2+\eta^2\ge k^2} \Big|\frac{e^{(d-z)\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}\Big| \|\widehat{g_{\delta,1}} - \widehat{g_{\delta,2}}\| + \sup_{\xi^2+\eta^2<k^2} \Big|\frac{e^{i(d-z)\sqrt{k^2-\xi^2-\eta^2}}}{1+\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}\Big| \|\widehat{g_{\delta,1}} - \widehat{g_{\delta,2}}\| \\
\le& \sup_{\xi^2+\eta^2\ge k^2} \Big|\frac{e^{(d-z)\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}\Big|\varepsilon + \sup_{\xi^2+\eta^2<k^2} \Big|\frac{e^{i(d-z)\sqrt{k^2-\xi^2-\eta^2}}}{1+\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}\Big|\varepsilon \\
\le&\alpha^{\frac{z}{d}-1}\varepsilon + \frac{\varepsilon}{1-\alpha}.
\end{aligned} \tag{2.13}$$

Hence if $\alpha = O(\varepsilon)$, then

$$\|v_\delta^{\alpha,1} - v_\delta^{\alpha,2}\| \to 0, \text{ for } \varepsilon \to 0.$$

For the inverse problem, we can obtain the error estimate between the regularized solution $v_\delta^\alpha(x, y, z)$ and the exact solution $u(x, y, z)$.

Theorem 2.1. Suppose $u(x, y, z)$ is the solution of the problem (1.1)–(1.3) with the exact data $g(x, y) \in L^2(\mathbb{R}^2)$ and $v_\delta^\alpha(x, y, z)$ is the regularization solution whose Fourier transform is given by (2.9) with the noisy data $g_\delta(x, y) \in L^2(\mathbb{R}^2)$, let (2.1) and the a-priori condition (2.2) hold with $p = 0$. If $\alpha = \frac{\delta}{E}$, then for a fixed $0 < z < d$ we have the error estimate

$$\|v_\delta^\alpha(\cdot, \cdot, z) - u(\cdot, \cdot, z)\| \le 2\delta^{\frac{z}{d}} E^{1-\frac{z}{d}}(1 + o(1)), \ for\, \delta \to 0. \tag{2.14}$$

Proof. We take two steps to prove it.

Step I: Convergence. We need to prove the regularization solution v^α whose Fourier transform is given by (2.9) approaches the exact solution u with the same exact data g.

By Parseval's identity, we have

$$\begin{aligned} &\|v^\alpha(\cdot, \cdot, z) - u(\cdot, \cdot, z)\| = \|\hat{v^\alpha}(\cdot, \cdot, z) - \hat{u}(\cdot, \cdot, z)\| \\ =&\|\frac{e^{(d-z)a(\xi,\eta)}}{1 + \alpha e^{da(\xi,\eta)}}\hat{g}(\xi, \eta) - e^{(d-z)a(\xi,\eta)}\hat{g}(\xi, \eta)\| \\ =&\|\alpha\frac{e^{(2d-z)a(\xi,\eta)}}{1 + \alpha e^{da(\xi,\eta)}}\hat{g}(\xi, \eta)\|. \end{aligned} \tag{2.15}$$

From (1.12), there holds

$$\hat{g}(\xi, \eta) = \frac{\hat{u}(\xi, \eta, 0)}{e^{da(\xi,\eta)}}.$$

Therefore,

$$\begin{aligned} &\|\hat{v^\alpha}(\cdot, \cdot, z) - \hat{u}(\cdot, \cdot, z)\| \\ =&\alpha\|\frac{e^{(d-z)a(\xi,\eta)}}{1 + \alpha e^{da(\xi,\eta)}}\hat{u}(\xi, \eta, 0)\| \\ =&\alpha\Big(\int_{\xi^2+\eta^2\ge k^2}|\frac{e^{(d-z)\sqrt{\xi^2+\eta^2-k^2}}}{1 + \alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}\hat{u}(\xi, \eta, 0)|^2 d\xi d\eta \\ &+ \int_{\xi^2+\eta^2<k^2}|\frac{e^{i(d-z)\sqrt{k^2-\xi^2-\eta^2}}}{1 + \alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}\hat{u}(\xi, \eta, 0)|^2 d\xi d\eta\Big)^{1/2} \\ \le&\alpha \sup_{\xi^2+\eta^2\ge k^2}|\frac{e^{(d-z)\sqrt{\xi^2+\eta^2-k^2}}}{1 + \alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}|\Big(\int_{(\xi,\eta)\in\mathbb{R}^2}|\hat{u}(\xi, \eta, 0)|^2 d\xi d\eta\Big)^{1/2} \\ &+ \alpha \sup_{\xi^2+\eta^2<k^2}|\frac{e^{i(d-z)\sqrt{k^2-\xi^2-\eta^2}}}{1 + \alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}|\Big(\int_{(\xi,\eta)\in\mathbb{R}^2}|\hat{u}(\xi, \eta, 0)|^2 d\xi d\eta\Big)^{1/2}. \end{aligned} \tag{2.16}$$

By the a-priori assumption (2.2) and Lemma 2.1, (2.16) yields

$$\|\hat{v^\alpha}(\cdot, \cdot, z) - \hat{u}(\cdot, \cdot, z)\| \le \alpha E\alpha^{\frac{z}{d}-1} + \frac{\alpha E}{1 - \alpha}.$$

If $\alpha = \delta/E$, we have

$$\|\hat{v^{\alpha}}(\cdot,\cdot,z) - \hat{u}(\cdot,\cdot,z)\| \le \delta^{\frac{z}{d}} E^{1-\frac{z}{d}} + \frac{\delta}{1-\delta/E} = \delta^{\frac{z}{d}} E^{1-\frac{z}{d}}(1+o(1)),\ for\ \delta \to 0. \tag{2.17}$$

Step II: Stability. We now prove that the regularization solution is dependent continuously on the data. Using the Parseval's equality and Lemma 2.1, we obtain

$$\begin{aligned}
&\|v_{\delta}^{\alpha}(\cdot,\cdot,z) - v^{\alpha}(\cdot,\cdot,z)\| \\
=&\|\hat{v_{\delta}^{\alpha}}(\cdot,\cdot,z) - \hat{v^{\alpha}}(\cdot,\cdot,z)\| = \|\frac{e^{(d-z)a(\xi,\eta)}}{1+\alpha e^{da(\xi,\eta)}}(\hat{g_{\delta}} - \hat{g})\| \\
=&\Big(\int_{\xi^2+\eta^2\ge k^2} |\frac{e^{(d-z)\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}(\hat{g_{\delta}} - \hat{g})|^2 d\xi d\eta \\
&+ \int_{\xi^2+\eta^2< k^2} |\frac{e^{i(d-z)\sqrt{k^2-\xi^2-\eta^2}}}{1+\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}(\hat{g_{\delta}} - \hat{g})|^2 d\xi d\eta\Big)^{1/2} \\
\le& \sup_{\xi^2+\eta^2\ge k^2} |\frac{e^{(d-z)\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}|\delta + \sup_{\xi^2+\eta^2< k^2} |\frac{e^{i(d-z)\sqrt{k^2-\xi^2-\eta^2}}}{1+\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}|\delta \\
\le&\alpha^{\frac{z}{d}-1}\delta + \frac{\delta}{1-\alpha}.
\end{aligned}$$

Noting $\alpha = \delta/E$, we have

$$\|\hat{v_{\delta}^{\alpha}}(\cdot,\cdot,z) - \hat{v^{\alpha}}(\cdot,\cdot,z)\| \le \delta^{\frac{z}{d}} E^{1-\frac{z}{d}} + \frac{\delta}{1-\delta/E} = \delta^{\frac{z}{d}} E^{1-\frac{z}{d}}(1+o(1)),\ for\ \delta \to 0. \tag{2.18}$$

The conclusion of the theorem now follows immediately by using the triangle inequalities (2.17) and (2.18).

We find that the error estimate (2.14) is not valid for the location at $z = 0$. This is common in ill-posed problems. If a stronger a-priori condition is added, for example, $\|u(\cdot,\cdot,0)\|_p \le E$, with $p > 0$, where $\|\cdot\|_p$ denotes the norm of Sobolev space $H^p(\mathbb{R}^2)$, then the convergence order is logarithmic.

Theorem 2.2. Suppose that $f(x,y) := u(x,y,0)$ is the exact solution with exact data $g(x,y) \in L^2(\mathbb{R}^2)$, (2.9) is the regularization solution with $z = 0$ and noisy data $g_{\delta}(x,y) \in L^2(\mathbb{R}^2)$, respectively, let (2.1) and (2.2) hold with $p > 0$, if $\alpha = (\frac{\delta}{E})^r$ with $0 < r < 1$, then we have for $p > 0$,

$$\|v_{\delta}^{\alpha}(\cdot,\cdot,0) - u(\cdot,\cdot,0)\| \le O((\ln\frac{E}{\delta})^{-p}),\quad for\ \delta \to 0. \tag{2.19}$$

Proof. (I) (Stability) From Parseval's identity and Lemma 2.1, we have

$$\begin{aligned}
&\|v_{\delta}^{\alpha}(\cdot,\cdot,0) - v^{\alpha}(\cdot,\cdot,0)\| = \|\hat{v}_{\delta}^{\alpha}(\cdot,\cdot,0) - \hat{v}^{\alpha}(\cdot,\cdot,0)\| \\
=&\|\frac{e^{da(\xi,\eta)}}{1+\alpha e^{da(\xi,\eta)}}\hat{g}_{\delta}(\xi) - \frac{e^{da(\xi,\eta)}}{1+\alpha e^{da(\xi,\eta)}}\hat{g}(\xi)\| = \|\frac{e^{da(\xi,\eta)}}{1+\alpha e^{da(\xi,\eta)}}(\hat{g}_{\delta}(\xi) - \hat{g}(\xi))\| \\
=&\Big(\int_{\xi^2+\eta^2\ge k^2} |\frac{e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}(\hat{g_{\delta}} - \hat{g})|^2 d\xi d\eta + \int_{\xi^2+\eta^2< k^2} |\frac{e^{id\sqrt{k^2-\xi^2-\eta^2}}}{1+\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}(\hat{g_{\delta}} - \hat{g})|^2 d\xi d\eta\Big)^{1/2}
\end{aligned}$$

$$\leq \sup_{\xi^2+\eta^2\geq k^2} |\frac{e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}| \|\hat{g}_\delta - \hat{g}\| + \sup_{\xi^2+\eta^2<k^2} |\frac{e^{id\sqrt{k^2-\xi^2-\eta^2}}}{1+\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}| \|\hat{g}_\delta - \hat{g}\|$$
$$\leq \sup_{\xi^2+\eta^2\geq k^2} |\frac{e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}|\delta + \sup_{\xi^2+\eta^2<k^2} |\frac{e^{id\sqrt{k^2-\xi^2-\eta^2}}}{1+\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}|\delta$$
$$\leq \frac{\delta}{\alpha} + \frac{\delta}{1-\alpha}.$$

If $\alpha = (\frac{\delta}{E})^r$ with $0 < r < 1$, then we have

$$\|v_\delta^\alpha(\cdot,\cdot,0) - v^\alpha(\cdot,\cdot,0)\| \leq \delta^{1-r}E^r + \frac{\delta}{1-\delta^r E^{-r}} = \delta^{1-r}E^r(1+o(1)), \quad for\ \delta \to 0. \tag{2.20}$$

(II) (Convergence) Via Parseval's identity, we have

$$\|v^\alpha(\cdot,\cdot,0) - u(\cdot,\cdot,0)\| = \|\hat{v}^\alpha(\cdot,\cdot,0) - \hat{u}(\cdot,\cdot,0)\|$$
$$= \|\frac{e^{da(\xi,\eta)}}{1+\alpha e^{da(\xi,\eta)}}\hat{g}(\xi,\eta) - e^{da(\xi,\eta)}\hat{g}(\xi,\eta)\| = \|\frac{\alpha e^{da(\xi,\eta)}}{1+\alpha e^{da(\xi,\eta)}}\hat{u}(\xi,\eta,0)\|$$
$$= \|\frac{\alpha e^{da(\xi,\eta)}(1+\xi^2+\eta^2)^{-p/2}}{1+\alpha e^{da(\xi,\eta)}}(1+\xi^2+\eta^2)^{p/2}\hat{u}(\xi,\eta,0)\|$$
$$= (\int_{\xi^2+\eta^2\geq k^2} |\frac{\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}(1+\xi^2+\eta^2)^{-p/2}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}(1+\xi^2+\eta^2)^{p/2}\hat{u}(\xi,\eta,0)|^2 d\xi d\eta$$
$$+ \int_{\xi^2+\eta^2<k^2} |\frac{\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}(1+\xi^2+\eta^2)^{-p/2}}{1+\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}(1+\xi^2+\eta^2)^{p/2}\hat{u}(\xi,\eta,0)|^2 d\xi d\eta)^{1/2}$$
$$\leq E \sup_{\xi^2+\eta^2\geq k^2} |\frac{\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}(1+\xi^2+\eta^2)^{-p/2}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}| + E \sup_{\xi^2+\eta^2<k^2} |\frac{\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}(1+\xi^2+\eta^2)^{-p/2}}{1+\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}|$$
$$:= I_1 + I_2.$$

For I_2, by (2.11) and $(1+\xi^2+\eta^2)^{-p/2} \leq 1$, we obtain

$$I_2 = E \sup_{\xi^2+\eta^2<k^2} |\frac{\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}(1+\xi^2+\eta^2)^{-p/2}}{1+\alpha e^{id\sqrt{k^2-\xi^2-\eta^2}}}| \leq \frac{\alpha E}{1-\alpha} = \frac{\delta^r E^{1-r}}{1-\delta^r E^{-r}}. \tag{2.21}$$

Next we estimate the term $I_1 = E \sup_{\xi^2+\eta^2\geq k^2} |\frac{\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}(1+\xi^2+\eta^2)^{-p/2}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}|$. Denote $\zeta := \sqrt{\xi^2+\eta^2-k^2}$, therefore, we need to estimate the term

$$|\frac{\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}(1+\xi^2+\eta^2)^{-p/2}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}| = \frac{\alpha e^{d\zeta}(1+k^2+\zeta^2)^{-p/2}}{1+\alpha e^{d\zeta}}.$$

It is divided into two cases for ζ.

Case I. $\zeta \leq \frac{1}{d}\ln(\frac{1}{\sqrt{\alpha}})$. It yields $\alpha e^{d\zeta} \leq \sqrt{\alpha}$. From

$$\frac{\alpha e^{d\zeta}(1+k^2+\zeta^2)^{-p/2}}{1+\alpha e^{d\zeta}} \leq \alpha e^{d\zeta} \leq \sqrt{\alpha} = \delta^{r/2}E^{-r/2}.$$

Case II. $\zeta \geq \frac{1}{d}\ln(\frac{1}{\sqrt{\alpha}})$. Firstly, we have

$$\frac{\alpha e^{d\zeta}(1+k^2+\zeta^2)^{-p/2}}{1+\alpha e^{d\zeta}} \leq \frac{\alpha e^{d\zeta}(1+k^2+\zeta^2)^{-p/2}}{\alpha e^{d\zeta}} \leq \zeta^{-p}.$$

Now we need to estimate ζ^{-p} under the condition $\zeta \geq \frac{1}{d}\ln(\frac{1}{\sqrt{\alpha}})$.

If $\zeta \geq \frac{1}{d}\ln(\frac{1}{\sqrt{\alpha}})$, we have

$$\zeta^{-p} \leq (\frac{1}{d}\ln(\frac{1}{\sqrt{\alpha}}))^{-p} = (\frac{r}{2d}\ln\frac{E}{\delta})^{-p}.$$

Therefore, combining (I) Stability and (II) Convergence, and using the triangle inequality, we have

$$\|v_\delta^\alpha(\cdot,\cdot,0) - u(\cdot,\cdot,0)\| \leq O((\ln\frac{E}{\delta})^{-p}), \quad for\ \delta \to 0.$$

Remark 2.1. In this paper, we only consider the reconstruction of $\chi_A(x,y)$, where the domain A is a two-dimensional cube-like shape and belongs to the Sobolev space $H^s(\mathbb{R}^2)$ with $s < 1/2$. Therefore, in this theorem when $0 < p < 1/2$, the result is convergence and when $p < 0$, the result is divergence.

3. Quasi-boundary methods versus Tikhonov method

In this section, we give the comparison between the quasi-boundary method and Tikhonov method in the situation of free noise. Firstly, we concentrate on the Tikhonov regularization method [34]. The method consists of looking for the solution for the inverse problem, and minimizes the quadratic functional, i.e., finding a minimizer $u_{\alpha,tik}^\delta(x,y,z)$ such that

$$u_{\alpha,tik}^\delta(x,y,z) = \text{argmin}_u\{\|F(z)u(\cdot,\cdot,z) - g_\delta(\cdot,\cdot)\|^2 + \alpha\|u(\cdot,\cdot,0)\|^2\}, \tag{3.1}$$

where $\alpha = \delta^2/E^2$ with δ and E given by (2.1) and (2.2), respectively, $F(z) : u(x,y,z) \to g(x,y)$ is the forward operator of the problem. By Parseval's identity and according to (1.12), the variational problem becomes

$$\text{argmin}_u\{\|e^{-(d-z)a(\xi,\eta)}\hat{u}(\xi,\eta,z) - \hat{g}_\delta(\xi,\eta)\|^2 + \alpha\|e^{za(\xi,\eta)}\hat{u}(\xi,\eta,z)\|^2\}. \tag{3.2}$$

Let $\hat{u}_{\alpha,tik}^\delta(\xi,\eta,z)$ be the solution of above problem, then it satisfies the Euler equation

$$[e^{-2(d-z)a(\xi,\eta)} + \alpha e^{2za(\xi,\eta)}]\hat{u}_{\alpha,tik}^\delta(\xi,\eta,z) = e^{-(d-z)a(\xi,\eta)}\hat{g}_\delta(\xi,\eta). \tag{3.3}$$

The Tikhonov regularization solution $\widehat{u}_{\alpha,tik}^\delta(\xi,\eta,z)$ in frequency domain can be given:

$$\hat{u}_{\alpha,tik}^\delta(\xi,\eta,z) = \frac{e^{(d-z)a(\xi,\eta)}}{1+\alpha e^{2da(\xi,\eta)}}\hat{g}_\delta(\xi,\eta). \tag{3.4}$$

The restoration of Tikhonov method is then obtained by inverse Fourier transforming $\hat{u}_{\alpha,tik}^\delta(\xi,\eta,z)$. Under the conditions (2.1) and (2.2), we can prove the following error estimate for Tikhonov method in the same way:

$$\|u_{\alpha,tik}^\delta(\cdot,\cdot,z) - u(\cdot,\cdot,z)\| \leq 2\delta^{\frac{z}{d}}E^{\frac{d-z}{d}}(1+o(1)),\ for\ \delta \to 0. \tag{3.5}$$

It is natural for us to compare the quasi-boundary regularization method with the Tikhonov regularization when $\xi^2+\eta^2 \geq k^2$. Now denote the norm of the regularization error $\|v^\alpha - u\|^2$ associated with the quasi-boundary regularization as ε_1. Denote the norm of the regularization error $\|u_{\alpha,tik} - u\|^2$ associated with the Tikhonov regularization as ε. Thus, we have

$$\varepsilon_1 = \|v^\alpha - u\|^2 = \int_{\mathbb{R}^2} \left(\frac{\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}\right)^2 e^{2(d-z)\sqrt{\xi^2+\eta^2-k^2}} |\hat{g}(\xi,\eta)|^2 d\xi d\eta.$$

$$\varepsilon = \|u_{\alpha,tik} - u\|^2 = \int_{\mathbb{R}^2} \left(\frac{\alpha e^{2d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{2d\sqrt{\xi^2+\eta^2-k^2}}}\right)^2 e^{2(d-z)\sqrt{\xi^2+\eta^2-k^2}} |\hat{g}(\xi,\eta)|^2 d\xi d\eta.$$

Now, the difference in regularization errors for Tikhonov method and the quasi-boundary method is given by

$$\begin{aligned} d :&= \varepsilon - \varepsilon_1 \\ &= \int_{\mathbb{R}^2} \left[\left(\frac{\alpha e^{2d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{2d\sqrt{\xi^2+\eta^2-k^2}}}\right)^2 - \left(\frac{\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}\right)^2\right] e^{2(d-z)\sqrt{\xi^2+\eta^2-k^2}} |\hat{g}(\xi,\eta)|^2 d\xi d\eta. \end{aligned}$$

Obviously, the function $H(y) := \frac{y}{1+\alpha y}$ for a fixed $\alpha > 0$ is an increasing function with respect to $y > 0$. Therefore, $d > 0$.

Then we have the following result.

Conclusion 3.1. In the absence of noise in the data, the quasi-boundary regularization method is more accurate than Tikhonov method in the sense of L^2-error when the same parameter α is used.

Remark 3.1. From the above comparison, we can give a general regularization method with noisy data based on the quasi-boundary method

$$\widehat{u^\alpha_{\gamma,\delta}}(\xi,\eta,z) = \frac{e^{(d-z)a(\xi,\eta)}}{1+\alpha e^{\gamma d a(\xi,\eta)}} \hat{g}_\delta(\xi,\eta), \text{ with } 1 \leq \gamma \leq 2. \tag{3.6}$$

Obviously, for $\gamma = 1$, it is the quasi-boundary method. For $\gamma = 2$, it is the Tikhonov method. For $1 < \gamma < 2$, we call it the quasi-boundary-Tikhonov method.

4. Numerical examples

In this section in numerics, we consider the numerical computation in both cases of evanescent waves and homogeneous waves. In numerical computation, we refer to the following formulas. The quasi-boundary regularization solution:

$$\begin{aligned} (\hat{v^\alpha_\delta})(\xi,\eta,0) &= \tfrac{e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}} \hat{g}_\delta(\xi,\eta), \text{ if } \xi^2+\eta^2 \geq k^2, \\ (\hat{v^\alpha_\delta})(\xi,\eta,0) &= \tfrac{e^{di\sqrt{k^2-\xi^2-\eta^2}}}{1+\alpha e^{di\sqrt{k^2-\xi^2-\eta^2}}} \hat{g}_\delta(\xi,\eta), \text{ if } \xi^2+\eta^2 < k^2. \end{aligned} \tag{4.1}$$

The classical Tikhonov solution:

$$\hat{u}^\delta_{\alpha,tik}(\xi,\eta,0) = \tfrac{e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{2d\sqrt{\xi^2+\eta^2-k^2}}} \hat{g}_\delta(\xi,\eta), \text{ if } \xi^2+\eta^2 \geq k^2,$$

$$\hat{u}^{\delta}_{\alpha,tik}(\xi,\eta,0) = \frac{e^{di\sqrt{k^2-\xi^2-\eta^2}}}{1+\alpha e^{2di\sqrt{k^2-\xi^2-\eta^2}}}\hat{g}_{\delta}(\xi,\eta), \text{ if } \xi^2+\eta^2<k^2. \tag{4.2}$$

In this section, we describe an experiment to illustrate the ability of the quasi-boundary regularization method for solving the inverse problem. Although the inverse problem is formulated in an unbounded domain in the (x, y) plane, we are interesting in the domain $(x, y) \in [0, 1]\times[0, 1]$. This is reasonable because the problem can be solved by periodic extension to (x, y) plane.

Define the discrete L^2 norm of $f(x, y)$ by

$$\|f\| = \left(n^{-2}\sum_{j,k=1}^{n} f(x_j, y_k)^2\right)^{\frac{1}{2}}, \tag{4.3}$$

where n is the total number of sampled points. In this section, we take $n = 100$.

In order to measure the accuracy of numerical results, we define the discrete L^2-norm error e_f for the exact solution f between the approximate solution f_a as follow:

$$e_f = \left(n^{-2}\sum_{j,k=1}^{n} |f(x_j, y_k) - f_a(x_j, y_k)|^2\right)^{\frac{1}{2}}. \tag{4.4}$$

The noise was added to $g(x, y)$ by setting $\delta(x_j, t_k) = \sigma\, r_{jk} \max\{g(x_j, y_k)\})$, where σ denotes the noise level and r_{jk} a random number drawn from a uniform distribution in the range $[-1, 1]$. In numerical experiment, we use (4.3) to compute $\delta = \|g - g_\delta\|$. In this section, we used the discrete L^2-norm E of the exact solution $f(x, y)$ and the δ to compute the signal-to-noise ratio E/δ. In computation, according to the theoretical result the regularization parameters of the quasi-boundary method and Tikhonov method are chosen as $\alpha = \delta/E$ and $\alpha = \delta^2/E^2$ respectively when the noise is added if necessary.

We shall illustrate the reconstruction methods (4.1) and (4.2) with different numerical examples. The Fourier formulas (4.1) and (4.2) are based on FFT algorithm.

Example 1. We choose the aperture A in the shape of the letter 'L' for the test. The characteristic function as shown in Figure 1a on the the domain A is to be constructed.

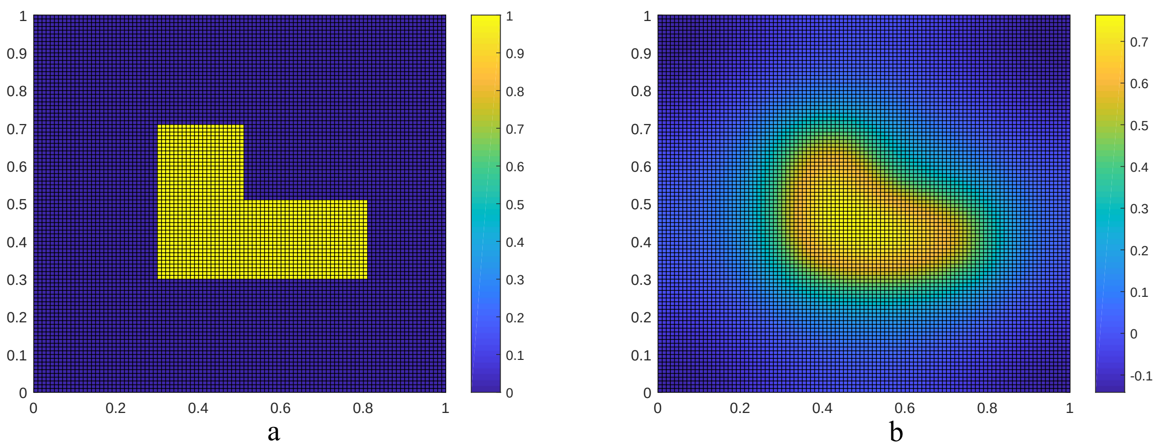


Figure 1. (a) The original A, (b) The data g.

Convergence of the proposed method in near field pattern: In this experiment where $d = 0.1$ and the wave-number $k = 2\pi$ and $E = 0.35$, first we investigate the effectiveness of the proposed method, i.e., the convergence of the quasi-boundary method. Figure 1b displays data g which is a very blurred picture in the diffraction field. In Figure 2a,b, the $\sigma = 0.01$ is changed to $\sigma = 0.0001$, we can see the reconstructed result become better.

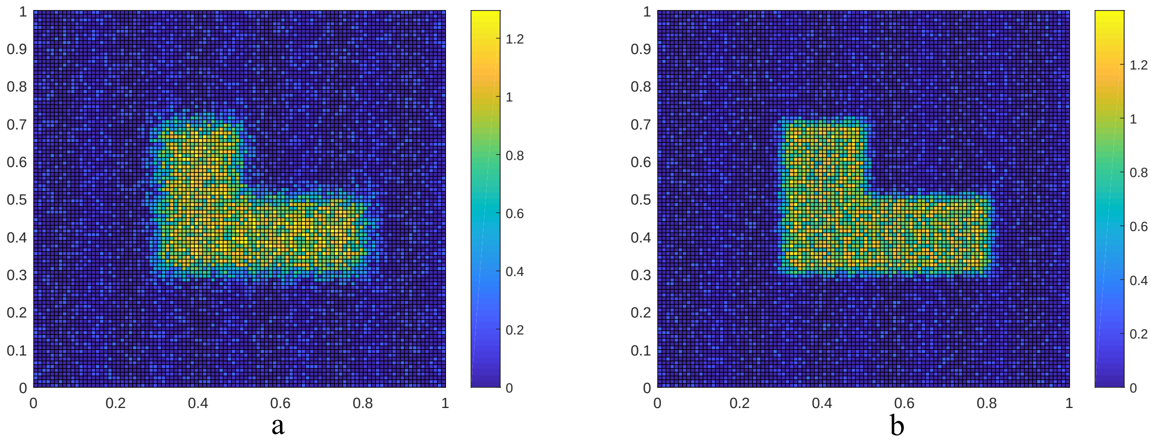


Figure 2. Quasi-boundary method (near field pattern): (a) $\alpha = 2 * 10^{-2}$, (b) $\alpha = 2 * 10^{-4}$.

Convergence of the proposed method in far field pattern: In this experiment where $d = 0.2$ and the wave-number $k = 20\pi$ and $E = 0.35$, we investigate the the convergence of the quasi-boundary method. In Figure 3a,b, the $\sigma = 0.01$ is changed to $\sigma = 0.0001$, we can see the reconstructed result become better, but the rate is very slow.

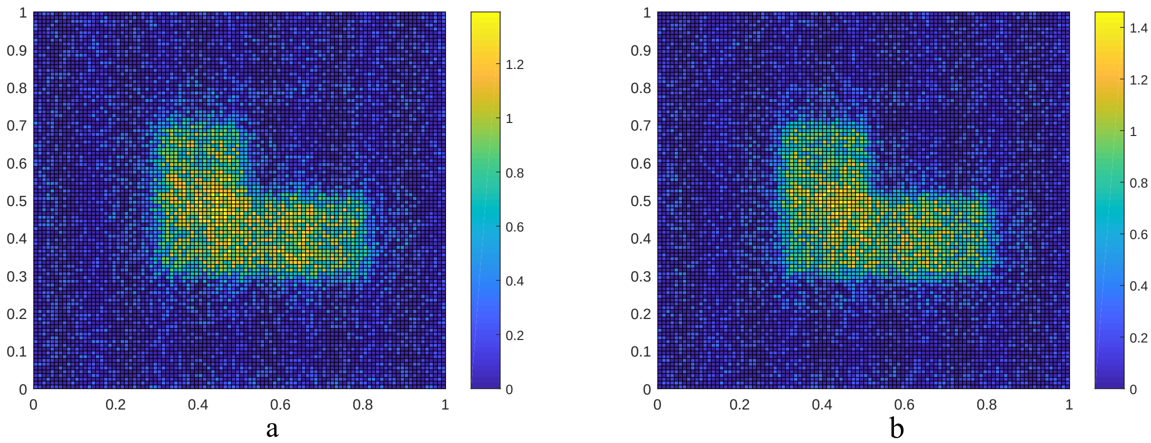
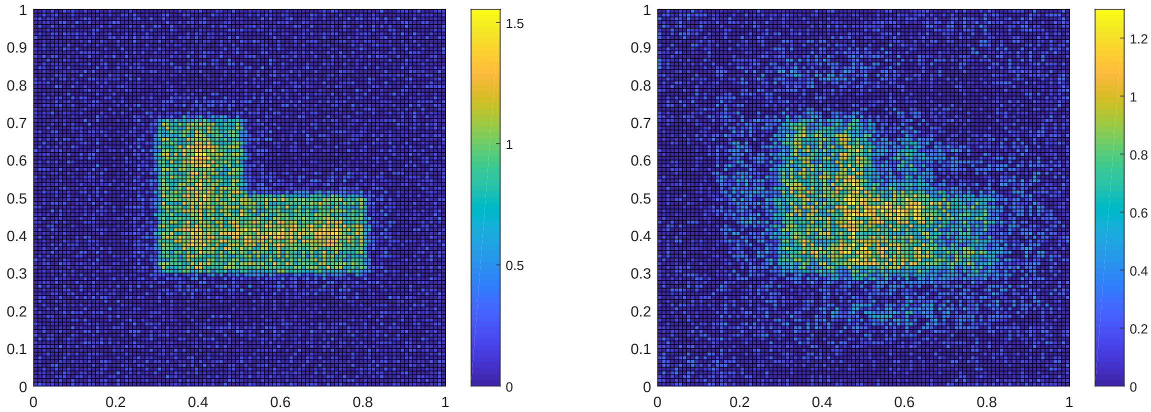


Figure 3. Quasi-boundary method (far field pattern): (a) $\alpha = 2 * 10^{-2}$, (b) $\alpha = 2 * 10^{-4}$.

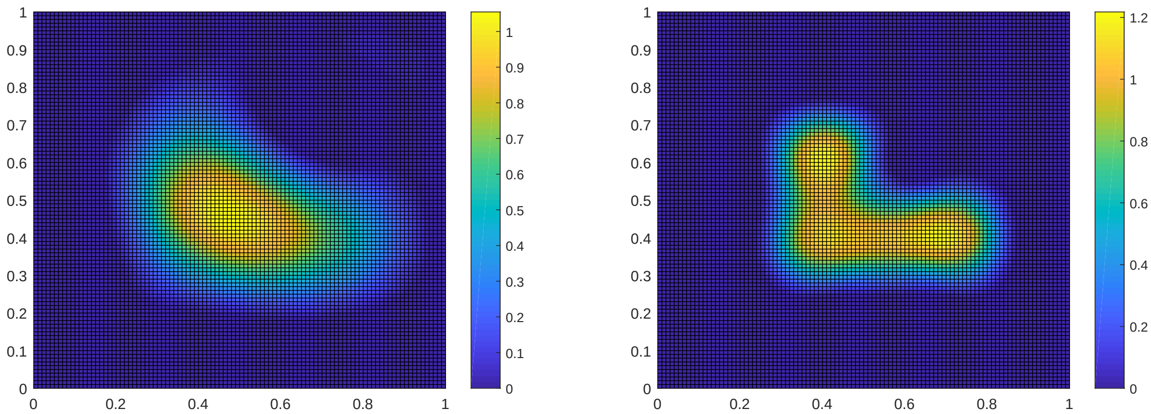
From near field pattern to far field pattern: In this experiment the wave-number $k = 20\pi$, $\lambda = 0.1$, $E = 0.35$ and $\sigma = 0.01$, we investigate the result when we change the d. In Figure 4a,b, the $d = 0.05$ (near field) is changed to $d = 0.5$ (far field), we can see the reconstructed result become worse.



a b

Figure 4. Quasi-boundary method: (a) Near field pattern: $\alpha = 2 * 10^{-2}$, (b) Far field pattern: $\alpha = 2 * 10^{-2}$.

Quasi-boundary regularization method vs Tikhonov method: In this experiment, we take $d = 0.6$ and the wave-number $k = 1$, $\sigma = 0.01$. In this case, to illustrate the Conclusion 3.1, firstly we add no perturbation to the data $g(x, y)$. For the sake of fairness, in the numerical tests, the regularization parameters α is taken as $\alpha = 1 * 10^{-8}$ in the methods. Figure 5 corresponds to the results by the Tikhonov method and quasi-boundary method. From Figure 5, obviously the quasi-boundary regularization method shows better approximation than Tikhonov method. A lot of experiments show the same result. Here, we don't list the plots.



a b

Figure 5. (a) Tikhonov regularization with $\alpha = 1 * 10^{-8}$, (b) Quasi-boundary method with $\alpha = 1 * 10^{-8}$.

Example 2. The aperture A consists of two independent squares. The characteristic function to be constructed and the data g is displayed in Figure 6. In this example, 1% relative error is added to the data $g(x, y)$ at the sampled points in the simulation as the Example 1.

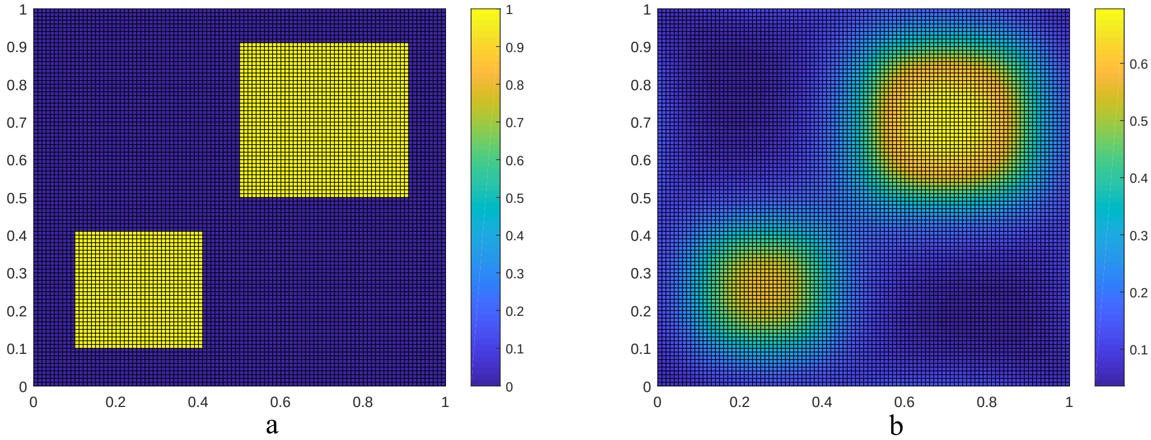


Figure 6. (a) The original A, (b) The data g.

According to [1], the Fourier truncated method is formulated as the following:

$$(\hat{w_\delta^\alpha})(\xi,\eta,0) = \frac{e^{d\sqrt{\xi^2+\eta^2-k^2}}}{1+\alpha e^{d\sqrt{\xi^2+\eta^2-k^2}}}\hat{g_\delta}(\xi,\eta), \text{ if } k \le \sqrt{\xi^2+\eta^2} \le k_{eff},$$
$$(\hat{w_\delta^\alpha})(\xi,\eta,0) = 0, \text{ if } \sqrt{\xi^2+\eta^2} > k_{eff}, \tag{4.5}$$

where $k_{eff} = k[1 + \frac{1}{(kd)^2}\ln^2(E/\delta)]$.

From this formula, we can see that the truncated Fourier regularized solution is a bandlimited function in the frequency, and the solution in the time field is analytic function.

Figure 7 corresponds to the results by the truncated Fourier method and quasi-boundary method, where the distance from receiving plane to the screen is $d = 0.1$ with the wave-number $k = 5.5$.

In this numerical test, according to the theoretical result the regularization parameters of the quasi-boundary method and truncated Fourier method are chosen as $\alpha = \delta/E$ and k_{eff}. From Figure 7, the truncated Fourier method shows its instability.

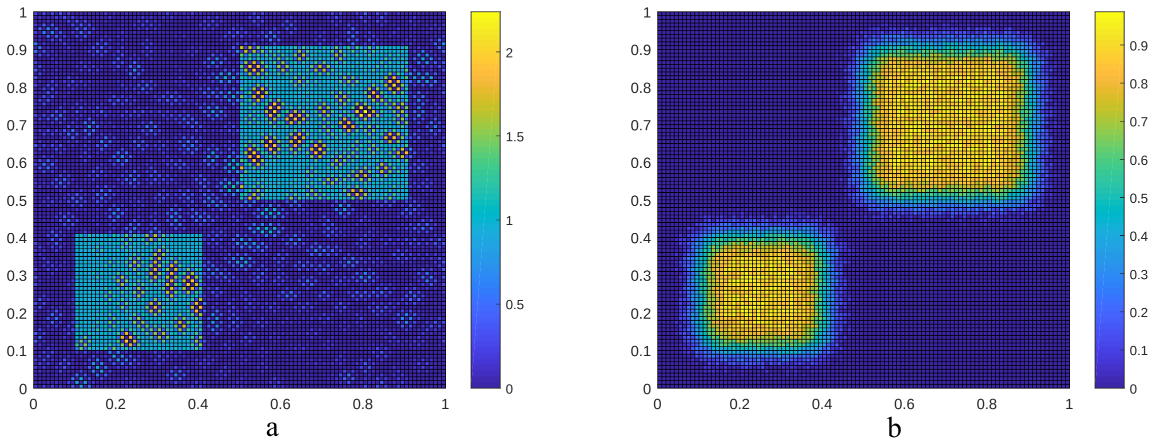


Figure 7. (a) Fourier regularization with $k_{eff} = 389$, (b) Quasi-boundary method with $\alpha = 5 * 10^{-2}$.

5. Conclusions

In this paper, we gave a linear regularization method for solving the the reconstruction problem of aperture in the plane from their diffraction patterns. In theory, we proved the convergence of quasi-boundary regularization method. Relative to the classical Tikhonov method, for non-smooth reconstructing object, the quasi-boundary regularization show better numerical results (it can recover the "corner" of the object better) because it has less smoothing effect.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. M. Bertero, P. Boccacci, M. Piana, Resolution and super-resolution in inverse diffraction, In: *Inverse problems of wave propagation and diffraction*, Lecture notes in physics, Vol. 486, Berlin, Heidelberg: Springer, 1997. https://doi.org/10.1007/BFb0105756
2. P. S. Carney, J. C. Schotland, Inverse scattering for near-field microscopy, *Appl. Phys. Lett.*, **77** (2000), 2798–2800. https://doi.org/10.1063/1.1320844
3. G. Bao, P. J. Li, H. J. Wu, A computational inverse diffraction grating problem, *J. Opt. Soc. Amer. A*, **29** (2012), 394–399. https://doi.org/10.1364/JOSAA.29.000394
4. G. Bao, P. J. Li, Near-field imaging of infinite rough surfaces, *SIAM J. Appl. Math.*, **73** (2013), 2162–2187. https://doi.org/10.1137/130916266
5. G. Bao, P. J. Li, Convergence analysis in near-field imaging, *Inverse Probl.*, **30** (2014), 1–26. https://doi.org/10.1088/0266-5611/30/8/085008
6. G. Bao, P. J. Li, Y. L. Wang, Near-field imaging with far-field data, *Appl. Math. Lett.*, **60** (2016), 36–42. https://doi.org/10.1016/j.aml.2016.03.023
7. P. J. Li, Y. L. Wang, Y. Zhao, Near-field imaging of biperiodic surfaces for elastic waves, *J. Comput. Phys.*, **324** (2016), 1–23. https://doi.org/10.1016/j.jcp.2016.07.030
8. W. S. Yin, W. H. Yang, H. Y. Liu, A neural network scheme for recovering scattering obstacles with limited phaseless far-field data, *J. Comput. Phys.*, **417** (2020), 109594. https://doi.org/10.1016/j.jcp.2020.109594
9. Y. Gao, H. Y. Liu, X. C. Wang, K. Zhang, On an artificial neural network for inverse scattering problems, *J. Comput. Phys.*, **448** (2022), 110771. https://doi.org/10.1016/j.jcp.2021.110771

10. R. Magnanini, G. Papi, An inverse problem for the Helmholtz equation, *Inverse Probl.*, **1** (1985), 357–370. https://doi.org/10.1088/0266-5611/1/4/007

11. H. H. Qin, T. Wei, R. Shi, Modified Tikhonov regularization method for the Cauchy problem of the Helmholtz equation, *J. Comput. Appl. Math.*, **224** (2009), 39–53. https://doi.org/10.1016/j.cam.2008.04.012

12. X. T. Xiong, C. L. Fu, Two approximate methods of a Cauchy problem for the Helmholtz equation, *Comput. Appl. Math.*, **26** (2007), 285–307. https://doi.org/10.1590/S0101-82052007000200006

13. X. L. Feng, C. L. Fu, H. Cheng, A regularization method for solving the Cauchy problem for the Helmholtz equation, *Appl. Math. Model.*, **35** (2011), 3301–3315. https://doi.org/10.1016/j.apm.2011.01.021

14. H. H. Qin, T. Wei, Two regularization methods for the Cauchy problems of the Helmholtz equation, *Appl. Math. Model.*, **34** (2010), 947–967. https://doi.org/10.1016/j.apm.2009.07.008

15. Y. X. Zhang, C. L. Fu, Z. L. Deng, An a posteriori truncation method for some Cauchy problems associated with Helmholtz-type equations, *Inverse Probl. Sci. En.*, **21** (2013), 1151–1168. https://doi.org/10.1080/17415977.2012.743538

16. X. T. Xiong, A regularization method for a Cauchy problem of the Helmholtz equation, *J. Comput. Appl. Math.*, **233** (2010), 1723–1732. https://doi.org/10.1016/j.cam.2009.09.001

17. H. W. Zhang, H. H. Qin, T. Wei, A quasi-reversibility regularization method for the Cauchy problem of the Helmholtz equation, *Int. J. Comput. Math.*, **88** (2011), 839–850. https://doi.org/10.1080/00207160.2010.482986

18. H. S. Kaveh, H. Adibi, Mapped regularization methods for the Cauchy problem of the Helmholtz and Laplace equations, *Iran. J. Sci. Technol. Trans. A Sci.*, **45** (2021), 669–682. https://doi.org/10.1007/s40995-020-01050-8

19. S. Q. He, X. F. Feng, A kind of operator regularization method for Cauchy problem of the Helmholtz equation in a multi-dimensional case, *Int. J. Comput. Math.*, **98** (2021), 1349–1364. https://doi.org/10.1080/00207160.2020.1819987

20. Z. P. Li, C. Xu, M. Lan, Z. Qian, A mollification method for a Cauchy problem for the Helmholtz equation, *Int. J. Comput. Math.*, **95** (2018), 2256–2268. https://doi.org/10.1080/00207160.2017.1380193

21. Z. Qian, X. L. Feng, A fractional Tikhonov method for solving a Cauchy problem of Helmholtz equation, *Appl. Anal.*, **96** (2017), 1656–1668. https://doi.org/10.1080/00036811.2016.1254776

22. Y. F. Kong, Z. P. Li, X. T. Xiong, An inverse diffraction problem: Shape reconstruction, *E. Asian J. Appl. Math.*, **5** (2015), 342–360. https://doi.org/10.4208/eajam.310315.250915a

23. C. L. Fu, Y. J. Ma, Y. X. Zhang, F. Yang, A a posteriori regularization for the Cauchy problem for the Helmholtz equation with inhomogeneous Neumann data, *Appl. Math. Model.*, **39** (2015), 4103–4120. https://doi.org/10.1016/j.apm.2014.12.030

24. X. T. Xiong, X. C. Zhao, J. X. Wang, Spectral Galerkin method and its application to a Cauchy problem of Helmholtz equation, *Numer. Algor.*, **63** (2013), 691–711. https://doi.org/10.1007/s11075-012-9648-9

25. H. W. Engl, M. Hanke, A. Neubauer, *Regularization of inverse problems*, Dordrecht: Kluwer Academic Publisher, 2000.

26. U. Tautenhahn, Optimal stable approximations for the sideways heat equation, *J. Inv. Ill-posed Problems*, **5** (1997), 287–307. https://doi.org/10.1515/jiip.1997.5.3.287

27. U. Tautenhahn, Optimal stable solution of Cauchy problems for elliptic equations, *Z. Anal. Anwend.*, **15** (1996), 961–984. https://doi.org/10.4171/ZAA/740

28. M. M. Sondhi, Reconstruction of objects from their sound-diffraction patterns, *J. Acoust. Soc. Amer.*, **46** (1969), 1158–1164. https://doi.org/10.1121/1.1911836

29. M. Bertero, C. De Mol, Stability problems in inverse diffraction, *IEEE Trans. Antenn. Propag.*, **29** (1981), 368–372. https://doi.org/10.1109/TAP.1981.1142558

30. F. Santosa, A level-set approach for inverse problems involving obstacles, *ESAIM Control Optim. Calc. Var.*, **1** (1996), 17–33. https://doi.org/10.1051/cocv:1996101

31. X. T. Xiong, X. Y. Fan, M. Li, Spectral method for ill-posed problems based on the balancing principle, *Inverse Probl. Sci. Eng.*, **23** (2015), 292–306. https://doi.org/10.1080/17415977.2014.894039

32. R. Acar, C. R. Vogel, Analysis of bounded variation penalty methods for ill-posed problems, *Inverse Probl.*, **10** (1994), 1217–1229. https://doi.org/10.1088/0266-5611/10/6/003

33. K. A. Ames, G. W. Clark, J. F. Epperson, S. F. Oppenheimer, A comparison of regularizations for an ill-posed problem, *Math. Comput.*, **67** (1998), 1451–1471. https://doi.org/10.1090/S0025-5718-98-01014-X

34. A. N. Tikhonov, V. Y. Arsenin, *Solutions of ill-posed problems*, Washington: Winston and Sons, 1977.