Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(6): 11015–11023.
DOI:10.3934/math.2022615
Received: 22 January 2022
Revised: 17 March 2022
Accepted: 27 March 2022
Published: 06 April 2022

Research article

Equivalent characterizations of harmonic Teichmüller mappings

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Abstract: In this paper, three equivalent conditions of ρ-harmonic Teichmüller mapping are given firstly. As an application, we investigate the relationship between a ρ-harmonic Teichmüller mapping and its associated holomorphic quadratic differential and obtain a relatively simple method to prove Theorem 2.1 in [1]. Furthermore, the representation theorem of $1/|\omega|^2$-harmonic Teichmüller mappings is given as a by-product. Our results extend the corresponding researches of harmonic Teichmüller mappings.

Keywords: flat harmonic mapping; Teichmüller mapping; associated holomorphic quadratic differential; quasiconformal mapping
Mathematics Subject Classification: Primary 30C62; Secondary 31A05, 31A35

1. Introduction and preliminaries

Let Ω and Ω' be two Jordan domains of the complex plane $\mathbb{C}$. $\rho(\omega)|d\omega|^2$ is a conformal metric of Ω'. A function $\omega = f(z)$ from Ω into Ω' is the so-called harmonic mapping with respect to ρ (or brieflyρ-*harmonic mapping*) if $f \in C^2$ satisfies the Euler-Lagrange equation

$$f_{z\bar{z}}(z) + (\log\rho)_\omega \circ f \cdot f_z(z) f_{\bar{z}}(z) = 0, \tag{1.1}$$

for $z \in \Omega$ and $\omega = f(z)$. Denote the Hopf differential of f by $\Phi(z)dz^2 := \rho(f) f_z \overline{f_{\bar{z}}} dz^2$. Then f is a ρ-harmonic mapping on Ω if and only if $\Phi(z)dz^2$ is a holomorphic quadratic differential on Ω.

The Gaussian curvature of $\rho(\omega)|d\omega|^2$ on Ω' is given by

$$K(\rho) = -\frac{1}{2}\frac{\Delta \log\rho}{\rho},$$

where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator. Then the solution of partial differential equation $K(\rho) = 0$ can be induced by a non-vanishing analytic function φ, that is, $\rho(\omega) = |\varphi(\omega)|$. Thus f is said to be a flat harmonic mapping (or briefly φ-harmonic mapping) if f is a ρ-harmonic mapping with $\rho = |\varphi|$, that is, f is the solution of the equation

$$\varphi(\omega) f_{z\bar{z}}(z) + 2\varphi'(\omega) f_z(z) f_{\bar{z}}(z) = 0, \quad z \in \Omega. \tag{1.2}$$

Taking some special values of ρ, we obtain some subclass of φ-harmonic mappings. Such as when $\rho = c$ is a positive constant, then f becomes an Euclidean harmonic mapping (or briefly π-harmonic mapping) which can be decomposed by the form as $f = h + \overline{g}$, where h and g are analytic in Ω; when $\rho(\omega) = 1/|\omega|^2$ and $0 \notin \Omega'$, then f just corresponds to a non-vanishing logharmonic mapping; when $\rho(\omega) = 1/|\omega|^4$ and $0 \notin \Omega'$, then $1/f$ happens to be an Euclidean harmonic mapping. For further details on φ-harmonic mapping, the reader can refer to the monographs [4, 14] and papers [2, 3, 5, 6, 10, 13].

In 2006, Kalaj and Matejević [6] found that f is a φ-harmonic mapping if and only if there exists a conformal mapping ψ and an Euclidean harmonic mapping f_1 such that $f = \psi \circ f_1$, where $\varphi = \left((\psi^{-1})'\right)^2$.

Function f is a ρ-harmonic quasiconformal mapping on Ω if f is a ρ-harmonic mapping from Ω onto Ω' and is also a quasiconformal mapping. Some basic concepts and properties of quasiconformal mapping can be found in [11, 12, 15]. Let μ_f be the Beltrami coefficient of f, then $\mu_f = f_{\bar{z}}/f_z$ with $\|\mu_f\|_\infty < 1$. Particularly, if there exists a constant $k \in (0, 1)$ such that $|\mu_f| = k$, then we call f is a ρ-harmonic Teichmüller mapping from Ω onto Ω'. Reich [12] obtained that the Beltrami coefficient of ρ-harmonic Teichmüller mapping has the expression

$$\mu_f(z) = k \frac{\overline{\phi(z)}}{|\phi(z)|}, \quad z \in \Omega, \tag{1.3}$$

where $\phi(z) := \rho(f) f_z \overline{f_{\bar{z}}}$. Then the quantity $\phi(z)dz^2$ is said to be the associated holomorphic quadratic differential of f. Notice that the Beltrami coefficient of a Teichmüller mapping f has the same representation as (1.3) (but the function ϕ is only holomorphic), thus we still define ϕ be its associated holomorphic quadratic differential of Teichmüller mappings. Teichmüller mappings play an important role in the theories of quasiconfromal mapping, Teichmüller space and so on. It is well known that the inverse of a Teichmüller mapping is also a Teichmüller mapping, and the detail is as follows.

Theorem A. ([11, p.116]) *If f is a Teichmüller mapping from the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto itself, then its inverse function $F = f^{-1}$ is a Teichmüller mapping on $\mathbb{D}$.*

Theorem A is also valid for every Teichmüller mapping defined on Ω. Using Theorem A, Chen and Fang find that ρ-harmonic Teichmüller mapping must be a certain φ-harmonic mapping in [1].

Theorem B. ([1, Theorem 2.1]) *If f is a C^2 Teichmüller mapping from Ω onto Ω' and the associated holomorphic quadratic differential of its inverse function $F = f^{-1}$ is $\phi(\omega)d\omega^2$, then f is a ρ-harmonic mapping if and only if $\rho = c|\phi|$, where c is a positive constant.*

By Theorems A and B, Chen [2] got that its inverse function F is also a φ-harmonic Teichmüller mapping. Meanwhile, applying Theorem B, Chen and Fang [1] assert that there does not exist a solution to the Schone conjecture in the class of C^2 Teichmüller mappings. In addition, they obtain the representation of π-harmonic Teichmüller mappings in [1]. One can refer to [7–9] for more details about the study on the Schone conjecture.

In this article, we examine some properties of harmonic Teichmüller mappings and obtain the explicit representation of $1/|\omega|^2$-harmonic Teichmüller mappings. The structure of the article is organized as follows. Firstly, we study on some properties of φ-harmonic Teichmüller mapping and find that φ-harmonic Teichmüller mapping is a solution of the partial differential Eq (2.1) in Theorem 1. Based on Theorem 1, a relatively simple method is given to prove Theorem B again. Last, as an application of Theorem 1, the representation of $1/|\omega|^2$-harmonic Teichmüller mapping is gained in Section 3 which is an extension of π-harmonic Teichmüller mappings.

2. Characterizations of ρ-harmonic Teichmüller mappings

In this section, we firstly investigate the decomposition of ρ-harmonic Teichmüller mappings and find that every ρ-harmonic Teichmüller mapping is the solution of a partial differential equation. Depending on these characterizations of ρ-harmonic Teichmüller mappings, we can prove Theorem B very simply.

Theorem 1. *Let $\omega = f(z) \in C^2$ be a sense preserving homeomorphic mapping from Ω onto Ω' and $\varphi(\omega)$ be an non-vanishing analytic function on Ω'. If $\phi(\omega) = \varphi^2(\omega)$, then the following three statements are equivalent.*

(1) f is a ρ-harmonic Teichmüller mapping and $\phi(\omega)d\omega^2$ is the associated holomorphic quadratic differential of $F = f^{-1}$;

(2) There exists a constant α with $|\alpha| = k \in (0,1)$ such that f is the solution of the following partial differential equation

$$\frac{\overline{f_{\bar{z}}(z)}}{f_z(z)} = \alpha\frac{\varphi(\omega)}{\overline{\varphi(\omega)}} \tag{2.1}$$

for all $z \in \Omega$;

(3) f can be decomposed as

$$f = \psi \circ \left(h + \alpha\overline{h}\right),$$

where ψ is conformal on Ω' and h is conformal on Ω satisfy

$$\psi'(\psi^{-1}(\omega)) \cdot \varphi(\omega) = 1, \ \ h'(z) = \varphi(\omega) \cdot f_z(z),$$

for all $\omega \in \Omega'$, $z \in \Omega$ and a constant α with $|\alpha| = k \in (0,1)$.

Proof. We first prove that (1) $\Longrightarrow$ (2). Since f is a ρ-harmonic Teichmüller mapping and $\phi(\omega)d\omega^2$ is the associated holomorphic quadratic differential of its inverse function, we see from Theorems A and B that $\rho = c|\phi|$ and F is also a Teichmüller mapping, i.e., there exists a constant $k \in (0,1)$ such that the Beltrami coefficient μ_F of F satisfies

$$\mu_F(\omega) = \frac{F_{\overline{\omega}}(\omega)}{F_\omega(\omega)} = k\frac{\overline{\phi(\omega)}}{|\phi(\omega)|} = k\frac{\overline{\varphi(\omega)}}{\varphi(\omega)}. \tag{2.2}$$

Differentiating the equation $F \circ f(z) = z$ with respect to z and $\overline{z}$ respectively, one has

$$\begin{cases} F_\omega f_z + F_{\overline{\omega}}\overline{f_{\bar{z}}} = 1, \\ F_\omega f_{\bar{z}} + F_{\overline{\omega}}\overline{f_z} = 0. \end{cases}$$

Then

$$f_z = \frac{\overline{F_\omega}}{J_F}, \quad f_{\bar{z}} = -\frac{F_{\overline{\omega}}}{J_F}, \tag{2.3}$$

where

$$J_F = |F_\omega|^2 - |F_{\overline{\omega}}|^2 = \left(1-k^2\right)|F_\omega|^2 > 0. \tag{2.4}$$

According to relations (2.2) and (2.3), we see that

$$\frac{\overline{f_{\overline{z}}(z)}}{f_z(z)} = -\frac{\overline{F_{\overline{\omega}}(\omega)}}{F_\omega(\omega)} = -k\frac{\varphi(\omega)}{\overline{\varphi(\omega)}} \tag{2.5}$$

and thus (2.1) holds true with $\alpha = -k$.

Now we prove (2) $\Longrightarrow$ (1). Suppose that the homeomorphic mapping f is a solution of the partial differential Eq (2.1). Then f is a Teichmüller mapping in Ω. Moreover, (2.1) is equivalent to

$$\overline{\varphi(\omega)f_{\overline{z}}(z)} = \alpha\varphi(\omega)f_z(z). \tag{2.6}$$

Differentiating the Eq (2.6) with respect to $\overline{z}$, we get

$$\overline{\varphi f_{z\overline{z}} + \varphi' f_z f_{\overline{z}}} = \alpha\left(\varphi f_{z\overline{z}} + \varphi' f_z f_{\overline{z}}\right), \tag{2.7}$$

by the fact that $\varphi(\omega)$ is analytic in Ω'. Since $0 < |\alpha| = k < 1$, we see from (2.7) that for all $z \in \Omega$

$$\varphi f_{z\overline{z}} + \varphi' f_z f_{\overline{z}} = 0,$$

which implies that f is a ρ-harmonic mapping on Ω and $\rho = c|\varphi|^2 = c|\phi|$ from the relation (1.2). Here c is a positive constant.

Let $F = f^{-1}$. Since f is a Teichmüller mapping which satisfies (2.1), we see from the relations (2.1) and (2.3) that for all $\omega \in \Omega$

$$\mu_F(\omega) = \frac{F_{\overline{\omega}}(\omega)}{F_\omega(\omega)} = -\frac{f_{\overline{z}}(z)}{\overline{f_z(z)}} = -\overline{\left(\frac{\overline{f_{\overline{z}}(z)}}{f_z(z)}\right)} = -\overline{\alpha}\frac{\overline{\varphi(\omega)}}{\varphi(\omega)} = -\overline{\alpha}\frac{\overline{\phi(\omega)}}{|\phi(\omega)|},$$

which implies that $\phi(\omega)d\omega^2$ is the associated holomorphic quadratic differential of F. Thus, the statements (1) and (2) are equivalent.

Next, we show that (1) and (3) are equivalent. We start from (1) $\Longrightarrow$ (3). Since f is a ρ-harmonic mapping with $\rho = c|\phi|$, where ρ is deduced by a non-vanishing analytic function, one has $f = \psi \circ f^*$, where ψ is a conformal mapping from $f^*(\Omega)$ onto Ω' and $f^*(z)$ is an Euclidean harmonic mapping from Ω onto $f^*(\Omega)$. Moreover,

$$\psi'\left(\psi^{-1}(\omega)\right) = \frac{1}{\varphi(\omega)}, \quad \omega \in \Omega'.$$

Note that f is a Teichmüller mapping on Ω, thus for all $z \in \Omega$ one has

$$k = |\mu_f| = \left|\frac{f_{\overline{z}}}{f_z}\right| = \left|\frac{f^*_{\overline{z}}}{f^*_z}\right|,$$

which implies that f^* is also a Teichmüller mapping on Ω. Therefore, there exists a conformal mapping h and a constant α with $|\alpha| = k$ such that $f^* = h + \alpha\overline{h}$ on Ω ([1, Theorem 4.1]).

Meanwhile, it follows from

$$f_z(z) = \psi'\left(\psi^{-1}(\omega)\right) \cdot h'(z)$$

that

$$h'(z) = \varphi(\omega) \cdot f_z(z), \quad z \in \Omega.$$

We now show that (3) $\Longrightarrow$ (1) as follows. If there exist two conformal mappings ψ and h such that $f = \psi \circ f^*$ on Ω, where $f^* = h + \alpha\overline{h}$, then the Beltrami coefficient of f satisfies

$$|\mu_f| = \left|\frac{f_{\overline{z}}}{f_z}\right| = \left|\frac{f^*_{\overline{z}}}{f^*_z}\right| = |\alpha| = k,$$

which deduces that f is a Teichmüller mapping on Ω. Moreover,

$$\frac{\overline{f_{\overline{z}}}}{f_z} = \overline{\alpha}\frac{\overline{\psi'(\xi)}}{\psi'(\xi)} = \overline{\alpha}\frac{\varphi(\omega)}{\overline{\varphi(\omega)}}.$$

Thus f is a $c|\phi|$-harmonic mapping and $\phi(\omega)d\omega^2 = \varphi^2(\omega)d\omega^2$ is the associated holomorphic quadratic differential of its inverse function $F = f^{-1}$, according to the proof of the part (2) $\Longrightarrow$ (1). □

By Theorem B, f is a π-harmonic Teichmüller mapping if and only if $\phi = c_1$, that is, $\varphi = c_2$, where $c_1,\ c_2$ are two positive constants. Therefore, from the relation (2.1), we have

$$\overline{f_{\overline{z}}(z)} = \alpha_0 f_z(z), \quad z \in \Omega. \tag{2.8}$$

Since f can be represented as $f = h + \overline{g}$, where h and g are analytic on Ω, we see (2.8) that there exists a conformal mapping h such that $f = h + \overline{\alpha_0}\overline{h}$. This is a coincident with [1, Theorem 4.1].

Meanwhile, from the proof of (1) $\Longleftrightarrow$ (2) in Theorem 1, the following theorem(that is Theorem B) can be directly obtained, which its proof process is relatively simple compared with [1].

Theorem 2. *If f is a C^2 Teichmuller mapping from Ω onto Ω' and $\phi(\omega)d\omega^2$ is the associated holomorphic quadratic differential of $F = f^{-1}$, then f is a ρ-harmonic mapping if and only if $\rho(\omega) = c|\phi(\omega)|$ for $\omega \in \Omega'$.*

Proof. Since f is a Teichmüller mapping on Ω, then its inverse function F is also a Teichmüller mapping by Theorem A, that is, there exists a constant $k \in (0, 1)$ such that

$$\mu_F = \frac{F_{\overline{\omega}}}{F_\omega} = k\frac{\overline{\phi}}{|\phi|} = k\frac{\overline{\varphi}}{\varphi},$$

where $\phi = \varphi^2$. By the relation (2.2), we yields

$$\frac{\overline{f_{\overline{z}}(z)}}{f_z(z)} = -\overline{\left(\frac{F_{\overline{\omega}}(\omega)}{F_\omega(\omega)}\right)} = -k\frac{\varphi(\omega)}{\overline{\varphi(\omega)}}$$

which is equivalent to

$$\overline{\varphi(\omega) f_{\overline{z}}(z)} = -k\varphi(\omega) f_z(z), \tag{2.9}$$

where $z \in \Omega$ and $\omega = f(z)$. Differentiating the Eq (2.9) with respect to $\overline{z}$, we get

$$\varphi(\omega) f_{z\overline{z}}(z) + \varphi'(\omega) f_z(z) f_{\overline{z}}(z) = 0,$$

which implies that f is a ρ-harmonic mapping if and only if $\rho = c|\varphi^2| = c|\phi(\omega)|$ from the relation (1.2), here c is a positive constant. □

Applying Theorem 1 and Theorem 2, the following conclusion can be drawn naturally.

Corollary 1. *Let f be a C^2 Teichmüller mapping from Ω onto Ω' and $\phi(\omega)$ be an analytic function on Ω'. Then f is a $c|\phi|$-harmonic mapping if and only if $\phi(\omega)d\omega^2$ is the associated holomorphic quadratic differential of $F = f^{-1}$.*

Proof. The sufficiency is directly obtained by Theorem 2. We only need to prove the necessity. Let $\phi = \varphi^2$. Since f is a $c|\phi|$-harmonic Teichmüller mapping, then there exists an analytic function $a(z)$ on Ω with $|a(z)| < 1$ such that

$$\frac{\overline{f_{\bar{z}}(z)}}{f_z(z)} = a(z)\frac{\varphi(\omega)}{\overline{\varphi(\omega)}}$$

and $\mathrm{esssup}|a(z)| = k \in (0, 1)$ from the relation (1.2). Thus $a(z)$ is a constant for $z \in \Omega$ by Liouville's theorem. Hence, by the proof of (2) $\Longrightarrow$ (1) in Theorem 1, we get that $\phi(\omega)d\omega^2$ is the associated holomorphic quadratic differential of $F = f^{-1}$. □

3. $1/|\omega|^2$-harmonic Teichmüller mappings

Applying Theorems 1 and 2, the representation theorem of $1/\omega|^2$-harmonic Teichmüller mapping is given below.

Theorem 3. *If $\omega = f(z)$ is a C^2 Teichmüller mapping from the unit disk $\mathbb{D}$ onto Ω with $0 \notin \Omega$, then the following statements are equivalent:*

(1) f is a $1/|\omega|^2$-harmonic mapping;

(2) f has the form as $f = h\overline{h^\alpha}$, where h is a non-vanishing conformal mapping in $\mathbb{D}$ and α is a constant which satisfies $0 < |\alpha| < 1$.

Proof. We first prove (2) $\Longrightarrow$ (1). Since $f = h\overline{h^\alpha}$ and h is non-vanishing conformal mapping, we see that $\omega = f(z) \neq 0$ in $\mathbb{D}$ and

$$f_z = h'\overline{h^\alpha}, \quad f_{\bar{z}} = \overline{\alpha} h \overline{h' h^{\alpha-1}}.$$

Therefore,

$$\frac{\overline{f_{\bar{z}}}}{f_z} = \frac{\alpha h' h^{\alpha-1}\overline{h}}{h'\overline{h^\alpha}} = \alpha\frac{\overline{\omega}}{\omega}$$

for all $z \in \mathbb{D}$. Let $\varphi(\omega) = 1/\omega$ and $\phi(\omega) = \varphi^2(\omega)$, then we have

$$\frac{\overline{f_{\bar{z}}(z)}}{f_z(z)} = \alpha\frac{\varphi(\omega)}{\overline{\varphi(\omega)}},$$

which implies that f is a $1/|\omega|^2$-harmonic Teichmüller mapping (ignoring multiplying a positive constant) and $\phi(\omega)d\omega^2$ is the associated holomorphic quadratic differential of $F = f^{-1}$ by Theorem 1 and Corollary 1 respectively.

Next we prove (1) $\Longrightarrow$ (2). Let $\varphi(\omega) = 1/\omega$. If f is a $1/|\omega|^2$-harmonic Teichmüller mapping on $\mathbb{D}$, then there exists an analytic function $a(z)$ on $\mathbb{D}$ with $|a(z)| < 1$ such that

$$\frac{\overline{f_{\bar{z}}}}{f_z} = a(z)\frac{\varphi}{\overline{\varphi}} = a(z)\frac{\overline{\omega}}{\omega} \tag{3.1}$$

and $|a(z)| = k \in (0, 1)$ by Properties 2.1 in [10]. Notice that by Liouville's theorem $a(z) = \alpha$ is a constant with $|\alpha| = k$ for all $z \in \mathbb{D}$.

It is well known that a non-vanishing logharmonic mapping F has the expression that $F = H\overline{G}$ on $\mathbb{D}$, where H and G are analytic on $\mathbb{D}$. Since f is a $1/|\omega|^2$-harmonic mapping if and only if f is a non-vanishing logharmonic mapping [10], we have $f(z) = H(z)\overline{G(z)}$ for all $z \in \mathbb{D}$. Hence,

$$f_z = H'\overline{G} = \frac{H'}{H}\omega, \quad f_{\bar{z}} = H\overline{G'} = \frac{\overline{G'}}{\overline{G}}\omega.$$

From the relation (3.1), we have

$$\frac{G'}{G} = \alpha\frac{H'}{H}$$

for all $z \in \mathbb{D}$, which yields to

$$\log f = \log H + \overline{\log G} = \log H + \overline{\alpha \log H + c},$$

where c is a constant. Let

$$\log h = \log H + \frac{\bar{c} - \overline{\alpha}c}{1 - |\alpha|^2}.$$

Then $\log f = \log h + \overline{\alpha}\overline{\log h}$, that is, $f = h\overline{h^{\alpha}}$ for $z \in \mathbb{D}$.

Finally, for any two points z_1 and z_2 in $\mathbb{D}$, we obtain

$$\log f(z_1) - \log f(z_2) = \log h(z_1) - \log h(z_2) + \overline{\alpha}\left(\overline{\log h(z_1) - \log h(z_2)}\right).$$

Since $|\alpha| = k < 1$, one has $\log f$ is univalent if and only if $\log h$ is a conformal function if and only if h is a conformal function in $\mathbb{D}$. Thus we see that (1) implies (2). □

Remark. The representation of $1/|\omega|^2$-harmonic Teichmüller mappings in Theorem 3 is coincident with that in Theorem 1. In fact, since

$$f = h\overline{h^{\alpha}} = e^{\log h + \overline{\alpha \log h}}$$

can be viewed as $f = \psi \circ f_1$, where

$$\omega = \psi(\xi) = e^{\xi}, \quad f_1(z) = \log h(z) + \overline{\alpha}\,\overline{\log h(z)}$$

for $\xi = f_1(z)$ and $z \in \mathbb{D}$. Moreover, f_1 is univalent in $\mathbb{D}$ by the fact that h is a conformal mapping in $\mathbb{D}$. It is easy to verify that ψ is just the univalent solution of equation $\psi'(\psi^{-1}) \circ \varphi = 1$ and h satisfies $(\log h)' = \varphi \cdot f_z$.

On the other hand, for a given non-vanishing conformal mapping h in $\mathbb{D}$, let $\psi(\xi) = e^{\xi}$ and $\xi = f_1(z) = \log h + \overline{\alpha}\,\overline{\log h}$. Then the composition function

$$\omega = f = \psi \circ f_1 = h\overline{h^{\alpha}}$$

is a $1/|\omega|^2$-harmonic mapping.

Acknowledgments

The research was partly supported by the Natural Science Foundation of Fujian Province (No. 2020J05157), the Research projects of Young and Middle-aged Teacher's Education of Fujian Province (No. JAT190508).

Conflict of interest

The author declares that there is no conflict of interests regarding the publication of this article.

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