Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(6): 10867–10891.
DOI: 10.3934/math.2022608
Received: 14 February 2022
Revised: 14 March 2022
Accepted: 16 March 2022
Published: 01 April 2022

Research article

Novel fixed point technique to coupled system of nonlinear implicit fractional differential equations in complex valued fuzzy rectangular b-metric spaces

Sumaiya Tasneem Zubair[1], Kalpana Gopalan[1,*], Thabet Abdeljawad[2,3,*] and Nabil Mlaiki[2]

[1] Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Kalavakkam, Chennai-603 110, India

[2] Department of Mathematics and Sciences, Prince Sultan University, P. O. Box 66833, Riyadh 11586, Saudi Arabia

[3] Department of Medical Research, China Medical University, Taichung 40402, Taiwan

* **Correspondence:** Email: tabdeljawad@psu.edu.sa, kalpanag@ssn.edu.in.

Abstract: The fundamental purpose of this research is to investigate the existence theory as well as the uniqueness of solutions to a coupled system of fractional order differential equations with Caputo derivatives. In this regard, we utilize the definition and properties of a newly developed conception of complex valued fuzzy rectangular b-metric spaces to explore the fuzzy form of some significant fixed point and coupled fixed point results. We further present certain examples and a core lemma in the case of complex valued fuzzy rectangular b-metric spaces.

Keywords: complex valued fuzzy rectangular b-metric spaces; fixed point; Green's function; implicit coupled system; existence theory
Mathematics Subject Classification: 47H10, 54H25, 26A33, 34B27

1. Introduction and preliminaries

Fuzzy set theory performs a significant role in our real-life challenges involving unpredictability. They enable us to depict a difficult and unpredictable situation in a simplistic manner. This fact sparked an intriguing research study that resulted in the advancement of fuzzy set theory. In 1965, Zadeh presented the idea of fuzzy sets [29], which sparked a flurry of study that led to the development of a more appealing fuzzy system theory. Kramosil and Michalek [17] pioneered the concept of fuzzy metric. They applied probabilistic metric space notions to the fuzzy case. The conventional fuzzy metric was modified by George and Veeramani [12] to obtain a Hausdorff topology started by fuzzy metric. In fuzzy metric spaces, Garbiec [13] developed a fuzzy variant of the Banach contraction principle which is given as follows: "Let $(Y, M, *)$ be a complete fuzzy metric space such that $\lim_{t\to\infty} M(\vartheta, \varkappa, t) = 1$ for all $\vartheta, \varkappa \in Y$ and $T : Y \to Y$ be a mapping satisfying $M(T\vartheta, T\varkappa, kt) \geq M(\vartheta, \varkappa, t)$ for all $\vartheta, \varkappa \in Y, 0 < k < 1$. Then T has a unique fixed point in Y".

The authors in [8, 19, 21] initiated the concepts of fuzzy b-metric spaces, fuzzy rectangular metric spaces and fuzzy rectangular b-metric spaces, as well as proving a banach fixed point theorem and other fixed point theorems in the underlying metric, which is discussed in detail in the following.

Definition 1.1. *[21] Let Y be a nonempty set, $b \geq 1$ a real number, $*$ a continuous t-norm, and $M_{\mathcal{B}}$ be a fuzzy set on $Y \times Y \times [0, \infty)$. Then, $M_{\mathcal{B}}$ is said to be a fuzzy b-metric on Y, if, for all $\vartheta, \varkappa, \tilde{\vartheta} \in Y$, $M_{\mathcal{B}}$ fulfills the following criteria:*

(1) $M_{\mathcal{B}}(\vartheta, \varkappa, 0) = 0$ *for* $t = 0$;
(2) $M_{\mathcal{B}}(\vartheta, \varkappa, t) = 1$ *if and only if* $\vartheta = \varkappa$ *for all* $t > 0$;
(3) $M_{\mathcal{B}}(\vartheta, \varkappa, t) = M_{\mathcal{B}}(\varkappa, \vartheta, t)$;
(4) $M_{\mathcal{B}}(\vartheta, \varkappa, b(t + t')) \geq M_{\mathcal{B}}(\vartheta, \tilde{\vartheta}, t) * M_{\mathcal{B}}(\tilde{\vartheta}, \varkappa, t')$ *for all* $t, t' > 0$;
(5) $M_{\mathcal{B}}(\vartheta, \varkappa, .) : (0, \infty) \to [0, 1]$ *is left continuous and* $\lim_{t\to\infty} M_{\mathcal{B}}(\vartheta, \varkappa, t) = 1$.

The quadruple $(Y, M_{\mathcal{B}}, *, b)$ is called fuzzy b-metric space.

Definition 1.2. *[8] Let Y be a nonempty set, $*$ a continuous t-norm, and $M_{\mathcal{R}}$ be a fuzzy set on $Y \times Y \times [0, \infty)$. Then, $M_{\mathcal{R}}$ is called fuzzy rectangular metric, if, for any $\vartheta, \varkappa \in Y$ and all distinct points $\tilde{\vartheta}, \tilde{\varkappa} \in Y \backslash \{\vartheta, \varkappa\}$, the following requirements are fulfilled:*

(1) $M_{\mathcal{R}}(\vartheta, \varkappa, 0) = 0$ *for* $t = 0$;
(2) $M_{\mathcal{R}}(\vartheta, \varkappa, t) = 1$ *if and only if* $\vartheta = \varkappa$ *for all* $t > 0$;
(3) $M_{\mathcal{R}}(\vartheta, \varkappa, t) = M_{\mathcal{R}}(\varkappa, \vartheta, t)$;
(4) $M_{\mathcal{R}}(\vartheta, \varkappa, t + t' + t'') \geq M_{\mathcal{R}b}(\vartheta, \tilde{\vartheta}, t) * M_{\mathcal{R}}(\tilde{\vartheta}, \tilde{\varkappa}, t') * M_{\mathcal{R}}(\tilde{\varkappa}, \varkappa, t'')$ *for all* $t, t', t'' > 0$;
(5) $M_{\mathcal{R}}(\vartheta, \varkappa, .) : (0, \infty) \to [0, 1]$ *is left continuous and* $\lim_{t\to\infty} M_{\mathcal{R}}(\vartheta, \varkappa, t) = 1$.

Then, $(Y, M_{\mathcal{R}}, *)$ is known as a fuzzy rectangular metric space.

Definition 1.3. *[19] Let Y be a nonempty set, $b \geq 1$ a real number, $*$ a continuous t-norm, and $M_{\mathcal{R}b}$ be a fuzzy set on $Y \times Y \times [0, \infty)$. Then, $M_{\mathcal{R}b}$ is called fuzzy rectangular b-metric, if, for any $\vartheta, \varkappa \in Y$ and all distinct points $\tilde{\vartheta}, \tilde{\varkappa} \in Y \backslash \{\vartheta, \varkappa\}$, the following requirements are fulfilled:*

(1) $M_{\mathcal{R}b}(\vartheta, \varkappa, 0) = 0$ *for* $t = 0$;
(2) $M_{\mathcal{R}b}(\vartheta, \varkappa, t) = 1$ *if and only if* $\vartheta = \varkappa$ *for all* $t > 0$;
(3) $M_{\mathcal{R}b}(\vartheta, \varkappa, t) = M_{\mathcal{R}b}(\varkappa, \vartheta, t)$;
(4) $M_{\mathcal{R}b}(\vartheta, \varkappa, b(t + t' + t'')) \geq M_{\mathcal{R}b}(\vartheta, \tilde{\vartheta}, t) * M_{\mathcal{R}b}(\tilde{\vartheta}, \tilde{\varkappa}, t') * M_{\mathcal{R}b}(\tilde{\varkappa}, \varkappa, t'')$ *for all* $t, t', t'' > 0$;
(5) $M_{\mathcal{R}b}(\vartheta, \varkappa, .) : (0, \infty) \to [0, 1]$ *is left continuous and* $\lim_{t\to\infty} M_{\mathcal{R}b}(\vartheta, \varkappa, t) = 1$.

Then, $(Y, M_{\mathcal{R}b}, *, b)$ is known as a fuzzy rectangular b-metric space.

Furthermore, researchers discovered that in cone metric spaces, rational type contraction is meaningless because of vector division. As a consequence, some results are not applicable to cone metric spaces. To deal with this issue, Azam et al. [6] proposed complex valued metric spaces as a novel setting of metric fixed point theory in 2011. Rather than using the set of positive real numbers as a foundation set, they utilized the set of complex numbers as a partial structural ground set. The researchers obtained fixed point findings that met rational contraction and explored their significance in this scenario. For more recent fixed point results, refer to [1–4, 15, 16, 20, 26]. The relevant concepts and results will be essential in what proceeds, in accordance with Shukla et al. [25].

The complex number over the field of real numbers is denoted by $\mathbb{C}$. Let $P = \{(c, d) : 0 \leq c < \infty, 0 \leq d < \infty\} \subset \mathbb{C}$. θ and l are used to represent the elements $(0, 0), (1, 1)$ in P, respectively. Define a partial ordering $\preceq$ on $\mathbb{C}$ by $t_1 \preceq t_2$ (or equivalently, $t_2 \succeq t_1$) if and only if $t_2 - t_1 \in P$. We write $t_1 \prec t_2$ (or equivalently, $t_2 \succ t_1$) to indicate $Re(t_1) < Re(t_2)$ and $Im(t_1) < Im(t_2)$. The sequence $\{t_n\}$ is said to be monotonic with respect to $\preceq$ if either $t_n \preceq t_{n+1}$ for all $n \in \mathbb{N}$ or $t_{n+1} \preceq t_n$ for all $n \in \mathbb{N}$.

The closed unit complex interval is defined as $I = \{(c, d) : 0 \leq c \leq 1, 0 \leq d \leq 1\}$, as well as the open unit complex interval $I_\theta = \{(c, d) : 0 < c < 1, 0 < d < 1\}$ and P_θ represents the set $\{(c, d) : 0 < c < \infty, 0 < d < \infty\}$, respectively. It needs to be noted that $t_1, t_2 \in \mathbb{C}, t_1 \prec t_2$ if and only if $t_2 - t_1 \in P_\theta$. For $A \subset \mathbb{C}$, if there is an element $infA \in \mathbb{C}$ such that it is a lower bound of A, that is $infA \preceq c$ for all $c \in A$ and $d \preceq infA$ for every lower bound $d \in \mathbb{C}$ of A, then $infA$ is called the greatest lower bound or infimum of A. In the same way, we define $supA$, the least upper bound or supremum of A, in the regular manner.

Remark 1.4. *[25] Let $t_n \in P$ for all $n \in \mathbb{N}$. Then*

(i) If the sequence $\{t_n\}$ is monotonic with respect to $\preceq$ and there exist $\gamma, \delta \in P$ such that $\gamma \preceq t_n \preceq \delta$, for all $n \in \mathbb{N}$, then there exists a $t \in P$ such that $\lim_{n\to\infty} t_n = t$.

(ii) Given the partial ordering $\preceq$ is not a linear (total) order on $\mathbb{C}$, the pair $(\mathbb{C}, \preceq)$ is a lattice.

(iii) If $A \subset \mathbb{C}$ is such that there exist $\gamma, \delta \in \mathbb{C}$ with $\gamma \preceq c\delta$ for all $c \in A$, then $infA$ and $supA$ both exist.

Remark 1.5. *[25] Let $t_n, t'_n, \vartheta \in P$ for all $n \in \mathbb{N}$. Then*

(i) If $t_n \preceq t'_n \preceq l$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} t_n = l$, then $\lim_{n\to\infty} t'_n = l$.

(ii) If $t_n \preceq \vartheta$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} t_n = \varkappa \in P$, then $\varkappa \preceq \vartheta$.

(iii) If $\vartheta \preceq t_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} t_n = \varkappa \in P$, then $\vartheta \preceq \varkappa$.

Shukla et al. [25] pioneered the idea of complex fuzzy set and complex valued t-norm which is presented below.

Definition 1.6. *[25] Let Y be a nonempty set. A complex fuzzy set M is characterized by a mapping with domain Y and values in the closed unit complex interval I.*

Definition 1.7. *[25] A binary operation $* : I \times I \to I$ is called as complex valued t-norm if for all $t, t_1, t_2, t_3, t_4 \in I$, the following requirements are met:*

(i) $*(t_1, t_2) = *(t_2, t_1)$;

(ii) $*(t_1, *(t_2, t_3)) = *(*(t_1, t_2), t_3)$;

(iii) $*(t_1, t_2) \preceq *(t_3, t_4)$ whenever $t_1 \preceq t_3, t_2 \preceq t_4$.

(iv) $t * \theta = \theta$, $t * l = l$.

Example 1.8. *[25] Let the binary opertions* $*_1, *_2, *_3 : I \times I \to I$ *is defined for all* $t_1 = (c_1, d_1), t_2 = (c_2, d_2) \in I$ *by*

(i) $t_1 *_1 t_2 = (c_1c_2, d_1d_2)$;

(ii) $t_1 *_2 t_2 = (min\{c_1, c_2\}, min\{d_1, d_2\})$;

(iii) $t_1 *_3 t_2 = (max\{c_1 + c_2 - 1, 0\}, max\{d_1 + d_2 - 1, 0\})$,

then $*_1, *_2$ *and* $*_3$ *are complex valued t-norms.*

Shukla et al. [25] recently proposed a novel idea of complex valued fuzzy metric spaces, specifying numerous associated topological properties for complex valued fuzzy metric spaces. Furthermore, they developed and studied the fuzzy form of the well-known Banach contraction principle in several perspectives.

Definition 1.9. *[25] Let* Y *be a nonempty set,* $*$ *a continuous complex valued t-norm, and* $M_{\mathfrak{C}}$ *a complex fuzzy set on* $Y^2 \times P_\theta$ *fulfills the following criteria:*

(1) $\theta \prec M_{\mathfrak{C}}(\vartheta, \varkappa, t)$;

(2) $M_{\mathfrak{C}}(\vartheta, \varkappa, t) = l$ *if and only if* $\vartheta = \varkappa$ *for all* $t \in P_\theta$;

(3) $M_{\mathfrak{C}}(\vartheta, \varkappa, t) = M_{\mathfrak{C}}(\varkappa, \vartheta, t)$;

(4) $M_{\mathfrak{C}}(\vartheta, \tilde{\vartheta}, t) * M_{\mathfrak{C}}(\tilde{\vartheta}, \varkappa, t') \preceq M_{\mathfrak{C}}(\vartheta, \varkappa, t + t')$;

(5) $M_{\mathfrak{C}}(\vartheta, \varkappa, t) : P_\theta \to I$ *is continuous*

for all $\vartheta, \varkappa, \tilde{\vartheta} \in Y$ *and* $t, t' \in P_\theta$. *Then* $(Y, M_{\mathfrak{C}}, *)$ *is said to be complex valued fuzzy metric space and* $M_{\mathfrak{C}}$ *is called a complex valued fuzzy metric on* Y. *The degree of closeness among two points of* Y *with regard to a complex parameter* $t \in P_\theta$ *can indeed be considered as a complex valued fuzzy metric.*

Example 1.10. *[25] Let* (Y, d) *be a metric space. The complex fuzzy set* $M_{\mathfrak{C}}$ *and * be defined by* $M_{\mathfrak{C}}(\vartheta, \varkappa, t) = \frac{e+f}{e+f+d(\vartheta,\varkappa)}$ *and* $t_1 * t_2 = (e_1e_2, f_1f_2)$, *for all* $\vartheta, \varkappa \in Y, t_1 = (e_1, f_1), t_2 = (e_2, f_2) \in I$,
$t = (e, f) \in P_\theta$. *Then* $(Y, M, *)$ *is a complex valued fuzzy metric space.*

Theorem 1.11. *[25] Let* $(Y, M_{\mathfrak{C}}, *)$ *be a complete complex valued fuzzy metric space and* $T : Y \to Y$ *be a mapping such that*

$$l - M_{\mathfrak{C}}(T\vartheta, T\varkappa, t) \preceq k[l - M_{\mathfrak{C}}(\vartheta, \varkappa, t)],\ \textit{for all}\ \vartheta, \varkappa \in Y, t \in P_\theta,$$

where $0 \leq k < 1$. *Then* T *has a unique fixed point in* Y.

Demir [10] introduced the theory of complex valued fuzzy b-metric spaces, relying on the results of Shukla et al. [25]. In the setting of complex valued fuzzy b-metric spaces, he delved into various topological aspects and established several fixed-point theorems.

Definition 1.12. *[10] Let* Y *be a nonempty set,* $b \geq 1$ *a given real number,* $*$ *a continuous complex valued t-norm, and* $M_{\mathfrak{C}b}$ *a complex fuzzy set on* $Y^2 \times P_\theta$ *fulfills the following criteria:*

(1) $\theta \prec M_{\mathfrak{C}\mathfrak{b}}(\vartheta, \varkappa, t)$;

(2) $M_{\mathfrak{C}\mathfrak{b}}(\vartheta, \varkappa, t) = l$ *if and only if* $\vartheta = \varkappa$ *for all* $t \in P_\theta$;

(3) $M_{\mathfrak{C}\mathfrak{b}}(\vartheta, \varkappa, t) = M_{\mathfrak{C}\mathfrak{b}}(\varkappa, \vartheta, t)$;

(4) $M_{\mathfrak{C}\mathfrak{b}}(\vartheta, \tilde{\vartheta}, t) * M_{\mathfrak{C}\mathfrak{b}}(\tilde{\vartheta}, \varkappa, t') \preceq M_{\mathfrak{C}\mathfrak{b}}(\vartheta, \varkappa, b(t + t'))$;

(5) $M_{\mathfrak{C}\mathfrak{b}}(\vartheta, \varkappa, t) : P_\theta \to I$ *is continuous*

for all $\vartheta, \varkappa, \tilde{\vartheta} \in Y$ *and* $t, t' \in P_\theta$. *The quadruple* $(Y, M_{\mathfrak{C}\mathfrak{b}}, *, b)$ *is called a complex valued fuzzy b-metric space and* $M_{\mathfrak{C}\mathfrak{b}}$ *is called a complex valued fuzzy b-metric on X.*

On the other hand, the kinematics of numerous complicated and transient systems can be modeled using fractional differential equations. They can be found in a wide range of scientific and engineering disciplines, including physics, chemistry, biology, biophysics, economics, control theory, signal and image processing, and so on. Fractional derivatives are very useful for modelling nonlinear systems that describe a variety of phenomena. The widespread use of FDEs in engineering and science has resulted in a huge increase in research in this area everywhere across the world.

Meanwhile, mathematicians assert that a complex structure and processes cannot be explained by a single differential equation. As a result, coupled systems with fractional differential equations have garnered considerable attention, and a lot of work has gone into studying. The analysis of coupled systems of differential equations of various orders is particularly important since this this sort of system emerges in a plethora of applied applications; see [7, 11, 14, 23, 27, 28] and the references given therein. The implicit FODEs are a kind of differential equation that is crucially significant. Several researchers have investigated on implicit equations; see [5, 9, 18, 22, 24] and the sources referenced therein for more information. Various fixed point methodologies have been used by many authors to extrapolate the preceding results to the ensuing equations.

Though there are only limited results in a fuzzy version of complex valued metric spaces, the eventual goal in this manuscript is to embrace the concept of a complex valued fuzzy rectangular b-metric spaces and illustrate various fixed point and coupled fixed point results. Ultimately, we delve at the existence and uniqueness of a coupled system of nonlinear implicit fractional differential equations, as seen below.

$$\begin{aligned}
&{}^cD^w\vartheta(t) - \chi_1(t, \varkappa(t), {}^cD^w\vartheta(t)) = 0; t \in \mathcal{J}\\
&{}^cD^z\varkappa(t) - \chi_2(t, \vartheta(t), {}^cD^z\varkappa(t)) = 0; t \in \mathcal{J}\\
&\vartheta(t)|_{t=0} = -\vartheta(t)|_{t=\mathsf{U}}, \vartheta'(t)|_{t=0} = -\vartheta'(t)|_{t=\mathsf{U}}, \vartheta''(t)|_{t=0} = -\vartheta'''(t)|_{t=\mathsf{U}},\\
&\varkappa(t)|_{t=0} = -\varkappa(t)|_{t=\mathsf{U}}, \varkappa'(t)|_{t=0} = -\varkappa'(t)|_{t=\mathsf{U}}, \varkappa''(t)|_{t=0} = -\varkappa'''(t)|_{t=\mathsf{U}},
\end{aligned} \tag{1.1}$$

where $3 < w, z \leq 4, \mathcal{J} = [0, \mathsf{U}]$ with $\mathsf{U} > 0$. The functions $\chi_1, \chi_2 : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.

The following is a description of the manuscript's structure: In Section 1, we provide some essential facts that will be used to explain our primary results. Section 2 presents the concept of a newly developed metric space, namely complex valued fuzzy rectangular b-metric spaces, as well as an important lemma for the metric defined is proved. In Section 3 and Section 4, various new and interesting fixed point and coupled fixed point results are illustrated. In Section 5, we establish some necessary conditions for the Green's function of the proposed Implicit FODE's. Under the appropriate conditions, we explore the existence theory as well as uniqueness of solutions to the stated problem in (1.1) respectively, by applying our coupled fixed point theorem derived in the previous section. The conclusion of the work is presented in Section 6.

2. Main results

The idea of complex valued fuzzy rectangular b-metric spaces (or simply *CVFRb-MS*) is presented in this main section, together with an example and a fundamental lemma for the underlying metric.

Definition 2.1. *Let Y be a nonempty set, $b \geq 1$ a given real number, $*$ a continuous complex valued t-norm, and $M_{\mathcal{R}}$ a complex fuzzy set on $Y^2 \times P_\theta$ fulfills the following criteria:*

(1) $\theta \prec M_{\mathcal{R}}(\vartheta, \varkappa, t)$;

(2) $M_{\mathcal{R}}(\vartheta, \varkappa, t) = l$ if and only if $\vartheta = \varkappa$, for all $t \in P_\theta$;

(3) $M_{\mathcal{R}}(\vartheta, \varkappa, t) = M_{\mathcal{R}}(\varkappa, \vartheta, t)$;

(4) $M_{\mathcal{R}}(\vartheta, \tilde{\vartheta}, t) * M_{\mathcal{R}}(\tilde{\vartheta}, \tilde{\varkappa}, t') * M_{\mathcal{R}}(\tilde{\varkappa}, \varkappa, t'') \leq M_{\mathcal{R}}(\vartheta, \varkappa, b(t + t' + t''))$;

(5) $M_{\mathcal{R}}(\vartheta, \varkappa, t) : P_\theta \to I$ is continuous

for all $t, t', t'' \in P_\theta$, $\vartheta, \varkappa \in Y$ and distinct $\tilde{\vartheta}, \tilde{\varkappa} \in Y \backslash \{\vartheta, \varkappa\}$. The quadruple $(Y, M_{\mathcal{R}}, *, b)$ is called a complex valued fuzzy rectangular b-metric space and $M_{\mathcal{R}}$ is called a complex valued fuzzy rectangular b-metric on X.

Example 2.2. *Let $Y = \mathbb{N}$ and $M_{\mathcal{R}} : Y \times Y \times P_\theta \to I$ be given by*

$$M_{\mathcal{R}}(\vartheta, \varkappa, t) = e^{-\frac{(\vartheta-\varkappa)^2}{c+d}} l$$

*for all $\vartheta, \varkappa \in Y, t = (c, d) \in P_\theta$. Define * by $a_1 * a_2 = (c_1c_2, d_1d_2)$ where $a_1 = (c_1, d_1), a_2 = (c_2, d_2) \in I$. Then $(X, M_{\mathcal{R}}, *, b)$ is a complex valued fuzzy rectangular b-metric space with $b = 3$.*

Example 2.3. *Let $Y = \mathbb{N}$ and the function $\mathcal{R}_b : Y \times Y \to \mathbb{R}$ such that $\mathcal{R}_b$ is defined by*

$$\mathcal{R}_b(\vartheta, \varkappa) = \begin{cases} 0, & \text{if } \vartheta = \varkappa \\ 4x, & \text{if } \vartheta, \varkappa \in \{1, 2\}, \vartheta \neq \varkappa \\ x, & \text{if } \vartheta \text{ or } \varkappa \notin \{1, 2\} \text{ and } \vartheta \neq \varkappa \end{cases}$$

where $x > 0$. Then $(Y, \mathcal{R}_b)$ is a rectangular b-metric space with $b = \frac{4}{3}$. Let $M_{\mathcal{R}} : Y^2 \times P_\theta \to I$ can be defined as

$$M_{\mathcal{R}}(\vartheta, \varkappa, t) = \begin{cases} \frac{ef}{ef + \mathcal{R}_b(\vartheta, \varkappa)} l, & \text{if } t \in P_\theta \\ \theta, & \text{if } t = \theta \end{cases}$$

where $t = (e, f) \in P_\theta$ and $t_1 * t_2 = (min\{e_1, f_1\}, min\{e_2, f_2\})$ for $t_1 = (e_1, e_2), t_2 = (f_1, f_2) \in I$. Hence $(X, M_r, *)$ is determined as a complex valued fuzzy rectangular b-metric space with $b = \frac{4}{3}$. The first, second, third, and fifth conditions of Definition 2.1 are all factually true. Consider the following for any $\vartheta, \varkappa \in Y$ and $t = (e_1, f_1), t' = (e_2, f_2)$ and $t'' = (e_3, f_3)$ to illustrate the forth condition

$$M_{\mathcal{R}}(\vartheta, \varkappa, b(t+t'+t'')) = M_{\mathcal{R}}\Big(\vartheta, \varkappa, \frac{4}{3}\Big((e_1, f_1)+(e_2, f_2)+(e_3, f_3)\Big)\Big) = \frac{\frac{4}{3}(e_1 + e_2 + e_3)(f_1 + f_2 + f_3)}{\frac{4}{3}(e_1 + e_2 + e_3)(f_1 + f_2 + f_3) + \mathcal{R}_b(\vartheta, \varkappa)} l.$$

For $\vartheta = 1, \varkappa = 2$

$$M_{\mathcal{R}}(1,2,b(t+t'+t'')) = \frac{\frac{4}{3}(e_1+e_2+e_3)(f_1+f_2+f_3)}{\frac{4}{3}(e_1+e_2+e_3)(f_1+f_2+f_3)+\mathcal{R}_b(1,2)}l$$
$$= \frac{\frac{4}{3}(e_1+e_2+e_3)(f_1+f_2+f_3)}{\frac{4}{3}(e_1+e_2+e_3)(f_1+f_2+f_3)+4x}l$$
$$= l - \frac{4x}{\frac{4}{3}(e_1+e_2+e_3)(f_1+f_2+f_3)+4x}l.$$

In a related manner, we can see that

$$M_{\mathcal{R}}(1,3,t) = \frac{e_1f_1}{e_1f_1+x}l = l - \frac{x}{e_1f_1+x}l, \quad M_{\mathcal{R}}(3,4,t') = \frac{e_2f_2}{e_2f_2+x}l = l - \frac{x}{e_2f_2+x}l,$$
$$M_{\mathcal{R}}(4,2,t'') = \frac{e_3f_3}{e_3f_3+x}l = l - \frac{x}{e_3f_3+x}l.$$

As a consequence, for all $t, t', t'' \in P_\theta$, *we get*

$$M_{\mathcal{R}}(1,2,b(t+t'+t'')) = l - \frac{4x}{\frac{4}{3}(e_1+e_2+e_3)(f_1+f_2+f_3)+4x}l$$
$$= l - \frac{x}{\frac{1}{3}(e_1+e_2+e_3)(f_1+f_2+f_3)+x}l$$
$$\succ l - \frac{x}{e_1f_1+x}l$$
$$= M_{\mathcal{R}}(1,3,t) = \left(\frac{e_1f_1}{e_1f_1+x}, \frac{e_1f_1}{e_1f_1+x}\right).$$

Likewise, it can be proven that

$$M_{\mathcal{R}}(1,2,b(t+t'+t'')) \succ M_{\mathcal{R}}(3,4,t') = \left(\frac{e_2f_2}{e_2f_2+x}, \frac{e_2f_2}{e_2f_2+x}\right)$$
$$\text{and } M_{\mathcal{R}}(1,2,b(t+t'+t'')) \succ M_{\mathcal{R}}(4,2,t'') = \left(\frac{e_3f_3}{e_3f_3+x}, \frac{e_3f_3}{e_3f_3+x}\right).$$

As a result, it follows that

$$M_{\mathcal{R}}(1,2,b(t+t'+t'')) \succeq \left(min\left\{\frac{e_1f_1}{e_1f_1+x}, \frac{e_2f_2}{e_2f_2+x}, \frac{e_3f_3}{e_3f_3+x}\right\}, min\left\{\frac{e_1f_1}{e_1f_1+x}, \frac{e_2f_2}{e_2f_2+x}, \frac{e_3f_3}{e_3f_3+x}\right\}\right).$$
$$= \left(\frac{e_1f_1}{e_1f_1+x}, \frac{e_1f_1}{e_1f_1+x}\right) * \left(\frac{e_2f_2}{e_2f_2+x}, \frac{e_2f_2}{e_2f_2+x}\right) * \left(\frac{e_3f_3}{e_3f_3+x}, \frac{e_3f_3}{e_3f_3+x}\right)$$
$$= M_{\mathcal{R}}(1,3,t) * M_{\mathcal{R}}(3,4,t') * M_{\mathcal{R}}(4,2,t'').$$

The subsequent cases can be checked in the same way. Hence, for all $t, t', t'' > 0$

$$M_{\mathcal{R}}(\vartheta,\varkappa,b(t+t'+t'')) \succeq M_{\mathcal{R}}(\vartheta,\tilde{\vartheta},t) * M_{\mathcal{R}}(\tilde{\vartheta},\tilde{\varkappa},t') * M_{\mathcal{R}}(\tilde{\varkappa},\varkappa,t'').$$

Therefore $(X, M_{\mathcal{R}}, *)$ *is a complex valued fuzzy rectangular b-metric space.*

Example 2.4. *Let $Y = U \cup W$, where $U = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $W = \mathbb{N}$. Define $M_{\mathcal{R}} : Y \times Y \times P_\theta \to I$ so that $M_{\mathcal{R}}(\vartheta, \varkappa, t) = M_{\mathcal{R}}(\varkappa, \vartheta, t)$ for all $\vartheta, \varkappa \in Y$ and*

$$M_{\mathcal{R}}(\vartheta, \varkappa, t) = \begin{cases} l, & \text{if } \vartheta = \varkappa \\ \frac{1}{c+d+\vartheta+8} l, & \text{if } \vartheta \in U, \varkappa \in \{6,7\}, \text{ or } \vartheta \in \{6,7\}, \varkappa \in U \\ \frac{1}{c+d+2} l, & \text{otherwise} \end{cases}$$

where $t = (c, d) \in P_\theta$. Define $*$ by $a_1 * a_2 = (c_1c_2, d_1d_2)$ where $a_1 = (c_1, d_1), a_2 = (c_2, d_2) \in I$. Then $(X, M_{\mathcal{R}}, *, b)$ is a complex valued fuzzy rectangular b-metric space with $b = 5$. However, we can infer that $(X, M_{\mathcal{R}}, *, b)$ is not a complex valued fuzzy b-metric space, because for $t = (1, 2)$ and $t' = (1, 3)$, we have

$$\begin{aligned} M_{\mathcal{R}}\left(\frac{1}{2}, \frac{1}{3}, b(t + t')\right) &= M_{\mathcal{R}}\left(\frac{1}{2}, \frac{1}{3}, 5((1, 2) + (1, 3))\right) \\ &= M_{\mathcal{R}}\left(\frac{1}{2}, \frac{1}{3}, (10, 25)\right) \\ &= (0.027, 0.027) \\ &\prec (0.033, 0.033) = M_{\mathcal{R}}\left(\frac{1}{2}, \frac{1}{4}, (1, 2)\right) * M_{\mathcal{R}}\left(\frac{1}{4}, \frac{1}{3}, (1, 3)\right). \end{aligned}$$

Lemma 2.5. *Let $(Y, M_{\mathcal{R}}, *, b)$ be a complex valued fuzzy rectangular b-metric space and $t_1, t_2 \in \mathbb{C}$. If $t_1 \prec t_2$ then $M_{\mathcal{R}}(\vartheta, \varkappa, t_1) \preceq M(\vartheta, \varkappa, bt_2)$ for all $\vartheta, \varkappa \in Y$.*

Proof. Let $t_1, t_2, t_3 \in P$ such that $t_1 \prec t_3 \prec t_2$. Therefore $t_3 - t_1 \in P_\theta$ and $t_2 - t_3 \in P_\theta$ so we have for all $\vartheta, \varkappa \in Y$,

$$\begin{aligned} M_{\mathcal{R}}(\vartheta, \varkappa, t_1) &= l * l * M_{\mathcal{R}}(\vartheta, \varkappa, t_1) \\ &= M_{\mathcal{R}}(\vartheta, \vartheta, t_3 - t_1) * M_{\mathcal{R}}(\vartheta, \vartheta, t_2 - t_3) * M_{\mathcal{R}}(\vartheta, \varkappa, t_1) \\ &\succeq M_{\mathcal{R}}(\vartheta, \varkappa, bt_2). \end{aligned}$$

□

Definition 2.6. *Let $(Y, M_{\mathcal{R}}, *, b)$ be a complex valued fuzzy rectangular b-metric space. An open ball $B(\vartheta, r, t)$ with center $\vartheta \in Y$ and radius $r \in I_\theta$, $t \in P_\theta$ is defined as*

$$B(\vartheta, r, t) = \{\varkappa \in X : l - r \prec M_{\mathcal{R}}(\vartheta, \varkappa, t)\}. \tag{2.1}$$

Definition 2.7. *Let $(Y, M_{\mathcal{R}}, *)$ be a complex valued fuzzy rectangular b-metric space. A sequence $\{\vartheta_n\}$ in Y is said to be*

(1) convergent, if for every $r \in I_\theta$ and $t \in P_\theta$, there exists $n_0 \in \mathbb{N}$ such that for all $l - r \prec M_{\mathcal{R}}(\vartheta, \varkappa, t)$. We denote it by $\lim_{n\to\infty} \vartheta_n = \vartheta$.

(2) Cauchy if for every $t \in P_\theta$, $\lim_{n\to\infty} \inf_{m>n} M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t) = l$.

(3) $(Y, M_{\mathcal{R}}, *)$ is called complete CVFRb-MS, if every Cauchy sequence in Y converges to some ϑ in Y.

We exclude the proof of the next lemma as it is similar to arguments of [25].

Lemma 2.8. *Let $(Y, M_{\mathcal{R}}, *, b)$ be a complex valued fuzzy rectangular b-metric space. A sequence $\{\vartheta_n\}$ in Y converges to $\vartheta \in Y$ if and only if $\lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_n, \vartheta, t) = l$ holds for all $t \in P_\theta$.*

3. Fixed point theorem on *CVFRb-MS*

Theorem 3.1. *Let $(Y, M_{\mathcal{R}}, *, b)$ be a complete complex valued fuzzy rectangular b-metric space and $\mathcal{P} : Y \to Y$ be a mapping that fulfills the criteria*

$$l - M_{\mathcal{R}}(\mathcal{P}\vartheta, \mathcal{P}\varkappa, \beta t) \prec \frac{1}{b}(l - \Delta(\vartheta, \varkappa, t)), \forall \vartheta, \varkappa \in Y \text{ and } t \in P_{\theta}, \tag{3.1}$$

where $\beta \in (0, 1)$ and

$$\begin{aligned} \Delta(\vartheta, \varkappa, t) = max\Big\{ & M_{\mathcal{R}}(\vartheta, \varkappa, 3b^2t), \frac{2M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, bt)(l + M_{\mathcal{R}}(\vartheta, \varkappa, 3b^2t))}{l + M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, bt)}, \\ & \frac{2M_{\mathcal{R}}(\varkappa, \mathcal{P}\varkappa, bt)(l + M_{\mathcal{R}}(\varkappa, \mathcal{P}^2\vartheta, 3b^2t))}{l + M_{\mathcal{R}}(\varkappa, \mathcal{P}\varkappa, bt)}, \\ & \frac{M_{\mathcal{R}}(\vartheta, \mathcal{P}^2\varkappa, 3b^2t)M_{\mathcal{R}}(\varkappa, \mathcal{P}^2\vartheta, bt)}{M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, bt) * M_{\mathcal{R}}(\varkappa, \mathcal{P}\varkappa, bt) * M_{\mathcal{R}}(\mathcal{P}\varkappa, \mathcal{P}^2\varkappa, bt)}\Big\}. \end{aligned} \tag{3.2}$$

Then $\mathcal{P}$ has a unique fixed point in Y.

Proof. Let $\vartheta_0 \in Y$ be the starting point for a sequence $\{\vartheta_n\}$ where $\vartheta_n = \mathcal{P}\vartheta_{n-1}$ for all $n \in \mathbb{N}$. Suppose, for the sake of generality $\vartheta_n \neq \vartheta_{n+1}, \forall n \geq 0$. Using Eq (3.2) and Lemma 2.5, we obtain

$$\begin{aligned} l - M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, \beta t) &= l - M_{\mathcal{R}}(\mathcal{P}\vartheta_n, \mathcal{P}\vartheta_{n+1}, \beta t) \\ &\prec \frac{1}{b}\Big(l - \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, 3b^2t), \frac{2M_{\mathcal{R}}(\vartheta_n, \mathcal{P}\vartheta_n, bt)(l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, 3b^2t))}{l + M_{\mathcal{R}}(\vartheta_n, \mathcal{P}\vartheta_n, bt)}, \\ &\quad \frac{2M_{\mathcal{R}}(\vartheta_{n+1}, \mathcal{P}\vartheta_{n+1}, bt)(l + M_{\mathcal{R}}(\vartheta_{n+1}, \mathcal{P}^2\vartheta_n, 3b^2t))}{l + M_{\mathcal{R}}(\vartheta_{n+1}, \mathcal{P}\vartheta_{n+1}, bt)}, \\ &\quad \frac{M_{\mathcal{R}}(\vartheta_n, \mathcal{P}^2\vartheta_{n+1}, 3b^2t)M_{\mathcal{R}}(\vartheta_{n+1}, \mathcal{P}^2\vartheta_n, bt)}{M_{\mathcal{R}}(\vartheta_n, \mathcal{P}\vartheta_n, bt) * M_{\mathcal{R}}(\vartheta_{n+1}, \mathcal{P}\vartheta_{n+1}, bt) * M_{\mathcal{R}}(\mathcal{P}\vartheta_{n+1}, \mathcal{P}^2\vartheta_{n+1}, bt)}\Big\}\Big) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{b}\Bigg(l - \max\bigg\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, 3b^2t), \frac{2M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt)(l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, 3b^2t))}{l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt)},\\
&\qquad \frac{2M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt)(l + M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, 3b^2t))}{l + M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt)},\\
&\qquad \frac{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+3}, 3b^2t)M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt)}{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt) * M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt) * M_{\mathcal{R}}(\vartheta_{n+2}, \vartheta_{n+3}, bt)}\bigg\}\Bigg)\\
&\prec \frac{1}{b}\Bigg(l - \max\bigg\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt), \frac{2M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt)(l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt))}{l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt)},\\
&\qquad \frac{2M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt)(l + M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt))}{l + M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt)},\\
&\qquad \frac{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+3}, bt)M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt)}{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt) * M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt) * M_{\mathcal{R}}(\vartheta_{n+2}, \vartheta_{n+3}, bt)}\bigg\}\Bigg) \qquad (3.3)\\
&\prec \frac{1}{b}\Big(l - \max\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt), M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt), M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt), M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt)\}\Big)\\
&\prec l - \max\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt), M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt), M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt), M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt)\}.
\end{aligned}
$$

This results in

$$
M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, \beta t) \succeq \max\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt), M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt)\}. \qquad (3.4)
$$

If $\max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt), M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt)\Big\} = M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt)$, then $M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, \beta t) \succeq M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt)$ which ends in a contradiction by Lemma 2.5. Hence

$$
M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, \beta t) \succeq M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt). \qquad (3.5)
$$

Since $\beta t \prec t$, by Lemma 2.5, we get $M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt) \succeq M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, \beta t)$. Then Eq (3.5) becomes

$$
M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt) \succeq M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, \beta t) \succeq M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt) \qquad (3.6)
$$

which yields $M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt) \succeq M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt)$ i.e., $M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, t) \succeq M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t)$. Now, let $\xi_n = M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t)$, we shall prove that $\lim_{n\to\infty} \xi_n = l$. We have, by definition

$$
l \succeq \xi_{n+1} \succeq \xi_n \succeq \theta. \qquad (3.7)
$$

As a result $\{\xi_n\}$ is a monotonic sequence in P, and by employing Eq (3.7) and Remark 1.4, there is an element $l' \in P$ such that

$$
\lim_{n\to\infty} \xi_n = l'. \qquad (3.8)
$$

Inequality (3.3) gives

$$
\begin{aligned}
l - M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, \beta t) &\prec \frac{1}{b}[l - \max\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t), M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt)\}]\\
\Longrightarrow \Big[1 - \frac{1}{b}\Big]l + \frac{1}{b}M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt) &\prec M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, \beta t) \preceq M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, bt).
\end{aligned} \qquad (3.9)
$$

Taking limit $n \to \infty$, we arrive at

$$
\Big[1 - \frac{1}{b}\Big](l - l') \preceq \theta. \qquad (3.10)
$$

As $\frac{1}{b} < 1$, we get $l = l'$, by utilizing Remark 1.5. Thus

$$\lim_{n\to\infty} \xi_n = l. \tag{3.11}$$

Consider

$$\begin{aligned}
l - M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, \beta t) &= l - M_{\mathcal{R}}(\mathcal{P}\vartheta_n, \mathcal{P}\vartheta_{n+2}, \beta t) \\
&\prec \frac{1}{b}\Bigg(l - \max\Bigg\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+2}, 3b^2t), \frac{2M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt)(l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+2}, 3b^2t))}{l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt)}, \\
&\qquad \frac{2M_{\mathcal{R}}(\vartheta_{n+2}, \vartheta_{n+3}, bt)(l + M_{\mathcal{R}}(\vartheta_{n+2}, \vartheta_{n+2}, 3b^2t))}{l + M_{\mathcal{R}}(\vartheta_{n+2}, \vartheta_{n+3}, bt)}, \\
&\qquad \frac{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+4}, 3b^2t)M_{\mathcal{R}}(\vartheta_{n+2}, \vartheta_{n+2}, bt)}{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt) * M_{\mathcal{R}}(\vartheta_{n+2}, \vartheta_{n+3}, bt) * M_{\mathcal{R}}(\vartheta_{n+3}, \vartheta_{n+4}, bt)}\Bigg\}\Bigg) \\
&\prec \frac{1}{b}\Bigg(l - \max\Bigg\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt) * M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, bt) * M_{\mathcal{R}}(\vartheta_{n+3}, \vartheta_{n+2}, bt), \\
&\qquad \frac{2M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt)(M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt) * M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, bt) * M_{\mathcal{R}}(\vartheta_{n+3}, \vartheta_{n+2}, bt))}{l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt)}, \\
&\qquad \frac{2lM_{\mathcal{R}}(\vartheta_{n+2}, \vartheta_{n+3}, bt)}{l + M_{\mathcal{R}}(\vartheta_{n+2}, \vartheta_{n+3}, bt)}, \frac{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt) * M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, bt) * M_{\mathcal{R}}(\vartheta_{n+3}, \vartheta_{n+4}, bt)}{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, bt) * M_{\mathcal{R}}(\vartheta_{n+2}, \vartheta_{n+3}, bt) * M_{\mathcal{R}}(\vartheta_{n+3}, \vartheta_{n+4}, bt)}\Bigg\}\Bigg).
\end{aligned} \tag{3.12}$$

Passing the limit $n \to \infty$ in the last inequality, we get

$$\begin{aligned}
l - \lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, \beta t) \prec \frac{1}{b}\Big(l - \max\{&\lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, bt), \lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, bt), \\
&l, \lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, bt)\}\Big).
\end{aligned} \tag{3.13}$$

If $\max\{\lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, bt), l\} = l$, then the inequality (3.13) becomes $\lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, bt) \succeq l$. If $\max\{\lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, bt), l\} = \lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, bt)$, inequality (3.13) then implies

$$\begin{aligned}
l - \lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, bt) &\prec l - \lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, \beta t) \prec \frac{1}{b}[l - \lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, \beta t)] \\
&\Longrightarrow \Big[1 - \frac{1}{b}\Big]l + \Big[1 - \frac{1}{b}\Big] \lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, \beta t) \prec \theta.
\end{aligned}$$

As $\frac{1}{b} < 1$, we get $\lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, bt) \succeq l$. Thus in both the cases

$$\lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, bt) \succeq l. \tag{3.14}$$

To verify that $\{\vartheta_n\}$ is Cauchy, define $R_n = \{M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t) : m > n\} \subset I$ for $n \in \mathbb{N}$ and fixed $t \in P_\theta$. Because $\theta \prec M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t) \prec l$, for all $n \in \mathbb{N}$, the infimum exists, by utilizing Remark 1.4. Consider

$M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t)$ in 2 cases. For odd m, we get

$$\begin{aligned} M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t) &\geq M_{\mathcal{R}}\Big(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\vartheta_{n+1}, \vartheta_{n+2}, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\vartheta_{n+2}, \vartheta_m, \frac{t}{3b}\Big) \\ &\geq M_{\mathcal{R}}\Big(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\vartheta_{n+1}, \vartheta_{n+2}, \frac{t}{3b}\Big) * \ldots * M_{\mathcal{R}}\Big(\vartheta_{m-3}, \vartheta_{m-2}, \frac{t}{(3b)^{\frac{m-n-1}{2}}}\Big) \\ &\quad * M_{\mathcal{R}}\Big(\vartheta_{m-2}, \vartheta_{m-1}, \frac{t}{(3b)^{\frac{m-n-1}{2}}}\Big) * M_{\mathcal{R}}\Big(\vartheta_{m-1}, \vartheta_m, \frac{t}{(3b)^{\frac{m-n-1}{2}}}\Big). \end{aligned}$$

For even m, we get through using the rectangular inequality

$$\begin{aligned} M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t) &\geq M_{\mathcal{R}}\Big(\vartheta_n, \vartheta_{n+2}, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\vartheta_{n+2}, \vartheta_{n+3}, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\vartheta_{n+3}, \vartheta_m, \frac{t}{3b}\Big) \\ &\geq M_{\mathcal{R}}\Big(\vartheta_n, \vartheta_{n+2}, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\vartheta_{n+2}, \vartheta_{n+3}, \frac{t}{3b}\Big) * \ldots * M_{\mathcal{R}}\Big(\vartheta_{m-3}, \vartheta_{m-2}, \frac{t}{(3b)^{\frac{m-n-2}{2}}}\Big) \\ &\quad * M_{\mathcal{R}}\Big(\vartheta_{m-2}, \vartheta_{m-1}, \frac{t}{(3b)^{\frac{m-n-2}{2}}}\Big) * M_{\mathcal{R}}(\vartheta_{m-1}, \vartheta_m, \frac{t}{(3b)^{\frac{m-n-2}{2}}}\Big). \end{aligned}$$

Thus for all m, it follows that $\lim_{n\to\infty} \inf_{m>n} M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t) \geq l * l * \ldots * l = l$, by using Eqs (3.11) and (3.14), respectively. Hence

$$\lim_{n\to\infty} \inf_{m>n} M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t) = l, \text{ for all } t \in P_\theta. \tag{3.15}$$

Thus we have proven that $\{\vartheta_n\}$ is a Cauchy sequence in Y. Due to the completeness of Y and Lemma 2.8, there is an element $\vartheta \in Y$ such that

$$\lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_n, \vartheta, t) = l \text{ for all } t \in P_\theta. \tag{3.16}$$

For any $t \in P_\theta$ and for $n \in \mathbb{N}$, we deduce using Eq (3.1) that

$$\begin{aligned} l - M_{\mathcal{R}}\Big(\mathcal{P}\vartheta_n, \mathcal{P}\vartheta, \frac{t}{3b^2}\Big) &\prec l - M_{\mathcal{R}}\Big(\mathcal{P}\vartheta_n, \mathcal{P}\vartheta, \frac{\beta t}{3b^2}\Big) \\ &\prec \frac{1}{b}\Bigg(l - \max\Bigg\{M_{\mathcal{R}}(\vartheta_n, \vartheta, t), \frac{2M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b})(l + M_{\mathcal{R}}(\vartheta_n, \vartheta, t))}{l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b})}, \\ &\qquad \frac{2M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, \frac{t}{3b})(l + M_{\mathcal{R}}(\vartheta, \vartheta_{n+2}, t))}{l + M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, \frac{t}{3b})}, \\ &\qquad \frac{M_{\mathcal{R}}(\vartheta_n, \mathcal{P}^2\vartheta, t) M_{\mathcal{R}}(\vartheta, \vartheta_{n+2}, \frac{t}{3b})}{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b}) * M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, \frac{t}{3b}) * M_{\mathcal{R}}(\mathcal{P}\vartheta, \mathcal{P}^2\vartheta, \frac{t}{3b})}\Bigg\}\Bigg) \end{aligned}$$

$$\prec \frac{1}{b}\Big(l - \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta, t), \frac{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b})(l + M_{\mathcal{R}}(\vartheta_n, \vartheta, t))}{l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b})}, \frac{2M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, \frac{t}{3b})(l + M_{\mathcal{R}}(\vartheta, \vartheta_{n+2}, t))}{l + M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, \frac{t}{3b})}, M_{\mathcal{R}}(\vartheta_n, \mathcal{P}^2\vartheta, t) M_{\mathcal{R}}\Big(\vartheta, \vartheta_{n+2}, \frac{t}{3b}\Big)\Big\}\Big)$$

$$\prec l - \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta, t), \frac{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b})(l + M_{\mathcal{R}}(\vartheta_n, \vartheta, t))}{l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b})}, \frac{2M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, \frac{t}{3b})(l + M_{\mathcal{R}}(\vartheta, \vartheta_{n+2}, t))}{l + M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, \frac{t}{3b})}, M_{\mathcal{R}}(\vartheta_n, \mathcal{P}^2\vartheta, t) M_{\mathcal{R}}\Big(\vartheta, \vartheta_{n+2}, \frac{t}{3b}\Big)\Big\}.$$

Thus, we get

$$M_{\mathcal{R}}\Big(\mathcal{P}\vartheta_n, \mathcal{P}\vartheta, \frac{t}{3b^2}\Big) \geq \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta, t), \frac{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b})(l + M_{\mathcal{R}}(\vartheta_n, \vartheta, t))}{l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b})}, \frac{2M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, \frac{t}{3b})(l + M_{\mathcal{R}}(\vartheta, \vartheta_{n+2}, t))}{l + M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, \frac{t}{3b})}, M_{\mathcal{R}}(\vartheta_n, \mathcal{P}^2\vartheta, t) M_{\mathcal{R}}\Big(\vartheta, \vartheta_{n+2}, \frac{t}{3b}\Big)\Big\} \tag{3.17}$$

Finally, for any $t \in P_\theta$, we have

$$\begin{aligned} M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, t) &\geq M_{\mathcal{R}}\Big(\vartheta, \vartheta_n, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\vartheta_n, \mathcal{P}\vartheta_n, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\mathcal{P}\vartheta_n, \mathcal{P}\vartheta, \frac{t}{3b}\Big) \\ &\geq M_{\mathcal{R}}\Big(\vartheta, \vartheta_n, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\vartheta_n, \mathcal{P}\vartheta_n, \frac{t}{3b}\Big) * \\ &\quad \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta, t), \frac{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b})(l + M_{\mathcal{R}}(\vartheta_n, \vartheta, t))}{l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b})}, \\ &\quad \frac{2M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, \frac{t}{3b})(l + M_{\mathcal{R}}(\vartheta, \vartheta_{n+2}, t))}{l + M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, \frac{t}{3b})}, M_{\mathcal{R}}(\vartheta_n, \mathcal{P}^2\vartheta, t) M_{\mathcal{R}}\Big(\vartheta, \vartheta_{n+2}, \frac{t}{3b}\Big)\Big\}. \end{aligned} \tag{3.18}$$

Taking limit $n \to \infty$, we get

$$M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, t) \geq l * l * \max\Big\{l, l, \frac{2M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, \frac{t}{3b})(l + l)}{2l}, M_{\mathcal{R}}(\vartheta, \mathcal{P}^2\vartheta, t)\Big\} = l.$$

As a result, we get $M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, t) = l$ for all $t \in P_\theta$, i.e., $\mathcal{P}\vartheta = \vartheta$. Suppose that $\varkappa$ is an another fixed point of $\mathcal{P}$ and there exists $t \in P_\theta$ with $M_{\mathcal{R}}(\vartheta, \varkappa, t) \neq l$, then

$$\begin{aligned}
l - M_{\mathcal{R}}(\vartheta, \varkappa, \beta t) &= l - M_{\mathcal{R}}(\mathcal{P}\vartheta, \mathcal{P}\varkappa, \beta t) \\
&\prec \frac{1}{b}\Bigg[l - \max\Bigg\{M_{\mathcal{R}}(\vartheta, \varkappa, 3b^2 t), \frac{2M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, bt)(l + M_{\mathcal{R}}(\vartheta, \varkappa, 3b^2 t))}{l + M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, bt)}, \\
&\qquad \frac{2M_{\mathcal{R}}(\varkappa, \mathcal{P}\varkappa, bt)(l + M_{\mathcal{R}}(\varkappa, \mathcal{P}^2\vartheta, 3b^2 t))}{l + M_{\mathcal{R}}(\varkappa, \mathcal{P}\varkappa, bt)}, \\
&\qquad \frac{M_{\mathcal{R}}(\vartheta, \mathcal{P}^2\varkappa, 3b^2 t) M_{\mathcal{R}}(\varkappa, \mathcal{P}^2\vartheta, bt)}{M_{\mathcal{R}}(\vartheta, \mathcal{P}\vartheta, bt) * M_{\mathcal{R}}(\varkappa, \mathcal{P}\varkappa, bt) * M_{\mathcal{R}}(\mathcal{P}\varkappa, \mathcal{P}^2\varkappa, bt)}\Bigg\}\Bigg] \\
&= \frac{1}{b}\Bigg[l - \max\Bigg\{M_{\mathcal{R}}(\vartheta, \varkappa, 3b^2 t), \frac{2l(l + M_{\mathcal{R}}(\vartheta, \varkappa, 3b^2 t))}{l + l}, \\
&\qquad \frac{2l(l + M_{\mathcal{R}}(\varkappa, \vartheta, 3b^2 t))}{l + l}, \frac{M_{\mathcal{R}}(\vartheta, \varkappa, 3b^2 t) M_{\mathcal{R}}(\varkappa, \vartheta, bt)}{l * l * l}\Bigg\}\Bigg] \\
&\prec l - \max\Big\{M_{\mathcal{R}}(\vartheta, \varkappa, 3b^2 t), M_{\mathcal{R}}(\vartheta, \varkappa, 3b^2 t), M_{\mathcal{R}}(\varkappa, \vartheta, 3b^2 t), \\
&\qquad M_{\mathcal{R}}(\vartheta, \varkappa, 3b^2 t) M_{\mathcal{R}}(\varkappa, \vartheta, bt)\Big\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
M_{\mathcal{R}}(\vartheta, \varkappa, \beta t) &\succeq \max\{M_{\mathcal{R}}(\vartheta, \varkappa, 3b^2 t), M_{\mathcal{R}}(\vartheta, \varkappa, 3b^2 t) M_{\mathcal{R}}(\varkappa, \vartheta, bt)\} \\
&= M_{\mathcal{R}}(\vartheta, \varkappa, 3b^2 t),
\end{aligned} \tag{3.19}$$

which ends in a contradiction, according to Lemma 2.5. Therefore $\vartheta = \varkappa$ is the unique fixed point of $\mathcal{P}$. □

4. Coupled fixed point theorems on *CVFRb-MS*

Theorem 4.1. *Let $(Y, M_{\mathcal{R}}, *, b)$ be a complete complex valued fuzzy rectangular b-metric space and $\mathcal{P} : Y \to Y$ be a mapping that fulfills the criteria*

$$l - M_{\mathcal{R}}(\mathcal{P}(\vartheta, \varkappa), \mathcal{P}(\tilde{\vartheta}, \tilde{\varkappa}), t) \prec \beta(l - \Delta((\vartheta, \varkappa), (\tilde{\vartheta}, \tilde{\varkappa}), t)), \forall (\vartheta, \varkappa), (\tilde{\vartheta}, \tilde{\varkappa}) \in Y \times Y \text{ and } t \in P_\theta, \tag{4.1}$$

where $\beta \in (0, 1)$ and

$$\begin{aligned}
\Delta((\vartheta, \varkappa), (\tilde{\vartheta}, \tilde{\varkappa}), t) = max\Bigg\{&M_{\mathcal{R}}(\vartheta, \tilde{\vartheta}, 3bt), M_{\mathcal{R}}(\varkappa, \tilde{\varkappa}, 3bt), \frac{(l + M_{\mathcal{R}}(\vartheta, \tilde{\vartheta}, 3bt))(l + M_{\mathcal{R}}(\vartheta, \mathcal{P}(\vartheta, \varkappa), 3bt))}{M_{\mathcal{R}}(\vartheta, \mathcal{P}(\vartheta, \varkappa), t)}, \\
&\frac{(l + M_{\mathcal{R}}(\varkappa, \tilde{\varkappa}, 3bt)(l + M_{\mathcal{R}}(\varkappa, \mathcal{P}(\varkappa, \vartheta), 3bt))}{M_{\mathcal{R}}(\varkappa, \mathcal{P}(\varkappa, \vartheta), t)}\Bigg\}.
\end{aligned} \tag{4.2}$$

Then $\mathcal{P}$ has a unique coupled fixed point of the form $(\tilde{\vartheta}, \tilde{\vartheta})$ which satisfies $\mathcal{P}\tilde{\vartheta} = (\tilde{\vartheta}, \tilde{\vartheta})$.

Proof. Let $(\vartheta_0, \varkappa_0) \in Y$ and define two sequences $\{\vartheta_n\}$ and $\{\varkappa_n\}$ by the iterative approach as

$$\vartheta_{n+1} = \mathcal{P}(\vartheta_n, \varkappa_n), \varkappa_{n+1} = \mathcal{P}(\varkappa_n, \vartheta_n) \text{ for all } n \in \mathbb{N}.$$

Suppose, for the sake of generality $\vartheta_n \neq \vartheta_{n+1}, \forall n \geq 0$. Due to Eq (4.2) and Lemma 2.5, we get

$$\begin{aligned}
l - M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, t) &= l - M_{\mathcal{R}}(\mathcal{P}(\vartheta_n, \varkappa_n), \mathcal{P}(\vartheta_{n+1}, \varkappa_{n+1}), t) \\
&\prec \beta\Big(l - \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, 3bt), M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, 3bt), \\
&\qquad \frac{(l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, 3bt))(l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, 3bt))}{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t)}, \\
&\qquad \frac{(l + M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, 3bt))(l + M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, 3bt))}{M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t)}\Big\}\Big) \\
&\prec \beta\Big(l - \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t), M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t), M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t), \\
&\qquad M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t)\Big\}\Big) \\
&\prec l - \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t), M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t)\Big\},
\end{aligned} \tag{4.3}$$

which gives

$$M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+2}, t) \succeq \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t), M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t)\Big\}. \tag{4.4}$$

Similarly, we can find

$$l - M_{\mathcal{R}}(\varkappa_{n+1}, \varkappa_{n+2}, t) \prec l - \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t), M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t)\Big\}, \tag{4.5}$$

which is equivalent to

$$M_{\mathcal{R}}(\varkappa_{n+1}, \varkappa_{n+2}, t) \succeq \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t), M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t)\Big\}. \tag{4.6}$$

Combining Eqs (4.4) and (4.6), we get $\phi_{n+1} \succeq \phi_n$ for all $n \in \mathbb{N}$, where $\phi_n = \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t), M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t)\Big\}$. By definition, we have $l \succeq \phi_{n+1} \succeq \phi_n \succeq \theta$. Thus $\{\phi_n\}$ is a monotonic sequence in Y and hence by Remark 1.4, there exists $l' \in P_\theta$ such that $\lim_{n\to\infty} \phi_n = l'$. From inequality (4.3) and (4.5), it implies that

$$(1-\beta)l + \beta\phi_n \prec \phi_{n+1}. \tag{4.7}$$

Letting limit $n \to \infty$ and by employing Remark 1.5, we get $l = l'$. Thus

$$\lim_{n\to\infty} \phi_n = l. \tag{4.8}$$

By considering the inequalities (4.4) and (4.6), we find

$$
\begin{aligned}
l - M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, t) &= l - M_{\mathcal{R}}(\mathcal{P}(\vartheta_n, \varkappa_n), \mathcal{P}(\vartheta_{n+2}, \varkappa_{n+2}), t) \\
&\prec \beta\Big(l - \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+2}, 3bt), M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+2}, 3bt), \\
&\qquad \frac{(l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+2}, 3bt))(l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, 3bt))}{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t)}, \\
&\qquad \frac{(l + M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+2}, 3bt))(l + M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, 3bt))}{M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t)}\Big\}\Big) \\
&\prec \beta\Big(l - \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t) * M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, t) * M_{\mathcal{R}}(\vartheta_{n+3}, \vartheta_{n+2}, t), \\
&\qquad M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t) * M_{\mathcal{R}}(\varkappa_{n+1}, \varkappa_{n+3}, t) * M_{\mathcal{R}}(\varkappa_{n+3}, \varkappa_{n+2}, t), \\
&\qquad \frac{[M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t) * M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, t) * M_{\mathcal{R}}(\vartheta_{n+3}, \vartheta_{n+2}, t)]M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, 3bt)}{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t)} \\
&\qquad \frac{[M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t) * M_{\mathcal{R}}(\varkappa_{n+1}, \varkappa_{n+3}, t) * M_{\mathcal{R}}(\varkappa_{n+3}, \varkappa_{n+2}, t)]M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, 3bt)}{M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t)}\Big\}\Big) \\
&\prec \beta\Big(l - \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t) * M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, t) * M_{\mathcal{R}}(\vartheta_{n+3}, \vartheta_{n+2}, t), \\
&\qquad M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t) * M_{\mathcal{R}}(\varkappa_{n+1}, \varkappa_{n+3}, t) * M_{\mathcal{R}}(\varkappa_{n+3}, \varkappa_{n+2}, t)\Big\}\Big) \\
&\prec \beta\Big(l - \max\Big\{\max\{M_{\mathcal{R}}(\vartheta_{n-1}, \vartheta_n, t), M_{\mathcal{R}}(\varkappa_{n-1}, \varkappa_n, t)\} * M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, t) \\
&\qquad * \max\{M_{\mathcal{R}}(\vartheta_{n+2}, \vartheta_{n+1}, t), M_{\mathcal{R}}(\varkappa_{n+2}, \varkappa_{n+1}, t)\}, \\
&\qquad \max\{M_{\mathcal{R}}(\varkappa_{n-1}, \varkappa_n, t), M_{\mathcal{R}}(\vartheta_{n-1}, \vartheta_n, t)\} * M_{\mathcal{R}}(\varkappa_{n+1}, \varkappa_{n+3}, t) \\
&\qquad * \max\{M_{\mathcal{R}}(\varkappa_{n+2}, \varkappa_{n+1}, t), M_{\mathcal{R}}(\vartheta_{n+2}, \vartheta_{n+1}, t)\}\Big\}\Big).
\end{aligned} \tag{4.9}
$$

By taking a limit on both sides, we arrive at

$$
\begin{aligned}
&l - \lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, t) \prec \beta\Big[l - \lim_{n\to\infty} \max\{M_{\mathcal{R}}(\varkappa_{n+1}, \varkappa_{n+3}, t), M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, t)\}\Big] \\
\Longrightarrow\ &(1-\beta)l + \beta \lim_{n\to\infty} \max\{M_{\mathcal{R}}(\varkappa_{n+1}, \varkappa_{n+3}, t), M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, t)\} \prec \lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, t).
\end{aligned}
$$

Likewise, we can prove

$$
(1-\beta)l + \beta \lim_{n\to\infty} \max\{M_{\mathcal{R}}(\varkappa_{n+1}, \varkappa_{n+3}, t), M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, t)\} \prec \lim_{n\to\infty} M_{\mathcal{R}}(\varkappa_{n+1}, \varkappa_{n+3}, t),
$$

which gives

$$
(1-\beta)l + \beta \lim_{n\to\infty} \phi^*_{n+1} \prec \lim_{n\to\infty} \phi^*_{n+1}, \tag{4.10}
$$

where $\phi^*_{n+1} = \max\{M_{\mathcal{R}}(\varkappa_{n+1}, \varkappa_{n+3}, t), M_{\mathcal{R}}(\vartheta_{n+1}, \vartheta_{n+3}, t)\}$. As $\beta < 1$, from inequality (4.10) we get

$$
\lim_{n\to\infty} \phi^*_{n+1} \succeq l, \text{ for all } n \in \mathbb{N}. \tag{4.11}
$$

To illustrate that $\{\vartheta_n\}$ and $\{\varkappa_n\}$ is Cauchy, define $R_n = \{M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t) : m > n\} \subset I$ and $S_n = \{M_{\mathcal{R}}(\varkappa_n, \varkappa_m, t) : m > n\} \subset I$ for $n \in \mathbb{N}$ and fixed $t \in P_\theta$. Due to the fact that $\theta \prec M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t) \prec l$, and

$\theta \prec M_{\mathcal{R}}(\varkappa_n, \varkappa_m, t) \prec l$ for all $n \in \mathbb{N}$, the infimum exists by utilizing Remark 1.4. Consider $M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t)$ in 2 cases, i.e., for both odd and even m and using Eqs (4.4) and (4.6), we get

$$\begin{aligned}
M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t) &\geq M_{\mathcal{R}}\Big(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\vartheta_{n+1}, \vartheta_{n+2}, \frac{t}{3b}\Big) * \ldots * M_{\mathcal{R}}\Big(\vartheta_{m-3}, \vartheta_{m-2}, \frac{t}{(3b)^{\frac{m-n-1}{2}}}\Big) \\
&\quad * M_{\mathcal{R}}\Big(\vartheta_{m-2}, \vartheta_{m-1}, \frac{t}{(3b)^{\frac{m-n-1}{2}}}\Big) * M_{\mathcal{R}}\Big(\vartheta_{m-1}, \vartheta_m, \frac{t}{(3b)^{\frac{m-n-1}{2}}}\Big) \\
&\geq \max\Big\{M_{\mathcal{R}}\Big(\vartheta_{n-1}, \vartheta_n, \frac{t}{3b}\Big), M_{\mathcal{R}}\Big(\varkappa_{n-1}, \varkappa_n, \frac{t}{3b}\Big)\Big\} * \\
&\quad \max\Big\{M_{\mathcal{R}}\Big(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b}\Big), M_{\mathcal{R}}\Big(\varkappa_n, \varkappa_{n+1}, \frac{t}{3b}\Big)\Big\} * \\
&\qquad \vdots \\
&\quad \max\Big\{M_{\mathcal{R}}\Big(\vartheta_{m-2}, \vartheta_{m-1}, \frac{t}{(3b)^{\frac{m-n-1}{2}}}\Big), M_{\mathcal{R}}\Big(\varkappa_{m-2}, \varkappa_{m-1}, \frac{t}{(3b)^{\frac{m-n-1}{2}}}\Big)\Big\}.
\end{aligned}$$

and

$$\begin{aligned}
M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t) &\geq M_{\mathcal{R}}\Big(\vartheta_n, \vartheta_{n+2}, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\vartheta_{n+2}, \vartheta_{n+3}, \frac{t}{3b}\Big) * \ldots * M_{\mathcal{R}}\Big(\vartheta_{m-3}, \vartheta_{m-2}, \frac{t}{(3b)^{\frac{m-n-2}{2}}}\Big) \\
&\quad * M_{\mathcal{R}}\Big(\vartheta_{m-2}, \vartheta_{m-1}, \frac{t}{(3b)^{\frac{m-n-2}{2}}}\Big) * M_{\mathcal{R}}\Big(\vartheta_{m-1}, \vartheta_m, \frac{t}{(3b)^{\frac{m-n-2}{2}}}\Big) \\
&\geq M_{\mathcal{R}}\Big(\vartheta_n, \vartheta_{n+2}, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\varkappa_n, \varkappa_{n+2}, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\vartheta_{n+2}, \vartheta_{n+3}, \frac{t}{3b}\Big) * \ldots * \\
&\quad M_{\mathcal{R}}\Big(\vartheta_{m-3}, \vartheta_{m-2}, \frac{t}{(3b)^{\frac{m-n-2}{2}}}\Big) * M_{\mathcal{R}}\Big(\vartheta_{m-2}, \vartheta_{m-1}, \frac{t}{(3b)^{\frac{m-n-2}{2}}}\Big) * \\
&\quad M_{\mathcal{R}}\Big(\vartheta_{m-1}, \vartheta_m, \frac{t}{(3b)^{\frac{m-n-2}{2}}}\Big) \\
&\geq \max\Big\{M_{\mathcal{R}}\Big(\vartheta_n, \vartheta_{n+2}, \frac{t}{3b}\Big), M_{\mathcal{R}}\Big(\varkappa_n, \varkappa_{n+2}, \frac{t}{3b}\Big)\Big\} * \\
&\quad \max\Big\{M_{\mathcal{R}}\Big(\vartheta_{n+1}, \vartheta_{n+2}, \frac{t}{3b}\Big), M_{\mathcal{R}}\Big(\varkappa_{n+1}, \varkappa_{n+2}, \frac{t}{3b}\Big)\Big\} * \\
&\qquad \vdots \\
&\quad \max\Big\{M_{\mathcal{R}}\Big(\vartheta_{m-2}, \vartheta_{m-1}, \frac{t}{(3b)^{\frac{m-n-2}{2}}}\Big), M_{\mathcal{R}}\Big(\varkappa_{m-2}, \varkappa_{m-1}, \frac{t}{(3b)^{\frac{m-n-2}{2}}}\Big)\Big\}.
\end{aligned}$$

In both the cases, it concludes from (4.8) and (4.11) that

$$\lim_{n\to\infty} \inf_{m>n} M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t) \geq l * l * \ldots * l = l.$$

Therefore, for all $m, n \in \mathbb{N}$ and $t \in P_\theta$

$$\lim_{n\to\infty} \inf_{m>n} M_{\mathcal{R}}(\vartheta_n, \vartheta_m, t) = l. \tag{4.12}$$

By following the same aforementioned steps, we get

$$\lim_{n\to\infty}\inf_{m>n} M_{\mathcal{R}}(\varkappa_n, \varkappa_m, t) = l. \tag{4.13}$$

Therefore, from (4.12) and (4.13) it is apparent that $\{\vartheta_n\}$ and $\{\varkappa_n\}$ are Cauchy sequences in Y. Due to the completeness of Y and Lemma 2.8, there exists elements $\vartheta, \varkappa \in Y$ so that

$$\lim_{n\to\infty} M_{\mathcal{R}}(\vartheta_n, \vartheta, t) = \lim_{n\to\infty} M_{\mathcal{R}}(\varkappa_n, \varkappa, t) = l, \text{ for all } t \in P_\theta. \tag{4.14}$$

For any $t \in P_\theta$ and for $n \in \mathbb{N}$, we deduce from Eq (4.1) that

$$\begin{aligned}
l - M_{\mathcal{R}}\Big(\mathcal{P}(\vartheta_n, \varkappa_n), \mathcal{P}(\vartheta, \varkappa), \frac{t}{3b}\Big) &\prec \beta\Big(l - \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta, t), M_{\mathcal{R}}(\varkappa_n, \varkappa, t),\\
&\qquad \frac{(l + M_{\mathcal{R}}(\vartheta_n, \vartheta, t))(l + M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t))}{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{b})},\\
&\qquad \frac{(l + M_{\mathcal{R}}(\varkappa_n, \varkappa, t))(l + M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t))}{M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, \frac{t}{b})}\Big\}\Big)\\
&\prec \beta\Big(l - \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta, t), M_{\mathcal{R}}(\varkappa_n, \varkappa, t),\\
&\qquad \frac{M_{\mathcal{R}}(\vartheta_n, \vartheta, t)\, M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t)}{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b})}, \frac{M_{\mathcal{R}}(\varkappa_n, \varkappa, t)\, M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t)}{M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, \frac{t}{3b})}\Big\}\Big)\\
&\prec l - \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta, t), M_{\mathcal{R}}(\varkappa_n, \varkappa, t),\\
&\qquad \frac{M_{\mathcal{R}}(\vartheta_n, \vartheta, t) M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t)}{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b})}, \frac{M_{\mathcal{R}}(\varkappa_n, \varkappa, t) M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t)}{M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, \frac{t}{3b})}\Big\}.
\end{aligned} \tag{4.15}$$

Therefore

$$\begin{aligned}
M_{\mathcal{R}}(\vartheta, \mathcal{P}(\vartheta, \varkappa), t) &\geq M_{\mathcal{R}}\Big(\vartheta, \vartheta_n, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\vartheta_{n+1}, \mathcal{P}(\vartheta, \varkappa), \frac{t}{3b}\Big)\\
&\geq M_{\mathcal{R}}\Big(\vartheta, \vartheta_n, \frac{t}{3b}\Big) * M_{\mathcal{R}}\Big(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b}\Big) *\\
&\quad \max\Big\{M_{\mathcal{R}}(\vartheta_n, \vartheta, t), M_{\mathcal{R}}(\varkappa_n, \varkappa, t), \frac{M_{\mathcal{R}}(\vartheta_n, \vartheta, t) M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, t)}{M_{\mathcal{R}}(\vartheta_n, \vartheta_{n+1}, \frac{t}{3b})}\\
&\quad \frac{M_{\mathcal{R}}(\varkappa_n, \varkappa, t) M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, t)}{M_{\mathcal{R}}(\varkappa_n, \varkappa_{n+1}, \frac{t}{3b})}\Big\}.
\end{aligned} \tag{4.16}$$

Taking limits $n \to \infty$, we get

$$M_{\mathcal{R}}(\vartheta, \mathcal{P}(\vartheta, \varkappa), t) \geq l * l * \max\{l, l, l, l\} = l,$$

i.e., $M_{\mathcal{R}}(\vartheta, \mathcal{P}(\vartheta, \varkappa), t) = l \implies \vartheta = \mathcal{P}(\vartheta, \varkappa)$. Similarly, we can prove $\varkappa = \mathcal{P}(\varkappa, \vartheta)$. As a result, the pair

$(\vartheta, \varkappa)$ is a coupled fixed point of $\mathcal{P}$. To demonstrate (ϑ, ϑ) is a unique coupled fixed point, consider

$$\begin{aligned}
l - M_{\mathcal{R}}(\vartheta, \varkappa, t) &= l - M_{\mathcal{R}}(\mathcal{P}(\vartheta, \varkappa), \mathcal{P}(\varkappa, \vartheta), t) \\
&\prec \beta\Big(l - \max\Big\{M_{\mathcal{R}}(\vartheta, \varkappa, 3bt), M_{\mathcal{R}}(\varkappa, \vartheta, 3bt), \frac{(l + M_{\mathcal{R}}(\vartheta, \varkappa, 3bt))(l + M_{\mathcal{R}}(\vartheta, \mathcal{P}(\vartheta, \varkappa), 3bt))}{M_{\mathcal{R}}(\vartheta, \mathcal{P}(\vartheta, \varkappa), t)}, \\
&\qquad \frac{(l + M_{\mathcal{R}}(\varkappa, \vartheta, 3bt))(l + M_{\mathcal{R}}(\varkappa, \mathcal{P}(\varkappa, \vartheta), 3bt))}{M_{\mathcal{R}}(\varkappa, \mathcal{P}(\varkappa, \vartheta), t)}\Big\}\Big) \\
&\prec \beta\Big(l - \max\Big\{M_{\mathcal{R}}(\vartheta, \varkappa, t), M_{\mathcal{R}}(\varkappa, \vartheta, t), \frac{(l + M_{\mathcal{R}}(\vartheta, \varkappa, t))M_{\mathcal{R}}(\vartheta, \mathcal{P}(\vartheta, \varkappa), t)}{M_{\mathcal{R}}(\vartheta, \mathcal{P}(\vartheta, \varkappa), t)}, \\
&\qquad \frac{(l + M_{\mathcal{R}}(\varkappa, \vartheta, t))M_{\mathcal{R}}(\varkappa, \mathcal{P}(\varkappa, \vartheta), t)}{M_{\mathcal{R}}(\varkappa, \mathcal{P}(\varkappa, \vartheta), t)}\Big\}\Big) \\
&\prec \beta\Big(l - \max\Big\{M_{\mathcal{R}}(\vartheta, \varkappa, t), M_{\mathcal{R}}(\varkappa, \vartheta, t), M_{\mathcal{R}}(\vartheta, \varkappa, t), M_{\mathcal{R}}(\varkappa, \vartheta, t)\Big\}\Big).
\end{aligned} \tag{4.17}$$

From the previous inequality, we can deduce

$$\begin{aligned}
l - M_{\mathcal{R}}(\vartheta, \varkappa, t) &\prec \beta(l - M_{\mathcal{R}}(\vartheta, \varkappa, t)) \\
\Longrightarrow (1 - \beta)l - (1 - \beta)M_{\mathcal{R}}(\vartheta, \varkappa, t) &\prec \theta.
\end{aligned}$$

As $\beta < 1$, we get $M_{\mathcal{R}}(\vartheta, \varkappa, t) \succeq l$, i.e., $M_{\mathcal{R}}(\vartheta, \varkappa, t) = l$ which implies $\vartheta = \varkappa$. Hence (ϑ, ϑ) is a unique coupled fixed point of $\mathcal{P}$. □

Theorem 4.2. *Let $(Y, M_{\mathcal{R}}, *, b)$ be a complete complex valued fuzzy rectangular b-metric space and $\mathcal{P} : Y \to Y$ be a mapping that fulfills the criteria*

$$l - M_{\mathcal{R}}\Big(\mathcal{P}(\vartheta, \varkappa), \mathcal{P}(\tilde{\vartheta}, \tilde{\varkappa}), t\Big) \prec \beta\Big(l - max\Big\{M_{\mathcal{R}}(\vartheta, \tilde{\vartheta}, t), M_{\mathcal{R}}(\varkappa, \tilde{\varkappa}, t)\Big\}\Big), \tag{4.18}$$

for all $(\vartheta, \varkappa), (\tilde{\vartheta}, \tilde{\varkappa}) \in Y \times Y$ and $t \in P_\theta$ and $\beta \in (0, 1)$. Then $\mathcal{P}$ has a unique coupled fixed point of the form $(\tilde{\vartheta}, \tilde{\vartheta})$ which satisfies $\mathcal{P}\tilde{\vartheta} = (\tilde{\vartheta}, \tilde{\vartheta})$.

5. Existence and uniqueness results for (1.1)

In this section, we examine at if a coupled system of nonlinear implicit fractional differential equations (1.1) exists and if it is unique. For a more extensive discussion of the problem's origin, the reader might refer to [27]. We prove the existence of solutions to problem (1.1) utilising our deduced fixed point theorems. Let $Y = \{\vartheta(t) | \vartheta \in C(\mathcal{J})\}$ be a Banach space with the property of having a norm defined as $\|\vartheta\|_Y = \max_{t \in \mathcal{J}} |\vartheta(t)|$. Correspondingly, over the product space, the norm is $\|(\vartheta, \varkappa)\|_{Y \times Y} = \|\vartheta\|_Y + \|\varkappa\|_Y$.

In the context of Lemma 5 in [27], the solution of the given system (1.1) is analogous to the subsequent integral equations,

$$\begin{aligned}
\vartheta(t) &= \int_0^{\mathrm{U}} G_w(t, s)\chi_1(s, \varkappa(s), {}^c D^w \vartheta(s))ds, t \in J, \\
\varkappa(t) &= \int_0^{\mathrm{U}} G_z(t, s)\chi_2(s, \vartheta(s), {}^c D^z \varkappa(s))ds, t \in J,
\end{aligned} \tag{5.1}$$

where G_w and G_z are Green's function given by

$$G_w(t,s) = \begin{cases} -\frac{(\mathsf{U}-s)^{w-1}}{2\Gamma(w)} + \frac{(\mathsf{U}-2t)(\mathsf{U}-s)^{w-2}}{4\Gamma(w-1)}, & \text{if } 0 \le t \le s \le \mathsf{U} \\ \frac{(t-s)^{w-1}-(\mathsf{U}-s)^{w-1}/2}{\Gamma(w)} + \frac{(\mathsf{U}-2t)(T-s)^{w-2}}{4\Gamma(w-1)}, & \text{if } 0 \le s \le t \le \mathsf{U} \end{cases}$$

$$G_z(t,s) = \begin{cases} -\frac{(\mathsf{U}-s)^{z-1}}{2\Gamma(w)} + \frac{(\mathsf{U}-2t)(\mathsf{U}-s)^{z-2}}{4\Gamma(z-1)}, & \text{if } 0 \le t \le s \le \mathsf{U} \\ \frac{(t-s)^{z-1}-(\mathsf{U}-s)^{z-1}/2}{\Gamma(z)} + \frac{(\mathsf{U}-2t)(\mathsf{U}-s)^{z-2}}{4\Gamma(z-1)}, & \text{if } 0 \le s \le t \le \mathsf{U}. \end{cases}$$

The system (1.1) turns into a fixed point problem. In the sense of Lemma 5 in [27], an operator $\mathcal{P} : Y \times Y \to Y \times Y$ is defined as

$$\mathcal{P}(\vartheta,\varkappa)(t) = \begin{pmatrix} \int_0^{\mathsf{U}} G_w(t,s)\chi_1(s,\varkappa(s),{}^c D^w\vartheta(s))ds \\ \int_0^{\mathsf{U}} G_z(t,s)\chi_2(s,\vartheta(s),{}^c D^z\varkappa(s))ds \end{pmatrix} = \begin{pmatrix} \mathcal{P}_w(\vartheta,\varkappa) \\ \mathcal{P}_z(\vartheta,\varkappa) \end{pmatrix}. \tag{5.2}$$

The solution of the suggested system (1.1) then corresponds to the fixed point of the operator $\mathcal{P}$. The L^2 norm of the Green's function of a coupled system of implicit fractional differential equations provided in (1.1) is approximated herein.

Lemma 5.1. *Let $3 < w, z \le 4$. For every $t, s \in (0,1)$, the Green's function $G_{w,z}(t,.) = (G_w(t,.), G_z(t,.))$ where $G_w(t,.), G_z(t,.) \in L_2$, obeys*

$$\begin{aligned} \int_0^{\mathsf{U}} |G_w(t,s)|^2 ds &< \frac{U^7}{4}\left[\frac{1}{5(\Gamma(w))^2} + \frac{1}{12(\Gamma(w-1))^2} + \frac{3}{4\Gamma(w)\Gamma(w-1)}\right], \\ \int_0^{\mathsf{U}} |G_z(t,s)|^2 ds &< \frac{U^7}{4}\left[\frac{1}{5(\Gamma(z))^2} + \frac{1}{12(\Gamma(z-1))^2} + \frac{3}{4\Gamma(z)\Gamma(z-1)}\right]. \end{aligned} \tag{5.3}$$

Proof. (1) If $3 < w, z \le 4$, then for $0 \le s \le t \le \mathsf{U}$, we obtain

$$\begin{aligned} |G_w(t,s)| &\le \frac{(t-s)^{w-1} + (\mathsf{U}-s)^{w-1}/2}{\Gamma(w)} + \frac{|\mathsf{U}-2t|(\mathsf{U}-s)^{w-2}}{4\Gamma(w-1)} \\ &\le \frac{3(\mathsf{U}-s)^{w-1}}{2\Gamma(w)} + \frac{\mathsf{U}(\mathsf{U}-s)^{w-2}}{4\Gamma(w-1)}. \end{aligned} \tag{5.4}$$

(2) For $0 \le t \le s \le \mathsf{U}$,

$$\begin{aligned} |G_w(t,s)| &\le \frac{(\mathsf{U}-s)^{w-1}}{2\Gamma(z)} + \frac{|\mathsf{U}-2t|(\mathsf{U}-s)^{w-2}}{4\Gamma(w-1)} \\ &\le \frac{3(\mathsf{U}-s)^{w-1}}{2\Gamma(w)} + \frac{\mathsf{U}(\mathsf{U}-s)^{w-2}}{4\Gamma(w-1)}. \end{aligned} \tag{5.5}$$

Therefore, by Eqs (5.4) and (5.5), for every $t, s \in (0,1)$, we have

$$|G_w(t,s)|^2 \le \frac{9(\mathsf{U}-s)^{2w-2}}{4(\Gamma(w))^2} + \frac{\mathsf{U}^2(\mathsf{U}-s)^{2w-4}}{16(\Gamma(w-1))^2} + \frac{3\mathsf{U}(\mathsf{U}-s)^{2w-3}}{4\Gamma(w)\Gamma(w-1)} \tag{5.6}$$

which yields

$$\begin{aligned}\int_0^{\mathsf{U}} |G_w(t,s)|^2 ds &\le \int_0^{\mathsf{U}} \frac{9(\mathsf{U}-s)^{2w-2}}{4(\Gamma(w))^2} ds + \int_0^{\mathsf{U}} \frac{\mathsf{U}^2(\mathsf{U}-s)^{2w-4}}{16(\Gamma(w-1))^2} ds + \int_0^{\mathsf{U}} \frac{3\mathsf{U}(\mathsf{U}-s)^{2w-3}}{4\Gamma(w)\Gamma(w-1)} ds \\ &= \frac{1}{4}\Big[\frac{\mathsf{U}^{2w-1}}{(2w-1)(\Gamma(w))^2} + \frac{\mathsf{U}^2(\mathsf{U})^{2w-3}}{4(2w-3)(\Gamma(w-1))^2} + \frac{(3\mathsf{U})\mathsf{U}^{2w-2}}{\Gamma(w)\Gamma(w-1)(2w-2)}\Big] \\ &< \frac{\mathsf{U}^7}{4}\Big[\frac{1}{5(\Gamma(w))^2} + \frac{1}{12(\Gamma(w-1))^2} + \frac{3}{4\Gamma(w)\Gamma(w-1)}\Big].\end{aligned}$$

In a similar fashion, it can be proved that

$$\int_0^{\mathsf{U}} |G_z(t,s)|^2 ds < \frac{\mathsf{U}^7}{4}\Big[\frac{1}{5(\Gamma(z))^2} + \frac{1}{12(\Gamma(z-1))^2} + \frac{3}{4\Gamma(z)\Gamma(z-1)}\Big].$$

□

Theorem 5.2. *Let the assertions (H1)–(H3) hold, then the boundary value problem (1.1) will have a unique solution in Y.*

(H1) Let $3 < w, z \le 4$ and

$$\begin{aligned}\frac{\mathsf{U}^7}{4}\Big[\frac{1}{5(\Gamma(w))^2} + \frac{1}{12(\Gamma(w-1))^2} + \frac{3}{4\Gamma(w)\Gamma(w-1)}\Big] < 1,\\ \frac{\mathsf{U}^7}{4}\Big[\frac{1}{5(\Gamma(z))^2} + \frac{1}{12(\Gamma(z-1))^2} + \frac{3}{4\Gamma(z)\Gamma(z-1)}\Big] < 1,\end{aligned} \tag{5.7}$$

holds.

(H2) Suppose that the function $\chi_1(t,\varkappa(.),^c D^w\vartheta(.)), \chi_2(t,\vartheta(.),^c D^z\varkappa(.)) \in L_2$ for any $\vartheta, \varkappa \in C[0,1]$ and for all $t \in \mathcal{J} = [0,\mathsf{U}]$ and $i = 1,2$, there exist some positive constants C_i, D_i such that $\sqrt{\frac{\mathsf{U}C_1}{1-D_1}} + \sqrt{\frac{\mathsf{U}C_2}{1-D_2}} < 1$ and the inequality

$$\begin{aligned}\big|\chi_1(t,\varkappa,p) - \chi_1(t,\tilde{\varkappa},\tilde{p})\big| \le C_1|\varkappa-\tilde{\varkappa}| + D_1|p-\tilde{p}|,\ \forall t\in\mathcal{J},\\ \big|\chi_2(t,\vartheta,q) - \chi_2(t,\tilde{\vartheta},\tilde{q})\big| \le C_2|\vartheta-\tilde{\vartheta}| + D_2|q-\tilde{q}|,\ \forall t\in\mathcal{J},\end{aligned} \tag{5.8}$$

where $\tilde{p}(t) =^c D^w\vartheta(t)$ and $\tilde{q}(t) =^c D^z\eta(t)$.

(H3) For $\Lambda_1 = \sqrt{\frac{\mathsf{U}C_1}{1-D_1}}$ and $\Lambda_2 = \sqrt{\frac{\mathsf{U}C_2}{1-D_2}}$, consider

$$\big\|(\vartheta-\tilde{\vartheta}),(\varkappa-\tilde{\varkappa})\big\| < \Big\|\Big(\frac{\vartheta}{\sqrt{\Lambda_1+\Lambda_2}}, \frac{\tilde{\vartheta}}{\sqrt{\Lambda_1+\Lambda_2}}\Big)\Big\|.$$

Hence the map $\mathcal{P}$ defined in Eq (5.2) has a unique fixed point, and accordingly, the problem (1.1) has a unique solution.

Proof. It is simple to verify that $(Y, M_{\mathcal{R}}, *, b)$ is a *CVFRb-MS* with the complex valued fuzzy rectangular b-metric specified by

$$M_{\mathcal{R}}((\vartheta,\varkappa),(\tilde{\vartheta},\tilde{\varkappa}),t) = l - \frac{\|(\vartheta,\varkappa)-(\tilde{\vartheta},\tilde{\varkappa})\|^2}{1+cd} l. \tag{5.9}$$

Let $p(t) = \chi_1(t,\varkappa(t),p(t))$ and $\tilde{p}(t) = \chi_1(t,\tilde{\varkappa}(t),\tilde{p}(t))$. Then

$$\begin{aligned} |p(t)-\tilde{p}(t)| &= |\chi_1(t,\varkappa(t),p(t)) - \chi_1(t,\varkappa(t),\tilde{p}(t))| \\ &\le C_1|\varkappa(t)-\tilde{\varkappa}(t)| + D_1|p(t)-\tilde{p}(t)|. \end{aligned} \tag{5.10}$$

Thus

$$|p(t)-\tilde{p}(t)| \le \frac{C_1}{1-D_1}|\varkappa(t)-\tilde{\varkappa}(t)|.$$

For any $\vartheta,\varkappa \in Y$, we find by considering Cauchy–Schwarz inequality and the formulation of the map $\mathcal{P}$ in Eq (5.2)

$$\begin{aligned} |\mathcal{P}_w(\vartheta,\varkappa) - \mathcal{P}_w(\tilde{\vartheta},\tilde{\varkappa})|^2 &= \Big|\int_0^{\mathsf{U}} G_w(t,s)[\chi_1(t,\varkappa(s),{}^c D^w\vartheta(s)) - \chi_1(t,\tilde{\varkappa}(s),{}^c D^w\tilde{\vartheta}(s)]ds\Big|^2 \\ &= \Big|\int_0^{\mathsf{U}} G_w(t,s)[\chi_1(t,\varkappa(s),p(s)) - \chi_1(t,\tilde{\varkappa}(s),\tilde{p}(s))]ds\Big|^2 \\ &\le \left(\int_0^{\mathsf{U}} \big|G(t,s)\big|^2 ds\right)\left(\int_0^{\mathsf{U}} \big|p(s)-\tilde{p}(s)\big|^2 ds\right) \\ &< \int_0^{\mathsf{U}} \big|p(s)-\tilde{p}(s)\big|^2 ds \\ &\le \int_0^{\mathsf{U}} \frac{C_1}{1-D_1}|\varkappa(s)-\tilde{\varkappa}(s)|^2\, ds \\ &\le \frac{C_1}{1-D_1}\max_t |\varkappa(t)-\tilde{\varkappa}(t)|^2\left(\int_0^{\mathsf{U}} ds\right) \\ &\le \left(\sqrt{\frac{\mathsf{U}C_1}{1-D_1}}\max_t |\varkappa(t)-\tilde{\varkappa}(t)|\right)^2 \end{aligned} \tag{5.11}$$

which results in

$$|\mathcal{P}_w(\vartheta,\varkappa) - \mathcal{P}_w(\tilde{\vartheta},\tilde{\varkappa})| \le \Lambda_1 \max|\varkappa(t)-\tilde{\varkappa}(t)|, \tag{5.12}$$

where $\Lambda_1 = \sqrt{\frac{\mathsf{U}C_1}{1-D_1}}$. As a result of the above inequality, we arrive at

$$\|\mathcal{P}_w(\vartheta,\varkappa) - \mathcal{P}_w(\tilde{\vartheta},\tilde{\varkappa})\| \le \Lambda_1\|\varkappa-\tilde{\varkappa}\|.$$

Similarly, we can find

$$\|\mathcal{P}_z(\vartheta,\varkappa) - \mathcal{P}_z(\tilde{\vartheta},\tilde{\varkappa})\| \le \Lambda_2\|\vartheta-\tilde{\vartheta}\|,$$

with $\Lambda_2 = \sqrt{\frac{\mathsf{U}C_2}{1-D_2}}$. Through combining the preceding two equations, we get

$$\|\mathcal{P}(\vartheta,\varkappa) - \mathcal{P}(\tilde{\vartheta},\tilde{\varkappa})\| \le (\Lambda_1+\Lambda_2)\|(\vartheta,\varkappa)-(\tilde{\vartheta},\tilde{\varkappa})\|. \tag{5.13}$$

Consider

$$
\begin{aligned}
l - M_{\mathcal{R}}(\mathcal{P}(\vartheta,\varkappa),\mathcal{P}(\tilde{\vartheta},\tilde{\varkappa}),t) &= l - \left[l - \frac{\|\mathcal{P}(\vartheta,\varkappa) - \mathcal{P}(\tilde{\vartheta},\tilde{\varkappa})\|^2}{1+cd} l \right] = \frac{\|\mathcal{P}(\vartheta,\varkappa) - \mathcal{P}(\tilde{\vartheta},\tilde{\varkappa})\|^2}{1+cd} l * l \\
&= \frac{\|\mathcal{P}(\vartheta,\varkappa) - \mathcal{P}(\tilde{\vartheta},\tilde{\varkappa})\|^2}{1+cd} l \\
&\leq (\Lambda_1 + \Lambda_2)^2 \left(\frac{\|(\vartheta,\varkappa) - (\tilde{\vartheta},\tilde{\varkappa})\|^2}{1+cd} l \right) \\
&\prec (\Lambda_1 + \Lambda_2)^2 \left(\frac{\left\| \left(\frac{\vartheta}{\sqrt{\Lambda_1+\Lambda_2}}, \frac{\tilde{\vartheta}}{\sqrt{\Lambda_1+\Lambda_2}} \right) \right\|^2}{1+cd} l \right) \\
&= |\Lambda_1 + \Lambda_2| \left(\frac{\|(\vartheta,\tilde{\vartheta}\|^2}{1+cd} l \right) \\
&\leq (\Lambda_1 + \Lambda_2) \left(\max \left\{ \frac{\|(\vartheta,\varkappa)\|^2}{1+cd}, \frac{\|(\tilde{\vartheta},\tilde{\varkappa})\|^2}{1+cd} \right\} l \right) \\
&= (\Lambda_1 + \Lambda_2) \left(l - \max \left\{ 1 - \frac{\|(\vartheta,\varkappa)\|^2}{1+cd}, 1 - \frac{\|(\tilde{\vartheta},\tilde{\varkappa})\|^2}{1+cd} \right\} l \right) \\
&= (\Lambda_1 + \Lambda_2) \left(l - \max \left\{ M_{\mathcal{R}}(\vartheta,\varkappa,t), M_{\mathcal{R}}(\tilde{\vartheta},\tilde{\varkappa},t) \right\} \right) \\
&= \beta \Big(l - \Omega((\vartheta,\tilde{\vartheta}),(\varkappa,\tilde{\varkappa}),t) \Big),
\end{aligned}
$$

where $\Omega((\vartheta,\tilde{\vartheta}),(\varkappa,\tilde{\varkappa}),t) = \max\{M_{\mathcal{R}}(\vartheta,\varkappa,t), M_{\mathcal{R}}(\tilde{\vartheta},\tilde{\varkappa},t)\}$ and $\beta = \Lambda_1 + \Lambda_2 < 1$. As a result of the preceding inequality, we conclude that the conditions of the Theorem 4.2 are met. Hence $\mathcal{P}$ has a fixed point $(\vartheta^*(t),\varkappa^*(t)) \in Y$, that is unique. Therefore, problem (1.1) has a unique solution $(\vartheta^*(t),\varkappa^*(t))$. □

6. Conclusions

In this research study, we examined the existence and uniqueness of solution to a coupled system of nonlinear implicit fractional differential equations. The problem is investigated using a new notion known as complex valued fuzzy rectangular b-metric spaces. It utilizes a new approach of applying a coupled fixed point theorem for the associated fixed point approach depends on contractive iterates of the integral operator. We subsequently used an appropriate complex valued fuzzy rectangular b-metric space to determine the problem's uniqueness solvability within certain conditions.

Acknowledgments

The authors Sumaiya Tasneem Zubair and Kalpana Gopalan would like to thank the Management of Sri Sivasubramaniya Nadar College of Engineering for their continuous support and encouragement to carry out this research work. The authors Thabet Abdeljawad and Nabil Mlaiki would like to thank Prince Sultan University for paying the APC and for the support through the TAS research lab.

Conflict of interest

The authors declare no conflicts of interest.

References

1. H. Afshari, S. M. A. Aleomraninejad, Some fixed point results of F-contraction mapping in D-metric spaces by Samet's method, *J. Math. Anal. Model.*, **2** (2021), 1–8. https://doi.org/10.48185/jmam.v2i3.299
2. A. Ali, H. Işık, H. Aydi, E. Ameer, J. R. Lee, M. Arshad, On multivalued Suzuki-type Θ-contractions and related applications, *Open Math. J.*, **2020** (2020), 1–14. https://doi.org/10.1515/math-2020-0139
3. A. Ali, F. Uddin, M. Arshad, M. Rashid, Hybrid fixed point results via generalized dynamic process for F-HRS type contractions with application, *Phys. A: Stat. Mech. Appl.*, **538** (2020). https://doi.org/10.1016/j.physa.2019.122669
4. A. Ali, M. Arshad, A. Asif, E. Savas, C. Park, D. Y. Shin, On multi-valued maps for ϕ-contractions involving orbits with application, *AIMS Math.*, **6** (2021), 7532–7554. https://doi.org/10.3934/math.2021440
5. N. Ameth, A nonlinear implicit fractional equation with caputo derivative, *J. Math.*, **2021** (2021), 5547003. https://doi.org/10.1155/2021/5547003
6. A. Azam, B. Fisher, M. Khan, Common fixed point theorems in complex-valued metric spaces, *Numer. Func. Anal. Opt.,* **32** (2011), 243–253. https://doi.org/10.1080/01630563.2011.533046
7. Y. Chen, H. An, Numerical solutions of coupled Burgers equations with time and space fractional derivatives, *Appl. Math. Comput.*, **200** (2008), 87–95. https://doi.org/10.1016/j.aej.2016.03.028
8. R. Chugh, S. Kumar, Weakly compatible maps in generalized fuzzy metric spaces, *J. Anal.,* **10** (2002), 65–74.
9. L. Danfeng, A. Mehboob, Z. Akbar, R. Usman, L. Zhiguo, Existence and stability of implicit fractional differential equations with Stieltjes boundary conditions involving Hadamard derivatives, *Complexity*, **2021** (2021), 8824935. https://doi.org/10.1155/2021/8824935
10. I. Demir, Fixed point theorems in complex valued fuzzy b-metric spaces with application to integral equations, *Miskolc Math. Notes.*, **22** (2021), 153–171. https://doi.org/10.18514/MMN.2021.3173
11. V. Gafiychuk, B. Datsko, V. Meleshko, D. Blackmore, Analysis of the solutions of coupled nonlinear fractional reaction-difusion equations, *Chaos Soliton. Fract.*, **41** (2009), 1095–1104. https://doi.org/10.1016/j.chaos.2008.04.039
12. A. George, P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets Syst.*, **64** (1994), 395–399. https://doi.org/10.1016/0165-0114(94)90162-7
13. M. Grabiec, Fixed points in fuzzy metric spaces, *Fuzzy Sets Syst.,* **27** (1988), 385–389. https://doi.org/10.1016/0165-0114(88)90064-4
14. H. A. Hammad, H. Aydi, M. De la Sen, Solutions of fractional differential type equations by fixed point techniques for multi-valued contractions, *Complexity*, **2021** (2021), 5730853. https://doi.org/10.1155/2021/5730853

15. Humaira, H. A. Hammad, M. Sarwar, Manuel De la Sen, Existence theorem for a unique solution to a coupled system of impulsive fractional differential equations in complex-valued fuzzy metric spaces, *Adv. Differ. Equ.,* **242** (2021). https://doi.org/10.1186/s13662-021-03401-0
16. Humaira, M. Sarwar, T. Abdeljawad, Existence of solutions for nonlinear impulsive fractional differential equations via common fixed-point techniques in complex valued fuzzy metric spaces, *Math. Probl. Eng.*, **2020** (2020), 7042715. https://doi.org/10.1155/2020/7042715
17. I. Kramosil, J. Michalek, Fuzzy metrics and statistical metric spaces, *Kybernetika*, **11** (1975), 336–344.
18. K. D. Kucche, J. J. Nieto, V. Venktesh, Theory of nonlinear implicit fractional differential equations, *Differ. Equ. Dyn. Syst.*, **28** (2020), 1–17. https://doi.org/10.1007/s12591-016-0297-7
19. F. Mehmood, R. Ali, N. Hussain, Contractions in fuzzy rectangular b-metric spaces with application, *J. Intell. Fuzzy Syst.*, **37** (2019), 1275–1285. https://doi.org/10.3233/JIFS-182719
20. N. Mlaiki, D. Rizk, F. Azmi, Fixed points of (ψ, ϕ)-contractions and Fredholm type integral equation, *J. Math. Anal. Model.,* **2** (2021), 91–100. https://doi.org/10.48185/jmam.v2i1.194
21. S. Nădăban, Fuzzy b-metric spaces, *Int. J. Comput. Commun.*, **11** (2016), 273–281. https://doi.org/10.15837/ijccc.2016.2.2443
22. J. Patil, A. Chaudhari, M. S. Abdo, B. Hardan, A. Bachhav, Positive solution for a class of Caputo-type fractional differential equations, *J. Math. Anal. Model.*, **2** (2021), 16–29. https://doi.org/10.48185/jmam.v2i2.274
23. T. Qi, Y. Liu, Y. Zou, Existence result for a class of coupled fractional differential systems with integral boundary value conditions, *J. Nonlinear Sci. Appl*, **10** (2017), 4034–4045. http://dx.doi.org/10.22436/jnsa.010.07.52
24. K. Sathiyanathan, V. Krishnaveni, Nonlinear implicit caputo fractional differential equations with integral boundary bonditions in Banach space, *Glob. J. Pure Appl. Math.*, **13** (2017), 3895–3907.
25. S. Shukla, R. Rodríguez-Lopez, M. Abbas, Fixed point results for contractive mappings in complex-valued fuzzy metric spaces, *Fixed Point Theor.*, **19** (2018), 1–22. https://doi.org/10.24193/fpt-ro.2018.2.56
26. Z. Sumaiya Tasneem, G. Kalpana, T. Abdeljawad, B. Abdalla, On fuzzy extended hexagonal b-metric spaces with applications to nonlinear fractional differential equations, *Symmetry,* **13** (2021). https://doi.org/10.3390/sym13112032
27. A. Zada, Z. Ali, J. Xu, Y. Cul, Stability results for a coupled system of impulsive fractional differential equations, *Mathematics*, **7** (2019). https://doi.org/10.3390/math7100927
28. Y. Zhang, Z. Bai, T. Feng, Existence results for a coupled system of nonlinear fractional three-point boundary value problems at resonance, *Comput. Math. Appl.*, **61** (2011), 1032–1047. https://doi.org/10.1016/j.camwa.2010.12.053
29. S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional differential equations, *Electron. J. Differ. Equ.*, **36** (2006), 1–12. https://doi.org/10.1016/j.jmaa.2005.02.052

AIMS Press

©2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)