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Research article

Symmetry of large solutions for semilinear elliptic equations in a symmetric convex domain

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Abstract: In this paper, we consider the solutions of the boundary blow-up problem

$$\begin{cases} \Delta u = \frac{1}{u^\gamma} + f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where $\gamma > 0$, Ω is a bounded convex smooth domain and symmetric w.r.t. a direction. f is a locally Lipschitz continuous and non-decreasing function. We prove symmetry and monotonicity of solutions of the problem above by the moving planes method. A maximum principle in narrow domains plays an important role in proof of the main result.

Keywords: semilinear elliptic systems; moving plane method; maximum principle; symmetry of large solutions
Mathematics Subject Classification: 35A21, 35B06

1. Introduction

In this paper, we investigate symmetry and monotonicity of solutions to the problem

$$\begin{cases} \Delta u = \frac{1}{u^\gamma} + f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \lim\limits_{|x|\to\partial\Omega} u = +\infty, \end{cases} \tag{1.1}$$

where $\gamma > 0, \Omega$ is a bounded smooth domain, The boundary condition means that $u(x) \to +\infty$, as $x \to \partial\Omega$, $u \in C^2(\Omega)$, and we give the assumption

(H) f is locally Lipschitz continuous, non-decreasing, $f(s) > 0$ for $s > 0$, and $f(0) \geq 0$.

Our interest in this paper is motivated by symmetry of solutions of nonlinear elliptic equations with singular nonlinearities in [1], and symmetry of large solutions for nonlinear elliptic equations in a ball [2, 3]. In [1], the author studied the symmetry and monotonicity properties of positive solutions for the semilinear equation

$$\begin{cases} -\Delta u = \frac{1}{u^\gamma} + f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Omega, \end{cases} \tag{1.2}$$

where $\gamma > 0$, Ω is a convex bounded smooth domain and symmetric w.r.t. a direction, f is a locally Lipschitz continuous and non-decreasing function. As the singularities of the problem (1.2), some difficulties should be overcome. After introducing the new techniques based on decomposition in (1.2), providing some weak and strong maximum principles, the author proved the results in [1]. In our problem, the singularities of solutions near $\partial\Omega$ bring difficulties to use the moving plane method, which is a very useful tool to get the most of symmetry results ([4–7]).

It is well-known that the problem (1.1) admits a solution, which is usually called "a large solution", if and only if f satisfies the Kellar-Osserman condition([8–10]), that is, $\frac{1}{t^\gamma} + f(t) \geq h(t)$, $t \in [a, +\infty)$ for some $a > 0$, where $h(s)$ is nondecreasing and satisfies

$$\int_a^\infty \frac{dt}{\sqrt{H(t)}} < \infty, \quad \text{where } H(s) = \int_a^s h(t)dt. \tag{1.3}$$

Now, we give our result as follows:

Theorem 1.1. *Let $u \in C^2(\Omega)$ be a solution of (1.1), f satisfies (H) and Kellar-Osserman condition. Assume that the bounded domain Ω is strictly convex w.r.t. the e-direction ($e \in S^{N-1}$) and symmetric w.r.t. T_0^e, where*

$$T_0^e = \{x \in R^N | x \cdot e = 0\}.$$

Then, u is symmetric w.r.t. T_0^e and non-increasing w.r.t. the e-direction in Ω_0^e, where

$$\Omega_0^e = \{x \in \Omega | x \cdot e < 0\}.$$

Moreover, if Ω is a ball or an annulus, the following condition holds

$$\lim_{|x| \to R} \frac{\partial u}{\partial r}(x) = \infty \;\; and \;\; |\nabla_\tau u(x)| = o(\frac{\partial u}{\partial r}(x)) \;\; \text{as } |x| \to R, \tag{1.4}$$

where, $\partial_r u$ and $\nabla_\tau u$ are the radial derivative and tangential gradient of u, respectively. Then, u is radially symmetric and radially increasing.

Up to now, at the best of our knowledge, only partial results about symmetry and monotonicity of large solutions of nonlinear elliptic equations were known. In [11], the author conjectured that any solution of $-\Delta u + g(u) = 0$ in a ball is radially symmetry. This conjecture was proved in [3], where it was verified under assumptions of asymptotic convexity upon on g. In [2], for the large solutions of $-\Delta u + f(u) = 0$ in a ball, the restriction is considered: $f(t) + Kt^p$ is non-decreasing for large t, where $p > 1, K > 0$. In this paper, we consider the symmetry of large solutions of (1.1) corresponding to the semilinear equations in [1], and the symmetric convex domain Ω, which is more general than a ball. In [7, 12, 13], solutions have been proved to be radially symmetry and increasing under some restrictions at infinity. There is also some interest in such qualitative properties of large solutions raised from different problems ([14–18]).

The structure of the paper is arranged as follows. In Section 2, we give some notations in order to use the moving plane method. In Section 3, we give the proof of main result by three steps. Thank you for your cooperation.

2. Preliminaries of notations

To prove our results, we need some notations related to the moving plane method.

For a number $\lambda \in \mathbf{R}$, we denote

$$T_\lambda^e = \{x \in \Omega | x \cdot e = \lambda\}, \text{ and } \Omega_\lambda^e = \{x \in \Omega | x \cdot e < \lambda\}.$$

Next, we use the x_λ^e to denote the reflection of x through the hyperplane T_λ^e as follows

$$x_\lambda^e = R_\lambda^e(x) = x + 2(\lambda - x \cdot e)e.$$

Then, we naturally set

$$(\Omega_\lambda^e)' = R_\lambda^e(\Omega_\lambda^e),$$

which is the reflection Ω_λ^e *w.r.t.* T_λ^e. (Note that $(\Omega_\lambda^e)'$ may be not contained in Ω, for an example, if $\lambda > 0$, $(\Omega_\lambda^e)'$ is not contained in Ω.) In addition, we denote

$$a_1(e) := \inf_{x \in \Omega} x \cdot e.$$

For $\lambda > a_1(e)$, it's obvious that Ω_λ^e is nonempty. So we set

$$\Lambda_1(e) = \{\lambda | (\Omega_t^e)' \subset \Omega, \quad \text{for } t \in (a_1(e), \lambda]\}, \text{ and } \lambda_1(e) = \sup \Lambda_1(e).$$

At last, for $u \in C^2(\Omega)$, we also set

$$u_\lambda^e(x) = u(x_\lambda^e), \text{ and } w_\lambda^e(x) = u(x) - u_\lambda^e(x), \quad x \in \Omega_\lambda^e, \ \lambda \in (a_1(e), \lambda_1].$$

Remark 2.1. *By Theorem 1.1 , u is nondecreasing w.r.t. e-direction in* $(\Omega_0^e)' = R_0^e(\Omega_0^e)$.

3. Proof of the main result

In this section, we give the proof of Theorem 1.1, which is based on the moving planes method.

Proof of Theorem 1.1. To give a clear proof, we will divide it into three steps.

Step 1. We will prove that $u \geq u_\lambda^e$ in Ω_λ^e, if λ is enough close to $a_1(e)$.

Indeed, for $a_1(e) < \lambda \leq \lambda_1(e)$, it's obvious that $\Omega^e_\lambda \subset \Omega^e_0$. We consider the domain

$$D_\epsilon = \{x \in \Omega | \epsilon < \text{dist}(x, \partial\Omega)\} \cap \Omega^e_\lambda. \tag{3.1}$$

It's obvious that $w^e_\lambda(x) = u(x) - u(x^e_\lambda) \geq 0$, on ∂D_ϵ. In fact, for $x \in T^e_\lambda \cap \partial D_\epsilon, u(x) - u(x^e_\lambda) = 0$, and for $x \in (\partial D_\epsilon \setminus T^e_\lambda),\ u(x) - u(x^e_\lambda) > 0$, since u approaches to positive infinity at boundary and is finite in the interior for enough small ϵ.

So we have

$$\begin{cases} \Delta w^e_\lambda(x) = \frac{1}{u^\gamma} - \frac{1}{(u^e_\lambda)^\gamma} + \frac{f(u)-f(u^e_\lambda)}{u-u^e_\lambda}(u - u^e_\lambda) \\ \qquad\qquad = (-\gamma\xi(x)^{-\gamma-1} + c(x,\lambda))w^e_\lambda \qquad x \in D_\epsilon, \\ \quad w^e_\lambda \geq 0 \qquad\qquad\qquad\qquad\qquad\qquad x \in \partial D_\epsilon, \end{cases}$$

where $\min\{u(x), u^e_\lambda(x)\} \leq \xi(x) \leq \max\{u(x), u^e_\lambda(x)\}$, $c(x,\lambda) = \frac{f(u^e_\lambda)-f(u)}{u^e_\lambda - u}$, for $x \in D_\epsilon,\ \lambda \in (a(e), \lambda_1(e)]$. Since $\overline{D}_\epsilon$ is in the interior of Ω, $-\gamma\xi(x)^{-\gamma-1} + c(x,\lambda)$ is a bounded function in D_ϵ. So, by the strong maximum principle in narrow domains, as $\lambda(> a_1(e))$ is enough close to $a_1(e)$, we have

$$w^e_\lambda(x) = u(x) - u(x^e_\lambda) \geq 0, \text{ for } x \in (D_\epsilon \cap \Omega^e_\lambda). \tag{3.2}$$

Furthermore, since ϵ can be chosen arbitrary small, by (3.1) and (3.2), we get

$$w^e_\lambda(x) = u(x) - u(x^e_\lambda) \geq 0,\ x \in \Omega^e_\lambda.$$

So we obtain the start point λ in order to use the method of moving planes.

Step 2. We set

$$\overline{\lambda} = \sup\{\lambda | u(x) \geq u(x^e_\lambda),\ \forall\ x \in \Omega^e_s \text{ for } s \in (a_1(e), \lambda)\}.$$

Then, we will obtain $\overline{\lambda} = \lambda_1(e) = \sup \Lambda_1(e)$ by a contradiction. Now, we give the proof of this statement below.

We assume that $\overline{\lambda} < \lambda_1(e)$. Notice that, by continuity and the definition $\bar{\lambda}$, we get $u \geq u^e_{\bar{\lambda}}$ in $\Omega^e_{\bar{\lambda}}$. So we can write

$$\begin{cases} \Delta(u - u^e_{\bar{\lambda}}) = \frac{1}{u^\gamma} - \frac{1}{(u^e_{\bar{\lambda}})^\gamma} + \frac{f(u)-f(u^e_{\bar{\lambda}})}{u-u^e_{\bar{\lambda}}}(u - u^e_{\bar{\lambda}}) \\ \qquad\qquad = (-\gamma\xi(x)^{-\gamma-1} + a(x))(u - u_{\bar{\lambda}}) \qquad \text{in} \quad \Omega^e_{\bar{\lambda}}, \\ \quad u - u^e_{\bar{\lambda}} \geq 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad \Omega^e_{\bar{\lambda}}, \end{cases} \tag{3.3}$$

where $\xi(x) \in (\min\{u(x), u^e_{\bar{\lambda}}(x)\}, \max\{u(x), u^e_{\bar{\lambda}}(x)\}), a(x) = \frac{f(u(x))-f(u^e_{\bar{\lambda}}(x))}{u(x)-u^e_{\bar{\lambda}}(x)}$. Since f is locally Lipschitz continuous, we know $-\gamma\xi(x)^{-\gamma-1} + a(x)$ is locally bounded in $\Omega^e_{\bar{\lambda}}$. Then by the strong maximum principle, we deduce that

$$u \equiv u_{\bar{\lambda}},\ \text{or } u > u_{\bar{\lambda}} \ \text{ in }\ \Omega^e_{\bar{\lambda}}\ .$$

In fact, while u tends to infinity at the boundary and $u^e_{\bar{\lambda}}$ is finite in the interior of Ω, we have $u \not\equiv u^e_{\bar{\lambda}}$, in $\Omega^e_{\bar{\lambda}}$. Therefore, we conclude that

$$u > u^e_{\bar{\lambda}} \ \text{ in }\ \Omega^e_{\bar{\lambda}}, \tag{3.4}$$

and it follows from Hopf's lemma that

$$\frac{\partial w^e_{\bar\lambda}}{\partial e} = \frac{\partial u - u^e_{\bar\lambda}}{\partial e} < 0 \text{ on } T^e_{\bar\lambda}. \tag{3.5}$$

Next, by definition of $\overline{\lambda}$, there exists a decreasing sequence λ_n converging to $\overline{\lambda}$ and points $\{x_n\} \in \Omega^e_{\lambda_n}$ such that $u(x_n) \leq u_{\lambda_n}(x_n)$. Without loss of generality, up to subsequences, still denoted by $\{x_n\}$, will converge to a point $x_0 \in \overline{\Omega^e_{\bar\lambda}}$. Then three cases will be considered as follows

(1) For $x_0 \in \Omega^e_{\bar\lambda}$, by the continuity and the limitation of u, we get $u(x_0) \leq u^e_{\bar\lambda}(x_0)$, while $u > u^e_{\bar\lambda}$ in $\Omega^e_{\bar\lambda}$ by (3.4). It is a contradiction.

(2) For $\bar{x}_0 \in (\partial\{\overline{\Omega^e_{\bar\lambda}}\} \setminus T^e_{\bar\lambda})$, it is obvious that $u(x_n) - u^e_{\lambda_n}(x_n)$ would approach to infinity since u approaches to infinity and it is locally bounded in the interior, which is a contradiction to the choice of x_n.

(3) For $\bar{x}_0 \in T^e_{\bar\lambda} \cap \Omega$, we will get a contradiction again. Assuming that η_n is the projection of x_n on $T^e_{\lambda_n}$, so $\eta_n = \frac{x_n + (x_n)^e_{\lambda_n}}{2} = x_n + (\lambda_n - x_n \cdot e)e$. By $u(x_n) \leq u^e_{\lambda_n}(x_n)$ and $u(\eta_n) = u^e_{\lambda_n}(\eta_n)$, there is a point $\xi_n = [x_n + \theta(\eta_n - x_n)], \theta \in [0, 1]$(It's obvious that ξ is in the segment $[x_n, \eta_n]$ and $\lim_{n\to\infty} \xi_n = x_0$), such that

$$\begin{aligned}
0 &\geq [u(x_n) - u^e_{\lambda_n}(x_n)] - [u(\eta_n) - u^e_{\lambda_n}(\eta_n)] \\
&= \frac{\partial[u(\xi_n) - u^e_{\lambda_n}(\xi_n)]}{\partial e}((x_n - \eta_n) \cdot e) \\
&= \frac{\partial[u(\xi_n) - u^e_{\lambda_n}(\xi_n)]}{\partial e}[(x_n - (x_n + (\lambda_n - x_n \cdot e)e)) \cdot e] \\
&= \frac{\partial[u(\xi_n) - u^e_{\lambda_n}(\xi_n)]}{\partial e}[(-(\lambda_n - x_n \cdot e)e) \cdot e] \\
&= \frac{\partial[u(\xi_n) - u^e_{\lambda_n}(\xi_n)]}{\partial e}(x_n \cdot e - \lambda_n).
\end{aligned} \tag{3.6}$$

By the definition of λ_n and x_n, we have $x_n \cdot e - \lambda_n < 0$. So, at once, we have $\frac{\partial[u(\xi_n) - u^e_{\lambda_n}(\xi_n)]}{\partial e} \geq 0$. Therefore, we deduce

$$\begin{aligned}
\frac{\partial w^e_{\bar\lambda}(x_0)}{\partial e} &= \frac{\partial[u(x_0) - u^e_{\bar\lambda}(x_0)]}{\partial e} \\
&= \lim_{n\to\infty} \frac{\partial[u(\xi_n) - u^e_{\lambda_n}(\xi_n)]}{\partial e} \geq 0,
\end{aligned}$$

which is a contradiction to (3.5).

Step 3. Completing proof here. From the discussion above, we get $\bar\lambda \geq \lambda_1(e)$. From the other direction, using the method of moving planes again, we can also get $\bar\lambda \leq \lambda_1(e)$. Hence $\bar\lambda = \lambda_1(e)$.Observing the assumption $\lambda_1(e) = 0$, we directly get

$$u(x) = u^e_0(x), \text{ for } x \in \Omega^e_0,$$

which means that u is symmetric *w.r.t.* the direction $e \in S^{N-1}$. By the processing of using the method of moving planes, we know u non-decreases along the directions e in $(\Omega^e_0)'$, and $-e$ in Ω^e_0, respectively. So the u is non-increasing *w.r.t.* the e-direction in Ω^e_0. Furthermore, if Ω is a ball or an annulus, (1.4) holds, by the similar method in Theorem 2.1 in [3], we can easily prove that u is radially symmetric and radially increasing. The proof is complete. □

4. Conclusions

In this paper, we study symmetry and monotonicity of solutions to the boundary blow-up problem of nonlinear elliptic equations in a bounded smooth domain which is strictly convex w.r.t. a direction. We are inspired by some results about symmetry of large solutions for nonlinear elliptic equations in a ball and symmetry of solutions of some elliptic equations with singular nonlinearities. Corresponding to the equation in [6] with singular nonlinearities in a bounded Ω, where the solution $u = 0$, for $x \in \partial\Omega$, we get symmetry and monotonicity of large solutions of nonlinear elliptic equations in a general bounded convex domain under the condition that $u \to +\infty$, as $x \to \partial\Omega$.

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Conflict of interest

The authors declare no conflicts of interest.

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