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Research article

q-Noor integral operator associated with starlike functions and q-conic domains

Syed Ghoos Ali Shah[1,*], **Shahbaz Khan**[1], **Saqib Hussain**[1] **and Maslina Darus**[2]

[1] Department of Mathematics, COMSATS University Islamabad, Abbottabad Campus 22060, Pakistan

[2] Department of Mathematical Sciences, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor, Malaysia

* **Correspondence:** Email: alishahsyedghoos@gmail.com.

Abstract: In this paper, we will discuss some generalized classes of analytic functions related with conic domains in the context of q-calculus. In this work, we define and explore Janowski type q-starlike functions in q-conic domains. We investigate some important properties such as necessary and sufficient conditions, coefficient estimates, convolution results, linear combination, weighted mean, arithmetic mean, radii of starlikeness, growth and distortion results for these classes. It is important to mention that our results are generalization of number of existing results.

Keywords: analytic functions; subordination; Noor integral operator; q-conic domain; q-Janowski functions
Mathematics Subject Classification: Primary 30C45, Secondary 30C50

1. Introduction

The quantum (or q-) calculus is an important area of study in the field of traditional mathematical analysis. Quantum calculus is a fascinating area of mathematical science with historical background, as well as a revived focus in the modern era. Quantum calculus is the modern name for the investigation of calculus without notation of limit. The quantum calculus or q-calculus began with Jackson in the early twentieth century, but this type of calculus had already been investigated by Euler and Jacobi. Recently q-calculus attract researchers for its wide applications in mathematics and related areas, such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions and other sciences. In recent years, the topic of q-calculus has attracted the attention of several researchers, and a variety of new results can be found in the papers [1–4] and the references cited therein.

A function $\hat{g}$ is analytic at a point ξ_0 if $\hat{g}^{'}(\xi)$ exists at ξ_0 as well as in some neighborhood of

ξ_0. A function $\hat{g}(\xi)$ is analytic in $\mathbb{D}$ if $\hat{g}(\xi)$ is analytic at each point of $\mathbb{D}$. In most of the cases it is much harder to use a random domain, so Riemann mapping theorem allows us to replace it with open unit disk defined as:

$$\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}.$$

An analytic function $\hat{g}$ is univalent in $\mathbb{U}$, if $\hat{g}(\xi_1) = \hat{g}(\xi_2)$ then $\xi_1 = \xi_2$. A function $\hat{g}(\xi)$ is said to be the class $\mathfrak{A}$ if it has a Taylor series of the form

$$\hat{g}(\xi) = \xi + \sum_{t=2}^{\infty} a_t \xi^t, \quad \xi \in \mathbb{U}. \tag{1.1}$$

A collection of functions of the form (1.1), which are analytic and univalent in $\mathbb{U}$ are placed in the class $\mathfrak{S}$. An analytic function $p(\xi)$ having positive real part i.e., $\Re\{p(\xi)\} > 0$ and $p(0) = 1$ is placed in class $\mathfrak{P}$. Or equivalently

$$p \in \mathfrak{P} : p(\xi) = 1 + \sum_{t=1}^{\infty} a_t \xi^t \iff \Re\{p(\xi)\} > 0, \quad \xi \in \mathbb{U}. \tag{1.2}$$

The class of normalized convex functions is given by

$$C = \left\{\hat{g} : \hat{g} \in \mathfrak{S}; \Re\left(\frac{\left(\xi \hat{g}'(\xi)\right)'}{\hat{g}(\xi)}\right) > 0, \quad \xi \in \mathbb{U}\right\}.$$

Similarly, the class of normalized starlike functions with respect to origin is defined as:

$$\mathbb{S}^* = \left\{\hat{g} : \hat{g} \in \mathfrak{S}; \Re\left(\frac{\xi \hat{g}'(\xi)}{\hat{g}'(\xi)}\right) > 0, \quad \xi \in \mathbb{U}\right\},$$

for details, see [5]. A function $\hat{g}(\xi) \in QC$, the class of quasi-convex function if and only if there exists $\hat{h}(\xi) \in C$ such that $\Re\left(\frac{(\xi \hat{g}'(\xi))'}{\hat{h}'(\xi)}\right) > 0$. In 1952, Kaplan [6] introduced the class KC of close-to-convex function. A function is of the form (1.2) is in KC if and only if there exists $\hat{h}(\xi) \in \mathbb{S}^*$ such that $\Re\left(\frac{\xi \hat{g}'(\xi)}{\hat{h}(\xi)}\right) > 0$. Let $\hat{g}(\xi)$ is of the form (1.1) and $\hat{h}(\xi)$ is of the form

$$\hat{h}(\xi) = \xi + \sum_{t=2}^{\infty} b_t \xi^t, \ \xi \in \mathbb{U}. \tag{1.3}$$

Then the Hadamard product(convolution) of $\hat{g}$ and $\hat{h}$ is defined as:

$$\left(\hat{g} * \hat{h}\right)(\xi) = \xi + \sum_{t=2}^{\infty} a_t b_t \xi^t = \left(\hat{h} * \hat{g}\right)(\xi). \tag{1.4}$$

The q-derivative of a function $\hat{g}$ belonging to $\mathfrak{A}$ defined as:

$$D_q \hat{g}(\xi) = \frac{\hat{g}(q\xi) - \hat{g}(\xi)}{\xi(q-1)} \quad \text{for } \xi \neq 0, \tag{1.5}$$

for details, see [7], where $q \in (0,1)$ and $\xi \in \mathbb{U}$. For $\xi = 0$, (1.5) can be written as $\hat{g}^{'}(0)$ provided that the derivative exist. By using (1.1) and (1.5) the Maclaurin's series representation of $D_q\hat{g}$ is given by

$$D_q\hat{g}(\xi) = 1 + \sum_{t=0}^{\infty} [t,q]\, a_t \xi^{t-1},\ t \in \mathbb{N}. \tag{1.6}$$

It can be noted from (1.5) that

$$\lim_{q\to 1^-}\left(D_q\hat{g}(\xi)\right) = \lim_{q\to 1^-}\left(\frac{\hat{g}(q\xi)-\hat{g}(\xi)}{\xi(q-1)}\right) = \hat{g}^{'}(\xi), \quad \text{where } [t,q] = \frac{1-q^t}{1-q}.$$

For any non negative integer t, the q-number shift factorial is given by

$$[t,q]! = \begin{cases} 1, & t = 0 \\ [1,q]\,[2,q]\cdots[t,q]\,, & t \in \mathbb{N} \end{cases} \tag{1.7}$$

see [8]. For $y > 0$, the q-genralized Pochammar symbol is defined as:

$$[y,q]_t = \begin{cases} 1, & t = 0 \\ [y,q]\,[y+1,q]\cdots[y+t-1,q]\,, & t \in \mathbb{N} \end{cases} \tag{1.8}$$

For $\mu > -1$, we defined a function $\mathfrak{F}^{-1}_{q,1+\mu}(\xi)$ such that

$$\mathfrak{F}_{q,1+\mu}(\xi) * \mathfrak{F}^{-1}_{q,1+\mu}(\xi) = \xi D_q\hat{g}(\xi), \tag{1.9}$$

where

$$\mathfrak{F}_{q,1+\mu}(\xi) = \xi + \sum_{t=2}^{\infty}\left(\frac{[1+\mu,q]_{t-1}}{[t-1,q]!}\xi^t\right), \quad \text{for } \xi \in \mathbb{U}. \tag{1.10}$$

The study of operators plays an important role in the geometric function theory. Many differential and integral operators can be written in terms of convolution of certain analytic functions. In [8] q-analogue of Noor integral operator $\mathfrak{I}^{\mu}_q : \mathfrak{A} \to \mathfrak{A}$ is define as:

$$\mathfrak{I}^{\mu}_q\hat{g}(\xi) = \hat{g}(\xi) * \mathfrak{F}^{-1}_{q,1+\mu}(\xi) = \xi + \sum_{t=2}^{\infty}\psi_{t-1}a_t\xi^t, \tag{1.11}$$

where

$$\psi_{t-1} = \frac{[t,q]!}{[1+\mu,q]_{t-1}}. \tag{1.12}$$

From (1.11) we can easily obtain the following identity

$$[1+\mu,q]\,\mathfrak{I}^{\mu}_q\hat{g}(\xi) = [\mu,q]\,\mathfrak{I}^{\mu+1}_q\hat{g}(\xi) + q^{\mu}\xi D_q\left(\mathfrak{I}^{\mu+1}_q\hat{g}(\xi)\right), \tag{1.13}$$

from (1.11). It can be seen that $\mathfrak{I}^0_q\hat{g}(\xi) = \xi D_q\hat{g}(\xi)$, $\mathfrak{I}^1_q\hat{g}(\xi) = \hat{g}(\xi)$ and

$$\lim_{q\to 1^-}\left(\mathfrak{I}^{\mu}_q\hat{g}(\xi)\right) = \xi + \sum_{t=2}^{\infty}\frac{t!}{(1+\mu)_{t-1}}a_t\xi^t. \tag{1.14}$$

From (1.14), we can observe that by applying limit $q \to 1$, the operator defined in (1.11) reduces to well known Noor integral operator see ([9–12]).

In [13, 14], Kanas and Waniowska introduced the concept of a conic domain Ξ_l for $l \geq 0$ as:

$$\Xi_l = \left\{ U + iV : U > l\sqrt{V^2 + (U-1)^2} \right\}. \tag{1.15}$$

This domain merely represent the right half plane for $l = 0$, a hyperbola for $0 < l < 1$, parabola for $l = 1$ and ellipse for $l > 1$. The extremal functions ϖ_l for this conic region Ξ_l is given by

$$\varpi_l(\xi) = \begin{cases} \frac{1+\xi}{1-\xi} & l = 0, \\ 1 + \left\{ \frac{2}{\pi^2} \left(\log \frac{\sqrt{\xi}+1}{1-\sqrt{\xi}} \right)^2 \right\} & l = 1, \\ 1 + \frac{2}{1-l^2} \sinh^2 \left[\left(\frac{2}{\pi} \arccos l \right) \left(\arctan h \sqrt{\xi} \right) \right] & 0 < l < 1, \\ 1 + \frac{1}{l^2-1} \sin \left[\frac{\pi}{2R(n)} \int_0^{\frac{U(\xi)}{\sqrt{n}}} \left(\frac{1}{\sqrt{1-n^2y^2}\sqrt{1-x^2}} \right) dx \right] + \frac{1}{l^2-1} & l > 1, \end{cases} \tag{1.16}$$

where $U(\xi) = \frac{\xi - \sqrt{n}}{1-\sqrt{n\xi}}$, for all $\xi \in \mathbb{U}$, $0 < l < 1$ and $l = \cosh \left[\frac{\pi R'(n)}{4R(n)} \right]$ where $R(n)$ is Legendre's complete elliptic integral of first kind and $R'(n)$ is complementary integral of $R(n)$ for more details, see [13–16]. If we take $\varpi_l(\xi) = 1 + \delta(l)\xi + \delta_1(l)\xi^2 + \cdots$, then

$$\delta(l) = \begin{cases} \frac{8(\arccos l)^2}{\pi^2(1-l^2)} & 0 \leq l < 1, \\ \frac{8}{\pi^2} & l = 1, \\ \frac{\pi^2}{4\sqrt{n}(l^2-1)(1+n)R^2(n)} & l > 1. \end{cases} \tag{1.17}$$

Let $\delta_1(l) = \delta_2(l)\delta(l)$, where

$$\delta_2(l) = \begin{cases} \frac{2+\left(\frac{2}{\pi}\arccos l\right)^2}{3} & 0 \leq l < 1, \\ \frac{2}{3} & l = 1, \\ \frac{4R^2(n)(1+n^2+6n)-\pi^2}{24(1+n)\sqrt{n}R^2(n)} & l > 1. \end{cases} \tag{1.18}$$

***Definition* 1.** *[17] Let p be a analytic function with $p(0) = 1$. Then $p \in \mathfrak{P}(\lambda, M)$ if and only if*

$$p(\xi) \prec \frac{\lambda\xi + 1}{M\xi + 1}, \qquad \textit{where } -1 \leq M < \lambda \leq 1. \tag{1.19}$$

In [17] it was shown that $p \in \mathfrak{P}(\lambda, M)$ if and only if there exists a function $p \in \mathfrak{P}$ such that

$$\frac{(1+\lambda)p(\xi) - (\lambda - 1)}{(1+M)p(\xi) - (M-1)} \prec \frac{\lambda\xi + 1}{M\xi + 1}.$$

***Definition* 2.** *[18] A function $\hat{g} \in \mathfrak{A}$ is in the class $k - ST_q(N, O)$ if and only if*

$$\Re \left[\frac{G(\xi)}{H(\xi)} \right] > k \left| \frac{G(\xi)}{H(\xi)} - 1 \right|, \tag{1.20}$$

where

$$\begin{aligned} G(\xi) &= (OL_1 - L_2)\left(\frac{\xi D_q(\hat{g}(\xi))}{\hat{g}(\xi)}\right) - (NL_1 - L_2), \\ H(\xi) &= (OL_1 + L_2)\left(\frac{\xi D_q(\hat{g}(\xi))}{\hat{g}(\xi)}\right) - (NL_1 + L_2), \end{aligned}$$

and $k \geq 0$, $-1 \leq O < N \leq 1$, $L_1 = q + 1$ *and* $L_2 = 3 - q$. *For* $q \to 1$, $k - ST_q(N, O)$ *was discussed in [21].*

2. Set of lemmas

***Lemma* 1.** *[22] Suppose* $d(\xi) = 1 + \sum_{t=1}^{\infty} c_t \xi^t \prec 1 + \sum_{t=1}^{\infty} C_t \xi^t = \mathbb{H}(\xi)$. *If* $\mathbb{H}(\mathbb{U})$ *is convex and* $\mathbb{H}(\xi) \in \mathfrak{A}$, *then*

$$|C_1| \geq |c_t|, \quad \text{for } 1 \leq t. \tag{2.1}$$

***Lemma* 2.** *[18] Suppose* $1 + \sum_{t=1}^{\infty} c_t \xi^t = d(\xi) \in k - ST_q(N, O)$, *then*

$$\frac{L_1(\lambda - M)}{4}\delta(l) = |\delta(l, \lambda, M)| \geq |c_t|, \tag{2.2}$$

where $\delta(l)$ *is given by (1.17).*

***Lemma* 3.** *[18] If* $d(\xi) = \xi + \sum_{t=1}^{\infty} b_t \xi^t \in k - ST_q(N, O)$ *for* $\xi \in \mathbb{U}$ *and* $k \geq 0$, *then*

$$|b_t| \leq \prod_{p=0}^{t-2}\left[\frac{\left|(N - O)\delta(l)L_1 - 4O[p, q]\right|}{4[p + 1, q]q}\right], \tag{2.3}$$

where $\delta(l)$ *is given by (1.17).*

***Lemma* 4.** *If* $d \in \mathbb{S}^*$, $\mathcal{G} \in \mathfrak{S}$ *and* $\hat{g} \in C$ *then*

$$\frac{\hat{g}(\xi) * d(\xi)\mathcal{G}(\xi)}{\hat{g}(\xi) * d(\xi)} \in c\bar{o}(\mathcal{G}(\mathcal{U})), \quad \text{for all } \xi \in \mathbb{U}. \tag{2.4}$$

Where $c\bar{o}(\mathcal{G}(\mathbb{U}))$ *is the closed convex hull* $\mathcal{G}(\mathbb{U})$.

***Lemma* 5.** *[18] A function* $\hat{g} \in \mathfrak{A}$ *will be in the class* $k - ST_q(N, O)$, *if*

$$\sum_{t=2}^{\infty}\left\{2(k + 1)L_2 q[t - 1, q] + \left|(OL_1 + L_2)[t, q] - (NL_1 + L_2)\right|\right\}|a_t| < L_1|O - N|.$$

Motivated by the work of Mahmood et al. [18], Noor and Malik [21] and Arif et al. [8], we define a new subclasses of Janowski type q-starlike functions associated with q-conic domains as:

***Definition* 3.** *A function* $\hat{g}(\xi) \in \mathfrak{A}$ *is apparently in the function class* $k - ST_q(\mu, N, O)$ *if and only if*

$$\Re\left[\frac{A(\xi)}{B(\xi)}\right] > k\left|\frac{A(\xi)}{B(\xi)} - 1\right|,$$

where

$$A(\xi) = (OL_1 - L_2)\left(\frac{\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)}{\Im_q^\mu \hat{g}(\xi)}\right) - (NL_1 - L_2),$$

$$B(\xi) = (OL_1 + L_2)\left(\frac{\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)}{\Im_q^\mu \hat{g}(\xi)}\right) - (NL_1 + L_2),$$

and $k \geq 0, -1 \leq O < N \leq 1, \mu > -1, L_1 = 1 + q$ *and* $L_2 = 3 - q$.

It is noted that, for $\mu = 1$, the function class $k - ST_q(\mu, N, O)$ reduces to well known class $k - ST_q(N, O)$. Also $0 - ST_q(1, N, O) = S^*(N, O)$ introduced by Srivastava et al. [19,23], further for $q \to 1$, $k - ST_{q\to1^-}(N, O) = k - ST(N, O)$ this class was studied by Noor and Malik [21] also see [20, 26].

3. Main results

Theorem 1. *A function $\hat{g}(\xi) \in \mathfrak{A}$ and of the form (1.1) is in the class $k - ST_q(\mu, N, O)$, if it fulfill the following restriction*

$$\sum_{t=2}^{\infty} \Lambda_t |a_t| < L_1 |O - N|,$$

where $\Lambda_t = \left\{2(k+1)L_2 q\,[t-1, q] + \left|(OL_1 + L_2)[t, q] - (NL_1 + L_2)\right|\right\}\psi_{t-1}$.

Proof. Assume that (1.20) hold, then it is suffices to prove that

$$k\left|\frac{(OL_1 - L_2)\left(\frac{\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)}{\Im_q^\mu \hat{g}(\xi)}\right) - (NL_1 - L_2)}{(OL_1 + L_2)\left(\frac{\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)}{\Im_q^\mu \hat{g}(\xi)}\right) - (NL_1 + L_2)} - 1\right| - \Re\left[\begin{array}{c}\frac{(OL_1 - L_2)\left(\frac{\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)}{\Im_q^\mu \hat{g}(\xi)}\right) - (NL_1 - L_2)}{(OL_1 + L_2)\left(\frac{\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)}{\Im_q^\mu \hat{g}(\xi)}\right) - (NL_1 + L_2)} \\ -1\end{array}\right] < 1.$$

We consider,

$$\begin{aligned}
& k\left|\frac{(OL_1 - L_2)\left(\frac{\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)}{\Im_q^\mu \hat{g}(\xi)}\right) - (NL_1 - L_2)}{(OL_1 + L_2)\left(\frac{\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)}{\Im_q^\mu \hat{g}(\xi)}\right) - (NL_1 + L_2)} - 1\right| - \Re\left[\begin{array}{c}\frac{(OL_1 - L_2)\left(\frac{\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)}{\Im_q^\mu \hat{g}(\xi)}\right) - (NL_1 - L_2)}{(OL_1 + L_2)\left(\frac{\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)}{\Im_q^\mu \hat{g}(\xi)}\right) - (NL_1 + L_2)} \\ -1\end{array}\right] \\
\leq\ & k\left|\frac{(OL_1 - L_2)\left(\frac{\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)}{\Im_q^\mu \hat{g}(\xi)}\right) - (NL_1 - L_2)}{(OL_1 + L_2)\left(\frac{\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)}{\Im_q^\mu \hat{g}(\xi)}\right) - (NL_1 + L_2)} - 1\right| \\
=\ & 2L_2(k+1)\left|\frac{\Im_q^\mu \hat{g}(\xi) - \xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)}{(OL_1 + L_2)\left(\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)\right) - (NL_1 + L_2)\Im_q^\mu \hat{g}(\xi)}\right| \\
\leq\ & \frac{2L_2(k+1)\sum_{t=2}^{\infty}\left|(1 - [t, q])\psi_{t-1}\right||a_t|}{L_1|O - N| - \sum_{t=2}^{\infty}\left|\{(OL_1 + L_2)[t, q] - (NL_1 + L_2)\}\psi_{t-1}\right||a_t|}.
\end{aligned}$$

The last inequalities is bounded above by 1 if

$$\begin{aligned} & 2L_2\,(1+k)\sum_{t=2}^{\infty}\left|(1-[t,q])\,\psi_{t-1}\right||a_t| \\ < \; & L_1\,|O-N| - \sum_{t=2}^{\infty}\left|\{(OL_1+L_2)\,[t,q]-(NL_1+L_2)\}\,\psi_{t-1}\right||a_t|\,, \end{aligned}$$

which reduces to

$$\sum_{t=2}^{\infty}\Lambda_t\,|a_t| < L_1\,|O-N|\,.$$

This complete the proof. □

For $\mu = 1$, we have the following corollary.

***Corollary* 1.** *[18] A function $\hat{g}(\xi) \in \mathfrak{A}$ and of the type (1.1) is considered to be in the function class $k-ST_q(1,N,O)$, if it fulfill the following criterion*

$$\sum_{t=2}^{\infty}\left\{2(1+k)L_2q\,[t-1,q]+\left|(OL_1+L_2)\,[t,q]-(NL_1+L_2)\right|\right\}|a_t| < L_1\,|O-N|\,.$$

If $q \to 1^-$, then corollary (3.2) reduces to:

***Corollary* 2.** *[21] A function $\hat{g}(\xi) \in \mathfrak{A}$ of the form (1.1) is considered to be in the class $k-ST(1,N,O)$, if it fulfill the following criterion*

$$\sum_{t=2}^{\infty}\{2(1+k)\,(t-1)+|t\,(O+1)-(N+1)|\}\,|a_t| < |O-N|\,.$$

Further if we take $N = 1$, $O = -1$ then we have $k-ST(1,-1)$.

***Corollary* 3.** *[14] A function $\hat{g}(\xi) \in \mathfrak{A}$ of the form (1.1) is considered to be in the function class $k-ST(1,1,-1)$, if it fulfill the following criterion*

$$\sum_{t=2}^{\infty}\{t+k\,(t-1)\}\,|a_t| < 1.$$

3.1. Coefficient bound for the class $k-ST_q(\mu,N,O)$

Theorem 2. *Let a function $\hat{g} \in k-ST_q(\mu,N,O)$ is of the form (1.1). Then*

$$|a_t| \le \prod_{j=0}^{t-2}\frac{\left|(N-O)\,(q+1)\,\delta_l\psi_{j-1}-4\,[j,q]\,qO\psi_j\right|}{4\,[j+1,q]\,q\psi_{j+1}}, \qquad (t\in\mathbb{N}\setminus\{1\})\,. \tag{3.1}$$

This result is sharp.

Proof. Since $\hat{g} \in k - ST_q(\mu, N, O)$, so let

$$\frac{\xi D_q\left(\mathfrak{I}_q^{\mu}\hat{g}(\xi)\right)}{\mathfrak{I}_q^{\mu}\hat{g}(\xi)} = \wp(\xi), \tag{3.2}$$

where

$$\wp(\xi) \prec \frac{(N(1+q)+L_2)\,\varpi_l(\xi) - ((q+1)N+L_2)}{(O(q+1)+L_2)\,\varpi_l(\xi) - ((q+1)O+L_2)}. \tag{3.3}$$

If

$$\varpi_l(\xi) = 1 + \delta_l(\xi) + (\delta_l(\xi))^2 + (\delta_l(\xi))^3 + \cdots,$$

then

$$\begin{aligned} & \frac{((q+1)N+L_2)\,\varpi_l(\xi) - ((q+1)N+L_2)}{((q+1)O+L_2)\,\varpi_l(\xi) - ((1+q)O+L_2)} \\ = \; & 1 + \frac{1}{4}(q+1)(N-O)\,\delta_l \\ & + \frac{1}{4}\left[\left(-\frac{1}{4}Nq - \frac{1}{4}N + \frac{1}{4}Oq + \frac{1}{4}O\right)((O+1)(1+q)+2-2q)\right]\delta_l^2 + \cdots. \end{aligned}$$

Let $\wp(\xi) = 1 + \sum_{t=1}^{\infty} c_t\xi^t$, then by Lemma 2.1 and relation (3.3), we get

$$c_t \leq \frac{1}{4}(N-O)(1+q)\,|\delta_l|. \tag{3.4}$$

Now from (3.2), we have

$$\xi D_q\left(\mathfrak{I}_q^{\mu}\hat{g}(\xi)\right) = \wp(\xi)\,\mathfrak{I}_q^{\mu}\hat{g}(\xi),$$

which implies that

$$\sum_{t=2}^{\infty}([t,\ q]-1)\,\psi_{t-1}a_t\xi^t = \left(\sum_{t=2}^{\infty}\psi_{t-1}a_t\xi^t\right)\left(\sum_{t=1}^{\infty}c_t\xi^t\right).$$

By comparing coefficient of ξ^t, we obtain

$$([t,q]-1)\,\psi_{t-1}a_t = \sum_{j=1}^{t-1}\left|a_{t-j}\right|\left|\psi_{j-1}\right|\left|c_j\right|, \quad (a_1 = 1),$$

which yields

$$|a_t| \leq \frac{1}{q\,[t-1,q]\,\psi_{t-1}}\sum_{j=1}^{t-1}\left|\psi_{j-1}\right|\left|a_{t-j}\right|\left|c_j\right|.$$

By using (3.4), above inequalities can be written as:

$$|a_t| \leq \frac{(N-O)(1+q)\,|\delta_l|\left|\psi_{j-1}\right|}{4q\,[t-1,q]\,\psi_{t-1}}\sum_{j=1}^{t-1}\left|a_j\right|. \tag{3.5}$$

Next we need to show that

$$\frac{(N-O)(1+q)\,|\delta_l|\left|\psi_{j-1}\right|}{4q\,[t-1,q]}\sum_{j=1}^{t-1}\left|a_j\right| \leq \prod_{j=0}^{t-2}\frac{\left|(N-O)(1+q)\,\delta_l\psi_{j-1} - 4qO\,[j,\ q]\,\psi_j\right|}{4\,[j+1,q]\,q\psi_{j+1}}. \tag{3.6}$$

To derive (3.6), we will utilize the principle of mathematical induction.

For $t = 2$, (3.5) become

$$|a_2| \leq \frac{(N-O)(q+1)|\delta_l|\left|\psi_{j-1}\right|}{4[1,q]q\psi_1}.$$

Which shows that (3.1) is true for $t = 2$.

For $t = 3$, (3.5) give us

$$\begin{aligned} |a_3| &\leq \frac{(N-O)(q+1)|\delta_l|\left|\psi_{j-1}\right|}{4q[2,q]\psi_2}(1+|a_2|) \\ &\leq \frac{(q+1)(N-O)|\delta_l|\left|\psi_{j-1}\right|}{4q[2,q]\psi_2}\left(1+\frac{(N-O)(1+q)\left|\psi_{j-1}\right||\delta_l|}{4q[1,q]\psi_1}\right). \end{aligned}$$

This shows that (3.1) is true for $t = 3$. Now suppose that (3.6), for $t = m$ that is

$$|a_m| \leq \frac{(N-O)(1+q)|\delta_l|\left|\psi_{j-1}\right|}{4[m-1,q]q\psi_{m-1}}\sum_{j=1}^{m-1}\left|a_j\right|. \tag{3.7}$$

On the other hand from (3.1), we have

$$|a_m| \leq \prod_{j=0}^{m-2}\frac{\left|(N-O)(1+q)\delta_l\psi_{j-1}-4qO[j,q]\psi_j\right|}{4q[j+1,q]\psi_{j+1}}, \quad (m \in \mathbb{N}\setminus\{1\}).$$

Using induction hypothesis on (3.6), we have

$$\frac{(q+1)(N-O)|\delta_l|\left|\psi_{j-1}\right|}{4\psi_{m-1}q[m-1,q]}\sum_{j=1}^{m-1}\left|a_j\right| \leq \prod_{j=0}^{m-2}\frac{\left|(N-O)(q+1)\delta_l\psi_{j-1}-4qO[j,q]\psi_j\right|}{4\psi_{j+1}q[j+1,q]}. \tag{3.8}$$

As

$$\begin{aligned} &\prod_{j=0}^{m-2}\frac{(N-O)(q+1)\delta_l\psi_{j-1}+4qO[j,q]}{4[j+1,q]\psi_{j+1}q} \\ \geq &\left(\frac{(N-O)(q+1)|\delta_l|\psi_{j-1}+4[m-1,q]q}{4\psi_j q[m,q]}\right)\left(\frac{(q+1)(N-O)\left|\psi_{j-1}\right||\delta_l|}{4q\psi_{m-1}[m-1,q]}\sum_{j=1}^{m-1}\left|a_j\right|\right) \\ = &\frac{(N-O)(1+q)|\delta_l|\psi_{j-1}}{4\psi_j[m,q]q}\left(\frac{(q+1)(N-O)|\delta_l|\psi_{j-1}}{4\psi_{j+1}[m-1,q]q}\sum_{j=1}^{m-1}\left|a_j\right|+\sum_{j=1}^{m}\left|a_j\right|\right) \\ = &\frac{(N-O)(1+q)|\delta_l|\psi_{j-1}}{4q[m,q]\psi_j}\sum_{j=1}^{m}\left|a_j\right|. \end{aligned}$$

Thus

$$\frac{(N-O)(1+q)|\delta_l|\psi_{j-1}}{4[m,q]q\psi_j}\sum_{j=1}^{m}\left|a_j\right| \leq \prod_{j=0}^{m-1}\frac{\left|(N-O)(1+q)\delta_l\psi_{j-1}-4q[j,q]\psi_j\right|}{4q[j+1,q]\psi_{j+1}},$$

which shows that inequality (3.8), is true for $t = m + 1$ and hence we obtained the required result. □

For $\mu = 1$, we have the following corollary.

***Corollary* 4.** *[18] Consider a function $\hat{g} \in k - ST_q(N, O)$ is of the form (1.1), then*

$$|a_t| \leq \prod_{j=0}^{t-2} \frac{\left|(1+q)(N-O)\delta_l - 4qO[j,\ q]\right|}{4[j+1,\ q]q}, \quad (t \in \mathbb{N} \setminus \{1\}).$$

If $k = 0$, then corollary (3.6) reduces to:

***Corollary* 5.** *[23] Consider a function $\hat{g} \in ST_q^*(N, O)$ is of the type (1.1), then*

$$|a_t| \leq \prod_{j=0}^{t-2} \frac{\left|(1+q)(N-O) - 2qO[j,\ q]\right|}{2[j+1,\ q]q}, \quad (t \in \mathbb{N} \setminus \{1\}).$$

Further if we take $q \to 1^-$ then we have $ST(N, O)$.

***Corollary* 6.** *[21] Consider a function $\hat{g} \in ST(N, O)$ is of the type (1.1), then*

$$|a_t| \leq \prod_{j=0}^{t-2} \frac{|(N-O)\delta_l - 2Oj|}{2(1+j)}, \quad (t \in \mathbb{N} \setminus \{1\}).$$

If we take $N = 1$ and $O = -1$ then we have $k - ST_q(N, O)$.

***Corollary* 7.** *[14] Consider a function $\hat{g} \in ST$ is of the type (1.1), then*

$$|a_t| \leq \prod_{j=0}^{t-2} \frac{|\delta_l + j|}{(j+1)}, \ (t \in \mathbb{N} \setminus \{1\}).$$

Further if we take $N = 1 - 2\alpha$, $O = -1$, $k = 0$ and $0 \leq \alpha < 1$ then we have $S^*(\alpha)$.

***Corollary* 8.** *[24] Let the function $\hat{g} \in S^*(\alpha)$ be of the form (1.1), then*

$$|a_t| \leq \prod_{j=0}^{t-2} \frac{|j - 2\alpha|}{(t-1)!}, \qquad (t \in \mathbb{N} \setminus \{1\}).$$

Next we show that the class $k - P_q(\mu, N, O)$ is closed under convolution with convex function.

Theorem 3. *If $\hat{g} \in k - P_q(\mu, N, O)$ and $\chi \in C$, then $\hat{g} * \chi \in k - P_q(\mu, N, O)$.*

Proof. We want to prove that

$$\frac{\xi D_q\left(\chi(\xi) * \Im_q^\mu \hat{g}(\xi)\right)}{\left(\chi(\xi) * \Im_q^\mu \hat{g}(\xi)\right)} \in k - P_q(\mu, N, O).$$

It can be easily seen that

$$\begin{aligned} \frac{\xi D_q\left[\chi(\xi) * \Im_q^\mu \hat{g}(\xi)\right]}{\left[\chi(\xi) * \Im_q^\mu \hat{g}(\xi)\right]} &= \frac{\Im_q^\mu \hat{g}(\xi) * \chi(\xi)\left(\frac{\left(\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)\right)}{\left(\Im_q^\mu \hat{g}(\xi)\right)}\right)}{\Im_q^\mu \hat{g}(\xi) * \chi(\xi)} \\ &= \frac{\chi(\xi) * \Im_q^\mu \hat{g}(\xi)\Psi(\xi)}{\chi(\xi) * \hat{g}(\xi)}, \end{aligned}$$

where, $\frac{\xi D_q\left(\Im_q^\mu \hat{g}(\xi)\right)}{\Im_q^\mu \hat{g}(\xi)} = \Psi(\xi) \in k - P_q(\mu, N, O)$. By using Lemma 4, we obtain the required result. □

For $\mu = 1$, we have the following corollary.

***Corollary* 9.** *[25] If $\hat{g} \in k - P_q(N, O)$ and $\chi \in C$, then $\hat{g} * \chi \in k - P_q(N, O)$.*

Theorem 4. *If $\hat{g} \in k - ST_q(\mu, N, O)$ and $\Phi \in C$, then $\hat{g} * \Phi \in k - ST_q(\mu, N, O)$.*

Proof. We want to prove that

$$\frac{\xi D_q\left(\Phi(\xi) * \Im_q^{\mu}\hat{g}(\xi)\right)}{\left(\Im_q^{\mu}\hat{g}(\xi) * \Phi(\xi)\right)} \in k - P_q(\mu, N, O).$$

It can be easily seen that

$$\begin{aligned}\frac{\xi D_q\left(\Phi(\xi) * \Im_q^{\mu}\hat{g}(\xi)\right)}{\left(\Phi(\xi) * \Im_q^{\mu}\hat{g}(\xi)\right)} &= \frac{\Im_q^{\mu}\hat{g}(\xi) * \chi(\xi)\left(\frac{\xi D_q\left(\Im_q^{\mu}\hat{g}(\xi)\right)}{\Im_q^{\mu}\hat{g}(\xi)}\right)}{\Im_q^{\mu}\hat{g}(\xi) * \Phi(\xi)} \\ &= \frac{\Phi(\xi) * \Im_q^{\mu}\hat{g}(\xi)\Psi(\xi)}{\Phi(\xi) * \Im_q^{\mu}\hat{g}(\xi)},\end{aligned}$$

where, $\frac{\xi D_q\left(\Im_q^{\mu}\hat{g}(\xi)\right)}{\Im_q^{\mu}\hat{g}(\xi)} = \Psi(\xi) \in k - P_q(\mu, N, O)$, by applying Lemma 4, we obtain the required result. □

For $\mu = 1$, we have the following corollary.

***Corollary* 10.** *[25] If $\hat{g} \in k - ST_q(N, O)$ and $\Phi \in C$, then $\hat{g} * \Phi \in k - ST_q(N, O)$.*

3.2. Linear combination

Linear combination for our defined classes are defined as following.

Theorem 5. *Let $\hat{g}_i \in k - ST_q(\mu, N, O)$ and have the form*

$$\hat{g}_i(\xi) = \xi + \sum_{t=1}^{\infty} a_{t,i}\xi^t, \quad for i = 1, 2, 3, \cdots, n.$$

Then

$$F \in k - ST_q(\mu, N, O), where\ F(\xi) = \sum_{i=1}^{n} c_i\hat{g}_i(\xi)\ \ with \sum_{i=1}^{n} c_i = 1.$$

Proof. By using (1.20), one can write

$$\sum_{t=2}^{\infty}\left[\frac{\left\{2(k+1)L_2 q\,[t-1, q] + \left|(OL_1 + L_2)\,[t, q] - (NL_1 + L_2)\right|\right\}\psi_{t-1}}{L_1\,|O - N|}\right]\left|a_{t,i}\right| < 1.$$

Therefore

$$F(\xi) = \sum_{i=2}^{n} c_i\left(\xi + \sum_{t=2}^{\infty} a_{t,i}\cdot\xi^t\right) = \xi + \sum_{t=2}^{\infty}\left(\sum_{i=2}^{n} c_i\cdot a_{t,i}\right)\xi^t,$$

however

$$\sum_{t=2}^{\infty}\left[\frac{\left\{2(k+1)L_2q\,[t-1,q]+\left|(OL_1+L_2)\,[t,q]-(NL_1+L_2)\right|\right\}\psi_{t-1}}{L_1\,|O-N|}\right]\left(\sum_{i=2}^{n}c_i a_{t,\,i}\right)$$

$$=\sum_{i=2}^{n}\left(\sum_{t=2}^{\infty}\left[\frac{\left\{2(k+1)L_2q\,[t-1,q]+\left|(OL_1+L_2)\,[t,q]-(NL_1+L_2)\right|\right\}\psi_{t-1}}{L_1\,|O-N|}\right]a_{t,\,i}\right)c_i\leq 1.$$

□

3.3. Weighted means

Theorem 6. *If $\hat{g}$ and $\hat{h}$ belongs to $k-ST_q(\mu,N,O)$ where,*

$$h_W(\xi)=\left\{\frac{(1-W)\,\hat{g}(\xi)+(1+W)\,\hat{h}(\xi)}{2}\right\}.$$

Proof. As

$$h_W(\xi)=\left\{\frac{(1-W)\,\hat{g}(\xi)+(1+W)\,\hat{h}(\xi)}{2}\right\}=\xi+\sum_{t=2}^{\infty}\left\{\frac{(1-W)\,a_t+\sum_{t=2}^{\infty}(1+W)\,b_t}{2}\right\}\xi^t.$$

To prove that $h_W(\xi)\in k-ST_q(\mu,N,O)$, we need to show

$$\textstyle\sum_{t=2}^{\infty}\left[\frac{\left\{2(k+1)L_2q[t-1,q]+\left|(OL_1+L_2)[t,q]-(NL_1+L_2)\right|\right\}\psi_{t-1}}{L_1|O-N|}\right]\left\{\frac{(1-W)a_t+(1+W)b_t}{2}\right\}<1.$$

For this, consider

$$\frac{(1-W)}{2}\sum_{t=2}^{\infty}\left\{\frac{\left\{2(k+1)L_2q\,[t-1,q]+\left|(OL_1+L_2)\,[t,q]-(NL_1+L_2)\right|\right\}}{L_1\,|O-N|}\right\}\psi_{t-1}a_t+$$
$$\frac{(1+W)}{2}\sum_{t=2}^{\infty}\left\{\frac{\left\{2(k+1)L_2q\,[t-1,q]+\left|(OL_1+L_2)\,[t,q]-(NL_1+L_2)\right|\right\}}{L_1\,|O-N|}\right\}\psi_{t-1}b_t$$
$$<\frac{(1-W)}{2}(1)+\frac{(1+W)}{2}(1)=1.$$

□

***Corollary* 11.** *If we take $\mu=1$, if $\hat{g}$ and $\hat{h}$ belongs to $k-ST_q(1,N,O)=k-ST_q(N,O)$, then their weighted mean h_W is also in $k-ST_q(N,O)$.*

Further for $q\to 1^-$, then their weighted mean h_W is also in $k-ST(N,O)$. Where,

$$h_W(\xi)=\left\{\frac{(1-W)\,\hat{g}(\xi)+(1+W)\,\hat{h}(\xi)}{2}\right\}.$$

3.4. Arithmetic means

Theorem 7. *Let $\hat{g}_i \in k - ST_q(\mu, N, O)$ where $i = 1, 2, \cdots, \nu$ then the arithmetic mean*

$$A_M(\xi) = \frac{1}{\nu} \sum_{i=1}^{\nu} \hat{g}_i(\xi),$$

also belongs to the class $k - ST_q(\mu, N, O)$.

Proof. As $A_M(\xi) = \frac{1}{\nu}\sum_{i=1}^{\nu} \hat{g}_i(\xi)$ and $\hat{g}_i(\xi) = \xi + \sum_{t=2}^{\infty} a_{t,i}\xi^t$ then we have

$$A_M(\xi) = \frac{1}{\nu} \sum_{i=1}^{\nu} \left(\xi + \sum_{t=2}^{\infty} a_{t,i}\xi^t \right) = \xi + \sum_{t=2}^{\infty} \left(\frac{1}{\nu} \sum_{i=1}^{\nu} a_{t,i} \right) \xi^t. \tag{3.9}$$

Since $\hat{g}_i \in k - ST_q(\mu, N, O)$ for every $i = 1, 2, \cdots, \nu$, so by using (1.20) and (3.9), we get

$$\begin{aligned} &\sum_{t=2}^{\infty} \psi_{t-1} \left\{ 2(k+1)L_2 q\,[t-1,q] + \left| (OL_1 + L_2)\,[t,q] - (NL_1 + L_2) \right| \right\} \left(\frac{1}{\nu} \sum_{i=1}^{\nu} a_{t,i} \right) \\ \leq \;& \frac{1}{\nu} \sum_{i=1}^{\nu} (L_1 |O - N|) = L_1 |O - N|, \end{aligned}$$

i.e.

$$\sum_{t=2}^{\infty} \psi_{t-1} \left\{ 2(k+1)L_2 q\,[t-1,q] + \left| (OL_1 + L_2)\,[t,q] - (NL_1 + L_2) \right| \right\} \left(\frac{1}{\nu} \sum_{i=1}^{\nu} a_{t,i} \right) \leq L_1 |O - N|.$$

This complete the proof. □

***Corollary* 12.** *If we take $\mu = 1$, $\hat{g}_i \in k - ST_q(N, O)$ with $i = 1, 2, \cdots, \nu$ then the arithmetic mean*

$$A_M(\xi) = \frac{1}{\nu} \sum_{i=1}^{\nu} \hat{g}_i(\xi)$$

this belongs to the class $k - ST_q(N, O)$.

Further, for $\mu = 1$, $\hat{g}_i \in k - ST_{q\to 1}(1, N, O) = k - ST(N, O)$ where $i = 1, 2, \cdots, \nu$ then the arithmetic mean $A_M(\xi)$ also belongs to the class $k - ST(N, O)$.

3.5. Radii of starlikeness

Theorem 8. *Let $\hat{g} \in k - ST_q(\mu, N, O)$, then $\hat{g}$ will belongs to the family $S^*(\alpha)$ called starlike functions of order α $(0 \leq \alpha < 1)$ for $|\xi| < \mathfrak{r}_1$, where*

$$\mathfrak{r}_1 = \left[\frac{(1-\alpha)\left\{ 2(1+k)L_2 q\,[t-1,q] + \left| (OL_1 + L_2)\,[t,q] - (NL_1 + L_2) \right| \right\} [t,q]!}{L_1 |O - N| (t-\alpha) [\mu+1, q]_{t-1}} \right]^{\left(\frac{1}{t-1}\right)}.$$

Proof. Let $\hat{g} \in k - ST_q(\mu, N, O)$. To prove $\hat{g} \in S^*(\alpha)$, we need to show

$$\left| \frac{\xi \hat{g}'(\xi)/\hat{g}(\xi) - 1}{\xi \hat{g}'(\xi)/\hat{g}(\xi) + 1 - 2\alpha} \right| < 1.$$

Using values of $\hat{g}(\xi)$ along with some staightforward calculations, we have

$$\sum_{t=2}^{\infty} \left(\frac{t-\alpha}{1-\alpha} \right) |a_t| |\xi|^{t-1} < 1. \tag{3.10}$$

Since $\hat{g} \in k - ST_q(\mu, N, O)$, so from (1.20), we can easily obtain

$$\sum_{t=2}^{\infty} \frac{[t,q]!}{[\mu+1,q]_{t-1}} \left(\frac{\left\{ 2(k+1)L_2 q [t-1,q] + \left| (OL_1 + L_2)[t,q] - (NL_1 + L_2) \right| \right\}}{L_1 |O-N|} \right) |a_t| < 1.$$

The inequality (3.10), holds if the following relation are true

$$\sum_{t=2}^{\infty} \left(\frac{t-\alpha}{1-\alpha} \right) |a_t| |\xi|^{t-1} < \sum_{t=2}^{\infty} \frac{[t,q]!}{[\mu+1,q]_{t-1}} \left(\frac{\left\{ 2(k+1)L_2 q [t-1,q] + \left| (OL_1 + L_2)[t,q] - (NL_1 + L_2) \right| \right\}}{L_1 |O-N|} \right) |a_t|,$$

which implies that

$$|\xi| < \left(\frac{(1-\alpha)\left\{ 2(k+1)L_2 q [t-1,q] + \left| (OL_1 + L_2)[t,q] - (NL_1 + L_2) \right| \right\} [t,q]!}{L_1 |O-N| (t-\alpha) [\mu+1,q]_{t-1}} \right)^{\left(\frac{1}{t-1}\right)}.$$

Which completes the proof. □

Corollary 13. *If we take* $\mu = 1$, *if* $\hat{g} \in k - ST_q(N, O)$, *then* $|\xi| < \mathfrak{r}_2$, *where*

$$\mathfrak{r}_2 = \left[\frac{(1-\alpha)\left\{ 2(1+k) L_2 q [t-1,q] + \left| (OL_1 + L_2)[t,q] - (NL_1 + L_2) \right| \right\} [t,q]!}{L_1 |O-N| (t-\alpha)} \right]^{\left(\frac{1}{t-1}\right)}.$$

Further, $\hat{g} \in k - ST_{q \to 1}(1, N, O) = k - ST(N, O)$, $\hat{g}$ *then* $|\xi| < \mathfrak{r}_3$, *where*

$$\mathfrak{r}_3 = \left[\frac{(1-\alpha)\{2(1+k)(t-1) + |(O+1)t - (N+1)|\}}{|O-N|(t-\alpha)} \right]^{\left(\frac{1}{t-1}\right)}.$$

3.6. Growth and distortion theorems

Theorem 9. *If* $\hat{g} \in k - ST_q(\mu, N, O)$ *has the form (1.1), then*

$$r(1-\zeta) \le |\hat{g}(\xi)| \le r(\zeta + 1),$$

where

$$\zeta = \frac{L_1 |O-N|}{\{2(k+1)L_2 q + |(OL_1 + L_2)(1+q) - (NL_1 + L_2)|\} \psi_1} \text{ with } |\xi| = r < 1.$$

Proof. Consider

$$|\hat{g}(\xi)| = \left|\xi + \sum_{t=2}^{\infty} a_t \xi^t\right| = r + \sum_{t=2}^{\infty} |a_t| r^t,$$

This implies

$$|\hat{g}(\xi)| \leq r + r\sum_{t=2}^{\infty} |a_t| = r\left(1 + \sum_{t=2}^{\infty} |a_t|\right). \tag{3.11}$$

Similarly,

$$|\hat{g}(\xi)| \geq r\left(1 - \sum_{t=2}^{\infty} |a_t|\right). \tag{3.12}$$

It can be easily observed that

$$\begin{aligned} &\left\{2(k+1)L_2 q\,[1,q] + \left|(OL_1 + L_2)\,[2,q] - (NL_1 + L_2)\right|\right\}\psi_1 \sum_{t=2}^{\infty} a_t \\ \leq\ & \sum_{t=2}^{\infty} \left\{2(k+1)L_2 q\,[t-1,q] + \left|(OL_1 + L_2)\,[t,q] - (NL_1 + L_2)\right|\right\}\psi_{t-1}\,|a_t|\,. \end{aligned}$$

By using (1.20), we obtain

$$\left\{2(k+1)L_2 q\,[1,q] + \left|(OL_1 + L_2)\,[2,q] - (NL_1 + L_2)\right|\right\}\psi_1 \sum_{t=2}^{\infty} |a_t| \leq L_1\,|O - N|\,,$$

which gives

$$\begin{aligned} \sum_{t=2}^{\infty} |a_t| &\leq \frac{L_1\,|O-N|}{\left\{2(k+1)L_2 q\,[1,q] + \left|(OL_1 + L_2)\,[2,q] - (NL_1 + L_2)\right|\right\}\psi_1} \\ &= \frac{L_1\,|O-N|}{\{2(k+1)L_2 q + |(OL_1 + L_2)\,(1+q) - (NL_1 + L_2)|\}\,\psi_1}, \end{aligned}$$

now using this relation in (3.11) and (3.12), we get

$$r\,(1-\zeta) \leq |\hat{g}(\xi)| \leq r\,(\zeta + 1)\,.$$

As required. □

***Corollary* 14.** *If we take $\mu = 1$, and $\hat{g} \in k - ST_q(N, O)$, has the form (1.1), then*

$$r\,(1-\zeta_1) \leq |\hat{g}(\xi)| \leq r\,(1+\zeta_1)\,,$$

where

$$\zeta_1 = \frac{L_1\,|O-N|}{\{2(k+1)L_2 q + |(OL_1 + L_2)\,(1+q) - (NL_1 + L_2)|\}}.$$

Further, $\hat{g} \in k - ST_{q\to1}(1, N, O) = k - ST(N, O)$, has the form (1.1), then

$$r\,(1-\zeta_2) \leq |\hat{g}(\xi)| \leq r\,(1+\zeta_2)\,,$$

where,

$$\zeta_2 = \frac{|O-N|}{\{2(k+1) + |2\,(O+1) - (N+1)|\}}.$$

***Corollary* 15.** *If $\hat{g} \in k - ST_q(N, O)$, has the form (1.1), then*

$$(1 - rt\varkappa_1) \leq \left|\hat{g}'(\xi)\right| \leq (1 + rt\varkappa_1),$$

where,

$$\varkappa_1 = \frac{L_1 |O - N|}{\{2(k+1)L_2 q + |(OL_1 + L_2)(1+q) - (NL_1 + L_2)|\}}.$$

Further, $\hat{g} \in k - ST_{q \to 1}(1, N, O) = k - ST(N, O)$, has the form (1.1), then

$$(1 - rt\varkappa_2) \leq \left|\hat{g}'(\xi)\right| \leq (1 + rt\varkappa_2),$$

where,

$$\varkappa_2 = \frac{|O - N|}{\{2(k+1) + |2(O+1) - (N+1)|\}}.$$

4. Conclusions

By using q-analogue of Noor integral operator, we studied various properties such as necessary and sufficient conditions, coefficient bounds, convolution properties, linear combinations, weighted means, arithmetic means, distortion and covering theorems and radii of starlikenss, for a newly define class of analytic functions in conic regions. We also pointed out many special cases in the form of corollaries by specializing the parameters.

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Conflict of interest

The authors declare that there is no conflict of interests in this paper.

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