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Research article

Acyclic edge coloring of planar graphs

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Abstract: An acyclic edge coloring of a graph G is a proper edge coloring such that no bichromatic cycles are produced. The acyclic chromatic index of G, denoted by $\chi'_a(G)$, is the smallest integer k such that G is acyclically edge k-colorable. In this paper, we consider the planar graphs without 3-cycles and intersecting 4-cycles, and prove that $\chi'_a(G) \leq \Delta(G) + 1$ if $\Delta(G) \geq 8$.

Keywords: acyclic edge coloring; planar graph; cycle; girth; maximum degree
Mathematics Subject Classification: 05C10, 05C15

1. Introduction

All graphs considered in this paper are finite simple graphs. For a graph G, we use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ (Δ for short and reserved), and $g(G)$ to denote the vertex set, edge set, face set, maximum degree and girth, respectively. A graph G is 2-connected if there are two paths between any two distinct vertices.

Let G be a planar graph. The acyclic edge k-coloring of graph G is a mapping $c : E(G) \to \{1, 2, \ldots, k\}$ such that any two adjacent edges receive different colors, and there are no bichromatic cycles in G. The acyclic chromatic index of G, denoted by $\chi'_a(G)$, is the smallest integer k such that G is acyclically edge k-colorable.

Fiamčik posed a famous conjecture for acyclic edge coloring of any graphs.

Conjecture 1.1. *[3] For any graph G, $\chi'_a(G) \leq \Delta(G) + 2$.*

The conjecture is still open.

For any graph G, Alon, McDiarmid and Reed [1] proved that $\chi'_a(G) \leq 64\Delta$. Molloy and Reed [8] improved this bound to 16Δ. Later, Fialho et al. [4] showed that $\chi'_a(G) \leq 3.569(\Delta - 1)$, and most recently to $2\Delta - 1$ by Kirousis and Livieratos [7].

There have been numerous investigations about acyclic edge coloring of planar graphs.

For any planar graph G, Basavaraju et al. [2] proved that $\chi'_a(G) \le \Delta + 12$, Wang et al. [9] proved that $\chi'_a(G) \le \Delta + 7$, Wang and Zhang [10] proved that $\chi'_a(G) \le \Delta + 6$.

Let G be a planar graph with small grith. Shu, Wang and Wang [11] proved that $\chi'_a(G) \le \Delta(G) + 2$ if $g(G) \ge 4$. For planar graph G with $g(G) \ge 5$, Hou et al. [6] proved that $\chi'_a(G) \le \Delta(G) + 1$; they also proved that such graph has $\chi'_a(G) = \Delta(G)$ if $\Delta(G) \ge 9$. For planar graph G without 4-cycles, Wang and Sheng [13] proved that $\chi'_a(G) \le \Delta(G) + 3$. Then Wang, Shu and Wang [14] improved this bound to $\Delta(G) + 2$ when $\Delta(G) \ge 5$. In 2012, Fiedorowicz [5] proved that the planar graph G without an i-cycle intersect to a j-cycle has $\chi'_a(G) \le \Delta(G) + 2$ for $i, j \in \{3, 4\}$. Most recently, Shu et al. [12] proved that the planar graph G without intersecting triangles has $\chi'_a(G) \le \Delta(G) + 2$.

In this paper, we consider the planar graph without 3-cycles and intersecting 4-cycles, and prove the following theorem:

Theorem 1.1. *Let G be a planar graph without 3-cycles and intersecting 4-cycles. If $\Delta(G) \ge 8$, then $\chi'_a(G) \le \Delta(G) + 1$.*

2. Notation

Let G be a simple planar graph. For a vertex $v \in V(G)$, $N(v)$ denotes the set of vertices adjacent to v, and $d(v) = |N(v)|$ denotes the degree of v. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1u_2 \dots u_n]$ if $u_1, u_2, \dots, u_n$ are the vertices on $b(f)$ enumerated in the clockwise direction. For $f = [u_1u_2 \dots u_n]$, let $\delta(f)$ denote the minimum degree of any vertex on $b(f)$. That is, $\delta(f) = \min\{d(u_i), i = 1, \dots, n\}$. The degree of a face f, denoted by $d(f)$, is the number of edges in its boundary walk.

For $f \in F(G)$, f is called a k-(or k^+-, or k^--) face if $d(f) = k$ (or $d(f) \ge k$, or $d(f) \le k$). For $v \in V(G)$, v is called a k-(or k^+-, or k^--) vertex if $d(v) = k$ (or $d(v) \ge k$, or $d(v) \le k$). If $u \in N(v)$ and $d(u) = k$, then u is called k-neighbor of v. Let $N_k(v) = \{x \in N(v) | d(x) = k\}$, and $n_k(v) = |N_k(v)|$.

Let c be an edge coloring of G and v be a vertex of G. Then, $C(v) = \{c(uv) : u \in N(v)\}$, $F^c_v(uv) = C(v) \setminus \{c(uv)\}$. Let α, β be two colors. An (α, β)-bichromatic path with respect to c is a path consisting of edges that are colored with α and β alternately. An (α, β)-bichromatic path which starts at the vertex u via an edge colored α and ends at v via an edge colored α is an $(\alpha, \beta)_{(u,v)}$-bichromatic path. We use "w.l.o.g." as a shorthand for "without loss of generality".

3. Proof of Theorem 1.1

We apply a discharging procedure to prove Theorem 1.1. Discharging is a tool in a two-pronged approach to inductive proofs. It can be viewed as an amortized counting argument used to prove that a global hypothesis guarantees the existence of some desirable local configurations. In an application of the resulting structure theorem, one shows that each such local configuration cannot occur in a minimal counterexample to the desired conclusion. Such local configurations are called reducible configurations. In this section, we give some reducible configurations.

Let G be a counterexample with minimum $|V(G)| + |E(G)|$ of Theorem 1.1. In other words, G is a connected simple planar graph without 3-cycles and intersecting 4-cycles, $\Delta = \Delta(G) \ge 8$, but $\chi'_a(G) \ge \Delta + 2$. Let C be a color set of G, $C = \{1, 2, \dots, \Delta + 1\}$. Now, we discuss the structures of G.

3.1. The properties of minimal counterexample

Lemma 3.1. *The graph G is 2-connected.*

Proof. By contradiction, suppose that v is a cut vertex of G. Let $C_1, C_2, \ldots, C_t (t \geq 2)$ be the connected components of $G \setminus v$. For each $1 \leq i \leq t$, there is an acyclic $(\Delta + 1)$-edge coloring c_i of $G_i = C_i \cup \{v\}$. We can adjust the colors in each c_i such that the colors appearing on the edges incident with v are all distinct. Now the union of these colorings is an acyclic $(\Delta + 1)$-edge coloring of G, a contradiction. □

Lemma 3.2. *The graph G does not contain a 2-vertex adjacent to a 3^--vertex.*

Proof. By contradiction, let $d(v) = 2$, $N(v) = \{u, w\}$, $d(u) \leq 3$ (See Figure 1(1)). We prove the case $d(u) = 3$, and $d(u) = 2$ can be proved in a similar way. Let $N(u) = \{v, u_1, u_2\}$, $G' = G - uv$. By the minimality of G, G' admits an acyclic $(\Delta + 1)$-edge coloring c. Suppose that $c(uu_i) = i$ for $i = 1, 2$. We consider the following two cases. □

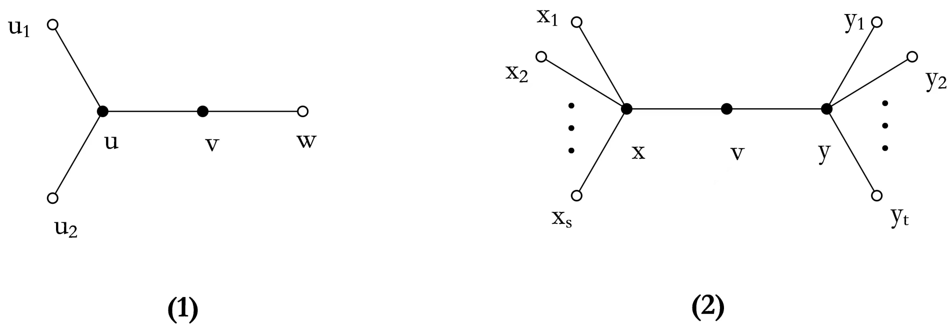

Figure 1. The configurations of Lemmas 3.2 and 3.4.

Case 3.1. $|C(u) \cap C(v)| = 0$.

Since $|C \setminus (C(u) \cup C(v))| = \Delta + 1 - 3 = \Delta - 2 > 0$, we can color uv with α for $\alpha \in C \setminus (C(u) \cup C(v))$. Therefore, c can be extended to be an acyclic $(\Delta + 1)$-edge coloring of G, a contradiction.

Case 3.2. $|C(u) \cap C(v)| = 1$. *W.l.o.g., assume that $c(vw) = c(uu_1) = 1$. If there exists a color $\gamma \in \{3, \ldots, \Delta + 1\}$ such that G contains no $(1, \gamma)_{(u,v)}$-bichromatic path via u_1 and w, then we can color uv with γ. In this way G is acyclically edge $(\Delta + 1)$-colorable, a contradiction. Thus, there must exist a $(1, \alpha)_{(u,v)}$-bichromatic path via u_1 and w for each $\alpha \in \{3, 4, \ldots, \Delta + 1\}$. So $C(u_1) = C(w) = \{1, 3, 4, \ldots, \Delta + 1\}$. Now we recolor vw with 2. Similarly, there must exist a $(2, \alpha)_{(u,v)}$-bichromatic path via u_2 and w for each $\alpha \in \{3, 4, \ldots, \Delta + 1\}$. So $C(u_2) = \{2, 3, 4, \ldots, \Delta + 1\}$. Now we exchange the colors between uu_1 and uu_2, color uv with 3, color vw with 1. Therefore, c can be extended to be an acyclic $(\Delta + 1)$-edge coloring of G, a contradiction.*

Next we discuss the degree sum of the neighbors of a particular vertex.

Lemma 3.3. *Let $d(v) = k \in \{2, 3\}$ and $N(v) = \{v_i, 1 \leq i \leq k\}$. Then $\sum_{i=1}^{k} d(v_i) \geq \Delta + k + 1$.*

Proof. By contradiction, assume that $\sum_{i=1}^{k} d(v_i) \leq \Delta + k$. Let $G' = G - vv_1$, by the minimality of G, G' admits an acyclic $(\Delta + 1)$-edge coloring c. Since the neighbors of each 2-vertex are both 4^+-vertices, we have $3 \leq d(v_i) \leq \Delta + k - 6$ when $1 \leq i \leq 3$ and $k = 3$. Suppose that $c(vv_i) = i - 1$ for $i = 2, \ldots, k$. We consider the following three cases. □

Case 3.3. $|C(v_1) \cap C(v)| = 0$.

Since $|C \setminus (C(v_1) \cup C(v))| = \Delta + 1 - (d(v_1) - 1) - (d(v) - 1) = \Delta + 3 - d(v_1) - d(v) \geq \Delta - d(v_1) > 0$, *we can color* vv_1 *with* α *for* $\alpha \in C \setminus (C(v_1) \cup C(v))$. *Therefore, c can be extended to be an acyclic* $(\Delta + 1)$*-edge coloring of G, a contradiction.*

Case 3.4. $|C(v_1) \cap C(v)| = 1$. *W.l.o.g., assume that* $1 \in C(v_1)$.

(i) $k = 2$.

Since $|C \setminus (C(v_1) \cup C(v_2))| \geq \Delta + 1 - (d(v_1) - 1) - (d(v_2) - 1) = \Delta + 3 - (d(v_1) + d(v_2)) \geq \Delta + 3 - (\Delta + 2) = 1$, *we can color* vv_1 *with* β *for* $\beta \in C \setminus (C(v_1) \cup C(v_2))$.*Therefore, c can be extended to be an acyclic* $(\Delta+1)$*-edge coloring of G, a contradiction.*

(ii) $k = 3$.

Since $|C \setminus (C(v_1) \cup C(v) \cup C(v_2))| \geq \Delta + 1 - (d(v_1) - 1) - 1 - (d(v_2) - 1) = \Delta - (d(v_1) + d(v_2)) + 2 \geq \Delta - \Delta + 2 = 2$, *we can color* vv_1 *with* α *for* $\alpha \in C \setminus (C(v_1) \cup C(v) \cup C(v_2))$, *c is an acyclic* $(\Delta + 1)$*-edge coloring of G, a contradiction.*

Case 3.5. $|C(v_1) \cap C(v)| = 2$. *Namely,* $d(v) = k = 3$.

Since $|C \setminus (C(v_1) \cup C(v_2) \cup C(v_3))| \geq \Delta + 1 - (d(v_1) - 1) - (d(v_2) - 1) - (d(v_3) - 1) = \Delta - (d(v_1) + d(v_2) + d(v_3)) + 4 \geq 1$, *we can color* vv_1 *with* α *for* $\alpha \in C \setminus (C(v_1) \cup C(v_2) \cup C(v_3))$, *c is an acyclic* $(\Delta + 1)$*-edge coloring of G, a contradiction.*

For a 2-vertex, the number of 3^--vertex and Δ-vertex of its neighbors is discussed below.

Lemma 3.4. *For a 2-vertex v, let* $N(v) = \{x, y\}$. *Then (1)* $n_2(x) \leq d(x) + d(y) - \Delta - 2$*; (2)* $n_2(x) + n_3(x) \leq d(x) + d(y) - \Delta - 1$*; (3)* $n_2(x) + n_3(x) \leq d(x) + d(y) - \Delta - 2$ *if* $d(x) < \Delta$.

Proof. Assume that $d(x) = s + 1$, $d(y) = t + 1$, $N(x) = \{v, x_1, x_2, \ldots, x_s\}$, $N(y) = \{v, y_1, y_2, \ldots, y_t\}$ (See Figure 1(2)). Let $G' = G - vy$, G' admits an acyclic $(\Delta + 1)$-edge coloring c. Let $c(yy_i) = i$ for $1 \leq i \leq t$, $T = C \setminus F^c_x(vx)$. It is clear that $|T| = \Delta + 1 - s \geq 2$.

If $T \setminus F^c_y(vy) \neq \emptyset$, then we can recolor vx with α for $\alpha \in T \setminus F^c_y(vy)$, color vy with β for $\beta \in \{t + 1, \ldots, \Delta + 1\}(\beta \neq \alpha)$. Now c can be extended to be an acyclic $(\Delta + 1)$-edge coloring of G, a contradiction. So $T \subseteq F^c_y(vy) = \{1, 2, \ldots, t\}$.

If there are $i_0 \in T$ and $j_0 \in \{t + 1, \ldots, \Delta + 1\}$, such that G contains no $(i_0, j_0)_{(v,y)}$-bichromatic path through x, then we can color vx with i_0, color vy with j_0, G is acyclically edge $(\Delta + 1)$-colorable, a contradiction. So G contains $(\alpha_0, j)_{(v,y)}$-bichromatic path for $c(vx) = \alpha_0, \alpha_0 \in T$ and $j \in \{t+1, \ldots, \Delta+1\}$. This means that $\{t+1, \ldots, \Delta+1\} \subseteq F^c_x(vx)$. W.l.o.g., let $c(xx_i) = t+i$ for $1 \leq i \leq \Delta+1-t$. Now recolor $c(vx) = i$ for any $i \in T$. Similarly, G contains $(i, j)_{(v,y)}$-bichromatic path for any $j \in \{t + 1, \ldots, \Delta + 1\}$. So $T \subseteq F^c_{x_l}(xx_l)$, which means that $d(x_l) \geq |T| + 1$ for $1 \leq l \leq \Delta + 1 - t$.

(1) Since $d(x) \leq \Delta$, $|T| = \Delta + 1 - s = \Delta + 1 - (d(x) - 1) = \Delta - d(x) + 2 \geq 2$. This means that $d(x_i) \geq 3$ for $1 \leq i \leq \Delta + 1 - t$. Therefore, $n_2(x) \leq d(x) - (\Delta + 1 - t) = d(x) - \Delta - 1 + d(y) - 1 = d(x) + d(y) - \Delta - 2$.

(2) If there are at least two 3-vertices in $\{x_1, \ldots, x_{\Delta+1-t}\}$, then assume that $d(x_1) = d(x_2) = 3$. Since $T \subseteq F^c_{x_l}(xx_l)(1 \leq l \leq \Delta + 1 - t)$, and $d(x_1) = d(x_2) = 3$, we have $T = \{1, 2\}$. So $F^c_{x_1}(xx_1) = F^c_{x_2}(xx_2) = \{1, 2\}$. Now we exchange the colors between xx_1 and xx_2, color vx with 1, color vy with $t + 1$, G is acyclically edge $(\Delta + 1)$-colorable, a contradiction. Therefore, $n_2(x) + n_3(x) \leq d(x) - (\Delta + 1 - t) + 1 = d(x) + d(y) - \Delta - 1$.

(3) If $d(x) < \Delta$, then $|T| = \Delta + 1 - s = \Delta + 1 - (d(x) - 1) = \Delta - d(x) + 2 \geq 3$. This implies that $d(x_l) \geq |T| + 1 \geq 4$ for $1 \leq l \leq \Delta + 1 - t$. Therefore, $n_2(x) + n_3(x) \leq d(x) - (\Delta + 1 - t) = d(x) - \Delta - 1 + d(y) - 1 = d(x) + d(y) - \Delta - 2$. □

Lemma 3.5. *For 2-vertex v, let $N(v) = \{x, y\}$. If $n_2(x) = d(x) - 2$, then $n_\Delta(x) = 2$, $n_2(y) = 1$, $n_3(y) \leq 1$, $d(y) = \Delta$.*

Proof. Let $N(x) = \{v, x_1, x_2, \ldots, x_s\}, N(y) = \{v, y_1, y_2, \ldots, y_t\}$, $d(x_i) = 2$, $N(x_i) = \{x, x_i^{'}\}$ for $3 \leq i \leq s$ (See Figure 2(3)). Observe that $n_2(x) \leq d(x) + d(y) - \Delta - 2$ by Lemma 3.4. If $n_2(x) = d(x) - 2$, then $d(y) = \Delta$, which means that $t = \Delta - 1$. Let $G^{'} = G - xv$, $G^{'}$ admits an acyclic $(\Delta + 1)$-edge coloring c. Suppose that $c(xx_i) = i$ for $1 \leq i \leq s$, $T = C \setminus F_y^c(vy)$. It is clear that $|T| = \Delta + 1 - (d(y) - 1) = 2$.

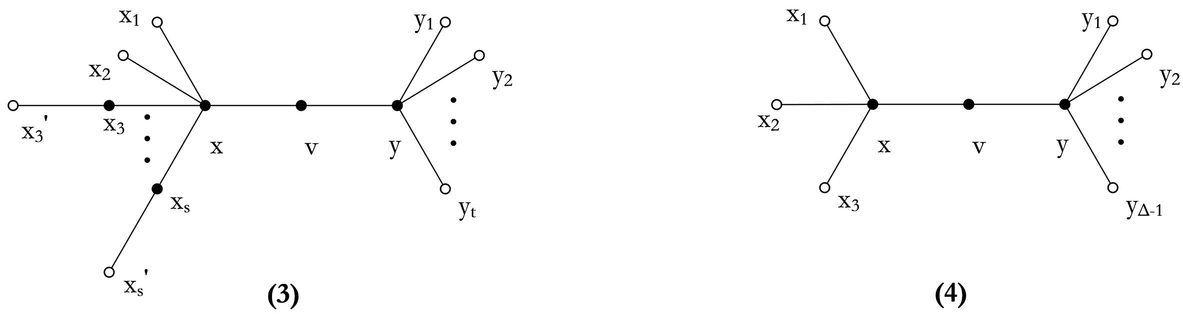

Figure 2. The configurations of Lemmas 3.5 and 3.6.

(1) $n_\Delta(x) = 2, n_2(y) = 1$.

We consider the following two cases.

Case 3.6. *$T \setminus F_x^c(vx) \neq \emptyset$.*

We can recolor vy with α for $\alpha \in T \setminus F_x^c(vx)$, color vx with β for $\beta \in \{s + 1, \ldots, \Delta + 1\}$($\beta \neq \alpha$). Now c can be extended to be an acyclic $(\Delta + 1)$-edge coloring of G, a contradiction.

Case 3.7. *$T \setminus F_x^c(vx) = \emptyset$.*

That is, $T \subseteq F_x^c(vx) = \{1, 2, \ldots, s\}$.

(i) $T \cap \{3, \ldots, s\} \neq \emptyset$.

Let $\alpha \in T \cap \{3, \ldots, s\}$. Since $|C \setminus (C(x) \cup C(x_\alpha))| \geq \Delta + 1 - (d(x) - 1) - 1 = \Delta + 1 - d(x) \geq 1$, we can color vy with α, color vx with β for $\beta \in C \setminus (C(x) \cup C(x_\alpha))$, c can be extended to be an acyclic $(\Delta + 1)$-edge coloring of G, a contradiction.

(ii) $T \cap \{3, \ldots, s\} = \emptyset$.

That is, T=$\{1, 2\}$ and $F_y^c(vy)$=$\{3, 4, \ldots, \Delta$+$1\}$. Note that $c(vy) \in \{1, 2\}$, w.l.o.g., suppose that $c(vy)$=1. If there is a color $\gamma \in \{s + 1, \ldots, \Delta + 1\}$ such that G contains no $(1, \gamma)_{(x,v)}$-bichromatic path via x_1 and y, then we can color vx with γ. In this way G is acyclically edge $(\Delta + 1)$-colorable, a contradiction. So there must exist a $(1, j)_{(x,v)}$-bichromatic path via x_1 and y for each j($j \in \{s + 1, \ldots, \Delta + 1\}$), which implies that $\{s + 1, \ldots, \Delta + 1\} \subseteq C(x_1) \cap C(y)$. Now we recolor vy with 2, and don't change the colors of the other edges, c is still an acyclic $(\Delta + 1)$-edge coloring of $G - xv$. As the same argument above, we have that $\{s + 1, \ldots, \Delta + 1\} \subseteq C(x_2)$. If there is $i \in \{3, 4, \ldots, s\}$, such that G contains no $(1, i)_{(x,v)}$-bichromatic path via x_1 and y, then we can color vx with i. Don't change the colors of the other edges, and remove the color of xx_i, c is still an acyclic $(\Delta + 1)$-edge coloring of $G - xx_i$, now we consider the color of xx_i. W.l.o.g., assume that $i = 3$.

If $c(x_3x_3^{'}) = j \geq s + 1$, then $|C(x) \cup C(x_3)| \leq s + 1 = d(x) \leq \Delta$, we can color xx_3 with α for $\alpha \notin (C(x) \cup C(x_3))$, G is acyclically edge $(\Delta + 1)$-colorable, a contradiction.

If $c(x_3x_3^{'}) = j \in \{1, 2\}$*, then it can be seen from the above argument that for each* $\alpha \in \{s+1, \ldots, \Delta+1\}$*, there are* $(i, \alpha)_{(x,v)}$*-bichromatic paths* $(i = 1, 2)$*. So* $G - xx_3$ *contains no* $(i, \alpha)_{(x,x_3)}$*-bichromatic path* $(i = 1, 2)$ *for each* $\alpha \in \{s + 1, \ldots, \Delta + 1\}$*, we can color* xx_3 *with* α*,* G *is acyclically edge* $(\Delta + 1)$*-colorable, a contradiction.*

If $c(x_3x_3^{'}) = j = 3$*, then we can color* xx_3 *with* α *for* $\alpha \notin (C(x) \cup C(v))$ *since* $|C(x) \cup C(v)| \leq s + 1 = d(x) < \Delta + 1$*,* G *is acyclically edge* $(\Delta + 1)$*-colorable, a contradiction.*

If $c(x_3x_3^{'}) = j \in \{4, \ldots, s\}$*, then we can recolor* xx_3 *with* α *for* $\alpha \notin (C(x) \cup C(x_j))$ *since* $|C(x) \cup C(x_j)| \leq s+1 = d(x) < \Delta+1$*,* G *is acyclically edge* $(\Delta+1)$*-colorable, a contradiction. Hence, there are* $(1, i)_{(x,v)}$*-bichromatic paths via* x_1 *and* y *for each* $i \in \{3, 4, \ldots, s\}$*, which implies that* $\{1, 3, 4, \ldots, s\} \subseteq C(x_1)$*.*

In conclusion, $C(x_1) = C(y) = \{1, 3, 4, \ldots, \Delta + 1\}$*. Now recolor* vy *with 2, and don't change the colors of the other edges,* c *is still an acyclic* $(\Delta + 1)$*-edge coloring of* $G - xv$*. As the same argument above, we have that* $C(x_2) = \{2, 3, 4, \ldots, \Delta + 1\}$*. It follows that* $F^c_{x_1}(xx_1) = F^c_{x_2}(xx_2) = F^c_y(vy) = \{3, 4, \ldots, \Delta + 1\}$ *and* $\{1, 2\} \subseteq F^c_{y_i}(yy_i)$ *for* $1 \leq i \leq t$*. Then* $d(x_1) = d(x_2) = \Delta + 1 - 2 + 1 = \Delta$*,* $n_2(y) = 1$*.*

(2) $n_3(y) \leq 1$.

By contradiction, assume that $d(y_1) = d(y_2) = 3$. Now exchange the colors between yy_1 and yy_2, and don't change the colors of the other edges, $G - xv$ has a new acyclic $(\Delta + 1)$-edge coloring ϕ. Let $\phi(yy_1) = \alpha$.

If $\alpha \in \{s + 1, \ldots, \Delta + 1\}$, then let $\phi(xv) = \alpha$. By symmetry, suppose that $\phi(vy) = 1$, G is acyclically edge $(\Delta + 1)$-colorable, a contradiction.

If $\alpha \in \{3, 4, \ldots, s\}$, assume that $\alpha = 3$, then let $\phi(xv) = 3$, remove the color of xx_3, $G - xx_3$ admits an acyclic $(\Delta + 1)$-edge coloring $\phi^{'}$. Let $\phi^{'}(x_3x_3^{'}) = \beta$. It follows from the above argument that there is a $(1, \Delta)_{(x,v)}$-bichromatic path through x_1, so $G - xx_3$ contains no $(1, \Delta)_{(x,x_3)}$-bichromatic path via x_1 and $x_3^{'}$. Similarly, $G - xx_3$ contains no $(2, \Delta)_{(x,x_3)}$-bichromatic path via x_2 and $x_3^{'}$. If $\beta \in \{1, 2\}$, then let $\phi^{'}(xx_3) = \Delta$, G is acyclically edge $(\Delta + 1)$-colorable, a contradiction. If $\beta \in \{4, \ldots, s\}$, then let $\phi^{'}(xx_3) = \gamma$ for $\gamma \notin (\phi^{'}(x) \cup \phi^{'}(x_\beta))$, G is acyclically edge $(\Delta + 1)$-colorable, a contradiction. If $\beta \geq s + 1$, then let $\phi^{'}(xx_3) = \gamma$ for $\gamma \notin (\phi^{'}(x) \cup \phi^{'}(x_3))$, G is acyclically edge $(\Delta + 1)$-colorable, a contradiction. This implies that $n_3(y) \leq 1$. □

Lemma 3.6. *For 2-vertex* v*, let* $N(v) = \{x, y\}$*. If* $d(x) = 4$ *and* $n_2(x) + n_3(x) = 2$*, then* $n_\Delta(x) = 2$*,* $n_2(y) = 1, n_3(y) \leq 1, d(y) = \Delta$*.*

Proof. Note that $n_2(x) + n_3(x) \leq d(x) + d(y) - \Delta - 2 = d(y) - \Delta + 2$ by Lemma 3.4. It is clear that $d(y) = \Delta$ since $n_2(x)+n_3(x) = 2$. Let $N(x) = \{v, x_1, x_2, x_3\}$, $N(y) = \{v, y_1, y_2, \ldots, y_{\Delta-1}\}$ (See Figure 2(4)). If $n_2(x) = d(x) - 2 = 2$, then $n_\Delta(x) = 2, n_2(y) = 1$ and $n_3(y) \leq 1$ by Lemma 3.5. We prove the case $n_2(x) = 1$ and $n_3(x) = 1$. Suppose that $d(x_3) = 3$. Let $G^{'} = G - xv$, by the minimality of G, $G^{'}$ admits an acyclic $(\Delta + 1)$-edge coloring c. Assume that $c(xx_i) = i$ for $1 \leq i \leq 3$, $T = C \setminus F^c_y(vy)$. Clearly, $|T| = \Delta + 1 - (\Delta - 1) = 2$.

(1) $n_\Delta(x) = 2, n_2(y) = 1$.

We consider the following two cases.

Case 3.8. $T \setminus F^c_x(vx) \neq \emptyset$.

We can recolor vy *with* α *for* $\alpha \in T \setminus F^c_x(vx)$*, color* vx *with* β *for* $\beta \in \{4, \ldots, \Delta + 1\}(\beta \neq \alpha)$*,* c *is an acyclic* $(\Delta + 1)$*-edge coloring of* G*, a contradiction.*

Case 3.9. $T \setminus F_x^c(vx) = \emptyset$.

That is, $T \subseteq F_x^c(vx) = \{1,2,3\}$. Since $|T| = \Delta + 1 - (\Delta - 1) = 2$, $\{1,2,3\} \setminus T \neq \emptyset$. As the same argument of Case 2 in Lemma 3.5, we have $n_\Delta(x) = 2$, $n_2(y) = 1$ by setting $s = 3$.

(2) $n_3(y) \leq 1$.

By contradiction, suppose that $d(y_1) = d(y_2) = 3$. It follows from the above argument that $F_{y_1}^c(yy_1) = F_{y_2}^c(yy_2) = \{1,2\}$. By the symmetry, assume that $c(vy) = 1$, exchange the colors between yy_1 and yy_2, $G - xv$ has a new acyclic $(\Delta + 1)$-edge coloring ϕ. Let $\phi(yy_1) = \alpha$.

If $\alpha \in \{4, \ldots, \Delta + 1\}$, then let $\phi(vx) = \alpha$, G is acyclically edge $(\Delta + 1)$-colorable, a contradiction.

If $\alpha = 3$, then let $\phi(xv) = 3$, and remove the color of xx_3, $G - xx_3$ has an acyclic $(\Delta + 1)$-edge coloring ϕ'. It follows from the above argument that there are $(i,\gamma)_{(x,v)}$-bichromatic paths under c of $G - xv$ for $i = 1,2$ and $\gamma \in \{4,5,\ldots,\Delta+1\}$. So $G - xx_3$ contains no $(i,\gamma)_{(x,x_3)}$-bichromatic path. If $|\phi'(x) \cap \phi'(x_3)| = 0$, then $|\phi'(x) \cup \phi'(x_3)| \leq 3 + 2 = 5 < \Delta + 1$. Let $\phi'(xx_3) = \beta$ for $\beta \in C \setminus (\phi'(x) \cup \phi'(x_3))$, ϕ' can be extended to G, a contradiction. If $|\phi'(x) \cap \phi'(x_3)| = 1$, then suppose $1 \in \phi'(x_3)$. Let $\phi'(xx_3) = \beta$ for $\beta \in \{4, \ldots, \Delta + 1\} \setminus \phi'(x_3)$, ϕ' can be extended to G, a contradiction. If $|\phi'(x) \cap \phi'(x_3)| = 2$, then $\phi'(x) \cap \phi'(x_3) = \{1,2\}$, we can let $\phi'(xx_3) = \beta$ for $\beta \in \{4, \ldots, \Delta + 1\}$, ϕ' can be extended to G, a contradiction. This implies that $n_3(y) \leq 1$. □

Next we discuss the number of 3^--vertex in the neighbors of Δ-vertex.

Lemma 3.7. *For $d(v) = \Delta$, if $n_2(v) \geq \Delta - 3$, then $n_2(v) + n_3(v) \leq \Delta - 2$.*

Proof. Assume that $v_4 \in N(v)$, $N(v_4) = \{v, v_4'\}$. By Lemma 3.4, $n_2(v) \leq d(v) + d(v_4') - \Delta - 2 = d(v_4') - 2 \leq \Delta - 2$, $n_2(v) + n_3(v) \leq d(v) + d(v_4') - \Delta - 1 = d(v_4') - 1 \leq \Delta - 1$; if $n_2(v) = \Delta - 2$, then $d(v_4') = \Delta$; if $n_2(v) + n_3(v) = \Delta - 1$, then $d(v_4') = \Delta$. Note that $n_\Delta(v) = 2$ when $n_2(v) = \Delta - 2$ by Lemma 3.5. Namely $n_3(v) = 0$, $n_2(v) + n_3(v) = \Delta - 2 + 0 = \Delta - 2$.

Otherwise $n_2(v) = \Delta - 3$. By contradiction, suppose that $n_2(v) + n_3(v) \geq \Delta - 1$. We have that $n_2(v) + n_3(v) = \Delta - 1$ since $n_2(v) + n_3(v) \leq \Delta - 1$. Let $N(v) = \{v_1, v_2, \ldots, v_\Delta\}$, $N(v_i) = \{v, x_i\}$ for $4 \leq i \leq \Delta$. Assume that $d(v_2) = d(v_3) = 3$ (See Figure 3(5)). Let $G' = G - vv_\Delta$, by the minimality of G, G' admits an acyclic $(\Delta + 1)$-edge coloring c. Let $c(vv_i) = i$ for $1 \leq i \leq \Delta - 1, T = C \setminus F_{x_\Delta}^c(v_\Delta x_\Delta)$. Since $n_2(v) + n_3(v) = \Delta - 1$, it follows from the above argument that $d(x_\Delta) = d(x_{\Delta-1}) = \ldots = d(x_4) = \Delta$, $|T| = \Delta + 1 - (\Delta - 1) = 2$. □

We consider the following two cases.

Case 3.10. $T \setminus C(v) \neq \emptyset$.

We can recolor $v_\Delta x_\Delta$ with α for $\alpha \in T \setminus C(v)$, color vv_Δ with β for $\beta \in \{\Delta, \Delta + 1\}(\beta \neq \alpha)$, c is an acyclic $(\Delta + 1)$-edge coloring of G, a contradiction.

Case 3.11. $T \setminus C(v) = \emptyset$. *That is, $T \subseteq C(v) = \{1, 2, \ldots, \Delta - 1\}$.*

(i) $T \cap \{4, 5, \ldots, \Delta - 1\} \neq \emptyset$.

We can color $v_\Delta x_\Delta$ with i for $i \in T \cap \{4,5,\ldots,\Delta - 1\}$, color vv_Δ with j for $j \in \{\Delta, \Delta + 1\} \setminus \{c(v_i x_i)\}$, G is acyclically edge $(\Delta + 1)$-colorable, a contradiction.

(ii) $T \cap \{4, 5, \ldots, \Delta - 1\} = \emptyset$.

That is, $T \subseteq \{1,2,3\}$. Note that $c(v_\Delta x_\Delta) \in T \subseteq \{1,2,3\}$. When $c(v_\Delta x_\Delta) = 1$, if there exists a color $\gamma \in \{\Delta, \Delta + 1\}$, such that G contains no $(1,\gamma)_{(v,v_\Delta)}$-bichromatic path via x_Δ and v_1, then let $c(vv_\Delta) = \gamma$, G is acyclically edge $(\Delta + 1)$-colorable, a contradiction. Hence, G contains $(1,i)_{(v,v_\Delta)}$-bichromatic paths

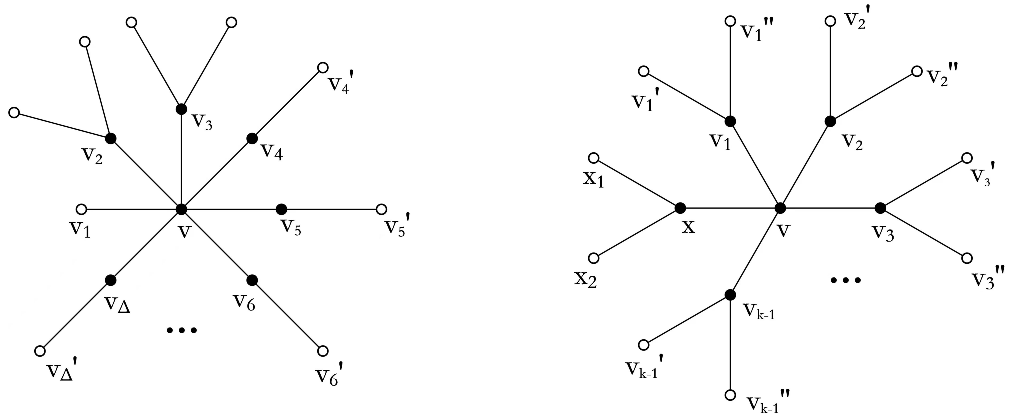

Figure 3. The configurations of Lemmas 3.7 and 3.8.

via x_Δ *and* v_1 *for each* $i \in \{\Delta, \Delta+1\}$. *This implies that* $\{\Delta, \Delta+1\} \subseteq C(v_1) \cap C(x_\Delta)$. *If* $2 \in T$, *then we can recolor* $v_\Delta x_\Delta$ *with 2, c is still an acyclic* $(\Delta+1)$*-edge coloring of* $G - vv_\Delta$, *the same argument shows that* $C(v_2) = \{2, \Delta, \Delta+1\}$. *If* $3 \in T$, *then we can recolor* $v_\Delta x_\Delta$ *with 3, c is still an acyclic* $(\Delta+1)$*-edge coloring of* $G - vv_\Delta$, *the same argument shows that* $C(v_3) = \{3, \Delta, \Delta+1\}$.

(a) $1 \notin T$. *Namely,* $T = \{2, 3\}$.

It follows from the above argument that $F^c_{v_2}(vv_2) = F^c_{v_3}(vv_3) = \{\Delta, \Delta+1\}$. *Now exchange the colors between* vv_2 *and* vv_3, *we can color* $v_\Delta x_\Delta$ *with* α *for* $\alpha \in T$, *color* vv_Δ *with* Δ, *G is acyclically edge* $(\Delta+1)$*-colorable, a contradiction.*

(b) $1 \in T$. *Suppose that* $T = \{1, 2\}$.

It follows from the above argument that $\{\Delta, \Delta+1\} \subseteq C(v_1)$ *and* $F^c_{v_2}(vv_2) = \{\Delta, \Delta+1\}$. *Now let* $c(v_\Delta x_\Delta) = 2, c(vv_\Delta) = 7$, *remove the color of* vv_7, *c is still an acyclic* $(\Delta+1)$*-edge coloring of* $G - vv_7$. *Let* $T' = C \setminus F^c_{x_7}(v_7x_7)$, *it is clear that* $|T'| = \Delta + 1 - (\Delta - 1) = 2$. *It follows from the above argument that* $T' \subseteq \{1, 2, 3\}$, *and* $1 \in T'$. *Assume that* $c(v_7x_7) = 1$. *Since there is a* $(1, \Delta)_{(v,v_\Delta)}$*-bichromatic path through* v_1, $G - vv_7$ *contains no* $(1, \Delta)_{(v,v_7)}$*-bichromatic path via* v_1 *and* x_7, *we can color* vv_7 *with* Δ, *G is acyclically edge* $(\Delta+1)$*-colorable, a contradiction.*

The following Lemma shows if a k-vertex($k \in \{4, 5, 6\}$) has no 2-neighbor, then $n_3(v) < k$.

Lemma 3.8. *Let v be a k-vertex, with* $k \in \{4, 5, 6\}$. *If* $n_2(v) = 0$, *then* $n_3(v) < k$.

Proof. By contradiction, suppose that $n_3(v) = k$. Let $N(v) = \{x, v_1, v_2, v_3, v_4, v_{k-1}\}$, $N(x) = \{v, x_1, x_2\}$ and $N(v_i) = \{v, v_i', v_i''\}$ $(1 \le i \le k-1)$ (See Figure 3(8)). Let $G' = G - xv$, by the minimality of G, G' admits an acyclic $(\Delta+1)$-edge coloring c. Let $c(vv_i) = i$ for $1 \le i \le k-1$. We consider the following three cases. □

Case 3.12. $|C(x) \cap C(v)| = 0$.

Note that $|C \setminus (C(x) \cup C(v))| = \Delta + 1 - 2 - (k-1) = \Delta - k > 0$, *we can color xv with* α *for* $\alpha \in C \setminus (C(x) \cup C(v))$, *c is an acyclic* $(\Delta+1)$*-edge coloring of G, a contradiction.*

Case 3.13. $|C(x) \cap C(v)| = 1$.

W.l.o.g., assume that $c(xx_1) = c(vv_1) = 1,\ c(xx_2) = k$. *If there exists a color* $\gamma \in \{k+1, \ldots, \Delta+1\}$ *such that G contains no* $(1, \gamma)_{(x,v)}$*-bichromatic path, then we can color xv with* γ. *In this way G is acyclically edge* $(\Delta + 1)$*-colorable, a contradiction. So there must exist* $(1, \alpha)_{(x,v)}$*-bichromatic paths for each* $\alpha \in \{k+1, \ldots, \Delta+1\}$*, which implies that* $\{k+1, \ldots, \Delta+1\} \subseteq C(x_1) \cap C(v_1)$. *Thus,* $d(v_1) \geq \Delta + 1 - k + 1 \geq \Delta - 4 \geq 4$*, a contradiction.*

Case 3.14. $|C(x) \cap C(v)| = 2$.

W.l.o.g., assume that $c(xx_1) = c(vv_1) = 1,\ c(xx_2) = c(vv_2) = 2$.

(i) $\Delta \geq 9$.

Since $d(v_1) = d(v_2) = 3$ *and* $\Delta \geq 9$*, we have that* $\{k, \ldots, \Delta+1\} \setminus (C(v_1) \cup C(v_2)) \neq \emptyset$. *Let* $\beta \in \{k, \ldots, \Delta+1\} \setminus (C(v_1) \cup C(v_2))$. *Note that G contains no* $(i, \beta)_{(x,v)}$*-bichromatic path for i* $= 1, 2$*, we can color xv with* β*, c is an acyclic* $(\Delta + 1)$*-edge coloring of G, a contradiction.*

(ii) $\Delta = 8$.

(a) $k \in \{4, 5\}$.

Since $d(v_1) = d(v_2) = 3$ *and* $\Delta \geq 8$*, we have that* $\{k, \ldots, \Delta+1\} \setminus (C(v_1) \cup C(v_2)) \neq \emptyset$. *Let* $\beta \in \{k, \ldots, \Delta+1\} \setminus (C(v_1) \cup C(v_2))$. *Note that G contains no* $(i, \beta)_{(x,v)}$*-bichromatic path for i* $= 1, 2$*, we can color xv with* β. *In this way c is an acyclic* $(\Delta + 1)$*-edge coloring of G, a contradiction.*

(b) $k = 6$.

If there exists a color $\gamma \in \{6, 7, 8, 9\}$ *such that G contains no* $(i, \gamma)_{(x,v)}$*-bichromatic path for i* $= 1, 2$*, then we can color xv with* γ. *In this way G is acyclically edge* $(\Delta + 1)$*-colorable, a contradiction. So* $\{6, 7, 8, 9\} \subseteq (C(v_1) \cup C(v_2))$. *Since* $d(v_1) = d(v_2) = 3$*, we have that* $C(v_1) \cup C(v_2) = \{1, 2, 6, 7, 8, 9\}$ *and* $c(v_1v_1^{'}) \notin C(v_2), c(v_1v_1^{''}) \notin C(v_2)$. *We can recolor* vv_2 *with* α *for* $\alpha \in \{c(v_1v_1^{'}), c(v_1v_1^{''})\}$*, don't change the colors of the other edges,* $G^{'}$ *has a new acyclic* $(\Delta+1)$*-edge coloring* $c^{'}$*, and* $|C^{'}(x) \cap C^{'}(v)| = 1$. *By the similar argument in case 2, we can get an acyclic* $(\Delta + 1)$*-edge coloring of G , a contradiction.*

3.2. Discharging

Note that G is a minimal counterexample to Theorem 1.1, and G is a connected planar graph. By Euler's formula $|V| + |F| - |E| = 2$ and the relation $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E(G)|$, we can derive the identity

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -8.$$

We define the initial charge function by $\omega(v) = d(v) - 4$ for $v \in V(G)$ and $\omega(f) = d(f) - 4$ for $f \in F(G)$. It follows from the identity that $\sum_{x \in V(G) \cup F(G)} \omega(x) = -8$. According to the structures of G, we design some discharging rules and redistribute charge such that the total amount of charge has not changed. Once the discharging is finished, a new charge function $\omega^{'}(x)$ is produced. Next, we prove $\omega^{'}(x) \geq 0$ for all $x \in V(G) \cup F(G)$. Therefore, we can get the following contradiction

$$0 \leq \sum_{x \in V(G) \cup F(G)} \omega^{'}(x) = \sum_{x \in V(G) \cup F(G)} \omega(x) = -8.$$

Hence, we demonstrate that the counterexample can not exist and Theorem 1.1 is proved.

Discharging rules:

(R1) Every 5^{+}-face f sends $\frac{d(f)-4}{d(f)}$ to each incident vertex.

(R2) Every 4-vertex v sends $\frac{1}{5}$ to each adjacent 3-vertex, and then distributes the remaining extra charge evenly among all adjacent 2-vertices.

(R3) Every 5-vertex v sends $\frac{2}{5}$ to each adjacent 3-vertex, and then distributes the remaining extra charge evenly among all adjacent 2-vertices.

(R4) Every 6^+-vertex v sends $\frac{3}{5}$ to each adjacent 3-vertex, and then distributes the remaining extra charge evenly among all adjacent 2-vertices.

In the following, we will prove that $\omega'(v) \geq 0$ for each $v \in V(G)$. Observe that $\delta(G) \geq 2$ by Lemma 3.1.

(1) $d(v) = 3$, $\omega(v) = -1$.

Let $N(v) = \{v_1, v_2, v_3\}$. We have that $d(v_1) + d(v_2) + d(v_3) \geq \Delta + 4 \geq 12$ by Lemma 3.3. Since the neighbors of 2-vertex are both 4^+-vertices, then $n_2(v) = 0$, that is $d(v_i) \geq 3$ for $i = 1, 2, 3$, and $n_3(v) \leq 2$.

If $n_3(v) = 2$, suppose that $d(v_1) = d(v_2) = 3$, then $d(v_3) \geq 6$. This implies that $\omega^{'}(v) \geq -1+2\times\frac{1}{5}+\frac{3}{5}=0$ by $R1, R4$. If $n_3(v) = 1$, then either $n_4(v) = 1$, $n_{5^+}(v) = 1$ or $n_4(v) = 0$, $n_{5^+}(v) = 2$. By $R1, R2, R3$, $R4$, we have that $\omega^{'}(v) \geq -1 + 2 \times \frac{1}{5} + \min\{\frac{1}{5} + \frac{2}{5}, 2 \times \frac{2}{5}, \frac{2}{5} + \frac{3}{5}, 2 \times \frac{3}{5}\} = 0$. If $n_3(v) = 0$, then $n_{4^+}(v) = 3$. By $R1$, $R4$, $\omega^{'}(v) \geq -1 + 2 \times \frac{1}{5} + 3 \times \frac{1}{5} = 0$.

(2) $d(v) = 4$, $\omega(v) = 0$.

If $n_2(v) \neq 0$, then $n_2(v) + n_3(v) \leq 2$ by Lemma 3.4.
This means that $\omega^{'}(v) \geq 0+3 \times \frac{1}{5} - \frac{1}{5}n_3(v) - \frac{0+3\times\frac{1}{5}-\frac{1}{5}n_3(v)}{n_2(v)} \times n_2(v) = 0$ by $R1, R2$.

If $n_2(v) = 0$, then $n_3(v) \leq 3$ by Lemma 3.8, which implies that $\omega^{'}(v) \geq 0 + 3 \times \frac{1}{5} - 3 \times \frac{1}{5} = 0$ by $R1, R4$.

(3) $d(v) = 5$, $\omega(v) = 1$.

If $n_2(v) \neq 0$, then $\omega^{'}(v) \geq 1 + 4 \times \frac{1}{5} - \frac{2}{5}n_3(v) - \frac{1+4\times\frac{1}{5}-\frac{2}{5}n_3(v)}{n_2(v)} \times n_2(v) = 0$ by $R1, R3$.

If $n_2(v) = 0$, then $n_3(v) \leq 4$ by Lemma 3.8. This implies that $\omega^{'}(v) \geq 1 + 4 \times \frac{1}{5} - 4 \times \frac{2}{5} = \frac{1}{5} > 0$ by $R1, R3$.

(4) $d(v) = 6$, $\omega(v) = 2$.

If $n_2(v) \neq 0$, then $\omega^{'}(v) \geq 2 + 5 \times \frac{1}{5} - \frac{3}{5}n_3(v) - \frac{2+5\times\frac{1}{5}-\frac{3}{5}n_3(v)}{n_2(v)} \times n_2(v) = 0$ by $R1, R4$.

If $n_2(v) = 0$, then $n_3(v) \leq 5$ by Lemma 3.8. This implies that $\omega^{'}(v) \geq 2+5\times\frac{1}{5}-5\times\frac{3}{5} = 0$ by $R1, R4$.

(5) $d(v) \geq 7$, $\omega(v) = d(v) - 4$.

If $n_2(v) \neq 0$, then $\omega^{'}(v) \geq d(v) - 4 + \frac{1}{5} \times (d(v) - 1) - \frac{3}{5}n_3(v) - \frac{d(v)-4+\frac{1}{5}\times(d(v)-1)-\frac{3}{5}n_3(v)}{n_2(v)} \times n_2(v) = 0$ by $R1, R4$.

If $n_2(v) = 0$, then $\omega^{'}(v) \geq d(v) - 4 + \frac{1}{5} \times (d(v) - 1) - \frac{3}{5}n_3(v) \geq \frac{6}{5}d(v) - \frac{21}{5} - \frac{3}{5}d(v) = \frac{3}{5}d(v) - \frac{21}{5} \geq 0$ by $R1, R4$.

(6) $d(v) = 2$, $\omega(v) = -2$.

For convenience, let $\tau(u \to v)$ denote the charge transferred out of u into v according to the above rules, $u, v \in V(G)$. Let $N(v) = \{x, y\}$, then $n_2(x) \leq d(x) - 2$, $n_2(y) \leq d(y) - 2$ by Lemma 3.4.

(6.1) If one of vertices in $\{x, y\}$, such as x, has $n_2(x)$=$d(x) - 2$, then $n_\Delta(x)$=2, $n_2(y)$=1, $n_3(y) \leq 1$ and $d(y)$=Δ by Lemma 3.5. By $R1$ and $R4$, we have that $\tau(y \to v) \geq \frac{\omega(y)+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}n_3(y)}{n_2(y)} \geq \frac{d(y)-4+\frac{1}{5}d(y)-\frac{1}{5}-1\times\frac{3}{5}}{1} = \frac{6}{5}d(y) - \frac{24}{5} = \frac{6}{5} \times (\Delta - 4) \geq \frac{24}{5}$. This implies that $\omega^{'}(v) \geq -2 + \frac{1}{5} + \frac{24}{5} = 3 > 0$ by $R1$.

(6.2) $n_2(x) \leq d(x) - 3$, $n_2(y) \leq d(y) - 3$. Suppose that $d(x) \leq d(y)$.

(6.2.1) $d(x) \geq 9$.

Observe that $n_2(x) + n_3(x) \leq d(x) + d(y) - \Delta - 1$ by Lemma 3.4. Note that $\tau(x \to v) \geq$

$\frac{\omega(x)+\frac{1}{5}\times(d(x)-1)-\frac{3}{5}n_3(x)}{n_2(x)} \geq \frac{d(x)-4+\frac{1}{5}d(x)-\frac{1}{5}-\frac{3}{5}\times(d(x)-1-n_2(x))}{n_2(x)} = \frac{\frac{3}{5}d(x)-\frac{18}{5}+\frac{3}{5}n_2(x)}{n_2(x)} = \frac{3}{5}\times(1+\frac{d(x)-6}{n_2(x)}) \geq \frac{3}{5}\times(1+\frac{d(x)-6}{d(x)-3})=\frac{3}{5}\times(2-\frac{3}{d(x)-3})$ by $R1, R4$.

Similarly, $\tau(y \to v) \geq \frac{3}{5}\times(2-\frac{3}{d(x)-3})$. Since $d(x) \geq 9$, $d(x) \leq d(y)$, we have $\tau(x \to v) \geq \frac{9}{10}$, $\tau(y \to v) \geq \frac{9}{10}$. This implies that $\omega^{'}(v) \geq -2+\frac{1}{5}+2\times\frac{9}{10} = 0$ by $R1$.

(6.2.2) $d(x) = 8$, then $n_2(x) \leq d(x)-3 = 5$.

(a) $\Delta \geq 9$.

Observe that $n_2(x)+n_3(x) \leq d(x)+d(y)-\Delta-2 \leq d(x)-2$ by Lemma 3.4. Note that $\tau(x \to v) \geq \frac{\omega(x)+\frac{1}{5}\times(d(x)-1)-\frac{3}{5}n_3(x)}{n_2(x)} \geq \frac{d(x)-4+\frac{1}{5}\times(d(x)-1)-\frac{3}{5}\times(d(x)-2-n_2(x))}{n_2(x)} = \frac{3}{5}\times(1+\frac{d(x)-5}{n_2(x)}) \geq \frac{3}{5}\times(1+\frac{d(x)-5}{d(x)-3}) = \frac{3}{5}\times(2-\frac{2}{d(x)-3}) = \frac{24}{25}$ by $R1, R4$.

If $d(y) = d(x) < \Delta$, then we have $\tau(y \to v) \geq \frac{3}{5}\times(1+\frac{3}{n_2(y)}) \geq \frac{3}{5}\times(1+\frac{3}{5}) = \frac{24}{25}$ by the similar argument above. This implies that $\omega^{'}(v) \geq -2+\frac{1}{5}+2\times\frac{24}{25} = \frac{3}{25} > 0$ by $R4$.

If $d(y) \geq 9$, then we have $n_2(y)+n_3(y) \leq d(x)+d(y)-\Delta-1 \leq d(y)-2$, $n_2(y) \leq d(y)-3$ by Lemma 3.4. This means that $\tau(y \to v) \geq \frac{3}{5}\times(1+\frac{d(y)-5}{n_2(y)}) \geq \frac{3}{5}\times(2-\frac{2}{d(y)-3})$ by $R1$ and $R4$. $\tau(y \to v) \geq 1$ since $d(y) \geq 9$. So $\omega^{'}(v) \geq -2+\frac{1}{5}+\frac{24}{25}+1 = \frac{2}{25} > 0$ by $R4$.

(b) $\Delta = 8$.

Now we have $d(x) = d(y) = \Delta = 8$. Observe that $n_2(x)+n_3(x) \leq d(x)+d(y)-\Delta-1 \leq d(x)-1=7$ by Lemma 3.4. If $n_2(x) \leq 4$, then $\tau(x \to v) \geq \frac{3}{5}\times(1+\frac{2d(x)-14}{n_2(x)}) \geq \frac{3}{5}\times(1+\frac{2}{4}) = \frac{9}{10}$ by $R1$, $R4$. Similarly, if $n_2(y) \leq 4$, then $\tau(y \to v) \geq \frac{9}{10}$. If $n_2(x) \geq 5$, then we have $n_2(x) = 5$ due to $n_2(x) \leq d(x)-3 = 5$. Observe that $n_2(x)+n_3(x) \leq \Delta-2 = 6$ by Lemma 3.7. This implies that $\tau(x \to v) \geq \frac{3}{5}\times(1+\frac{2d(x)-13}{n_2(x)}) = \frac{3}{5}\times(1+\frac{3}{5}) = \frac{24}{25}$ by $R1$, $R4$. Similarly, if $n_2(y) \leq 5$, then $\tau(y \to v) \geq \frac{24}{25}$. Hence, $\omega^{'}(v) \geq -2+\frac{1}{5}+\min\{2\times\frac{9}{10}, 2\times\frac{24}{25}, \frac{9}{10}+\frac{24}{25}\} = 0$ by $R1$.

(6.2.3) $d(x) = 7$, then $n_2(x) \leq d(x)-3 = 4$.

Observe that $n_2(x)+n_3(x) \leq d(x)+d(y)-\Delta-2 \leq 5$ by Lemma 3.4. Note that $\tau(x \to v) \geq \frac{3}{5}\times(1+\frac{2d(x)-12}{n_2(x)}) \geq \frac{9}{10}$ by $R1, R4$.

If $d(y) = 7$, then we have that $n_2(y)+n_3(y) \leq d(x)+d(y)-\Delta-2 = 12-\Delta \leq 4$, $n_2(y) \leq d(y)-3 = 4$ by Lemma 3.4. Note that $\tau(y \to v) \geq \frac{\omega(y)+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}n_3(y)}{n_2(y)} \geq \frac{d(y)-4+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}\times(4-n_2(y))}{n_2(y)} = \frac{3}{5}\times(1+\frac{2d(y)-11}{n_2(y)}) \geq \frac{3}{5}\times(1+\frac{3}{4}) = \frac{21}{20}$ by $R1, R4$. We have that $\omega^{'}(v) \geq -2+\frac{1}{5}+\frac{9}{10}+\frac{21}{20} = \frac{3}{20} > 0$ by $R1$.

If $d(y) \geq 8$, then we have that $n_2(y)+n_3(y) \leq d(x)+d(y)-\Delta-1 \leq d(y)-2$, $n_2(y) \leq d(y)-3$ by Lemma 3.4.

Note that $\tau(y \to v) \geq \frac{\omega(y)+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}n_3(y)}{n_2(y)} \geq \frac{d(y)-4+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}\times(d(y)-2-n_2(y))}{n_2(y)} \geq \frac{3}{5}\times(1+\frac{d(y)-5}{n_2(y)}) \geq \frac{3}{5}\times(2-\frac{2}{d(y)-3})$ by $R1$, $R4$. Since $d(y) \geq 8$, $\tau(y \to v) \geq \frac{24}{25}$, we have that $\omega^{'}(v) \geq -2+\frac{1}{5}+\frac{9}{10}+\frac{24}{25} = \frac{3}{50} > 0$ by $R1$.

(6.2.4) $d(x) = 6$, then $n_2(x) \leq d(x)-3 = 3$.

By Lemma 3.4, we have that $n_2(x)+n_3(x) \leq d(x)+d(y)-\Delta-2 \leq 4$. Note that $\tau(x \to v) \geq \frac{3}{5}\times(1+\frac{2d(x)-11}{n_2(x)}) \geq \frac{4}{5}$ by $R1, R4$.

If $d(y) = 6$, then we have that $n_2(y)+n_3(y) \leq d(x)+d(y)-\Delta-2 = 10-\Delta \leq 2$ by Lemma 3.4. So $n_2(y) \leq 2$. Note that $\tau(y \to v) \geq \frac{\omega(y)+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}n_3(y)}{n_2(y)} \geq \frac{d(y)-4+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}\times(2-n_2(y))}{n_2(y)} = \frac{3}{5}\times(1+\frac{2d(y)-9}{n_2(y)}) \geq \frac{3}{5}\times(1+\frac{3}{2}) = \frac{3}{2}$ by $R1, R4$. This means that $\omega^{'}(v) \geq -2+\frac{1}{5}+\frac{4}{5}+\frac{3}{2} = \frac{1}{2} > 0$ by $R1$.

If $d(y) = 7$, then we have that $n_2(y)+n_3(y) \leq d(x)+d(y)-\Delta-2 = 11-\Delta \leq 3$ by Lemma 3.4. So $n_2(y) \leq 3$. Note that $\tau(y \to v) \geq \frac{\omega(y)+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}n_3(y)}{n_2(y)} \geq \frac{d(y)-4+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}\times(3-n_2(y))}{n_2(y)} = \frac{3}{5}\times(1+\frac{2d(y)-10}{n_2(y)}) \geq \frac{3}{5}\times(1+\frac{4}{3}) = \frac{7}{5}$ by $R1, R4$. This means that $\omega^{'}(v) \geq -2+\frac{1}{5}+\frac{4}{5}+\frac{7}{5} = \frac{2}{5} > 0$ by $R1$.

If $d(y) \geq 8$, then we have that $n_2(y) + n_3(y) \leq d(x) + d(y) - \Delta - 1 \leq d(y) - 3,\ n_2(y) \leq d(y) - 3$ by Lemma 3.4.

Note that $\tau(y \to v) \geq \frac{\omega(y)+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}n_3(y)}{n_2(y)} \geq \frac{d(y)-4+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}\times(d(y)-3-n_2(y))}{n_2(y)} = \frac{3}{5}\times(1+\frac{d(y)-4}{n_2(y)}) \geq \frac{3}{5}\times(2-\frac{1}{d(y)-3})$ by $R1, R4$. Since $d(y) \geq 8$, $\tau(y \to v) \geq \frac{27}{25}$, we have that $\omega'(v) \geq -2 + \frac{1}{5} + \frac{4}{5} + \frac{27}{25} = \frac{2}{25} > 0$ by $R1$.

(6.2.5) $d(x) = 5$, then $n_2(x) \leq d(x) - 3 = 2$.

By Lemma 3.4, we have that $n_2(x) + n_3(x) \leq d(x) + d(y) - \Delta - 2 \leq 3$. Note that $\tau(x \to v) \geq \frac{\omega(x)+\frac{1}{5}\times(d(x)-1)-\frac{2}{5}n_3(x)}{n_2(x)} \geq \frac{1+4\times\frac{1}{5}-\frac{2}{5}\times(3-n_2(x))}{n_2(x)} = \frac{2}{5} + \frac{3}{5}\times\frac{1}{n_2(x)} \geq \frac{2}{5} + \frac{3}{5}\times\frac{1}{2} = \frac{7}{10}$ by $R1, R3$. Observe that $d(x) + d(y) \geq \Delta + 3$ by Lemma 3.3, so $d(y) \geq \Delta + 3 - d(x) \geq 6$.

If $d(y) = 6$, then we have that $n_2(y) + n_3(y) \leq d(x) + d(y) - \Delta - 2 = 9 - \Delta \leq 1$ by Lemma 3.4, which means that $n_2(y) = 1,\ n_3(y) = 0$. Note that $\tau(y \to v) \geq \frac{\omega(y)+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}n_3(y)}{n_2(y)} = \frac{2+5\times\frac{1}{5}-0\times\frac{3}{5}}{1} = 3$ by $R1, R4$. So $\omega'(v) \geq -2 + \frac{1}{5} + \frac{7}{10} + 3 = \frac{19}{10} > 0$ by $R1$.

If $d(y) = 7$, then we have that $n_2(y) + n_3(y) \leq d(x) + d(y) - \Delta - 2 = 10 - \Delta \leq 2$ by Lemma 3.4. So $n_2(y) \leq 2$. Note that $\tau(y \to v) \geq \frac{\omega(y)+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}n_3(y)}{n_2(y)} \geq \frac{d(y)-4+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}\times(2-n_2(y))}{n_2(y)} = \frac{3}{5}\times(1+\frac{5}{n_2(y)}) \geq \frac{3}{5}\times(1+\frac{5}{2}) = \frac{21}{10}$ by $R1, R4$. Hence, $\omega'(v) \geq -2 + \frac{1}{5} + \frac{7}{10} + \frac{21}{10} = 1 > 0$ by $R1$.

If $d(y) \geq 8$, then we have that $n_2(y)+n_3(y) \leq d(x)+d(y)-\Delta-1 \leq d(y)-4$ by Lemma 3.4. So $n_2(y) \leq d(y) - 4$. Note that $\tau(y \to v) \geq \frac{\omega(y)+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}n_3(y)}{n_2(y)} \geq \frac{d(y)-4+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}\times(d(y)-4-n_2(y))}{n_2(y)} = \frac{3}{5}\times(1+\frac{d(y)-3}{n_2(y)}) \geq \frac{3}{5}\times(2+\frac{1}{d(y)-4})$ by $R1, R4$. Observe that $\tau(y \to v) > \frac{6}{5}$ since $d(y) \geq 8$. Then $\omega'(v) > -2+\frac{1}{5}+\frac{7}{10}+\frac{6}{5} = \frac{1}{10} > 0$ by $R1$.

(6.2.6) $d(x) = 4$

By Lemma 3.4, we have that $n_2(x) + n_3(x) \leq d(x) + d(y) - \Delta - 2 \leq 2$.

(a) $n_2(x) + n_3(x) = 2$.

By Lemma 3.6, we have that $n_\Delta(x) = 2,\ n_2(y) = 1,\ n_3(y) \leq 1$, and $d(y) = \Delta$. Note that $\tau(y \to v) \geq \frac{\omega(y)+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}n_3(y)}{n_2(y)} \geq \frac{d(y)-4+\frac{1}{5}d(y)-\frac{1}{5}-1\times\frac{3}{5}}{1} = \frac{6}{5}d(y) - \frac{24}{5} = \frac{6}{5}(\Delta - 4) \geq \frac{24}{5}$ by $R1, R4$. So $\omega'(v) \geq -2 + \frac{1}{5} + \frac{24}{5}=3 > 0$ by $R1$.

(b) $n_2(x) + n_3(x) \leq 1$. That is $n_2(x) = 1,\ n_3(x) = 0$.

Note that $\tau(x \to v) \geq \frac{\omega(x)+\frac{1}{5}\times(d(x)-1)-\frac{1}{5}n_3(x)}{n_2(x)} \geq \frac{0+3\times\frac{1}{5}-0\times\frac{1}{5}}{1} = \frac{3}{5}$ by $R1, R2$. Observe that $d(x)+d(y) \geq \Delta+3$ by Lemma 3.3, so $d(y) \geq \Delta + 3 - d(x) \geq 7$.

If $d(y) = 7$, then we have that $n_2(y) + n_3(y) \leq d(x) + d(y) - \Delta - 2 = 9 - \Delta \leq 1$ by Lemma 3.4, which means that $n_2(y) = 1,\ n_3(y) = 0$. Note that $\tau(y \to v) \geq \frac{\omega(y)+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}n_3(y)}{n_2(y)} = \frac{3+6\times\frac{1}{5}-0\times\frac{3}{5}}{1} = \frac{21}{5}$ by $R1, R4$. So $\omega'(v) \geq -2 + \frac{1}{5} + \frac{3}{5} + \frac{21}{5} = 3 > 0$ by $R1$.

If $d(y) \geq 8$, then we have that $n_2(y) + n_3(y) \leq d(x) + d(y) - \Delta - 1 \leq d(y) - 5$ by Lemma 3.4. Hence $n_2(y) \leq d(y) - 5$.

Note that $\tau(y \to v) \geq \frac{\omega(y)+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}n_3(y)}{n_2(y)} \geq \frac{d(y)-4+\frac{1}{5}\times(d(y)-1)-\frac{3}{5}\times(d(y)-5-n_2(y))}{n_2(y)} = \frac{3}{5}\times(1+\frac{d(y)-2}{n_2(y)}) \geq \frac{3}{5}\times(2+\frac{3}{d(y)-5})$ by $R1, R4$. We have that $\tau(y \to v) > \frac{6}{5}$ since $d(y) \geq 8$. Hence, $\omega'(v) > -2 + \frac{1}{5} + \frac{3}{5} + \frac{6}{5} = 0$ by $R1$.

After $R1 - R4$, we get $\omega'(v) \geq 0$, for all $v \in V(G)$. For all $f \in F(G)$, if $d(f) = 4$, then $\omega'(f) = \omega(f) = d(f) - 4 = 0$. If $d(f) \geq 5$, then we have $\omega'(f) = d(f) - 4 - \frac{d(f)-4}{d(f)} \times d(f) = 0$ by $R4$.

In summary, we get the following contradictory formula:

$$-8 = \sum_{x\in V(G)\cup F(G)} \omega(x) = \sum_{x\in V(G)\cup F(G)} \omega'(x) \geq 0.$$

The above contradiction indicates that G does not exist, so Theorem 1.1 is true.

4. Conclusions

In this paper, we consider the acyclic chromatic index of planar graphs without 3-cycles and intersecting 4-cycles and proved that such graphs have $\chi'_a(G) \leq \Delta + 1$ if $\Delta(G) \geq 8$. A natural problem in context of our main result is the following:

What is the optimal constant c such that $\chi'_a(G) \leq \Delta(G) + 1$ for every planar graph G with $g(G) \geq c$?

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Conflict of interest

The authors declare no conflicts of interest.

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