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Research article

Normalized ground states for fractional Kirchhoff equations with critical or supercritical nonlinearity

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Abstract: The aim of this paper is to study the existence of ground states for a class of fractional Kirchhoff type equations with critical or supercritical nonlinearity

$$(a + b\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2dx)(-\triangle)^s u = \lambda u + |u|^{q-2}u + \mu|u|^{p-2}u,\ x \in \mathbb{R}^3,$$

with prescribed L^2-norm mass

$$\int_{\mathbb{R}^3} u^2dx = c^2$$

where $s \in (\frac{3}{4},\ 1)$, $a, b, c > 0$, $\frac{6+8s}{3} < q < 2_s^*$, $p \geq 2_s^*$ ($2_s^* = \frac{6}{3-2s}$), $\mu > 0$ and $\lambda \in \mathbb{R}$ as a Langrange multiplier. By combining an appropriate truncation argument with Moser iteration method, we prove that the existence of normalized solutions for the above equation when the parameter μ is sufficiently small.

Keywords: normalized solution; fractional Kirchhoff equation; Pohozaev manifold; Moser iteration method; supercritical growth
Mathematics Subject Classification: 35J65, 47J05, 47J30

1. Introduction

In this paper, we study mainly the existence of ground states to the Kirchhoff type problem with critical or supercritical nonlinearity

$$\begin{cases} (a + b\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2dx)(-\triangle)^s u = \lambda u + |u|^{q-2}u + \mu|u|^{p-2}u,\ x \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} u^2 = c^2, \end{cases} \tag{1.1}$$

where $s \in (\frac{3}{4}, 1)$, $a, b, c > 0$, $\frac{6+8s}{3} < q < 2_s^*$, $p \geq 2_s^*$ ($2_s^* = \frac{6}{3-2s}$), $\mu > 0$ is a real parameter, and $(-\triangle)^s$ denotes the fractional Laplacian operator.

The operator $(-\triangle)^s$ can be seen as the infinitesimal generators of Lévy stable diffusion processes, see [1,2] for example. This operator appears in several areas such as biology, chemistry and physics (see [3–6]). Problem (1.1) is viewed as being nonlocal because of the appearance of the term $b\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u|^2$, which implies that Eq (1.1) is no longer a pointwise identity. This also results in lack of weak sequential continuity of the energy function associated to (1.1), so it make the study of (1.1) particularly interesting. Over the last decade, many mathematicians were particularly keen on the study of nonlinear equations involving nonlocal operators, we can look it up in [7–14] and the references therein.

It is well known that problem (1.1) arises from looking for the standing wave type solutions $\varphi(x,t) = e^{-i\lambda t}u(x)$, $\lambda \in \mathbb{R}$ for the following time-dependent nonlinear fractional Kirchhoff equation

$$i\frac{\partial\varphi}{\partial t} = (a + b\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}\varphi|^2dx)(-\triangle)^s\varphi - f(|\varphi|)\varphi,\ x \in \mathbb{R}^3, \tag{1.2}$$

where $0 < s < 1$, i denotes the imaginary unit. The stationary case of (1.2) is the following equation

$$(a + b\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}\varphi|^2dx)(-\triangle)^s\varphi = f(|\varphi|)\varphi,\ x \in \mathbb{R}^3. \tag{1.3}$$

Clearly, φ solves (1.2) if and only if the stand wave $u(x)$ satisfies (1.1) with $f(u) = |u|^{q-2} + \mu|u|^{p-2}$. Alternatively one can consider the existence of normalized solutions to (1.1), that is, solutions with prescribed L^2-norm. Since solutions $\varphi \in C([0,1],\ H^s(\mathbb{R}^3))$ to (1.2) maintain their mass along time (In fact, multiplying (1.2) by the conjugate $\overline{\varphi}$ of φ, integrating over $\mathbb{R}^3$, and taking the imaginary part, we get $\frac{d}{dt}|\varphi(t)|_2^2 = 0$, $t \in [0,T]$.), it is natural and interesting, from a physical point view, to search for such solutions.

When $s = 1$, Problem (1.3) becomes the Kirchhoff equation. In the past several years, the Kirchhoff type equations has been studied extensively by many researchers(see [15–23]). For all we know, the existence results to problem (1.1) have been mostly available for the case where $p, q \in (2, 2_s^*)$ and λ is fixed and assigned. When $a = 1$, $b = 0$, $s = 1$ and $\mu = 0$, i.e., for the Laplacian operator, Jeanjean's [24] was the first paper to prove existence of normalized solutions in purely L^2-supercritical case. Li and Ye in [25] considered problem (1.1) with $s = 1, \mu = 0, N = 3, \lambda = -1, q \in (3,6)$ and proved that (1.1) has at least one least energy solution by dealing with a constrained minimization problem on a manifold of $H^1(\mathbb{R}^3)$, which is obtained by combining the Nehari manifold and the corresponding Pohozaev identity. Liu, Chen and Yang in [26] considered problem (1.1) with $2 < q < p < 2_s^*$ and proved some existence results about the normalized solutions. However, there is few literature concerned about the normalized solutions for fractional Kirchhoff equation with critical or supercritical nonlinearity. With regard to the point, we attempt to study this kind of problem in this paper.

It is well known that the fractional order Sobolev space $H := H^s(\mathbb{R}^3)$ can be defined as follows

$$H^s(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \int\int_{\mathbb{R}^6}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}dxdy < +\infty\},$$

endowed with the norm

$$\|u\|_H = (\int\int_{\mathbb{R}^6}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}dxdy + \int_{\mathbb{R}^3}|u|^2dx)^{\frac{1}{2}},$$

and the inner product is

$$(u,v)_H=\int\int_{\mathbb{R}^6}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}}dxdy+\int_{\mathbb{R}^3}uvdx.$$

According to [26], we know that

$$\|u\|=(\int_{\mathbb{R}^3}(|(-\triangle)^{\frac{s}{2}}u|^2dx+\int_{\mathbb{R}^3}|u|^2dx)^{\frac{1}{2}}$$

is also a norm on $H^s(\mathbb{R}^3)$ which is equivalent to $\|u\|_H$. Moreover, we define $H_r^s(\mathbb{R}^3):=\{u\in H^s(\mathbb{R}^3):u(x)=u(|x|),x\in\mathbb{R}^3\}$.

Let $\mathbb{H}=H\times\mathbb{R}$ with the scalar product

$$(\cdot,\cdot)_{\mathbb{H}}=(\cdot,\cdot)_H+(\cdot,\cdot)_{\mathbb{R}}$$

and the corresponding norm

$$\|\cdot\|_{\mathbb{H}}^2=\|\cdot\|_H^2+|\cdot|_{\mathbb{R}}^2.$$

Let $\|\cdot\|_t$ be the usual norm of space $L^t(\mathbb{R}^3)$ where $2\le t\le\infty$. H is continuously embedding into $L^t(\mathbb{R}^3)$ for $t\in[2,2_s^*]$ and there exists a best constant S_s^* such that

$$S_s^*=\inf_{u\in H,u\neq0}\frac{\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u|^2dx}{\|u\|_{2_s^*}^2}.\tag{1.4}$$

The normalized weak solution for the problem (1.1) is obtained by looking for critical points of the following C^1 functional

$$\mathcal{J}_\mu(u)=\frac{a}{2}\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u|^2+\frac{b}{4}(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u|^2)^2-\frac{1}{q}\int_{\mathbb{R}^3}|u|^q-\frac{\mu}{p}\int_{\mathbb{R}^3}|u|^p$$

constrained on the L^2-spheres in H:

$$S(c)=\{u\in H|\ \|u\|_2=c>0\}.$$

u_c is called a ground state of (1.1) on $S(c)$ if

$$d\mathcal{J}_\mu|_{S(c)}(u_c)=0\text{ and }\mathcal{J}_\mu(u_c)=\inf_{u\in S(c)}\{\mathcal{J}_\mu(u):\ d\mathcal{J}_\mu|_{S(c)}(u)=0\}.$$

Since $p\ge2_s^*$, the functional $\mathcal{J}_\mu$ is not well defined on $H^s(\mathbb{R}^3)$ unless $p=2_s^*$. Moreover, we need to overcome the lack of compactness in studying critical and supercritical growth. Hence, we cannot directly use variational methods to prove the existence of normalized solutions. To overcome these difficulties, we use a new method, which came from [14,18]. The main idea of this method is to reduce the supercritical problem into a subcritical one. In comparison with previous works, this paper has several new features. Firstly, we consider the nonlinear term with supcritical growth. Secondly, we give the existence of normalized solution for the appropriate truncation problem of (1.1). Finally, the existence of a normalized ground state solution is obtained by Moser iteration method. The results in this paper extend the results in paper [4,24,26]. There have been no previous studies considering the existence of normalized ground state solutions for problem (1.1) involving supcritical growth to the best of our knowledge.

Our main result is the following:

Theorem 1.1. For any $c > 0$, there exists a $\mu^* > 0$ such that, problem (1.1) has a couple of solutions $(u_c,\ \lambda_c) \in H^s(\mathbb{R}^3) \times \mathbb{R}$ for any $\mu \in (0, \mu^*]$. Moreover, u_c is a positive ground state, radially symmetric function and $\lambda_c < 0$.

Remark 1.2. When $\frac{6+8s}{3} < q < 2_s^*$, $\mathcal{J}_\mu$ is not bounded from below on $S(c)$, i.e., $\inf\limits_{u\in S(c)} \mathcal{J}_\mu(u) = -\infty$. So, the minimization problem constrained on $S(c)$ does not work. We try to look for a critical point with a minimax characterization. Although $\mathcal{J}_\mu$ has a mountain-pass geometry on $S(c)$, the boundedness of the obtained Palais-Smale sequence is not yet clear. Motivated by [4], we try to construct an auxiliary map I_μ, which on $S(c)\times\mathbb{R}$ has the same type of geometric structure as $\mathcal{J}_\mu$ on $S(c)$. Besides, the Palais-Smale sequence of I_μ satisfies the additional condition, which is the key point to obtain the boundedness of the Palais-Smale sequence.

2. Preliminaries

In this section, we give a truncation argument in order to overcome the lack of compactness in studying critical and supercritical growth. Let $M > 0$ be a constant. For fixed $c > 0,\ \mu > 0,\ M > 0$, we investigate the existence of ground state for the following truncation problem

$$\begin{cases} (a + b \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 dx)(-\triangle)^s u = \lambda u + |u|^{q-2}u + \mu d_M(u),\ x \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} u^2 = c^2, \end{cases} \tag{2.1}$$

where $s \in (\frac{3}{4},\ 1),\ a, b > 0,\ \frac{6+8s}{3} < q < 2_s^*,\ p \geq 2_s^*\ (2_s^* = \frac{6}{3-2s}),$ and

$$d_M(t) = \begin{cases} |t|^{p-2}t, & |t| \leq M, \\ M^{p-q}|t|^{q-2}t, & |t| > M, \end{cases}$$

To investigate (2.1), we define the the energy functional $E_\mu : H \to \mathbb{R}$ by

$$E_\mu(u) = \frac{a}{2}\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 dx + \frac{b}{4}(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 dx)^2 - \frac{1}{q}\int_{\mathbb{R}^3} |u|^q dx - \mu \int_{\mathbb{R}^3} D_M(u) dx, \tag{2.2}$$

where $D_M(t) \doteq \int_0^t d_M(\tau)d\tau$. It is easy to obtain that $E_\mu \in C^1(H, \mathbb{R})$ and

$$\langle E'_\mu(u), v\rangle = (a + b\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 dx)\int_{\mathbb{R}^3} (-\triangle)^{\frac{s}{2}} u (-\triangle)^{\frac{s}{2}} v dx - \int_{\mathbb{R}^3} |u|^{q-2}uv dx - \mu \int_{\mathbb{R}^3} d_M(u) v dx \tag{2.3}$$

for all $u, v \in H$.

Theorem 2.1. For any $c > 0$ and $M > 0$, there exists a $\mu_1 > 0$, such that, problem (2.1) has a couple of solutions $(u_c,\ \lambda_c) \in H_r^s(\mathbb{R}^3) \times \mathbb{R}$ for any $\mu \in (0, \mu_1]$. Moreover, u_c is a positive ground state, $\lambda_c < 0$ and $E_\mu(u_c) = m_{c,\mu}$, where

$$m_{c,\mu} := \inf_{u\in V(c)} E_\mu(u)$$

and $V(c)$ is the Pohozaev manifold defined in lemma 2.4.

Next, we give some useful preliminary lemmas to prove Theorem 2.1.

Lemma 2.1. [8] If $\alpha \in (2,\ 2_s^*)$, there exists an optimal constant $C(s,\alpha)$ such that for any $u \in H$,

$$\int_{\mathbb{R}^3} |u|^\alpha \leq C(s,\alpha)(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2)^{\frac{\alpha\beta_\alpha}{s}} (\int_{\mathbb{R}^3} |u|^2)^{\alpha-\frac{\alpha\beta_\alpha}{s}}, \tag{2.4}$$

where $\beta_\alpha := \frac{3(\alpha-2)}{2\alpha}$.

Lemma 2.2. [19] $H_r^s(\mathbb{R}^3)$ is compactly embedding into $L^t(\mathbb{R}^3)$ for $t \in (2, 2_s^*)$.

As in [4], we introduce the useful fiber map preserving the L^2-norm, that is,

$$(\tau \star u)(x) := e^{\frac{3}{2}\tau} u(e^\tau x), \quad for\ a.e.\ x \in \mathbb{R}^3. \tag{2.5}$$

Define the auxiliary functional $I : \mathbb{H} \to \mathbb{R}$ by

$$\begin{aligned} I_\mu(u,\tau) := E_\mu(\tau \star u) \ &= \frac{a}{2} e^{2s\tau} \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 + \frac{b}{4} e^{4s\tau} (\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2)^2 \\ &-\frac{1}{q} e^{q\beta_q\tau} \int_{\mathbb{R}^3} |u|^q dx - \mu e^{r\beta_r\tau} \int_{\mathbb{R}^3} D_M(u) dx, \end{aligned} \tag{2.6}$$

where

$$r = \begin{cases} p, & |u| \leq M, \\ q, & |u| > M, \end{cases} \tag{2.7}$$

then we can obtain that I_μ is a C^1-functional.

Lemma 2.3. [13] The map $(u,\tau) \in \mathbb{H} \mapsto \tau \star u \in H$ is continuous.

Similar to Lemma 2.1 in [4], we can easily get the following lemma.

Lemma 2.4. Let $(u,\lambda) \in S(c) \times \mathbb{R}$ be a weak solution of Eq (2.2). Then u belongs to the set

$$V(c) := \{u \in S(c) :\ P_\mu(u) = 0\}$$

where

$$P_\mu(u) = a\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 dx + b(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 dx)^2 - \frac{\beta_q}{s} \int_{\mathbb{R}^3} |u|^q dx - \frac{\mu\beta_r r}{s} \int_{\mathbb{R}^3} D_M(u) dx. \tag{2.8}$$

Lemma 2.5. For any $u \in S(c)$, $\tau \in \mathbb{R}$ is a critical point for $\Phi_u(\tau) := I_\mu(u,\tau)$ if and only if $\tau \star u \in V(c)$.

Proof. For any $u \in S(c)$ and $\tau \in \mathbb{R}$, we have

$$\begin{aligned} &(\Phi_u)'(\tau) \\ &= ase^{2s\tau} \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 + bse^{4s\tau} (\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2)^2 - \beta_q e^{\beta_q q\tau} \int_{\mathbb{R}^3} |u|^q - \mu\beta_r r e^{\beta_r r\tau} \int_{\mathbb{R}^3} D_M(u) \\ &= as\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} (\tau \star u)|^2 + bs(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} (\tau \star u)|^2)^2 - \beta_q \int_{\mathbb{R}^3} |(\tau \star u)|^q - \mu\beta_r r \int_{\mathbb{R}^3} D_M(\tau \star u) \\ &= sP_\mu(\tau \star u). \end{aligned} \tag{2.9}$$

It is easy to see that Lemma 2.5 holds. □

Lemma 2.6. Let $u \in S(c)$ be arbitrary fixed, then
(1) $\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}(\tau \star u)|^2 \to 0$ and $I_\mu(u,\tau) \to 0$ as $\tau \to -\infty$;
(2) $\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}(\tau \star u)|^2 \to +\infty$ and $I_\mu(u,\tau) \to -\infty$ as $\tau \to +\infty$.
Proof. For fixed $u \in S(c)$, we can easily get the conclusions (1) and (2) from the facts

$$\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}(\tau \star u)|^2 = e^{2s\tau} \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2,$$

$$I_\mu(u,\tau) = \frac{a}{2} e^{2s\tau} \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 + \frac{b}{4} e^{4s\tau} (\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2)^2 - \frac{e^{\beta_q q\tau}}{q} \int_{\mathbb{R}^3} |u|^q - \mu e^{\beta_r r\tau} \int_{\mathbb{R}^3} D_M(u) dx$$

and $\beta_r r \geq \beta_q q > 4s$. □

Lemma 2.7. For every $u \in S(c)$, there exists a unique $\tau_u \in \mathbb{R}$ such that $\tau_u \star u \in V(c)$, where τ_u is a strict maximum point for $\Phi_u(\tau)$ and $\Phi_u(\tau_u) > 0$.
Proof. For $u \in S(c)$ and $\tau \in \mathbb{R}$, by (2.9) we have

$$\begin{aligned}(\Phi_u)''(\tau) &= 2as^2 \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}(\tau \star u)|^2 + 4bs^2 (\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}(\tau \star u)|^2)^2 \\ &\quad -q\beta_q^2 \int_{\mathbb{R}^3} |(\tau \star u)|^q - \mu(\beta_r r)^2 \int_{\mathbb{R}^3} D_M(\tau \star u).\end{aligned} \tag{2.10}$$

Since $r\beta_r \geq q\beta_q > 4s$, it is easy to see that $(\Phi_u)'(\tau) > 0$ as $\tau \to -\infty$, and $(\Phi_u)'(\tau) < 0$ as $\tau \to \infty$. So, there exists $\tau_u \in \mathbb{R}$ such that $(\Phi_u)'(\tau_u) = 0$. From Lemma 2.5, $\tau_u \star u \in V(c)$.

Combining with $(\Phi_u)'(\tau_u) = 0$, (2.9) and (2.10), we have

$$(\Phi_u)''(\tau_u) = -2as^2 \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}(\tau \star u)|^2 - \beta_q(q\beta_q - 4s) \int_{\mathbb{R}^3} |(\tau \star u)|^q - \mu r\beta_r(r\beta_r - 4s) \int_{\mathbb{R}^3} D_M(\tau \star u) < 0,$$

which together with Lemma 2.6 implies that τ_u is unique and it is a strict global maximum point for $\Phi_u(\tau)$ and $\Phi_u(\tau_u) > 0$. □

3. Characterization of mountain pass level

As in [4], firstly, we prove that $E_\mu(u)$ has the mountain pass geometry on $S(c) \times \mathbb{R}$ in the following lemma.

Lemma 3.1. There exists $k_c > 0$ such that

$$P_\mu(u),\ E_\mu(u) > 0 \text{ for all } u \in \mathbb{A}_c, \quad \text{and} \quad 0 < \sup_{u \in \mathbb{A}_c} E_\mu(u) < \inf_{u \in \mathbb{B}_c} E_\mu(u)$$

with

$$\mathbb{A}_c = \{u \in S(c) : \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 \leq k_c\},\ \mathbb{B}_c = \{u \in S(c) : \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 = 2k_c\}.$$

Proof. Let $k > 0$ be arbitrary fixed and suppose $u, v \in S(c)$ are such that

$$\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 \leq k,\ \textit{and} \ \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} v|^2 = 2k.$$

Then for k small enough, by (2.4) and $\frac{3(r-2)}{2s} \geq \frac{3(q-2)}{2s} > 4$, there exist constants C_1 and C_2 such that

$$P_\mu(u) \geq a\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2 + b(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2)^2 - C_1(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2)^{\frac{3(q-2)}{2s}} - C_2(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2)^{\frac{3(r-2)}{2s}},$$

$$E_\mu(u) \geq \frac{a}{2}\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2 + \frac{b}{4}(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2)^2 - C_1(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2)^{\frac{3(q-2)}{2s}} - C_2(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2)^{\frac{3(r-2)}{2s}}$$

and

$$\begin{aligned} E_\mu(v) - E_\mu(u) \quad & \geq \frac{a}{2}\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}v|^2 + \frac{b}{4}(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}v|^2)^2 - \frac{1}{q}\int_{\mathbb{R}^3} |v|^q - \mu\int_{\mathbb{R}^3} D_M(v) \\ & -\frac{a}{2}\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2 - \frac{b}{4}(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2)^2 \\ & \geq ak + bk^2 - \frac{a}{2}k - \frac{b}{4}k^2 - C_1(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}v|^2)^{\frac{3(q-2)}{2s}} - C_2(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}v|^2)^{\frac{3(r-2)}{2s}} \\ & \geq \frac{ak}{2} + \frac{ak^2}{2} - C_1 k^{\frac{3(q-2)}{2s}} - C_2 k^{\frac{3(r-2)}{2s}}. \end{aligned}$$

By the above inequalities, we can obtain that there exists $k_c > 0$ sufficiently small such that Lemma 3.2 holds. □

Next, we need to construct the minimax characterization of I_μ and E_μ.

Lemma 3.2. Let

$$\tilde{\gamma}_{c,\mu} := \inf_{\tilde{h}\in\tilde{\Gamma}_c} \max_{t\in[0,1]} I_\mu(\tilde{h}(t))$$

with

$$\tilde{\Gamma}_c = \{\tilde{h} \in C([0,1], S(c)\times\mathbb{R}) : \tilde{h}(0) \in (\mathbb{A}_c, 0), \tilde{h}(1) \in (E_\mu^0, 0)\},\ E_\mu^0 := \{u \in S(c) :\ E_\mu(u) \leq 0\}$$

and

$$\gamma_{c,\mu} := \inf_{h\in\Gamma_c} \max_{t\in[0,1]} E_\mu(h(t))$$

with

$$\Gamma_c = \{h \in C([0,1], S(c)) : h(0) \in \mathbb{A}_c,\ h(1) \in E_\mu^0\},$$

then we have

$$\tilde{\gamma}_{c,\mu} = \gamma_{c,\mu} = m_{c,\mu} > 0.$$

Proof. Firstly, we prove that $\tilde{\gamma}_{c,\mu} = \gamma_{c,\mu}$.
For any $\tilde{h} \in \tilde{\Gamma}_c$,we can write it into

$$\tilde{h}(t) = (\tilde{h}_1(t), \tilde{h}_2(t)) \in S(c)\times\mathbb{R}.$$

We set $h(t) = \tilde{h}_2(t) \star \tilde{h}_1(t)$, then $h(t) \in \Gamma_c$, and

$$\max_{t\in[0,1]} I_\mu(\tilde{h}(t)) = \max_{t\in[0,1]} E_\mu(\tilde{h}_2(t) \star \tilde{h}_1(t)) = \max_{t\in[0,1]} E_\mu(h(t)),$$

which implies $\tilde{\gamma}_{c,\mu} \geq \gamma_{c,\mu}$. On the other hand, for any $h \in \Gamma_c$, if we set $\tilde{h}(t) = (h(t), 0)$, then we get $\tilde{h} \in \tilde{\Gamma}_c$ and

$$\max_{t\in[0,1]} I_\mu(\tilde{h}(t)) = \max_{t\in[0,1]} E_\mu(h(t)).$$

This infers that $\tilde{\gamma}_{c,\mu} \leq \gamma_{c,\mu}$. So, $\tilde{\gamma}_{c,\mu} = \gamma_{c,\mu}$.

Secondly, we claim that for $u \in S(c)$, $E_\mu(u) \leq 0$ implies $P_\mu(u) < 0$.
For $u \in S(c)$, if $E_\mu(u) \leq 0$, then $\Phi_u(0) \leq 0$. By the proof of Lemma 2.7 and Lemma 2.6, we easily see that $\tau_u < 0$, so

$$P_\mu(u) = P_\mu(0 \star u) = \frac{1}{s}(\Phi_u)'(0) < \frac{1}{s}(\Phi_u)'(\tau_u) = 0.$$

That is

$$E_\mu(u) \leq 0 \Rightarrow P_\mu(u) < 0. \tag{3.1}$$

Next, we prove that $m_{c,\mu} = \gamma_{c,\mu}$.
For any $u \in V(c)$, by Lemma 2.6 and Lemma 2.3, there exists $t^- \ll -1$ and $t^+ \gg 1$ such that

$$h_u : \tau \in [0,1] \to ((1-\tau)t^- + \tau t^+) \star u \text{ and } \quad h_u \in \Gamma_c.$$

By Lemma 2.7, we have $\max_{\tau\in[0,1]} E_\mu(h_u(\tau)) = E_\mu(u)$. So we have $m_{c,\mu} \geq \gamma_{c,\mu}$. On the other hand, for any $\tilde{h}(\tau) = (\tilde{h}_1(\tau), \tilde{h}_2(\tau)) \in \tilde{\Gamma}_c$, we know that $\tilde{h}_2(0) \star \tilde{h}_1(0) = \tilde{h}_1(0) \in \mathbb{A}_c$, $\tilde{h}_2(1) \star \tilde{h}_1(1) = \tilde{h}_1(1) \in E^0_\mu$. Hence by Lemma 3.1, we can deduce that

$$P_\mu(\tilde{h}_2(0) \star \tilde{h}_1(0)) > 0,$$

and using (3.1),

$$P_\mu(\tilde{h}_2(1) \star \tilde{h}_1(1)) < 0.$$

From Lemma 2.3, the function $\tilde{P}_\mu(\tau) := P_\mu(\tilde{h}_2(\tau) \star \tilde{h}_1(\tau))$ is continuous in $[0,1]$. Therefore, there exists $\bar{\tau} \in (0,1)$ such that $\tilde{P}_\mu(\bar{\tau}) = 0$, which implies that $\tilde{h}_2(\bar{\tau}) \star \tilde{h}_1(\bar{\tau}) \in V(c)$, and

$$\max_{\tau\in[0,1]} I(\tilde{h}(\tau)) = \max_{\tau\in[0,1]} E_\mu(\tilde{h}_2(\tau) \star \tilde{h}_1(\tau)) \geq \inf_{u\in V(c)} E_\mu(u),$$

which implies that $\gamma_{c,\mu} = \tilde{\gamma}_{c,\mu} \geq m_{c,\mu}$. So, $m_{c,\mu} = \gamma_{c,\mu}$.

Finally, we prove that $m_{c,\mu} > 0$.
For any $u \in V(c)$, then $P_\mu(u) = 0$. By (2.4), we have

$$a\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2dx + b(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2dx)^2 \leq C_1(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2dx)^{\frac{q\beta_q}{s}} + C_2(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2dx)^{\frac{r\beta_r}{s}},$$

noticing that $r\beta_r \geq q\beta_q > 4s$, there exists $\delta > 0$ such that $\inf_{u\in V(c)} \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2dx \geq \delta$, and

$$\begin{aligned} E_\mu(u) &= E_\mu(u) - \frac{1}{4}P_\mu(u) \\ &= \frac{a}{4}\int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2dx + (\frac{\beta_q}{4s} - \frac{1}{q})\int_{\mathbb{R}^3} |u|^q dx + \mu(\frac{r\beta_r}{4s} - 1)\int_{\mathbb{R}^3} D_M(u)dx \; . \\ &\geq \frac{a}{4}\delta \end{aligned}$$

Thus, $m_{c,\mu} > 0$. □

Remark 3.3. Let

$$\rho_c := \inf_{h\in\Gamma_c^0} \max_{t\in[0,1]} E_0(h(t))$$

with

$$\Gamma_c^0 = \{h \in C([0,1], S(c)) : h(0) \in \mathbb{A}_c,\ h(1) \in E^0\},\ E^0 := \{u \in S(c) :\ E_0(u) \le 0\}.$$

Obviously, $\Gamma_c^0 \subset \Gamma_c,\ E_0(u) \ge E_\mu(u)$ for $u \in S(c)$. Thus we can deduce that ρ_c is independent of positive numbers μ, M and $\rho_c \ge \gamma_{c,\mu}$ for any $\mu > 0$.

In the following lemma, we give the relationship between the Palais-Smale sequence for I and that of E_μ.

Lemma 3.4. Let $\tilde{\gamma}_{c,\mu}$ and $\gamma_{c,\mu}$ be defined in Lemma 3.2. Then there exist a sequence $\{(v_n, \tau_n)\} \subset S(c)\times\mathbb{R}$ such that for $n \to \infty$, we have
(1) $I_\mu(v_n, \tau_n) \to \tilde{\gamma}_{c,\mu}$,
(2) $(I_\mu)'|_{S(c)\times\mathbb{R}}(v_n, \tau_n) \to 0$, i.e., it holds that

$$\partial_\tau I_\mu(v_n, \tau_n) \to 0$$

and

$$\langle \partial_u I_\mu(v_n, \tau_n), \tilde{\varphi}\rangle \to 0$$

with

$$\tilde{\varphi} \in T_{v_n} := \{\varphi \in H : \int_{\mathbb{R}^3} v_n\varphi = 0\}.$$

In addition, setting $u_n(x) = \tau_n \star v_n(x)$, then for $n \to \infty$, we get
(i) $E_\mu(u_n) \to \gamma_{c,\mu}$,
(ii) $P_\mu(u_n) \to 0$,
(iii) $E'_\mu|_{S(c)}(u_n) \to 0$, i.e., it holds that

$$\langle E'_\mu(u_n), \varphi\rangle \to 0$$

with

$$\varphi \in T_{u_n} := \{\varphi \in H : \int_{\mathbb{R}^3} u_n\varphi = 0\}.$$

Proof. According to the construction of $\tilde{\gamma_{c,\mu}}$, we know that the conclusions (1) and (2) follow directly from the Ekeland's Variational Principle [8, Proposition 2.2]. Next we mainly show (i)–(iii).

For (i), by Lemma 3.2, $\tilde{\gamma}_{c,\mu} = \gamma_{c,\mu}$. we notice that

$$E_\mu(u_n) = E_\mu(\tau_n \star v_n) = I_\mu(v_n, \tau_n),$$

thus (i) holds.

By (2.9), we can get that $\partial_\tau I_\mu(v_n, \tau_n) = sP_\mu(\tau_n \star v_n)$. Thus, (ii) is a consequence of $\partial_\tau I_\mu(v_n, \tau_n) \to 0$ as $n \to \infty$.

For the proof of (iii), by the definition of I_μ, we have

$$\begin{aligned}\langle \partial_u I_\mu(v_n, \tau_n), \tilde{\varphi}\rangle &= ae^{2s\tau_n} \int\int_{\mathbb{R}^6} \frac{(v_n(x) - v_n(y))(\tilde{\varphi}(x) - \tilde{\varphi}(y))}{|x-y|^{3+2s}dxdy} \\ &+be^{4s\tau_n} \int\int_{\mathbb{R}^6} \frac{(v_n(x) - v_n(y))^2}{|x-y|^{3+2s}dxdy} \int\int_{\mathbb{R}^6} \frac{(v_n(x) - v_n(y))(\tilde{\varphi}(x) - \tilde{\varphi}(y))}{|x-y|^{3+2s}dxdy} \\ &-e^{\beta_q q\tau_n} \int_{\mathbb{R}^3} |v_n|^{q-2}v_n\tilde{\varphi} - \mu e^{\beta_r r\tau} \int_{\mathbb{R}^3} d_M(v_n)\tilde{\varphi},\end{aligned}$$

where $\tilde{\varphi} \in T_{v_n}$.
On the other hand, for any φ with satisfying $\varphi \in T_{u_n}$, by using (2.3), we have

$$\begin{aligned}
&\langle E'_\mu(u_n), \varphi\rangle \\
&= a\int\int_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}}dxdy \\
&+b\int\int_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))^2}{|x-y|^{3+2s}}dxdy\int\int_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}}dxdy \\
&-\int_{\mathbb{R}^3}|u_n|^{q-2}u_n\varphi dx-\mu\int_{\mathbb{R}^3}d_M(u_n)\varphi dx \\
&= a\int\int_{\mathbb{R}^6} \frac{e^{\frac{3}{2}\tau_n}(v_n(e^{\tau_n}x)-v_n(e^{\tau_n}y))(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}}dxdy \\
&+b\int\int_{\mathbb{R}^6} \frac{e^{3\tau_n}(v_n(e^{\tau_n}x)-v_n(e^{\tau_n}y))^2}{|x-y|^{3+2s}}dxdy\int\int_{\mathbb{R}^6} \frac{e^{\frac{3}{2}\tau_n}(v_n(e^{\tau_n}x)-v_n(e^{\tau_n}y))(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}}dxdy \\
&-\int_{\mathbb{R}^3}|e^{\frac{3}{2}\tau_n}v_n(e^{\tau_n}x)|^{q-2}e^{\frac{3}{2}\tau_n}v_n(e^{\tau_n}x)\varphi(x)dx-\mu\int_{\mathbb{R}^3}d_M(e^{\frac{3}{2}\tau_n}v_n(e^{\tau_n}x))\varphi(x)dx \\
&= ae^{2s\tau_n}\int\int_{\mathbb{R}^6} \frac{(v_n(x)-v_n(y))(e^{-\frac{3}{2}\tau_n}\varphi(e^{-\tau_n}x)-e^{-\frac{3}{2}\tau_n}\varphi(e^{-\tau_n}y))}{|x-y|^{3+2s}}dxdy \\
&+be^{4s\tau_n}\int\int_{\mathbb{R}^6} \frac{(v_n(x)-v_n(y))^2}{|x-y|^{3+2s}}dxdy\int\int_{\mathbb{R}^6} \frac{(v_n(x)-v_n(y))(e^{-\frac{3}{2}\tau_n}\varphi(e^{-\tau_n}x)-e^{-\frac{3}{2}\tau_n}\varphi(e^{-\tau_n}y))}{|x-y|^{3+2s}}dxdy \\
&-e^{q\beta_q\tau_n}\int_{\mathbb{R}^3}|v_n(x)|^{q-2}v_n(x)e^{-\frac{3}{2}\tau_n}\varphi(e^{-\tau_n}x)dx-\mu e^{r\beta_r\tau_n}\int_{\mathbb{R}^3}d_M(v_n)e^{-\frac{3}{2}\tau_n}\varphi(e^{-\tau_n}x)dx
\end{aligned}$$

Setting

$$\tilde{\varphi}(x) = e^{-\frac{3}{2}\tau_n}\varphi(e^{-\tau_n}x),$$

we get (iii) if we could show that $\tilde{\varphi} \in T_{v_n}$. In fact, $\tilde{\varphi} \in T_{v_n}$ follows from the following equalities

$$\begin{aligned}
0 &= \int_{\mathbb{R}^3} u_n\varphi = \int_{\mathbb{R}^3} e^{\frac{3}{2}\tau_n}v_n(e^{\tau_n}x)\varphi(x) \\
&= \int_{\mathbb{R}^3} v_n(x)e^{-\frac{3}{2}\tau_n}\varphi(e^{-\tau_n}x) = \int_{\mathbb{R}^3} v_n\tilde{\varphi}.
\end{aligned}$$

□

4. Proof of Theorem 2.1

According to Lemma 3.4 and Lemma 3.2, there exist a Palais-Smale sequence $\{u_n\} \subset S(c)$ for $E_\mu|_{S(c)}$ at level $\gamma_{c,\mu} > 0$, and it satisfies $P_\mu(u_n) \to 0$ as $n \to \infty$. By applying the Lagrange multipliers rule there exists $\{\lambda_n\} \subset \mathbb{R}$ such that

$$(a + b\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_n|^2)(-\triangle)^s u_n - |u_n|^{q-2}u_n - \mu d_M(u) = \lambda_n u_n + o(1),\ \text{as } n \to \infty. \tag{4.1}$$

(1). As $P_\mu(u_n) \to 0$, we have

$$a\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_n|^2 + b(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_n|^2)^2 = \frac{\beta_q}{s}\int_{\mathbb{R}^3}|u_n|^q + \frac{\mu\beta_r r}{s}\int_{\mathbb{R}^3}D_M(u_n) + o(1) \text{ as } n \to \infty. \tag{4.2}$$

Thus, by (4.2) we deduce that

$$\begin{aligned} E_\mu(u_n)+o_n(1) &= \frac{a}{2}\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_n|^2+\frac{b}{4}(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_n|^2)^2-\mu\int_{\mathbb{R}^3}D_M(u_n)\\ &-\frac{1}{q}\frac{s}{\beta_q}[a\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_n|^2+b(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_n|^2)^2-\frac{\mu\beta_r r}{s}\int_{\mathbb{R}^3}D_M(u_n)]+o(1)\\ &= a(\frac{1}{2}-\frac{s}{\beta_q q})\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_n|^2+b(\frac{1}{4}-\frac{s}{\beta_q q})(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_n|^2)^2\\ &+\mu(\frac{\beta_r r}{\beta_q q}-1)\int_{\mathbb{R}^3}D_M(u_n)\\ &\le \gamma_{c,\mu}+1 \end{aligned} \tag{4.3}$$

Since $\frac{6+8s}{3}<q<2_s^*$ and (2.7), it implies that $\frac{s}{\beta_q q}<\frac{1}{4}$ and $\beta_r r\ge\beta_q q$. According to (4.3), we can deduce the boundedness of $\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_n|^2$, thus $\{u_n\}$ is bounded in H.

(2). According to Lemma 2.2, we know that the embedding $H_r^s(\mathbb{R}^3)\hookrightarrow L^t(\mathbb{R}^3)$ is compact for $t\in(2,\ 2_s^*)$, and we can deduce that there exists $u_c\in H_r^s(\mathbb{R}^3)$ such that, up to a subsequence, $u_n\rightharpoonup u_c$ weakly in H, $u_n\to u_c$ strongly in $L^q(\mathbb{R}^3)$ for $q\in(\frac{6+8s}{3},2_s^*)$. since $\{u_n\}\subset S(c)$ is bounded in H. By (4.1), we obtain that

$$\lambda_n c^2=a\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_n|^2+b(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_n|^2)^2-\int_{\mathbb{R}^3}|u_n|^q-\mu r\int_{\mathbb{R}^3}D_M(u_n)+o_n(1). \tag{4.4}$$

Using the fact that the boundedness of $\{u_n\}$ in H and (4.2), we can deduce that $\{\lambda_n\}$ is bounded. Hence, up to a subsequence $\lambda_n\to\lambda_c\in\mathbb{R}$.

(3). We claim that $u_c\not\equiv 0$. We assume by contradiction that $u_c\equiv 0$, by (4.2) we deduce that $\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_n|^2\to 0$. Recalling that $P_\mu(u_n)\to 0$, according to (4.3), we have $E_\mu(u_n)\to 0$, which is a contradiction to the assumption that $E_\mu(u_n)\to\gamma_{c,\mu}\neq 0$. Now, since $\lambda_n\to\lambda_c$ and $u_n\to u_c\neq 0$ weakly in H, together with (4.1), we know $(u_c,\ \lambda_c)$ is a couple of solutions to (2.1). By the Pohozaev identity, we obtain

$$\frac{3-2s}{2}a\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_c|^2+\frac{3-2s}{2}b(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_c|^2)^2=3(\int_{\mathbb{R}^3}\frac{1}{2}\lambda_c|u_c|^2+\frac{1}{q}|u_c|^q+\mu D_M(u_c)).$$

Combining with the (4.4) for u_c, we get

$$\lambda_c c^2=\lambda_c|u_c|_2^2=\frac{(3-2s)(q-2_s^*)}{3(q-2)}[a\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_c|^2+b(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u_c|^2)^2]+\mu\frac{2(r-q)}{(q-2)}\int_{\mathbb{R}^3}D_M(u_c) \tag{4.5}$$

Since $\frac{6+8s}{3}<q<2_s^*$ and (4.5), there exists $\mu_1>0$ such that $\lambda_c<0$ for $\mu\in(0,\mu_1]$.

(4). Testing (4.1) and (2.1) with u_n-u_c, we can obtain that

$$\langle E'_\mu(u_n)-E'_\mu(u_c),u_n-u_c\rangle-\lambda_c\int_{\mathbb{R}^3}|u_n-u_c|^2=o_n(1).$$

Using the strong L^p convergence of u_n, we infer that

$$a\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}(u_n-u_c)|^2+b(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}(u_n-u_c)|^2)^2-\lambda_c\int_{\mathbb{R}^3}|u_n-u_c|^2=o_n(1),$$

which, being $\lambda_c<0$, implies $u_n\to u_c$ strongly in H. Therefore, $E_\mu(u_n)\to E_\mu(u_c)$, as $n\to\infty$. From Lemma 2.4 and Lemma 3.2, we easily obtain that u_c is a ground state of (2.1) and $E_\mu(u_c)=m_{c,\mu}$. □

5. Proof of main result

In this section, we devote to complete the proof of Theorem 1.1. From the truncation argument in Sections 2–4, we can see that if the ground state u_c of (2.1) satisfy $\|u_c\|_\infty \le M$. Then $u_c \in H$ is a ground state of (1.1).

Lemma 5.1. Let$(u_c,\ \lambda_c)$ be a couple of solutions of problem (2.1) for $\mu \in (0, \mu_1]$, then there exists a constant $K_c > 0$ independent of $\mu, M > 0$ such that $\|u_c\| \le K_c$.

Proof. By Theorem 2.1 and Lemma 2.4, it is easy to see that

$$E_\mu(u_c) = \gamma_{c,\mu} \ \text{ and } \ P_\mu(u_c) = 0, \tag{5.1}$$

It follows from (5.1) and Remark 3.3 that

$$\rho_c \ge \gamma_{c,\mu} \ge E_\mu(u_c) - \frac{1}{4}P_\mu(u_c) \ge \frac{a}{4}\|u_c\|^2$$

Consequently, there exists a constant $K_c > 0$ independent of $\mu, M > 0$ such that $\|u_c\| \le K_c$. □

Lemma 5.2. If $(u_c,\ \lambda_c)$ be a couple of solutions of problem (2.1) for $\mu \in (0, \mu_1]$, then $u_c \in L^\infty(\mathbb{R}^3)$, and there exists a constant $B_c > 0$ independent $\mu, M > 0$ such that

$$\|u_c\|_\infty \le B_c(1 + \mu^{\frac{1}{2_s^*-q}} M^{\frac{p-q}{2_s^*-q}}).$$

Proof. For convenience, we replace u_c with u in the following. Let $L > 0$ and $\beta > 1$, we first define the following functions:

$$\Upsilon(u) = u u_L^{2(\beta-1)} \in H,$$

where $u_L = min\{u, L\}$. Since Υ is an increasing function,we have

$$(x - y)[\Upsilon(x) - \Upsilon(y] \ge 0, \quad \forall x, y \in \mathbb{R}.$$

Let $\Phi(t) = \frac{1}{2}|t|^2$ and $\Psi(t) = \int_0^t (\Upsilon'(\tau))^{\frac{1}{2}} d\tau$.Then,if $x > y$, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\Phi'(x - y)[\Upsilon(x) - \Upsilon(y)] &= (x - y)[\Upsilon(x) - \Upsilon(y)] \\
&= (x - y)\int_y^x \Upsilon'(t)dt \\
&= (x - y)\int_y^x (\Psi'(t))^2 dt \\
&\ge (\int_y^x \Psi'(t)dt)^2 \\
&= |\Psi(x) - \Psi(y)|^2.
\end{aligned}$$

The same arguments hold for $x \le y$. Therefore,

$$\Phi'(x - y)[\Upsilon(x) - \Upsilon(y)] \ge |\Psi(x) - \Psi(y)|^2, \quad \forall x, y \in \mathbb{R}. \tag{5.2}$$

By the definition of u_L, it is easy to see that $|uu_L^{2(\beta-1)}| \leq L^{2(\beta-1)}u$ and $\Upsilon(u) \in H$. Taking $\Upsilon(u)$ as a test function in Eq (2.1), and let $g_{\mu,M}(x,t) = |t|^{q-2}t + \mu d_M(t)$, we obtain

$$\begin{aligned}&(a + b \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}}u|^2) \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(u(x)-u(y))(uu_L^{2(\beta-1)}(x)-uu_L^{2(\beta-1)}(y))}{|x-y|^{3+2s}}dxdy\\ &= \lambda \int_{\mathbb{R}^3} u(x)uu_L^{2(\beta-1)}(x)dx + \int_{\mathbb{R}^3} g_{\mu,M}(x,u(x))uu_L^{2(\beta-1)}(x)dx.\end{aligned} \tag{5.3}$$

Since $\Psi(u) \geq \frac{1}{\beta}uu_L^{\beta-1}$ and (5.2), we get

$$\begin{aligned}&a\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(u(x)-u(y))(uu_L^{2(\beta-1)}(x)-uu_L^{2(\beta-1)}(y))}{|x-y|^{3+2s}}dxdy\\ &\geq \frac{a}{\beta^2}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|uu_L^{\beta-1}(x)-uu_L^{\beta-1}(y)|^2}{|x-y|^{3+2s}}dxdy.\end{aligned} \tag{5.4}$$

For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|g_{\mu,M}(x,t)| \leq \varepsilon|t| + C_\varepsilon(1 + \mu M^{p-q})|t|^{q-1}. \tag{5.5}$$

Let $\omega_L = uu_L^{\beta-1}$. By employing Hölder's inequality and (5.3)–(5.5), we have

$$\begin{aligned}&\frac{a}{\beta^2}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\omega_L(x)-\omega_L(y)|^2}{|x-y|^{3+2s}}dxdy\\ &\leq \varepsilon\int_{\mathbb{R}^3}(\omega_L)^2dx + C_\varepsilon(1+\mu M^{p-q})\int_{\mathbb{R}^3}|u(x)|^{q-2}(\omega_L)^2dx\\ &\leq \varepsilon\int_{\mathbb{R}^3}(\omega_L)^2dx + C_\varepsilon(1+\mu M^{p-q})(\int_{\mathbb{R}^3}|u(x)|^{2_s^*}dx)^{\frac{q-2}{2_s^*}}((\omega_L)^{2t}dx)^{\frac{1}{t}},\end{aligned} \tag{5.6}$$

where $\frac{q-2}{2_s^*} + \frac{1}{t} = 1$ and $2t \in (2, 2_s^*)$. Moreover, it follows from (1.4) that

$$S_s^*\|u\|_{2_s^*}^2 \leq \int_{\mathbb{R}^3}|(-\triangle)^{\frac{s}{2}}u|^2. \tag{5.7}$$

Therefore, we deduce from (5.6) and (5.7) that

$$\|\omega_L\|_{2_s^*}^2 \leq C\beta^2[\|\omega_L\|_2^2 + (1+\mu M^{p-q})\|u\|_{2_s^*}^{q-2}\|\omega_L\|_{2t}^2],$$

where $C > 0$ is a constant. From the definition of u_L, we have $u_L \leq u$ in $\mathbb{R}^3$. Letting $L \to +\infty$, using the Fatou's Lemma, one has

$$\|u\|_{\beta 2_s^*}^{2\beta} \leq C\beta^2[\|u\|_{2\beta}^{2\beta} + (1+\mu M^{p-q})\|u\|_{2_s^*}^{q-2}\|u\|_{2\beta t}^{2\beta}]. \tag{5.8}$$

By the interpolation inequality, we get $\|u\|_{2\beta} \leq \|u\|_2^{1-\sigma}\|u\|_{2\beta t}^{\sigma}$, where $\sigma \in (0,1)$ satisfies $\frac{1}{2\beta} = \frac{1-\sigma}{2} + \frac{\sigma}{2\beta t}$. Thus, $\sigma = \frac{t(\beta-1)}{t\beta-1}$, which shows that $\sigma \to 1$ as $\beta \to +\infty$. Since $2\beta(1-\sigma) = 2 + \frac{2(1-\beta)}{t\beta-1} < 2$, we get

$$\|u\|_{2\beta}^{2\beta} \leq \|u\|_2^{2\beta(1-\sigma)}\|u\|_{2\beta t}^{2\beta\sigma} \leq (1+\|u\|_2)^2\|u\|_{2\beta t}^{2\beta\sigma},$$

which together with (5.8) yields

$$\|u\|_{\beta 2_s^*} \leq C_1^{\frac{1}{2\beta}}\beta^{\frac{1}{\beta}}[1 + \|u\|_2^2 + (1+\mu M^{p-q})\|u\|_{2_s^*}^{q-2}]^{\frac{1}{2\beta}}\|u\|_{2\beta t}^{k}. \tag{5.9}$$

where $k \in \{\sigma, 1\}$ and $C_1 > 0$ is a constant. Let $\theta := \frac{2_s^*}{2t}$, then $\theta > 1$. Taking $\beta = \theta$ in (3.13), we deduce that

$$\|u\|_{\theta 2_s^*} \le C_1^{\frac{1}{2\theta}} \theta^{\frac{1}{\theta}} [1 + \|u\|_2^2 + (1 + \mu M^{p-q}) \|u\|_{2_s^*}^{q-2}]^{\frac{1}{2\theta}} \|u\|_{2_s^*}^{k_1}, \tag{5.10}$$

where $k_1 \in \{\sigma_1, 1\}$ and $\sigma_1 = \frac{t(\theta-1)}{t\theta-1}$. Taking $\beta = \theta^2$ in (5.9), we get

$$\|u\|_{\theta^2 2_s^*} \le C_1^{\frac{1}{2\theta^2}} \theta^{\frac{2}{\theta^2}} [1 + \|u\|_2^2 + (1 + \mu M^{p-q}) \|u\|_{2_s^*}^{q-2}]^{\frac{1}{2\theta^2}} \|u\|_{2_s^*\theta}^{k_2}, \tag{5.11}$$

where $k_2 \in \{\sigma_2, 1\}$ and $\sigma_2 = \frac{t(\theta^2-1)}{t\theta^2-1}$. Combining (5.10) with (5.11), we have

$$\|u\|_{\theta^2 2_s^*} \le C_1^{\frac{1}{2\theta}+\frac{1}{2\theta^2}} \theta^{\frac{1}{\theta}+\frac{2}{\theta^2}} [1 + \|u\|_2^2 + (1 + \mu M^{p-q}) \|u\|_{2_s^*}^{q-2}]^{\frac{1}{2\theta}+\frac{1}{2\theta^2}} \|u\|_{2_s^*}^{k_1 k_2}.$$

Taking $\beta = \theta^i, i \in \mathbb{N}$, one has

$$\|u\|_{\theta^i 2_s^*} \le C_1^{\sum_{m=1}^{i} \frac{1}{2\theta^m}} \theta^{\sum_{m=1}^{i} \frac{m}{\theta^m}} [1 + \|u\|_2^2 + (1 + \mu M^{p-q}) \|u\|_{2_s^*}^{q-2}]^{\sum_{m=1}^{i} \frac{1}{2\theta^m}} \|u\|_{2_s^*}^{k_1 k_2 \ldots k_i}. \tag{5.12}$$

where $k_i \in \{\sigma_i, 1\}$ and $\sigma_i = \frac{t(\theta^i-1)}{t\theta^i-1}$.

Next, we divide into two cases: $\|u\|_{2_s^*} \ge 1$ and $\|u\|_{2_s^*} < 1$.

(1) Assume that $\|u\|_{2_s^*} \ge 1$ is in force. In view of $k_1 k_2 \ldots k_i \le 1$, we have $\|u\|_{2_s^*}^{k_1 k_2 \ldots k_i} \le \|u\|_{2_s^*}$. Letting $i \to +\infty$ in (5.12), we can know that

$$\|u\|_\infty \le C_1^{\frac{1}{2(\theta-1)}} \theta^{\frac{\theta}{(\theta-1)^2}} [1 + \|u\|_2^2 + (1 + \mu M^{p-q}) \|u\|_{2_s^*}^{q-2}]^{\frac{1}{2(\theta-1)}} \|u\|_{2_s^*}.$$

(2) Assume that $\|u\|_{2_s^*} < 1$ is true. By $\sigma_i = \frac{t(\theta^i-1)}{t\theta^i-1} = 1 - \frac{t-1}{t\theta^i-1}$ and $k_i \in \sigma_i, 1$, we have $0 < \sigma_1 \sigma_2 \ldots \sigma_i \le k_1 k_2 \ldots k_i \le 1$, which shows that $\sum_{m=1}^{i} ln\sigma_m \le \sum_{m=1}^{i} lnk_m \le 0$. From the fact that $ln(1-s) \ge \frac{-s}{1-s}$ for all $s \in (0, 1)$, one has

$$\sum_{m=1}^{i} lnk_m \ge \sum_{m=1}^{i} ln\sigma_m = \sum_{m=1}^{i} ln(1 - \frac{t-1}{t\theta^m - 1}) \ge \frac{1-t}{t} \sum_{m=1}^{i} \frac{1}{\theta^m - 1} := A,$$

which implies that

$$k_1 k_2 \ldots k_i \ge e^A \text{ for all } i \in \mathbb{N}.$$

By $\|u\|_{2_s^*} < 1$, we have $\|u\|_{2_s^*}^{k_1 k_2 \ldots k_i} \le \|u\|_{2_s^*}^{e^A}$. Similarly, letting $i \to +\infty$ in (3.16), we reach

$$\|u\|_\infty \le C_1^{\frac{1}{2(\theta-1)}} \theta^{\frac{\theta}{(\theta-1)^2}} [1 + \|u\|_2^2 + (1 + \mu M^{p-q}) \|u\|_{2_s^*}^{q-2}]^{\frac{1}{2(\theta-1)}} \|u\|_{2_s^*}^{e^A}.$$

Consequently, we have $u \in L^\infty(\mathbb{R}^3)$ and

$$\|u\|_\infty \le C_1^{\frac{1}{2(\theta-1)}} \theta^{\frac{\theta}{(\theta-1)^2}} [1 + c^2 + (1 + \mu M^{p-q}) \|u\|_{2_s^*}^{q-2}]^{\frac{1}{2(\theta-1)}} \|u\|_{2_s^*}^{\tau}, \tag{5.13}$$

where $\tau = 1$ or $\tau = e^A \le 1$.

Finally, by (1.4) and Lemma 5.1, there exists $C_2 > 0$ such that $\|u\|_{2_s^*} \le C_2$. Therefore, it follows from (5.13) and $\theta = \frac{2_s^*-q+2}{2}$, there exists a constant $B_c > 0$ independent μ, $M > 0$ such that

$$\|u\|_\infty \le B_c(1 + \mu^{\frac{1}{2_s^*-q}} M^{\frac{p-q}{2_s^*-q}}).$$

□

Proof of the Theorem 1.1. By Lemma 5.2, for any $c > 0$, there exists a constant $B_c > 0$ independent on μ and M such that

$$\|u_c\|_\infty \le B_c(1 + \mu^{\frac{1}{2_s^*-q}} M^{\frac{p-q}{2_s^*-q}}).$$

Thus, for large $M > 0$, we can choose small $\mu^* > 0$ with $\mu^* \le \mu_1$ such that $\|u_c\|_\infty \le M$ for all $\mu \in (0, \mu^*]$. By Theorem 2.1, problem (1.1) has a couple of solutions $(u_c,\ \lambda_c) \in H_r^s(\mathbb{R}^3) \times \mathbb{R}$ for any $\mu \in (0, \mu^*]$. Moreover, u_c is a positive ground state, radially symmetric function and $\lambda_c < 0$. □

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Conflict of interest.

All authors declare no conflicts of interest in this paper.

References

1. D. Applebaum, *Lévy processes and stochastic calculus*, 2 Eds., Cambridge: Cambridge University Press, 2009. https://doi.org/10.1017/CBO9780511809781
2. D. Applebaum, Lévy processes-from probability to finance and quantum groups, *Notices of the AMS*, **51** (2004), 1336–1347.
3. R. Servadei, E. Valdinoci, Fractional Laplacian equations with critivcal Sobolev exponent, *Rev. Mat. Complut.*, **28** (2015), 655–676. https://doi.org/10.1007/s13163-015-0170-1
4. H. Luo, Z. Zhang, Normalized solutions to the fractional Schrödinger equations with combined nonlinearities, *Calc. Var.*, **59** (2020), 143. https://doi.org/10.1007/s00526-020-01814-5
5. H. Lu, X. Zhang, Positive solution for a class of nonlocal elliptic equations, *Appl. Math. Lett.*, **88** (2019), 125–131. https://doi.org/10.1016/j.aml.2018.08.019
6. Y. Liu, Bifurcation techniques for a class of boundary value problems of fractional impulsive differential equations, *J. Nonlinear Sci. Appl.*, **8** (2015), 340–353. http://doi.org/10.22436/jnsa.008.04.07
7. B. Yan, C. An, The sign-changing solutions for a class of nonlocal elliptic problem in an annulus, *Topol. Methods Nonlinear Anal.*, **55** (2020), 1–18. http://doi.org/10.12775/TMNA.2019.081

8. R. Frank, E. Lenzmann, L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian, *Commun. Pure Appl. Math.*, **69** (2016), 1671–1726. https://doi.org/10.1002/cpa.21591

9. F. Jin, B. Yan, The sign-changing solutions for nonlinear elliptic problem with Carrier type, *J. Math. Anal. Appl.*, **487** (2020), 124002. https://doi.org/10.1016/j.jmaa.2020.124002

10. M. Wang, X. Qu, H. Lu, Ground state sign-changing solutions for fractional Laplacian equations with critical nonlinearity, *AIMS Mathematics*, **6** (2021), 5028–5039. https://doi.org/10.3934/math.2021297

11. H. Lu, X. Qu, J. Wang, Sign-changing and constant-sign solutions for elliptic problems involving nonlocal integro-differential operators, *SN Partial Differ. Equ. Appl.*, **1** (2020), 33. https://doi.org/10.1007/s42985-020-00028-w

12. K. Teng, Multiple solutions for a class of fractional Schrödinger equations in $\mathbb{R}^N$, *Nonlinear Anal. Real*, **21** (2015), 76–86. https://doi.org/10.1016/j.nonrwa.2014.06.008

13. T. Bartsch, N. Soave, Multiple normalized solutions for a competing system of Schrödinger equations, *Calc. Var.*, **58** (2019), 22. https://doi.org/10.1007/s00526-018-1476-x

14. G. Gu, X. Tang, J. Shen, Multiple solutions for fractional Schrödinger-Poisson system with critical or supercritical nonlinearity, *Appl. Math. Lett.*, **111** (2021), 106605. https://doi.org/10.1016/j.aml.2020.106605

15. A. Mao, Z. Zhang, Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition, *Nonlinear Anal. Theor.*, **70** (2009), 1275–1287. https://doi.org/10.1016/j.na.2008.02.011

16. H. Ye, The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations, *Math. Method. Appl. Sci.*, **38** (2015), 2663–2679. https://doi.org/10.1002/mma.3247

17. M. Mu, H. Lu, Existence and multiplicity of positive solutions for Schrodinger-Kirchhoff-Poisson system with Singularity, *J. Funct. Space.*, **2017** (2017), 5985962. https://doi.org/10.1155/2017/5985962

18. L. Gao, C. Chen, C. Zhu, Existence of sign-changning solutions for Kirchhoff equations with critical or supercritical nonlinearity, *Appl. Math. Lett.*, **107** (2020), 106424. https://doi.org/10.1016/j.aml.2020.106424

19. P. L. Lions, Symétrie et compacité dans les espaces de Sobolev, *J. Funct. Anal.*, **3** (1982), 315–334. https://doi.org/10.1016/0022-1236(82)90072-6

20. B. Yan, D. Wang, The multiplicity of positive solutions for a class of nonlocal elliptic problem, *J. Math. Anal. Appl.*, **442** (2016), 72–102. https://doi.org/10.1016/j.jmaa.2016.04.023

21. Z. Zhang, K. Perera, Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow, *J. Math. Anal. Appl.*, **317** (2006), 456–463. https://doi.org/10.1016/j.jmaa.2005.06.102

22. Y. Wang, Y. Liu, Y. Cui, Multiple sign-changing solutions for nonlinear fractional Kirchhoff equations, *Bound. Value Probl.*, **2018** (2018), 193. https://doi.org/10.1186/s13661-018-1114-8

23. X. He, W. Zou, Ground state solutions for a class of fractional Kirchhoff equations with critical growth, *Sci. China Math.*, **62** (2019), 853–890. https://doi.org/10.1007/s11425-017-9399-6

24. L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal. Theor.*, **28** (1997), 1633–1659. https://doi.org/10.1016/S0362-546X(96)00021-1

25. G. Li, H. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, *J. Differ. Equations*, **257** (2014), 378–396. https://doi.org/10.1016/j.jde.2014.04.011

26. L. Liu, H. Chen, J. Yang, Normalized solutions to the fractional Kirchhoff equation with combined nonlinearities, 2021, arXiv:2104.06053v1.