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Research article

Best proximity point results for Prešić type nonself operators in b-metric spaces

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Abstract: The present work is about the existence of best proximity points for Prešić type nonself operators in b-metric spaces. In order to elaborate the results an example is presented. Moreover, some interesting formulations of Prešić fixed point results are also established. In addition a result in double controlled metric type space is also formulated.

Keywords: b-metric space; Best proximity point; equilibrium point; Prešić operator; double controlled metric type space
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1. Introduction

In 1922, Banach proved his famous result known as the Banach contraction principle, which is a simple and powerful result with a wide range of applications [12]. Many generalizations of Banach contraction principle can be seen in the literature, see e.g., [5, 9, 16, 17, 19, 21–24].

Consider the kth order nonlinear difference equation

$$x_n = f(x_{n-1}, x_{n-2}, \dots, x_{n-k}), \quad n = k, k+1, \dots \tag{1.1}$$

with initial values $x_0, \dots, x_{k-1} \in X$, where $k \geq 1$ is a positive integer and $f : X^k \to X$. This difference equation can be discussed with the perspective of fixed point theory by considering the fact that x^* is a fixed point of f if and only if it is the solution of (1.1) exist, that is,

$$x^* = f(x^*, x^*, \ldots, x^*).$$

The first step in this direction is taken by Preŝić in 1965 by establishing a generalization of the Banach contraction principle in the following manner:

Theorem 1.1. *[28] Let (X, d) be a complete metric space. Given $k \geq 1$ a positive integer and $f : X^k \to X$. Assume also that*

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots x_k)) \leq \sum_{i=0}^{k} \eta_i d(x_{i-1}, x_i), \ \forall \ x_0, \ldots, x_{k-1} \in X,$$

where $\eta_1, \eta_2, \ldots, \eta_k$ are positive constants such that $\sum_{i=1}^{k} \eta_i \in (0, 1)$. Then there exists a unique $x^* \in X$ such that $x^* = f(x^*, x^*, \ldots x^*)$, that is, f has a unique fixed point $x^* \in X$. Moreover, for any initial values $x_0, \ldots, x_{k-1} \in X$ the iterative sequence given by (6.1) *converges to x^*.*

Note that for $k = 1$, the map $f : X \to X$ becomes a self map and hence the above Theorem is the generalization of Banach contraction principle (for contractions defined on X^k). In [14], Theorem 1.1 is generalized by Ćirić and Preŝić in the following way:

Theorem 1.2. *[14] Let (X, d) be a complete metric space. Given $k \geq 1$ a positive integer and $f : X^k \to X$. Suppose that*

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots x_k)) \leq \mu \max\{d(x_0, x_1), \ldots, d(x_{k-1}, x_k)\}, \ \forall \ x_0, \ldots, x_{k-1} \in X,$$

where $\mu \in (0, 1)$ is a constant. Then there exists a unique $x^* \in X$ such that $x^* = f(x^*, x^*, \ldots x^*)$, that is, f has a unique fixed point $x^* \in X$. Moreover, for any initial values $x_0, \ldots, x_{k-1} \in X$, the iterative sequence given by (6.1) *converges to x^*.*

Note that a fixed point of the operator $f : X^k \to X$ can be considered as the equilibrium point of the kth order nonlinear difference Eq (6.1). Therefore, the above theorems can be taken as a tool to ensure the existence and uniqueness of the kth order nonlinear difference equation. Some other generalizations are obtained by Pâcurar in [13, 27]. Recently, Ali et al. [3] studied the existence of an approximate solution of the equation $x = f(x, x, \ldots, x)$, where $f : H^k \to K$. This equation has a solution if H and K have some common element, but has no solution otherwise. Hence in that case we can only get the approximate solution of the equation. The approximate solution of $x = f(x, x, \ldots, x)$, with the error term $d(H, K)$ is called a best proximity point of $f : H^k \to K$. The classical result of approximation theory given by Fan [18] is a great source of inspiration for various researchers in study of approximate solutions of $x = f(x)$. This result is given as follows:

Theorem 1.3. *Let H be a nonempty compact convex subset of a normed linear space X and $f : H \to X$ be a continuous function. Then there exists $x \in X$ such that*

$$\|x - f(x)\| = \inf_{a \in A} \{\|f(x) - a\|\}.$$

Recently Altun et al. [7, 8] investigated certain best proximity points results on KW-type nonlinear contractions and fractals. Furthermore Ali et al. [3] used the metric space (X, d) endowed with a graph and proved some best proximity results. These results are the generalizations of already existing results which are stated earlier.

Czerwik [15] gave a generalization of the famous Banach fixed point theorem in so-called b-metric spaces. For some important results on b-metric spaces, we refer the reader to [2, 4, 10, 11, 25, 26].

The purpose of present research is to extend the results of Ali et al. [3], in the setting of b-metric spaces equipped with an order. Hence, many results in literature become special cases of results presented in this article. Our paper also contains some examples for the validation of presented results and an application for further authentication.

2. Preliminaries

We include the following definitions before giving the main results.

Definition 2.1. [6] Consider a metric space (X, d). Suppose H and K are two non-empty subsets of X. An element $x \in H$ is said to be a best proximity point of the mapping $T : H \to K$ if

$$d(x, Tx) = d(H, K).$$

Remark 2.1. From the above definition, it is obvious that a best proximity point reduces to a fixed point for self-mappings.

Basha and Shahzad [29] have presented the following definition:

Definition 2.2. Consider a complete metric space (X, d). Suppose that H, K are non empty subsets of X. If each sequence $\{k_n\}$ in K with $d(h, k_n) \to d(h, K)$, for some $h \in H$, has a convergent subsequence. Then, K is called approximately compact with respect to H.

Ali et al. [3] introduced path admissible mappings as follows:

Definition 2.3. Suppose that H, K are nonempty subsets of a metric space (X, d) endowed with a binary relation $\mathcal{R}$. Then $T : H \times H \to K$ is said to be path admissible, if

$$\begin{cases} d(w_1, T(h_1, h_2)) = d(H, K), \\ d(w_2, T(h_2, h_3)) = d(H, K), \\ h_1 P h_3, \end{cases} \Rightarrow w_1 \mathcal{R} w_2$$

where $h_1, h_2, h_3, w_1, w_2 \in H$.

Here, by $w_1 \mathcal{R} w_2$ we mean that w_1 and w_2 are related with each other under the binary relation $\mathcal{R}$ and $h_1 P h_3$, we mean that for above mentioned $h_1, h_2, h_3 \in X$ we have $h_1 \mathcal{R} h_2$ and $h_2 \mathcal{R} h_3$.

Definition 2.4. Suppose H, K are non empty subsets of a metric space (X, d). An element $h_* \in H$ is said to be a best proximity point of $T : H \times H \to K$ if

$$d(h_*, T(h_*, h_*)) = d(H, K), \tag{2.1}$$

where

$$d(H, K) = \inf\{d(h, k) : h \in H, k \in K\}.$$

3. Main results

First, we recall some definitions which are used in the sequel. Let (X, d_b) be a b-metric space with coefficient $b \geq 1$. Suppose that H and K are two nonempty subsets of X, then define the following sets:

$$\begin{gathered} d_b(H,K) = \inf\{d_b(h,k) : h \in H, k \in K\}, \\ d_b(x_0,K) = \inf\{d_b(x_0,k) : k \in K\}, \\ H_0 = \{h \in H : d_b(h,k) = d_b(H,K) \text{ for some } k \in K\}, \\ K_0 = \{k \in K : d_b(h,k) = d_b(H,K) \text{ for some } h \in H\}. \end{gathered}$$

Definition 3.1. Consider a b-metric space (X, d_b) with coefficient $b \geq 1$. Suppose that H and K are nonempty subsets of X. The element $h_* \in H$ is said to be a best proximity point of the mapping $T : H \to K$ if

$$d_b(h_*, T(h_*)) = d_b(H,K). \tag{3.1}$$

Definition 3.2. Consider a b-metric space (X, d_b) with coefficient $b \geq 1$ and let H and K be two nonempty subsets of X. Then K is said to be approximately compact with respect to H, if each sequence $\{k_n\} \subseteq K$ with $d_b(h, k_n) \to d_b(h, K)$ for some $h \in H$, has a convergent subsequence.

Definition 3.3. Let (X, d_b) be a b-metric space with coefficient $b \geq 1$ and $\mathcal{R}$ is the binary relation on X. Suppose H, K are nonempty subsets of X. A mapping $T : H \times H \to K$ is called path admissible, whenever $\forall\ h_1, h_2, h_3, w_1, w_2 \in H$ we have

$$\begin{cases} d_b(w_1, T(h_1,h_2)) = d_b(H,K), \\ d_b(w_2, T(h_2,h_3)) = d_b(H,K), \\ h_1 P h_3, \end{cases} \Rightarrow w_1 \mathcal{R} w_2,$$

here, by $w_1 \mathcal{R} w_2$ mean that w_1 and w_2 are related with each other under the binary relation $\mathcal{R}$ and $h_1 P h_3$ we mean that for above mentioned $h_1, h_2, h_3 \in H$, we have $h_1 \mathcal{R} h_2$ and $h_2 \mathcal{R} h_3$.

Theorem 3.1. *Suppose that (X, d_b) is a complete b-metric space with coefficient $b \geq 1$ endowed with a binary relation $\mathcal{R}$, where d_b is a continuous functional. Assume that H and K are nonempty closed subsets of X. Consider a mapping $T : H \times H \to K$ such that for each $h_1, h_2, h_3, w_1, w_2 \in H$ with $h_1 P h_3$ that is $h_1 \mathcal{R} h_2$, $h_2 \mathcal{R} h_3$ and $d_b(w_1, T(h_1,h_2)) = d_b(H,K) = d_b(w_2, T(h_2,h_3))$, we have:*

$$d_b(w_1, w_2) \leq \Gamma \max\{d_b(h_1,h_2), d_b(h_2,h_3)\}, \tag{3.2}$$

where $\Gamma \in [0,1)$ such that $b\Gamma < 1$. Furthermore, suppose that the subsequent conditions are true:

(1) T is path admissible;
(2) $\exists\ h_0, h_1, h_2 \in H$ which satisfy $d_b(h_2, T(h_0,h_1)) = d_b(H,K)$ and $h_0 P h_2$;
(3) $T(H \times H_0) \subseteq K_0$;
(4) K is approximately compact with respect to H;
(5) If $\{h_j\} \subseteq X$ such that $h_j P h_{j+2}$ for each $j \in \mathbb{N}$ and $h_j \to x_*$ as $j \to \infty$, then $h_j \mathcal{R} x_*$ for all $j \in \mathbb{N}$ and $x_* \mathcal{R} x_*$.

Then T has a best proximity point.

Proof. Using condition (ii), we have $h_0, h_1, h_2 \in H$ satisfying

$$d_b(h_2, T(h_0, h_1) = d_b(H, K), \text{ and } h_0 P h_2,$$

that is, $h_0 \mathcal{R} h_1, h_1 \mathcal{R} h_2$. From condition ($iv$), $T(h_1, h_2) \in K_0$, and by the definition of K_0, we have $h_3 \in H$ which satisfies

$$d_b(h_3, T(h_1, h_2)) = d_b(H, K).$$

Due to condition (i), we have $h_2 \mathcal{R} h_3$. Hence, $h_1 P h_3$. By continuing same process, we build a sequence $\{h_j\}_{\geq 2} \subseteq H$ which satisfies

$$d_b(h_{j+1}, T(h_{j-1}, h_j)) = d_b(H, K) \text{ for each } j \in \mathbb{N}, \tag{3.3}$$

and $h_{j-1} P h_{j+1}$. That is, $h_{j-1} \mathcal{R} h_j, h_j \mathcal{R} h_{j+1}$ $\forall$ $j \in \mathbb{N}$. From (3.2), we have

$$d_b(h_j, h_{j+1}) \leq \Gamma \max\{d_b(h_{j-2}, h_{j-1}), d_b(h_{j-1}, h_j)\} \text{ for each } j = 2, 3, 4, \ldots. \tag{3.4}$$

For convenience, we take $c_j = d_b(h_j, h_{j+1})$ for each $j \in \mathbb{N} \cup \{0\}$. Then we can rewrite (3.4) as

$$c_j \leq \Gamma \max\{c_{j-2}, c_{j-1}\} \text{ for each } j = 2, 3, 4, \ldots.$$

By using induction, we can get $c_{n-1} \leq Z\psi^n$ where $\psi = \Gamma^{1/2}$. It is obviously true for $j = 0, 1$ by considering

$$Z = \max\{c_0/\psi, c_1/\psi^2\},$$

since Z is $\max\{c_0/\psi, c_1/\psi^2\}$, one writes

$$c_0 \leq Z\psi \quad \text{and} \quad c_1 \leq Z\psi^2.$$

We obtain

$$\begin{aligned} c_2 &\leq \Gamma \max\{c_0, c_1\} \leq \Gamma \max\{Z\psi, Z\psi^2\} \leq \Gamma Z\psi = Z\psi^3, \\ &\vdots \\ c_j &\leq \Gamma \max\{c_{j-1}, c_{j-2}\} \leq \Gamma \max\{Z\psi^j, Z\psi^{j-1}\} \leq \Gamma Z\psi^{j-1} \\ &= Z\psi^{j+1}. \end{aligned}$$

Therefore, we have

$$c_{j-1} \leq Z\psi^j \quad \forall \quad j \in \mathbb{N}.$$

Hence,

$$d_b(h_{j-1}, h_j) \leq Z\psi^j \quad \forall \quad j \in \mathbb{N}. \tag{3.5}$$

By using triangle inequality, we get

$$\begin{aligned}
d_b(h_j, h_{j+q}) &\le b\{d_b(h_j, h_{j+1}) + d_b(h_{j+1}, h_{j+q})\},\\
&= bd_b(h_j, h_{j+1}) + bd_b(h_{j+1}, h_{j+q}),\\
&\le bd_b(h_j, h_{j+1}) + bb\{d_b(h_{j+1}, h_{j+2}) + d_b(h_{j+2}, h_{j+q})\},\\
&= bd_b(h_j, h_{j+1}) + b^2\{d_b(h_{j+1}, h_{j+2}) + d_b(h_{j+2}, h_{j+q})\},\\
&= bd_b(h_j, h_{j+1}) + b^2 d_b(h_{j+1}, h_{j+2}) + b^2 d_b(h_{j+2}, h_{j+q}),\\
&\le bd_b(h_j, h_{j+1}) + b^2 d_b(h_{j+1}, h_{j+2}) + \cdots + b^q d_b(h_{j+q-1}, h_{j+q}),\\
&\le bZ\psi^{j+1} + b^2 Z\psi^{j+2} + b^3 Z\psi^{j+3} + \cdots + b^q Z\psi^{j+q},\\
&\le b\psi^{j+1}\{1 + b\psi + b^2\psi^2 + \cdots + b^{q-1}\psi^{q-1}\}Z,\\
&\le \frac{1-(b\psi)^q}{1-b\psi} Zb\psi^{j+1},\\
&< \frac{b\psi^{j+1}}{1-b\psi} Z.
\end{aligned}$$

Thus, $\{h_j\}$ is a Cauchy sequence in H, so there is an element $h_* \in H$ such that $h_j \to h_*$ and $h_j \in H_0$ which satisfies

$$d_b(H, K) = d_b(h_*, T(h_{j-1}, h_j)),$$

that is, $h_{j-1}\mathcal{R}h_*$.

Furthermore, we have to prove that $d_b(h_*, T(h_{j-1}, h_j)) \to d_b(h_*, K) \quad as \quad j \to \infty$. Consider,

$$\begin{aligned}
d_b(h_*, K) &\le d_b(h_*, T(h_{j-1}, h_j))\\
&= \lim_{n\to\infty} d_b(h_{j+1}, T(h_{j-1}, h_j))\\
&= d_b(H, K)\\
&\le d_b(h_*, K)
\end{aligned}$$

Therefore,

$$d_b(h_*, T(h_{j-1}, h_j)) \to d_b(h_*, K) \quad as \quad j \to \infty \tag{3.6}$$

Since T is approximately compact with respect to H, the sequence $\{T(h_{j-1}, h_j)\}$ has a subsequence $\{T(h_{j_{m-1}}, h_{j_m})\}$, which converges to a point $k_* \in K$. That is,

$$d_b(h_*, k_*) = \lim_{m\to\infty} d_b(h_{j_{m+1}}, T(h_{j_{m-1}}, h_{j_m}) = d_b(H, K).$$

Hence, $h_* \in H_0$. As we know $T(h_j, h_*) \in K_0$, we have $g \in H$ satisfying

$$d_b(g, T(h_j, h_*)) = d_b(H, K). \tag{3.7}$$

By assumption (vi), we have $h_j\mathcal{R}h_*$ for all $j \in \mathbb{N}$. Thus, we have

$$d_b(h_*, T(h_{j-1}, h_j)) = d_b(H, K), \quad \text{and} \quad d_b(g, T(h_j, h_*)) = d_b(H, K) \ \forall \ j \in \mathbb{N}.$$

Hence, we get $h_{j-1}Ph_*$. Also, $h_{j-1}\mathcal{R}h_j$, and $h_j\mathcal{R}h_*$ for all $j \in \mathbb{N}$. Hence, from (6.2),

$$d_b(h_{j+1}, g) \le \Gamma \max\{d_b(h_{j-1}, h_j), d_b(h_j, h_*)\} \text{ for each } j = 2, 3, 4, \ldots.$$

Taking $j \to \infty$, we obtain $d_b(h_*, g) = 0$, that is $g = h_*$. Putting $g = h_*$ in (3.7), we have

$$d_b(h_*, T(h_j, h_*)) = d_b(H, K).$$

That is, $h_* \mathcal{R} h_*$. Furthermore, we know that $T(h_*, h_*) \in K_0$, and we have an element $t \in H$ which satisfies

$$d_b(t, T(h_*, h_*)) = d_b(H, K). \tag{3.8}$$

Condition (vi) implies that $h_* \mathcal{R} h_*$. Hence,

$$d_b(t, T(h_*, h_*)) = d_b(H, K), \text{ and } d_b(h_*, T(h_j, h_*)) = d_b(H, K) \text{ for each } j \in \mathbb{N}.$$

Therefore, $h_j P h_*$ for each $j \in \mathbb{N}$, that is, $h_j \mathcal{R} h_*, h_* \mathcal{R} h_*$ for each $j \in \mathbb{N}$. Thus, from (3.2),

$$d_b(h_*, t) \leq \Gamma \max\{d_b(h_j, h_*), d_b(h_*, h_*)\} \text{ for each } j \in \mathbb{N},$$

Taking limit as $j \to \infty$, we have $d_b(h_*, t) = 0$, that is $t = h_*$. Putting $t = h_*$ in (3.8), we have

$$d_b(h_*, T(h_*, h_*)) = d_b(H, K).$$

□

Theorem 3.2. *Let H and K be nonempty subsets of a complete b-metric space (X, d_b) endowed with binary relation $\mathcal{R}$, where b-metric is a continuous functional. Consider a mapping $T : H \times H \to K$ such that for each $h_1, h_2, h_3, w_1, w_2 \in H$ with $h_1 P h_3$, that is, $h_1 \mathcal{R} h_2$, $h_2 \mathcal{R} h_3$ and $d_b(w_1, T(h_1, h_2)) = d_b(H, K) = d_b(w_2, T(h_2, h_3))$, we have*

$$d_b(h_3, w_2) \leq \Gamma \max\{d_b(h_1, h_2), d_b(h_2, w_1)\}, \tag{3.9}$$

where $\Gamma \in [0, 1)$ such that $b\Gamma < 1$.

Furthermore, suppose that the subsequent conditions are true

(1) T is path admissible;
(2) $\exists\, h_0, h_1, h_2 \in H$ which satisfy $d_b(h_2, T(h_0, h_1)) = d_b(H, K)$ and $h_0 P h_2$;
(3) $T(H \times H_0) \subseteq K_0$;
(4) K is approximately compact with respect to H;
(5) When $\{h_j\} \subseteq X$ such that $h_j P h_{j+2}$ for each $j \in \mathbb{N}$ and $h_j \to x_*$ as $n \to \infty$, then $h_j \mathcal{R} x_*$ for all $j \in \mathbb{N}$ and $x_* \mathcal{R} x_*$.

Then there exists a point $h_* \in H$ which satisfies

$$d_b(h_*, T(h_*, h_*)) = d_b(H, K),$$

that is, T has a best proximity point.

Proof. Proceeding as in Theorem 3.1, we obtain a sequence $\{h_j : j \in \mathbb{N} - 1\}$ in H_0 satisfying

$$d_b(h_{j+1}, T(h_{j-1}, h_j)) = d_b(H, K) \text{ for each } j \in \mathbb{N},$$

and $h_{j-1}Ph_{j+1}$, that is $h_{j-1}\mathcal{R}h_j, h_j\mathcal{R}h_{j+1}\ \forall\ j \in \mathbb{N}$.
From (3.9), we have

$$d_b(h_j, h_{j+1}) \leq \Gamma\ \max\{d_b(h_{j-2}, h_{j-1}), d_b(h_{j-1}, h_j)\} \text{ for each } j = 2, 3, 4, \ldots.$$

Following the proof of Theorem 3.1 and above inequality , $\{h_j\}$ is a Cauchy sequence in H such that $h_j \to h_*$ and $h_* \in H_0$. As $T(h_j, h_*) \in K_0$, we have $w \in H$ satisfying

$$d_b(w, T(h_j, h_*)) = d_b(H, K). \tag{3.10}$$

From assumption (vi), we get $h_j\mathcal{R}h_*$ for all $j \in \mathbb{N}$. We already have

$$d_b(h_*, T(h_{j-1}, h_j)) = d_b(H, K).$$

Thus, we get $h_{j-1}Ph_*$, that is $h_{j-1}\mathcal{R}h_j$ and $h_j\mathcal{R}h_*$ for all $j \in \mathbb{N}$. Hence, from (3.9), we get

$$d_b(h_*, w) \leq \Gamma \max\{d_b(h_{j-1}, h_j), d_b(h_j, h_{j+1})\} \text{ for each } j \in \mathbb{N}.$$

Taking limit as $j \to \infty$ in above inequality, we get $d_b(h_*, w) = 0$, that is, $h_* = w$. Using $w = h_*$ in (3.10),

$$d_b(h_*, T(h_j, h_*)) = d_b(H, K).$$

Further, note that $T(h_*, h_*) \in K_0$, and there is $q \in H$ which satisfies

$$d_b(q, T(h_*, h_*)) = d_b(H, K).$$

Hypothesis (vi) implies $h_*\mathcal{R}h_*$. Hence, we have

$$d_b(h_*, T(h_j, h_*)) = d_b(H, K), \quad \text{and} \quad d_b(q, T(h_*, h_*)) = d_b(H, K),$$

$$\text{and } h_jPh_*, \quad \text{that is } h_j\mathcal{R}h_* \text{ and } h_*\mathcal{R}h_*.$$

Thus, from (3.9),

$$d_b(h_*, q) \leq \Gamma \max\ \{d_b(h_j, h_*), d(h_*, h_*)\} \text{ for each } j \in \mathbb{N}.$$

Letting $j \to \infty$, we have $q = h_*$. Thus, we have

$$d_b(h_*, T(h_*, h_*)) = d_b(H, K).$$

□

Example 3.1. Consider $X = \mathbb{R}^2$ endowed with the b-metric given by

$$d_b((s_1, s_2), (c_1, c_2)) = |s_1 - c_1|^2 + |s_2 - c_2|^2 \text{ for each } s = (s_1, s_2), c = (c_1, c_2) \in \mathbb{R}^2.$$

Define a binary relation $\mathcal{R}$ on $\mathbb{R}^2$ as $s\mathcal{R}c$ if and only if $s_1 \leq c_1$ *and* $s_2 \leq c_2$. Take

$$H = \{(0, s) : s \in [-2, 2]\}, \quad \text{and} \quad K = \{(1, s) : s \in [-2, 2]\}.$$

Define

$$T : H \times H \to K,\ T((0, s), (0, c)) = (1, c)\ \forall\ (0, s), (0, c) \in H.$$

Let $\overline{h_1} = (0, h_1)$, $\overline{h_2} = (0, h_2)$, $\overline{h_3} = (0, h_3) \in [-2, 2]$. To find w_1 and w_2, we have

$$d_b(\overline{w_1}, T(\overline{h_1}, \overline{h_2})) = d_b(H, K) = d_b(\overline{w_2}, T(\overline{h_2}, \overline{h_3})). \tag{3.11}$$

For this, consider

$$\begin{aligned} d_b(H, K) &= \inf\{d_b(\overline{h}, \overline{k}) : \overline{h} \in H, \overline{k} \in K\}, \\ &= \inf\{d_b((0, s), (1, s)) : \text{ where } s \in [-2, 2]\}, \\ &= \inf\{\mid 0 - 1 \mid^2 + \mid s - s \mid^2 : \text{ where } s \in [-2, 2]\}, \\ &= 1. \end{aligned}$$

That is,

$$d_b(H, K) = 1. \tag{3.12}$$

$$\begin{aligned} d_b(\overline{w_1}, T(\overline{h_1}, \overline{h_2}) &= d_b((0, w_1), T((0, h_1), (0, h_2))), \\ &= d_b((0, w_1), (1, h_2)), \\ &= \mid 0 - 1 \mid^2 + \mid w_1 - h_2 \mid^2, \\ &= 1 + (w_1 - h_2)^2. \end{aligned}$$

Then

$$d_b(\overline{w_1}, T(\overline{h_1}, \overline{h_2})) = 1 + (w_1 - h_2)^2. \tag{3.13}$$

Using (3.12) and (3.13) in (3.11), we obtain

$$1 = 1 + (w_1 - h_2)^2.$$

That is,

$$w_1 = h_2.$$

Similarly,

$$\begin{aligned} d_b(\overline{w_2}, T(\overline{h_2}, \overline{h_3})) &= d_b((0, w_2), T((0, h_2), (0, h_3))), \\ &= d_b((0, w_2), (1, h_3)), \\ &= \mid 0 - 1 \mid^2 + \mid w_2 - h_3 \mid^2, \\ &= 1 + (w_2 - h_3)^2. \end{aligned}$$

From (3.11), we obtain

$$w_2 = h_3.$$

$$\overline{w_1} = (0, w_1) = (0, h_2), \quad \overline{w_2} = (0, w_2) = (0, h_3).$$

Thus, $\overline{h_1}, \overline{h_2}, \overline{h_3}, \overline{w_1}, \overline{w_2} \in H$, with $\overline{h_1} P \overline{h_3}$.
Also, we have

$$d_b(\overline{h_3}, \overline{w_2}) \leq \Gamma \max\{d_b(\overline{h}_1, \overline{h}_2), d(\overline{h}_2, \overline{w}_1)\}, \tag{3.14}$$

where

$$d_b(\overline{h}_3, \overline{w}_2) = d_b((0, h_3), (0, w_2)),$$

$$
\begin{aligned}
&=\mid 0-0 \mid + \mid h_3 - w_2 \mid^2,\\
&=\mid h_3 - h_3 \mid^2,\\
&= 0.
\end{aligned}
$$

Using above equation in (3.14), we get

$$d_b(\overline{h_3}, \overline{w_2}) = \ 0 \ = \psi \max\{d_b(\overline{h}_1, \overline{h}_2), d_b(\overline{h}_2, \overline{w}_1)\}.$$

Here, we say $\psi = \Gamma^{\frac{1}{2}} = \frac{1}{2} \in [0, 1)$. Now, we will prove condition (i) of Theorem 3.2. Consider

$$\overline{h_1} = (0, h_1), \ \ \overline{h_2} = (0, h_2), \ \ \overline{h_3} = (0, h_3) \in H \quad \text{such that} \quad \overline{h_1}P\overline{h_3}.$$

Since $\overline{w}_1 = (0, w_1) = (0, h_2)$ and $w_2 = (0, w_2) = (0, h_3)$, we now prove

$$d_b((0, w_1), T((0, h_1), (0, h_2))) = d_b(H, K) \ \text{ and } \ d_b(H, K) = d_b((0, w_2), T((0, h_2), (0, h_3))),$$

$$
\begin{aligned}
d(\overline{w_1}, T(\overline{h_1}, \overline{h_2})) &= d_b((0, w_1), T((0, h_1), (0, h_2))),\\
&= d_b((0, h_2), (1, h_2)),\\
&=\mid 0-1 \mid^2 + \mid h_2 - h_2 \mid^2,\\
&= 1 = d_b(H, K).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
d_b(\overline{w_2}, T(\overline{h_2}, \overline{h_3})) &= d_b((0, w_2), T((0, h_2), (0, h_3))),\\
&= d_b((0, h_2), (1, h_3)),\\
&=\mid 0-1 \mid^2 + \mid h_3 - h_3 \mid^2,\\
&= 1 = d_b(H, K).
\end{aligned}
$$

This implies that $\overline{w_1}\ \mathcal{R}\ \overline{w_2}$. Thus, T is path admissible. Now, we will prove condition (ii):

$$d_b(h_2, T(h_0, h_1)) = d_b(H, K), \quad \text{and} \quad h_0 P h_2.$$

We need to consider

$$\overline{h_1} = (0, 0), \ \ \overline{h_2} = (0, \frac{1}{2}), \ \ \overline{h_3} = (0, \frac{5}{8}) \in H,$$

such that

$$
\begin{aligned}
d_b((0, \frac{5}{8}), T((0, 0), (0, \frac{1}{2}))) &= d_b((0, \frac{5}{8}) - (1, \frac{0 + \frac{1}{2} + 2}{4})),\\
&= \mid (0-1 \mid^2 + \mid \frac{5}{8} - \frac{5}{8} \mid^2,\\
&= 1,\\
&= d_b(H, K),
\end{aligned}
$$

and $(0, 0)P(0, \frac{5}{8})$. Moreover, assumption (v) holds, that is, $h_j P h_{j+2}$ for all $j \in \mathbb{N}$, and $h_j \to a$ as $j \to \infty$, then $h_j \mathcal{R} a$ for each $j \in \mathbb{N}$ and $a\mathcal{R}a$. Therefore, all axioms are true. Hence, T has a best proximity point.

Theorem 3.3. *Let H and K be nonempty closed subsets of a complete b-metric space (X, d_b) with coefficient $b \geq 1$ endowed with a binary relation $\mathcal{R}$, where the b-metric is a continuous functional. Consider a mapping $T : H \times H \to K$ such that for each $h_1, h_2, h_3, w_1, w_2 \in H$ with $h_1 P h_3$, that is, $h_1 \mathcal{R} h_2, h_2 \mathcal{R} h_3$, and*
$d_b(w_1, T(h_1, h_2)) = d_b(H, K) = d_b(w_2, T(h_2, h_3))$, we have

$$d_b(T(h_2, w_1), T(h_3, w_2)) \leq \Gamma\{d_b(T(h_1, h_2), T(h_2, h_3))\}, \tag{3.15}$$

where $\Gamma \in [0, 1)$ such that $b\Gamma < 1$.

Furthermore, suppose that the subsequent conditions are true

(1) T is path admissible;
(2) $\exists\, h_0, h_1, h_2 \in H$ which satisfy $d_b(h_2, T(h_0, h_1)) = d_b(H, K)$ and $h_0 P h_2$;
(3) $T(H \times H_0) \subseteq K_0$;
(4) K is approximately compact with respect to H;
(5) if $\{h_j\}$ and $\{\overline{h_j}\}$ are in X such that $h_j \to h$ and $\overline{h_j} \to \overline{h}$, then $T(h_j, \overline{h_j}) \to T(h, \overline{h})$.

Then there exists a point $h_* \in H$ so that

$$d_b(h_*, T(h_*, h_*)) = d_b(H, K),$$

that is, T has a best proximity point.

Proof. By using a similar argument as in Theorem 3.1, we build a sequence $\{h_{j\geq 2}\} \subseteq H$ which satisfies

$$d_b(h_{j+1}, T(h_{j-1}, h_j)) = d_b(H, K) \;\; \forall \;\; j \in \mathbb{N},$$

and $h_{j-1} P h_{j+1}$, that is, $h_{j-1} \mathcal{R} h_j, h_j \mathcal{R} h_{j+1} \;\; \forall \;\; j \in \mathbb{N}$.

From (3.15), we have

$$d_b(T(h_{j-1}, h_j), T(h_j, h_{j+1})) \;\leq \Gamma \;\max\{d_b(T(h_{j-2}, h_{j-1})), d_b(T(h_{j-1}, h_j))\} \text{ for each } j = 2, 3, 4, \ldots.$$

Inductively, we get

$$d_b(T(h_{j-1}, h_j), T(h_j, h_{j+1})) \;\leq \Gamma^{j-1} \;\max\{d_b(T(h_0, h_1), d_b(h_1, h_2))\}.$$

By using triangle inequality and above inequality for each $j \in \mathbb{N}$, we have

$$d_b(T(h_j, h_{j+1}), T(h_{j+1}, h_{j+p})) \;\leq\; \sum_{i=j}^{j+p-1} d_b(T(h_i, h_{i+1}), T(h_{i+1}, h_{i+2})).$$

This proves that $T(h_{j-1}, h_j)$ is a Cauchy sequence in the closed subset K. Since X is complete, there exists $k_* \in K$ such that $T(h_{j-1}, h_j) \to k_*$.

Moreover,

$$\begin{aligned} d_b(h_*, K) &\leq d_b(h_*, T(h_{j-1}, h_j)) \\ &= \lim_{n\to\infty} d_b(h_{j+1}, T(h_{j-1}, h_j)) \end{aligned}$$

$$= d_b(H, K)$$
$$\leq d_b(h_*, K).$$

Therefore, $d_b(k_*, h_{j+1}) \rightarrow d_b(k_*, H)$ as $j \rightarrow \infty$.

Using hypothesis (v), $\{h_j\}$ has a subsequence $\{h_{j_l}\}$ that converges to an element $h_* \subseteq H$ such that

$$d_b(h_*, T(h_*, h_*)) = \lim_{l\to\infty} d_b(h_{j_{l+1}}, T(h_{j_{l-1}}, h_{j_l})) = d_b(H, K).$$

Then

$$d_b(h_*, T(h_*, h_*)) = d_b(H, K).$$

□

Theorem 3.4. *Let H and K be closed nonempty subsets of a complete b-metric space (X, d_b) endowed with a binary relation $\mathcal{R}$ where the b-metric is a continuous functional. Consider a mapping T : $H \times H \rightarrow K$ such that for each $h_1, h_2, h_3, w_1, w_2 \in H$ with $h_1 P h_3$, that is, $h_1 \mathcal{R} h_2$ and $h_2 \mathcal{R} h_3$, and $d_b(w_1, T(h_1, h_2)) = d_b(H, K) = d_b(w_2, T(h_2, h_3))$, we have*

$$d_b(T(h_2, w_1), T(h_3, w_2)) \leq \Gamma \max\{d_b(T(h_1, h_2), T(h_2, h_3)), d_b(T(h_2, h_3), T(w_1, w_2))\}, \tag{3.16}$$

where $\Gamma \in [0, 1)$ such that $b\Gamma < 1$.

Furthermore, suppose that the subsequent conditions are true:

(1) T is path admissible;
(2) $\exists\, h_0, h_1, h_2 \in H$ which satisfy $d_b(h_2, T(h_0, h_1)) = d_b(H, K)$ and $h_0 P h_2$;
(3) $T(H \times H_0) \subseteq K_0$;
(4) K is approximately compact with respect to H;
(5) if $\{h_j\}$, $\{\overline{h_j}\}$ in X such that $h_j \rightarrow h$ and $\overline{h_j} \rightarrow \overline{h}$, then $T(h_j, \overline{h_j}) \rightarrow T(h, \overline{h})$.

Then there exists a point $h_* \in H$ which satisfies

$$d(h_*, T(h_*, h_*)) = d(H, K),$$

that is, T has a best proximity point.

Proof. Using the assumptions, we can build a sequence $\{h_j\}_{\geq 2}$ in H_0 which satisfies

$$d_b(h_{j+1}, T(h_{j-1}, h_j)) = d_b(H, K) \ \ \forall \ \ j \in \mathbb{N}, \tag{3.17}$$

and $h_{j-1} P h_{j+1}$, that is, $h_{j-1} \mathcal{R} h_j$ and $h_j \mathcal{R} h_{j+1}$ for all $j \in \mathbb{N}$. From (3.16), we have

$$\begin{aligned} d_b(T(h_{j-1}, h_j), T(h_j, h_{j+1})) &\leq \Gamma \max\{d_b(T(h_{j-2}, h_{j-1})), T(h_{j-1}, h_j)),\\ &\quad d_b(T(h_{j-1}, h_j), T(h_j, h_{j+1}))\}\\ &= d_b(T(h_{j-2}, h_{j-1})), T(h_{j-1}, h_j)) \text{ for each } j = 2, 3, \ldots. \end{aligned}$$

Therefore, we have

$$d_b(T_{j-1}, T(h_j) \ \leq \ \Gamma d_b(T_{j-2}, T_{j-1}) \ \text{ for each } \ j = 2, 3, 4, \ldots.$$

By using induction, we get

$$\begin{aligned} d_b(T_{j-1}, T_j) &\leq \Gamma\ d_b(T_{j-2}, T_{j-1}), \\ &\leq \Gamma(\Gamma\ d_b(T_{j-3}, T_{j-2})), \\ &= \Gamma^2 d_b(T_{j-3}, T_{j-2}), \\ &\leq \Gamma^2 \Gamma d_b(T_{j-4}, T_{j-3}), \\ &= \Gamma^3 d_b(T_{j-4}, T_{j-3}), \\ &\ \vdots \\ &\leq \Gamma^{j-1} d_b(T_1, T_0) \ \text{ for } \ j = 2, 3, 4, \ldots. \end{aligned}$$

Hence,

$$d_b(T_j, T_{j+1}) \leq \Gamma^j d_b(T_0, T_1) \quad \text{for} \quad j = 1, 2, 3, \ldots. \tag{3.18}$$

By using triangle inequality,

$$\begin{aligned} d_b(T_j, T_{j+p}) &\leq b\{d_b(T_j, T_{j+1}) + d_b(T_{j+1}, T_{j+p})\}, \\ &= bd_b(T_j, T_{j+1}) + bd_b(T_{j+1}, T_{j+p}), \\ &\leq bd_b(T_j, T_{j+1}) + bb\{d_b(T_{j+1}, T_{j+2}) + d_b(T_{j+2}T_{j+p})\}, \\ &= bd_b(T_j, T_{j+1}) + b^2 d_b(T_{j+1}, T_{j+2}) + b^2 d_b(T_{j+2}T_{j+p}), \\ &\leq bd_b(T_j, T_{j+1}) + b^2 d_b(T_{j+1}, T_{j+2}) + \cdots + d_b(T_{j+p-1}T_{j+p}). \end{aligned} \tag{3.19}$$

By using (3.18) in (3.19), we get

$$\begin{aligned} d_b(T_j, T_{j+p}) \leq & b\Gamma^j d_b(T_0, T_1) + b^2 \Gamma^{j+1} d_b(T_0, T_1) + b^3 \Gamma^{j+2} \\ & d_b(T_0, T_1) + \cdots + b^p \Gamma^{j+p-1} d_b(T_0, T_1), \\ =& b\Gamma^j d_b(T_0, T_1)(1 + b\Gamma + b^2\Gamma^2 + \cdots + b^{p-1}\Gamma^{p-1}), \\ \leq & b\Gamma^{j+1} d_b(T_0, T_1) \frac{1 - (b\Gamma)^p}{1 - \Gamma}, \\ <& b\Gamma^{j+1} d_b(T_0, T_1) \frac{1}{1 - \Gamma}. \end{aligned}$$

Letting $j \to \infty$ in above inequality, we have

$$\lim_{j\to\infty} d_b(T(h_j, T_{j+1}), T(h_{j+p}, h_{j+p+1}) \leq 0.$$

That is,

$$\lim_{j\to\infty} d_b(T(h_j, h_{j+1}), T(h_{j+p}, h_{j+p+1}) = 0.$$

We get a Cauchy sequence $T(h_{j-1}, h_j)$ in the closed subset K. Since X is complete, consider $k_* \in K$ such that $T(h_{j-1}, h_j) \to k_*$. Moreover, Consider,

$$\begin{aligned} d_b(h_*, K) &\leq d_b(h_*, T(h_{j-1}, h_j)) \\ &= \lim_{n\to\infty} d_b(h_{j+1}, T(h_{j-1}, h_j)) \end{aligned}$$

$$= d_b(H,K)$$
$$\leq d_b(h_*,K).$$

Therefore, $d_b(k_*, h_{j+1}) \rightarrow d_b(k_*, H)$ as $j \to \infty$.

Condition (iv) implies, $\{h_j\}$ has a subsequence $\{h_{j_l}\}$ that converges to an element $h_* \in H$ such that

$$d_b(h_*, T(h_*, h_*)) = \lim_{l\to\infty} d_b(h_{j_{l+1}}, T(h_{j_{l-1}}, h_{j_l})) = d_b(H,K).$$

Hence,

$$d_b(h_*, T(h_*, h_*)) = d_b(H,K).$$

□

Theorem 3.5. *Consider a complete b-metric space* (X, d_b) *with a coefficient* $b \geq 1$ *endowed with a binary relation* $\mathcal{R}$*, where b-metric is continuous. Suppose that H and K are non empty closed subsets of X. Consider a mapping* $T : H \times H \to K$ *such that for each* $h_1, h_2, h_3, w_1, w_2 \in H$ *with* $h_1 P h_3$ *such that* $h_1 \mathcal{R} h_2$*, and* $h_2 \mathcal{R} h_3$*, and* $d_b(w_1, T(h_1, h_2)) = d_b(H,K) = d_b(w_2, T(h_2, h_3))$*, we have*

$$d_b(T(h_2,h_3), T(w_1,w_2)) \leq \Gamma \max\{d_b(T(h_1,h_2), T(h_2,h_3)), d_b(T(h_2,w_1), T(h_3,w_2))\},$$

where $\Gamma \in [0,1)$ *with* $b\Gamma < 1$.

Furthermore, suppose that the subsequent conditions are true:

(1) T is path admissible;
(2) There exist $h_0, h_1, h_2 \in H$ *which satisfy* $d_b(h_2, T(h_0, h_1)) = d_b(H,K)$ *and* $h_0 P h_2$*;*
(3) $T(H \times H_0) \subseteq K_0$*;*
(4) K is approximately compact with respect to H;
(5) When $\{h_j\}$, $\{\overline{h_j}\} \subseteq X$ *such that* $h_j \to h$ *and* $\overline{h_j} \to \overline{h}$*, then* $T(h_j, \overline{h_j}) \to T(h, \overline{h})$*.*

Then there exists a point $h_* \in H$ *which satisfies*

$$d(h_*, T(h_*, h_*)) = d(H,K),$$

that is, T has a best proximity point.

Proof. This theorem can be proved by using similar argument as in Theorem 3.4. □

4. On metric spaces endowed with a graph

In order to generalize the idea of partial ordering in metric spaces and partially ordered metric spaces, Jachymski [20] in 2008 has introduced the idea of a metric space endowed with a graph. This section is about a consequence of our results in the setting of metric spaces endowed with a graph.

Theorem 4.1. *Let* (X, d_b) *be a complete b-metric space endowed with a graph G, where* d_b *is a continuous functional. Suppose that H and K are non empty closed subsets of X. Consider a mapping* $T : H \times H \to K$ *such that for each* $h_1, h_2, h_3, w_1, w_2 \in H$ *with* $h_1 P h_3$*, that is,* $h_1 \mathcal{R} h_2$*,* $h_2 \mathcal{R} h_3$ *and* $d_b(w_1, T(h_1, h_2)) = d_b(H,K) = d_b(w_2, T(h_2, h_3))$*, we have either*

$$d_b(w_1, w_2) \leq \Gamma \max\{d_b(h_1,h_2), d_b(h_2,h_3)\},$$

or

$$d_b(h_3, w_2) \le \Gamma \max\{d_b(h_1, h_2), d_b(h_2, w_1)\},$$

where $\Gamma \in [0, 1)$ such that $b\Gamma < 1$. Furthermore, assume that all the conditions of Theorem 3.1 are satisfied. Then T has a best proximity point.

Proof. It follows by using the same procedure as in Theorems 3.1 and 3.2. □

Theorem 4.2. *Let H and K be closed nonempty subsets of a complete b-metric space (X, d_b) endowed with a graph $G = (V(G), E)$ where the b-metric is a continuous functional. Consider a mapping $T : H \times H \to K$ such that for each $h_1, h_2, h_3, w_1, w_2 \in H$ with $h_1 P h_3$, that is, $(h_1, h_2) \in E$ and $(h_2, h_3) \in E$, and $d_b(w_1, T(h_1, h_2)) = d_b(H, K) = d_b(w_2, T(h_2, h_3))$, we have either*

$$d_b(T(h_2, w_1), T(h_3, w_2)) \le \Gamma\{d_b(T(h_1, h_2), T(h_2, h_3))\},$$

or

$$d_b(T(h_2, w_1), T(h_3, w_2)) \le \Gamma \max\{d_b(T(h_1, h_2), T(h_2, h_3)), d_b(T(h_2, h_3), T(w_1, w_2))\},$$

or

$$d_b(T(h_2, h_3), T(w_1, w_2)) \le \Gamma \max\{d_b(T(h_1, h_2), T(h_2, h_3)), d_b(T(h_2, w_1), T(h_3, w_2))\},$$

where $\Gamma \in [0, 1)$ such that $b\Gamma < 1$. Furthermore, assume that all the conditions of Theorem 3.3 are satisfied. Then T has a best proximity point.

Proof. It follows by using the same arguments given in Theorem 3.3, Theorem 3.4 and refthm5a. □

5. Application

Taking $A = B = X$ in Theorems 4.1 and 4.2, we obtain the following results, which guarantee the existence of a fixed point of the mapping $T : X \times X \to X$.

Theorem 5.1. *Let (X, d_b) be a complete b-metric space endowed with a graph G, where d_b is a continuous functional. Let $T : X \times X \to X$ be a mapping such that for each $h_1, h_2, h_3, w_1, w_2 \in X$ with $h_1 P h_3$ that is (h_1, h_2), $(h_2, h_3) \in E$ satisfies one of the following inequalities*

$$d_b(w_1, w_2) \le \Gamma \max\{d_b(h_1, h_2), d_b(h_2, h_3)\},$$

or

$$d_b(h_3, w_2) \le \Gamma \max\{d_b(h_1, h_2), d_b(h_2, w_1)\},$$

where $\Gamma \in [0, 1)$ such that $b\Gamma < 1$. Furthermore, assume that the following conditions are satisfied:

(1) T is path admissible;
(2) There exist $a_0, a_1, a_3 \in X$ with $a_3 = T(a_0, a_1)$ and $a_0 P a_3$;
(3) If $\{h_j\} \subseteq X$ such that $h_j P h_{j+2}$ for each $j \in \mathbb{N}$ and $h_j \to x_*$ as $j \to \infty$, then $(h_j, x_*) \in E$ for all $j \in \mathbb{N}$ and $(x_*, x_*) \in E$.

Then T has a fixed point in X.

Theorem 5.2. *Let (X, d_b) be a complete b-metric space endowed with a graph G, where d_b is a continuous functional. Let $T : X \times X \to X$ be a mapping such that for each $h_1, h_2, h_3, w_1, w_2 \in X$ with $h_1 P h_3$ that is (h_1, h_2), $(h_2, h_3) \in E$ satisfies one of the following inequalities:*

$$d_b(T(h_2, w_1), T(h_3, w_2)) \leq \Gamma\{d_b(T(h_1, h_2), T(h_2, h_3))\},$$

$$d_b(T(h_2, w_1), T(h_3, w_2)) \leq \Gamma \max\{d_b(T(h_1, h_2), T(h_2, h_3)), d_b(T(h_2, h_3), T(w_1, w_2))\},$$

$$d_b(T(h_2, h_3), T(w_1, w_2)) \leq \Gamma \max\{d_b(T(h_1, h_2), T(h_2, h_3)), d_b(T(h_2, w_1), T(h_3, w_2))\},$$

where $\Gamma \in [0, 1)$ such that $b\Gamma < 1$. Furthermore, assume that the following conditions are satisfied:

(1) T is path admissible;
(2) There exist $a_0, a_1, a_3 \in X$ with $a_3 = T(a_0, a_1)$ and $a_0 P a_3$;
(3) T is continuous with respect to each coordinate.

Then, T has a fixed point in X.

Suppose that $G = (V, E)$ where $V = X$ and $E = X \times X$, then Theorems 5.1 and 5.2 give rise to the following corollaries, respectively.

Corollary 5.1. Let (X, d_b) be a complete b-metric space, where d_b is a continuous functional and consider $T : X \times X \to X$ a mapping such that for each $h_1, h_2, h_3, w_1, w_2 \in X$, we have either

$$d_b(w_1, w_2) \leq \Gamma \max\{d_b(h_1, h_2), d_b(h_2, h_3)\},$$

or

$$d_b(h_3, w_2) \leq \Gamma \max\{d_b(h_1, h_2), d_b(h_2, w_1)\},$$

where $\Gamma \in [0, 1)$ such that $b\Gamma < 1$. Then T has a fixed point.

Corollary 5.2. Let (X, d_b) be a complete b-metric space, where d_b is a continuous functional and consider $T : X \times X \to X$ as a mapping such that for each $h_1, h_2, h_3, w_1, w_2 \in X$ we have either

$$d_b(T(h_2, w_1), T(h_3, w_2)) \leq \Gamma\{d_b(T(h_1, h_2), T(h_2, h_3))\},$$

or

$$d_b(T(h_2, w_1), T(h_3, w_2)) \leq \Gamma \max\{d_b(T(h_1, h_2), T(h_2, h_3)), d_b(T(h_2, h_3), T(w_1, w_2))\},$$

or

$$d_b(T(h_2, h_3), T(w_1, w_2)) \leq \Gamma \max\{d_b(T(h_1, h_2), T(h_2, h_3)), d_b(T(h_2, w_1), T(h_3, w_2))\},$$

where $\Gamma \in [0, 1)$ such that $b\Gamma < 1$. Then T has a fixed point.

6. Doubled controlled metric space

In this section Theorem 3.1 has been proved in the setting of double controlled metric spaces introduced by Abdeljawad et al. [1]. In [1] the notion of doubled controlled metric type space is given as follows.

Definition 6.1. Given non comparable functions $\alpha, \mu : X \times X \to [1, \infty)$. If $d_b : X \times X \to (0, \infty)$ satisfies

(1) $d_b(w_1, w_2) = 0 \Longleftrightarrow w_1 = w_2$
(2) $d_b(w_1, w_2) = d_b(w_2, w_1)$
(3) $d_b(w_1, w_3) \leq \alpha(w_1, w_2) d_b(w_1, w_2) + \mu(w_2, w_3) d_b(w_2, w_3)$ for all $w_1, w_2, w_3 \in X$.
Then (X, d_b) is called a double controlled metric type by α and μ.

Remark 6.1. The class of double controlled metric is larger than b-metric. If $\alpha(w) = \mu(w) = b \geq 1$ for all $w \in X$ then, double controlled metric type is a b-metric with coefficient b.

The notion of convergence, Cauchyness and completeness can be extended naturally in the setting of double controlled metric type space as in [1] .

Theorem 6.1. *Suppose that (X, d_b) be a complete double controlled metric type space by the functions $\alpha, \mu : X \times X \to [1, \infty)$ such that*

$$\sup_{m>1} \lim_{i\to\infty} \frac{\alpha(w_{i+1}, w_{i+2})}{\alpha(w_i, w_{i+1})} \mu(w_i, w_m) < \frac{1}{\Gamma^{\frac{1}{2}}} \tag{6.1}$$

and $\lim_{n\to\infty} \alpha(u, u_n)$ and $\lim_{n\to\infty} \mu(u, u_n)$ exist and are finite. Let $\mathcal{R}$ be a binary relation on X, where d_b is a continuous functional. Assume that H and K are nonempty closed subsets of X. Consider a mapping $T : H \times H \to K$ such that for each $h_1, h_2, h_3, w_1, w_2 \in H$ with $h_1 P h_3$ that is $h_1 \mathcal{R} h_2$, $h_2 \mathcal{R} h_3$ and $d_b(w_1, T(h_1, h_2)) = d_b(H, K) = d_b(w_2, T(h_2, h_3))$, we have:

$$d_b(w_1, w_2) \leq \Gamma \max \{d_b(h_1, h_2), d_b(h_2, h_3)\}, \tag{6.2}$$

where $\Gamma \in [0, 1)$ such that $b\Gamma < 1$. Furthermore, suppose that the subsequent conditions are true:

(1) T is path admissible;
(2) $\exists\, h_0, h_1, h_2 \in H$ which satisfy $d_b(h_2, T(h_0, h_1)) = d_b(H, K)$ and $h_0 P h_2$;
(3) $T(H \times H_0) \subseteq K_0$;
(4) K is approximately compact with respect to H;
(5) If $\{h_j\} \subseteq X$ such that $h_j P h_{j+2}$ for each $j \in \mathbb{N}$ and $h_j \to x_*$ as $j \to \infty$, then $h_j \mathcal{R} x_*$ for all $j \in \mathbb{N}$ and $x_* \mathcal{R} x_*$.

Then T has a best proximity point.

Proof. Proceeding as in Theorem 3.1 till Eq (3.5) we obtain
$d_b(h_{j-1}, h_j) \ \leq \ Z\psi^j \quad \forall \quad j \in \mathbb{N}$. Now for $m > n$

$$\begin{aligned} d_b(h_n, h_m) &\leq \alpha(h_n, h_{n+1}) d_b(h_n, h_{n+1}) + \mu(h_{n+1}, h_m) d_b(h_{n+1}, h_m) \\ &= \alpha(h_n, h_{n+1}) d_b(h_n, h_{n+1}) + \mu(h_{n+1}, h_m) d_b(h_{n+1}, h_m) \end{aligned}$$

$$
\begin{aligned}
&\leq \alpha(h_n, h_{n+1})d_b(h_n, h_{n+1})+\\
&\mu(h_{n+1}, h_m)\left[\alpha((h_{n+1}, (h_{n+2})d_b(h_{n+1}, h_{n+2}) + \mu(h_{n+1}, h_m)d_b(h_{n+2}, h_m)\right]\\
&= \alpha(h_n, h_{n+1})d_b(h_n, h_{n+1}) + \mu(h_{n+1}, h_m)\alpha(h_{n+1}, h_{n+2})d_b(h_{n+1}, h_{n+2})\\
&+ \mu(h_{n+1}, h_m)\mu(h_{n+2}, h_m)d_b(h_{n+2}, h_m)\\
&\leq \alpha(h_n, h_{n+1})d_b(h_n, h_{n+1}) + \mu(h_{n+1}, h_m)\alpha(h_{n+1}, h_{n+2})d_b(h_{n+1}, h_{n+2})\\
&+ \cdots + \mu(h_{n+1}, h_m)\mu(h_{n+2}, h_m)\cdots\mu(h_{m-2}, h_{m-1})\alpha(h_{m-2}, h_{m-1})d_b(h_{m-2}, h_{m-1})\\
&+ \mu(h_{n+1}, h_m)\mu(h_{n+2}, h_m)\cdots\mu(h_{m-2}, h_{m-1})\mu(h_{m-1}, h_m)d_b(h_{m-1}, h_m)\\
&\leq \alpha(h_n, h_{n+1})d_b(h_n, h_{n+1}) + \mu(h_{n+1}, h_m)\alpha(h_{n+1}, h_{n+2})d_b(h_{n+1}, h_{n+2})\\
&+ \cdots + \mu(h_{n+1}, h_m)\mu(h_{n+2}, h_m)\cdots\mu(h_{m-2}, h_{m-1})\alpha(h_{m-2}, h_{m-1})d_b(h_{m-2}, h_{m-1})\\
&+ \mu(h_{n+1}, h_m)\mu(h_{n+2}, h_m)\cdots\mu(h_{m-2}, h_{m-1})\mu(h_{m-1}, h_m)\alpha(h_{m-1}, h_m)d_b(h_{m-1}, h_m)\\
&\leq \alpha(h_n, h_{n+1})Z\psi^{n+1} + \mu(h_{n+1}, h_m)\alpha(h_{n+1}, h_{n+2})Z\psi^{n+2}\\
&+ \cdots + \mu(h_{n+1}, h_m)\mu(h_{n+2}, h_m)\cdots\mu(h_{m-2}, h_{m-1})\alpha(h_{m-2}, h_{m-1})Z\psi^{m-1}\\
&+ \mu(h_{n+1}, h_m)\mu(h_{n+2}, h_m)\cdots\mu(h_{m-2}, h_{m-1})\mu(h_{m-1}, h_m)\alpha(h_{m-1}, h_m)Z\psi^{m}\\
&= Z\psi^{n+1}\left[\alpha(h_n, h_{n+1}) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\mu(h_j, h_m)\right)\alpha(h_i, h_{i+1})\psi^{i-n}\right]
\end{aligned}
$$

Denoting $\mathbb{S}_q = \sum_{i=0}^{q}\left(\prod_{j=0}^{i}\mu(h_j, h_m)\right)\alpha(h_i, h_{i+1})\psi^i$, we have

$$d_b(h_n, h_m) \leq Z\psi^{n+1}\left[\alpha(h_n, h_{n+1}) + (\mathbb{S}_{m-1} - \mathbb{S}_n)\right]$$

The ratio test combined with (6.1) imply that the limit of the sequence $\{\mathbb{S}_n\}$ exists. Hence

$$\lim_{n,m\to\infty} d_b(h_n, h_m) = 0, \tag{6.3}$$

implies that $\{h_n\}$ is a Cauchy sequence in H. Since H is complete, there exists some $h_* \in H$ such that $h_n \to h_*$. Hence by (vi) $h_n\mathcal{R}h_*$ $\forall n \in \mathbb{N}$. Furthermore, we have to prove that $d_b(h_*, T(h_{n-1}, h_n)) \to d_b(h_*, K)\quad as\quad j \to \infty$. Consider,

$$
\begin{aligned}
d_b(h_*, K) &\leq d_b(h_*, T(h_{n-1}, h_n))\\
&= \lim_{n\to\infty} d_b(h_{n+1}, T(h_{n-1}, h_n))\\
&= d_b(H, K)\\
&\leq d_b(h_*, K)
\end{aligned}
$$

Therefore $d_b(h_*, T(h_{n-1}, h_n)) \to d_b(h_*, K)$ as $n \to \infty$. The rest of the proof can be carried out in the same way as in Theorem 3.1 after Equation (3.6). □

Remark 6.2. Note that Theorem 3.1 becomes a special case of Theorem 6.1 by taking $\alpha(w) = \mu(w) = b \geq 1$ for all $w \in X$.

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Conflict of interest

The authors declare that they have no competing interests.

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