AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(6): 10651–10670.
DOI: 10.3934/math.2022594
Received: 24 December 2021
Revised: 10 March 2022
Accepted: 15 March 2022
Published: 30 March 2022

Research article

A new application of the Legendre reproducing kernel method

Mohammad Reza Foroutan[1], **Mir Sajjad Hashemi**[2], **Leila Gholizadeh**[2], **Ali Akgül**[3,*] **and Fahd Jarad**[4,5,*]

[1] Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran
[2] Department of Mathematics, University of Bonab, P.O. Box 55513-95133, Bonab, Iran
[3] Art and Science Faculty, Department of Mathematics, Siirt University, TR-56100, Siirt, Turkey
[4] Department of Mathematics, Çankaya University, Ankara 06790, Turkey
[5] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

* **Correspondence:** Email: aliakgul00727@gmail.com, fahd@cankaya.edu.tr.

Abstract: In this work, we apply the reproducing kernel method to coupled system of second and fourth order boundary value problems. We construct a novel algorithm to acquire the numerical results of the nonlinear boundary-value problems. We also use the Legendre polynomials. Additionally, we discuss the convergence analysis and error estimates. We demonstrate the numerical simulations to prove the efficiency of the presented method.

Keywords: Legendre polynomials; reproducing kernel functions; approximate solution; convergence analysis; boundary value problem
Mathematics Subject Classification: 35A24, 46E20, 47B32

1. Introduction

In this work, we take into consideration the following nonlinear system of ordinary differential equation [21]:

$$f'''' - S(xf''' + 3f'' - 2ff'') - M^2 f'' = 0, \quad x \in (0,1), \tag{1.1}$$

$$\theta'' + P_1(2f\theta' - x\theta') + P_2(f''^2 + 12\delta^2 f'^2) = 0, \tag{1.2}$$

subject to the boundary conditions

$$f(0) = 0, \quad f'(0) - \beta f''(0) = 0, \quad \theta(0) - \gamma\theta'(0) = 0, \tag{1.3}$$

$$f(1) = \frac{1}{2}, \quad f^{'}(1) + \beta f^{''}(1) = 0, \quad \theta(1) + \gamma\theta^{'}(1) = 1, \tag{1.4}$$

where S, P_1, P_2, δ, β and γ are real finite constants.

We can see these problems in paper production, polymer extraction, aerodynamics, reaction-diffusion processes, fluid dynamics, biology and rheometry domains. These problems show up mainly due to the suction and injection effects on the unsteady magneto-hydrodynamic flow [24].

Many methods have been improved for the analytical and approximate solution of nonlinear ordinary differential systems. These techniques contain finite-difference methods [5, 31–33], Adams-Bashforth method [20, 23], B-spline approximation method [8], Chebyshev finite difference method [28], finite element method [6], He's homotopy perturbation method [27], G'/G- method [22], multi-step methods [14].

In recent years, much attempt has been done to the newly developed methods to introduce an analytic and approximate solution of nonlinear boundary value problems [10–13, 15–19, 25]. For more details see [1–3, 26, 34–37]. In this work, we present an approximate-analytical technique for solving a coupled system of second and fourth order boundary value problems.

The rest of this paper is organized as follows. In Section 2, an overview of shifted Legendre polynomials and their relevant properties required henceforward are presented. Also in this section, we will recall a brief review of the reproducing kernel spaces. In Section 3, we construct an orthogonal basis in the Legendre reproducing kernel space and construct a reproducing kernel space which includes boundary conditions. In Section 4, our method to approximate the solution of nonlinear system via shifted Legendre reproducing kernel basis function is considered. We present the convergence analysis and error estimation in Section 5. We demonstrate the numerical results in Section 6. We give the conclusion in the last section.

2. Legendre reproducing kernel functions

In this section, we will recall some basic polynomial functionals and define some new reproducing kernel functions. The well-known shifted Legendre polynomials are described on [0, 1] and can be obtained by the following iterative formula

$$\begin{gathered} P_0(x) = 1, \qquad P_1(x) = 2x - 1, \\ (n+1)P_{n+1}(x) = (2n+1)(2x-1)P_n(x) - nP_{n-1}(x), \quad n \geq 1. \end{gathered} \tag{2.1}$$

The polynomials $P_n(x)$ are orthogonal on $[0, 1]$ with $\rho(x) = 1$, in the sense that

$$\int_0^1 P_n(x)P_m(x)\rho(x)dx = \gamma_{m,n}\delta_{m,n}, \tag{2.2}$$

where

$$\gamma_{m,n} = \begin{cases} 0, & \text{if} \quad m \neq n, \\ 1, & \text{if} \quad m = n = 0, \\ \frac{1}{2n+1}, & \text{if} \quad m = n \neq 0. \end{cases}$$

We use shifted Jacobi basis functions which provide the homogeneous boundary conditions as:

$$f(0) = 0 \quad \text{and} \quad f(1) = 0.$$

Lemma 2.1. *Let* $\alpha, \beta \geq 1$ *and* $\alpha, \beta \in \mathbb{Z}$. *We have* $\{a_j\}$ *such that*

$$J_n^{-\alpha,-\beta}(x) = \sum_{j=n-\alpha-\beta}^{n} a_j P_j(x), \quad n \geq \alpha + \beta, \tag{2.3}$$

where $P_j(x)$ *are the shifted Legender polynomial of degree* j *and* $J_n^{-\alpha,-\beta}(x)$ *is the shifted Jacobi polynomial on* $[0, 1]$. *Then, we have*

$$J_n^{-1,-1}(x) = \frac{2(n-1)}{2n-1}(P_{n-2}(x) - P_n(x)), \quad n \geq 2. \tag{2.4}$$

Proof. For the proof of Lemma 2.1 (see [30], Lemma 1.4.3). □

Now, by utilizing the shifted Jacobi basis function and shifted Legendre functions, we will introduce a reproducing kernel Hilbert space method.

$$f(0) = f(1) = 0.$$

Since $P_n(1) = 1$ and $P_n(0) = (-1)^n$, we have

$$J_n^{-1,-1}(0) = J_n^{-1,-1}(1) = 0.$$

Therefore, we describe

$$u_n(x) = \sqrt{\frac{(n+2)(2n+3)}{(n+1)}} J_{n+2}^{-1,-1}(x), \quad n = 0, 1, 2, \ldots \,, \tag{2.5}$$

and

$$v_n(x) = \sqrt{\frac{2n+1}{2}} P_n(x), \quad n = 0, 1, 2, \ldots \,. \tag{2.6}$$

Definition 2.2. [10] For a nonempty set E, let H be a Hilbert space of real value functions on some set E. A function $K : E \times E \longrightarrow \mathbb{R}$ is said to be the reproducing kernel function of H if and only if:

(i) For every $y \in E$, $K(\cdot, y) \in H$.

(ii) For every $y \in E$ and $f \in H$, $\langle f(\cdot), K(\cdot, y)\rangle = f(y)$.

Also, a Hilbert space of function H that possesses a reproducing kernel K is a reproducing kernel Hilbert space; we represent the reproducing kernel Hilbert space and it's kernel by $H_K(E)$ and K_y respectively.

Theorem 2.3. *[7] Let H be n-dimensional Hilbert space,* $\{w_i\}_{i=1}^n$ *is an orthonormal basis of H, then the reproducing kernel of H as:*

$$K_n(x, y) = \sum_{j=0}^{n} w_j(x) w_j(y), \quad x, y \in [0, 1]. \tag{2.7}$$

Theorem 2.4. *([29] Theorem 1.24) For the orthonormal system* $\{w_n\}_{n=1}^{\infty}$, *formula* (2.7) *yields the Christoffel-Darboux formula:*

$$K_n(x, y) = \frac{k_n(w_{n+1}(x)w_n(y) - w_n(x)w_{n+1}(y))}{k_{n+1}(x - y)}. \tag{2.8}$$

Where, $k_n > 0$ *is the coefficient of* x^n *in* $w_n(x)$*. We get*

$$K_n(x,x) = \frac{k_n}{k_{n+1}}(w'_{n+1}(x)w_n(x) - w'_n(x)w_{n+1}(x)). \tag{2.9}$$

Definition 2.5. Let $H_{\omega,K_{1,n}}[0,1]$ be the weighted inner product space of Jacobi functions described as (2.5) on $[0,1]$ with degree less than or equal to n. The inner product and norm are given respectively by

$$\langle u_1, u_2\rangle_{H_{\omega,K_{1,n}}} = \int_0^1 u_1(x)u_2(x)\omega(x)dx, \quad \forall u_1, u_2 \in H_{\omega,K_{1,n}}[0,1],$$

$$\|u\|_{H_{\omega,K_{1,n}}} = \langle u, u\rangle^{\frac{1}{2}}_{H_{\omega,K_{1,n}}}, \quad \forall u \in H_{\omega,K_{1,n}}[0,1],$$

where $\omega(x) = (1-x)^{-1}(1+x)^{-1}$ and $K_{1,n}(x,y)$ the reproducing kernel of $H_{\omega,K_{1,n}}[0,1]$ is constructed using (2.8) with $w_n(x) := u_n(x)$. From definition

$$L^2_\omega[0,1] = \{u : \int_0^1 |u(x)|^2\omega(x)dx < \infty\},$$

for any fixed n, $H_{\omega,K_{1,n}}[0,1]$ is a subspace of $L^2_\omega[0,1]$ and

$$\langle u_1, u_2\rangle_{H_{\omega,K_{1,n}}} = \langle u_1, u_2\rangle_{L^2_\omega}, \quad \forall u_1, u_2 \in H_{\omega,K_{1,n}}[0,1].$$

From Definition 2.5, $H_{\omega,K_{1,n}}[0,1]$ is a finite dimensional inner product space. Every finite dimensional inner product space is a Hilbert space. Therefore, from this result and Theorem 2.3, $H_{\omega,K_{1,n}}[0,1]$ is a reproducing kernel Hilbert space.

Definition 2.6. Let $H_{K_{2,n}}[0,1]$ be the inner product space of Legendre functions described as (2.6) on $[0,1]$ with degree less than or equal to n. The inner product and norm are given respectively by

$$\langle v_1, v_2\rangle_{H_{K_{2,n}}} = \int_0^1 v_1(x)v_2(x)dx, \quad \forall v_1, v_2 \in H_{K_{2,n}}[0,1],$$

$$\|v\|_{H_{K_{2,n}}} = \langle v, v\rangle^{\frac{1}{2}}_{H_{K_{2,n}}}, \quad \forall v \in H_{K_{2,n}}[0,1],$$

where the reproducing kernel $K_{2,n}(x,y)$ of $H_{K_{2,n}}[0,1]$ is constructed using (2.8) with $w_n(x) := v_n(x)$. From definition

$$L^2[0,1] = \{v : \int_0^1 |v(x)|^2dx < \infty\},$$

for any fixed n, $H_{K_{2,n}}[0,1]$ is a subspace of $L^2[0,1]$ and

$$\langle v_1, v_2\rangle_{H_{K_{2,n}}} = \langle v_1, v_2\rangle_{L^2}, \quad \forall v_1, v_2 \in H_{K_{2,n}}[0,1].$$

3. Construction of reproducing kernel spaces

$H_{\omega,R_1}[0,1]$ and $H_{R_2}[0,1]$ are described by:

$$H_{\omega,R_1}[0,1] = \{f(x) : f(x) \in H_{\omega,K_{1,n}}[0,1], f'(0) - \beta f''(0) = 0, f'(1) + \beta f''(1) = 0\},$$
$$H_{R_2}[0,1] = \{\theta(x) : \theta(x) \in H_{K_{2,n}}[0,1], \theta(0) - \gamma\theta'(0) = 0, \theta(1) + \gamma\theta'(1) = 0\}.$$

Clearly, $H_{\omega,R_1}[0,1]$ and $H_{R_2}[0,1]$ are closed subspaces of $H_{\omega,K_{1,n}}[0,1]$ and $H_{K_{2,n}}[0,1]$, respectively.

Let $T_1 f = f'(0) - \beta f''(0)$ and $T_2 f = f'(1) + \beta f''(1)$ be the boundary condition of function $f(x)$. Put

$$R_{1,1}(x,y) = K_{1,n}(x,y) - \frac{T_{1,x}K_{1,n}(x,y)T_{1,y}K_{1,n}(x,y)}{T_{1,x}T_{1,y}K_{1,n}(x,y)}, \tag{3.1}$$

and

$$R_1(x,y) = R_{1,1}(x,y) - \frac{T_{2,x}R_{1,1}(x,y)T_{2,y}R_{1,1}(x,y)}{T_{2,x}T_{2,y}R_{1,1}(x,y)}, \tag{3.2}$$

where, the symbol $T_{1,x}$ shows that the operator T_1 implements to the function of x.

Theorem 3.1. *If $T_{1,x}T_{1,y}K_{1,n}(x,y) \neq 0$ and $T_{2,x}T_{2,y}R_{1,1}(x,y) \neq 0$, then $R_1(x,y)$ given by* (3.2) *satisfies the boundary conditions $T_1 f = 0$ and $T_2 f = 0$ exactly.*

Proof. By applying the operator $T_{1,x}$ to $R_{1,1}(x,y)$ in Eq (3.1), we get

$$T_{1,x}R_{1,1}(x,y) = T_{1,x}K_{1,n}(x,y) - \frac{T_{1,x}K_{1,n}(x,y)T_{1,x}T_{1,y}K_{1,n}(x,y)}{T_{1,x}T_{1,y}K_{1,n}(x,y)} = 0. \tag{3.3}$$

Furthermore, by applying the operator $T_{1,x}T_{2,y}$ to $R_{1,1}(x,y)$, we have

$$\begin{aligned} T_{1,x}T_{2,y}R_{1,1}(x,y) &= T_{1,x}T_{2,y}K_{1,n}(x,y) - \frac{T_{2,y}T_{1,x}K_{1,n}(x,y)T_{1,x}T_{1,y}K_{1,n}(x,y)}{T_{1,x}T_{1,y}K_{1,n}(x,y)} \\ &= T_{1,x}T_{2,y}K_{1,n}(x,y) - T_{2,y}T_{1,x}K_{1,n}(x,y) = 0. \end{aligned} \tag{3.4}$$

Then, by applying the operator $T_{1,x}$ to $R_1(x,y)$ in Eq (3.2) and using Eqs (3.3) and (3.4), we get

$$T_{1,x}R_1(x,y) = T_{1,x}R_{1,1}(x,y) - \frac{T_{2,x}R_{1,1}(x,y)T_{1,x}T_{2,y}R_{1,1}(x,y)}{T_{2,x}T_{2,y}R_{1,1}(x,y)} = 0.$$

Also, by applying the operator $T_{2,x}$ to $R_1(x,y)$ in Eq (3.2), we have

$$T_{2,x}R_1(x,y) = T_{2,x}R_{1,1}(x,y) - \frac{T_{2,x}R_{1,1}(x,y)T_{2,x}T_{2,y}R_{1,1}(x,y)}{T_{2,x}T_{2,y}R_{1,1}(x,y)} = 0.$$

□

Theorem 3.2. *[9] If $T_{1,x}T_{1,y}K_{1,n}(x,y) \neq 0$ and $T_{2,x}T_{2,y}R_{1,1}(x,y) \neq 0$, then, we obtain*

$$R_1(x,y) = R_{1,1}(x,y) - \frac{T_{2,x}R_{1,1}(x,y)T_{2,y}R_{1,1}(x,y)}{T_{2,x}T_{2,y}R_{1,1}(x,y)}.$$

Let $T_3\theta = \theta(0) - \gamma\theta'$ and $T_4\theta = \theta(1) + \gamma\theta'(1) = 0$ be the boundary condition of function $\theta(x)$. Put

$$R_{2,2}(x,y) = K_{2,n}(x,y) - \frac{T_{3,x}K_{2,n}(x,y)T_{3,y}K_{2,n}(x,y)}{T_{3,x}T_{3,y}K_{2,n}(x,y)}, \tag{3.5}$$

and

$$R_2(x,y) = R_{2,2}(x,y) - \frac{T_{4,x}R_{2,2}(x,y)T_{4,y}R_{2,2}(x,y)}{T_{4,x}T_{4,y}R_{2,2}(x,y)}. \tag{3.6}$$

Theorem 3.3. *If $T_{3,x}T_{3,y}K_{2,n}(x,y) \neq 0$ and $T_{4,x}T_{4,y}R_{2,2}(x,y) \neq 0$, then $R_2(x,y)$ given by* (3.6) *satisfies the boundary conditions $T_3\theta = 0$ and $T_4\theta = 0$ exactly.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.1. □

Theorem 3.4. *If $T_{3,x}T_{3,y}K_{2,n}(x,y) \neq 0$ and $T_{4,x}T_{4,y}R_{2,2}(x,y) \neq 0$, then $H_{R_2}[0,1]$ is a reproducing kernel space and its reproducing kernel is*

$$R_2(x,y) = R_{2,2}(x,y) - \frac{T_{4,x}R_{2,2}(x,y)T_{4,y}R_{2,2}(x,y)}{T_{4,x}T_{4,y}R_{2,2}(x,y)}.$$

Note that $R_x(y) = R(x,y)$, $R_{1,x}(y) = R_1(x,y)$ and $R_{2,x}(y) = R_2(x,y)$. Henceforth and not to conflict unless stated otherwise, we denote $H[0,1] = H_{\omega,R_1} \oplus H_{R_2}$, $L[0,1] = L^2_\omega[0,1] \oplus L^2[0,1]$ and $R_x(y) = (R_{1,x}(y), R_{2,x}(y))^T$.

Definition 3.5. (a) The Hilbert space $H[0,1]$ is described by:

$$H[0,1] = \{z = (z_1, z_2)^T : \ z_1 \in H_{\omega,R_1}[0,1] \text{ and } z_2 \in H_{R_2}[0,1]\}.$$

The inner product in $H[0,1]$ is building as

$$\langle z, w\rangle_H = \langle z_1, w_1\rangle_{H_{\omega,R_1}} + \langle z_2, w_2\rangle_{H_{R_2}}$$

and the norm is $\|z\|_H = \|z_1\|_{H_{\omega,K_n}} + \|z_2\|_{H_{K_n}}$ where $z, w \in H[0,1]$.

(b) The Hilbert space $L[0,1]$ is described by:

$$L[0,1] = \{z = (z_1, z_2)^T : \ z_1 \in L^2_\omega[0,1] \text{ and } z_2 \in L^2[0,1]\}.$$

The inner product in $L[0,1]$ is building as

$$\langle z, w\rangle_L = \langle z_1, w_1\rangle_{L^2_\omega} + \langle z_2, w_2\rangle_{L^2}$$

and the norm is $\|z\|_L = \|z_1\|_{L^2_\omega} + \|z_2\|_{L^2}$ where $z, w \in L[0,1]$.

4. Representation of approximate solutions

We assume

$$f(x) = F(x) + \beta_1 d_\beta(x) \tag{4.1}$$

and

$$\theta(x) = \Theta(x) + \gamma_1(x + \gamma), \tag{4.2}$$

where

$$\beta_1 = \frac{1}{10 + 24\beta + 24\beta^2}, \quad d_\beta(x) = 2x^3 + (3 + 6\beta)x^2 + (6\beta + 12\beta^2)x$$

and $\gamma_1 = \frac{1}{2\gamma+1}$, then Eqs (1.1) and (1.2) changes to the following problem:

$$\left\{ \begin{array}{l} F'''' - SxF''' + (2S\beta_1 d_\beta(x) - 3S - M^2)F'' + 2S\beta_1 c_\beta(x)F = -2SFF'' + g_1(x), \\ 2P_2\beta_1 c_\beta(x)F'' + 24P_2\delta^2\beta_1 d_\beta(x)F' + 2P_1\gamma_1 F + \Theta'' + P_1(2\beta_1 d_\beta(x) - x)\Theta' \\ \qquad\qquad = -2P_1 F\Theta' + P_2 F''^2 + 12P_2\delta^2 F'^2 + g_2(x), \end{array} \right. \tag{4.3}$$

and the boundary conditions changes to the following conditions:

$$F(0) = 0, \quad F'(0) - \beta F''(0) = 0, \quad \Theta(0) - \gamma\Theta'(0) = 0, \tag{4.4}$$

$$F(1) = 0, \quad F'(1) + \beta F''(1) = 0, \quad \Theta(1) + \gamma\Theta'(1) = 0, \tag{4.5}$$

where

$$\begin{aligned} c_\beta(x) &= 12x + 12\beta + 6, \\ g_1(x) &= -2S\beta_1^2 d_\beta(x)c_\beta(x) + 12S\beta_1 x + M^2\beta_1 c_\beta(x), \\ g_2(x) &= P_1\gamma_1 x - 2P_1\gamma_1\beta_1 d_\beta(x) - P_2\beta_1^2 c_\beta^2(x) - 12P_2\delta^2\beta_1^2 d_\beta^2(x). \end{aligned}$$

Put

$$\begin{aligned} &\mathbb{L}_{11}F = F'''' - SxF''' + (2S\beta_1 d_\beta(x) - 3S - M^2)F'' + 2S\beta_1 c_\beta(x)F, \\ &\mathbb{L}_{12}\Theta = 0, \\ &\mathbb{L}_{21}F = 2P_2\beta_1 c_\beta(x)F'' + 24P_2\delta^2\beta_1 d_\beta(x)F' + 2P_1\gamma_1 F, \\ &\mathbb{L}_{22}\Theta = \Theta'' + P_1(2\beta_1 d_\beta(x) - x)\Theta', \\ &\mathbb{L} = \begin{pmatrix} \mathbb{L}_{11} & \mathbb{L}_{12} \\ \mathbb{L}_{21} & \mathbb{L}_{22} \end{pmatrix}, \qquad N_1(F,\Theta) = 2SFF'', \\ &N_2(F,\Theta) = 2P_1 F\Theta' - P_2 F''^2 - 12P_2\delta^2 F'^2, \\ &\Phi = (F,\Theta)^T, \qquad \Phi^{'} = (F^{'},\Theta^{'})^T \qquad \text{and} \qquad \Phi^{''} = (F^{''},0)^T, \end{aligned}$$

then,the coupled differential systems of Eqs (1.1) and (1.2) can be written as follows:

$$\mathbb{L}\Phi(x) = g(x) - N(\Phi(x), \Phi^{'}(x), \Phi^{''}(x)), \tag{4.6}$$

with boundary conditions:

$$\left\{ \begin{array}{l} (e_1^T\Phi(0))e_1 = 0, \quad (e_1^T\Phi'(0))e_1 - \beta(e_1^T\Phi''(0))e_1 = 0, \\ (e_2^T\Phi(0))e_2 - \gamma(e_2^T\Phi'(0))e_2 = 0, \\ (e_1^T\Phi(1))e_1 = 0, \quad (e_1^T\Phi'(1))e_1 + \beta(e_1^T\Phi''(1))e_1 = 0, \\ (e_2^T\Phi(1))e_2 + \gamma(e_2^T\Phi'(1))e_2 = 0, \end{array} \right. \tag{4.7}$$

where $g = (g_1, g_2)^T$, $N = (N_1, N_2)^T$, $\Phi \in H[0,1]$, $g - N \in L[0,1]$, $e_1 = (1,0)^T$, $e_2 = (0,1)^T$ and $\mathbb{L} : H[0,1] \to L[0,1]$.

Here, $(e_1^T\Phi(i))e_1 = (F(i),0)^T$ and $(e_2^T\Phi(i))e_2 = (0,\Theta(i))^T$, $i = 0,1$.

Lemma 4.1. *([10], Lemma 4.1) The operators* $\mathbb{L}_{22} : H_{R_2}[0,1] \to L^2[0,1]$ *and* $\mathbb{L}_{i1}[0,1] : H_{\omega,R_1} \to L^2_\omega[0,1]$, $i = 1,2$, *are linear bounded operators.*

Theorem 4.2. *The operator* $\mathbb{L} : H[0,1] \to L[0,1]$ *is bounded linear operator.*

Proof. For each $\Phi \in H[0,1]$, using Definition 3.5, we have

$$\begin{aligned} \|\mathbb{L}\Phi\|_L &= \sqrt{\sum_{i=1}^{2} \|\mathbb{L}_{i1}F\|^2_{L_\omega} + \|\mathbb{L}_{22}\Theta\|^2_L} \\ &\le \sqrt{(\sum_{i=1}^{2} \|\mathbb{L}_{i1}\|\|F\|_{H_{\omega,R_1}})^2 + (\|\mathbb{L}_{22}\|\|\Theta\|_{H_{R_2}})^2} \\ &\le \sqrt{(\|\mathbb{L}_{11}\|^2 + \|\mathbb{L}_{21}\|^2 + \|\mathbb{L}_{22}\|^2)(\|F\|^2_{H_{\omega,R_1}} + \|\Theta\|^2_{H_{R_2}})} \\ &\le \sqrt{\|\mathbb{L}_{11}\|^2 + \|\mathbb{L}_{21}\|^2 + \|\mathbb{L}_{22}\|^2}\ \|\Phi\|_H. \end{aligned}$$

The boundedness of $\mathbb{L}_{22}$ and $\mathbb{L}_{i1}$ for $i = 1,2$, shows that $\mathbb{L}$ is bounded. The proof is complete. $\square$

Let $D = \{x_i\}_{i=1}^{\infty}$ is countable dense subset in the domain $[0,1]$, then for any fixed $x_i \in [0,1]$, we have

$$\begin{aligned} \Psi_{ij}(x) :=& \mathbb{L}^* R(x, x_i) e_j = \mathbb{L}^* R_{x_i}(x) e_j = \langle \mathbb{L}^* R_{x_i}(x), R_x(y) \rangle_H e_j \\ =& \langle R_{x_i}(x), \mathbb{L}_y R_x(y) \rangle_L e_j = \mathbb{L}_y R_x(y) e_j|_{y=x_i} \\ =& \mathbb{L}_y R_y(x) e_j|_{y=x_i}, \ \ j = 1,2, \ \ i = 1,2,3,\ldots, \end{aligned}$$

where $\mathbb{L}^* = \begin{pmatrix} \mathbb{L}^*_{11} & \mathbb{L}^*_{21} \\ 0 & \mathbb{L}^*_{22} \end{pmatrix}$ is the adjoint operator of $\mathbb{L}$ and the subscript y in the operator $\mathbb{L}_y$ indicates that the operator $\mathbb{L}$ applied to the function y. For any fixed $x_i \in (0,1)$, $\Psi_{ij}(x) \in H[0,1]$.

Theorem 4.3. *If* $\{x_i\}_{i=1}^{\infty}$ *is distinct points dense on* $[0,1]$ *and* $\mathbb{L}^{-1}$ *is existent, then*

$$Im\mathbb{L}^* = H_{\omega,R_1}([0,1]) \oplus H_{R_2}([0,1]), \quad (Ker\mathbb{L}^*)^\perp = Im\mathbb{L} = L^2_\omega([0,1]) \oplus L^2([0,1]).$$

Proof. Clearly $\psi_{ij}(x) \in H_{\omega,R_1}([0,1]) \oplus H_{R_2}([0,1])$. For any $\Phi \in (Im\mathbb{L}^*)^\perp$, since $\psi_{ij}(x) = \mathbb{L}^* R_{x_i}(x) e_j$, we have

$$\langle \Phi(x), \psi_{ij}(x) \rangle_H = 0. \tag{4.8}$$

On the other hand,

$$\Phi(x) = F(x)e_1 + \Theta(x)e_2 = \sum_{j=1}^{2} \langle \Phi(.), R_x(.)e_j \rangle_H e_j.$$

Thus, by Eq (4.8), we get

$$\mathbb{L}\Phi(x_i) = \sum_{j=1}^{2} \langle \mathbb{L}\Phi(y), R_x(y) e_j \rangle_H e_j = 0, \quad i = 1,2,\ldots\ .$$

Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$. Hence $(\mathbb{L}\Phi)(x) = 0$. So from the existence $\mathbb{L}^{-1}$, we have $\Phi(x) = 0$. That is $(Im\mathbb{L}^*)^\perp = 0$. Therefore $Im\mathbb{L}^* = H_{\omega,R_1}([0,1]) \oplus H_{R_2}([0,1])$. Similarly, we can show $(Ker\mathbb{L}^*)^\perp = L^2_\omega([0,1]) \oplus L^2([0,1])$. $\square$

Corollary 4.4. *For Eqs* (1.1)–(1.4)*, if* $\{x_i\}_{i=1}^{\infty}$ *is distinct points dense on* $[0,1]$ *and* $\mathbb{L}^{-1}$ *is existent, then* $\{\psi_{ij}(x)\}_{(i,j)=(1,1)}^{(\infty,2)}$ *is the complete function system of the space* $H([0,1])$.

By Gram-Schmidt process, we acquire an orthogonal basis $\{\overline{\psi}_{ij}(x)\}_{(i,j)=(1,1)}^{(\infty,2)}$ of $H([0,1])$, such that

$$\overline{\psi}_{ij}(x) = \sum_{l=1}^{i}\sum_{k=1}^{j} \alpha_{lk}^{ij}\psi_{ij}(x),\ \ i = 1,2,\dots,\ j = 1,2,$$

where α_{lk}^{ij} represents orthogonal coefficients, which are given by the following relations [4]:

$$\alpha_{1k}^{1j} = \frac{1}{\|\psi_{1k}\|_H},\quad \alpha_{lk}^{ij} = \frac{1}{a_{lk}^{ij}},\ \ l = i \neq 1,$$
$$\alpha_{lk}^{ij} = -\frac{1}{a_{lk}^{ij}}\sum_{s=i}^{l-1} c_{lk}^{sj}\alpha_{sk}^{ij},\ \ l < i,$$

such that $a_{lk}^{ij} = \sqrt{\|\psi_{lk}\|_H^2 - \sum_{s=i}^{l-1}(c_{lk}^{sj})^2}$ and $c_{lk}^{sj} = \langle \psi_{lk}, \overline{\psi}_{sk}\rangle_H^2$.

Lemma 4.5. *Let* $\{\overline{\psi}_{ij}(x)\}_{(i,j)=(1,1)}^{(\infty,2)}$ *be an orthonormal basis of H then we have*

$$R(x,y) = \sum_{i=1}^{\infty}\sum_{j=1}^{2} \overline{\psi}_{ij}(x)\overline{\psi}_{ij}(y).$$

Proof. Let $g \in H$, then

$$\begin{aligned}\langle g(y), R(x,y)\rangle_H &= \langle g(y), \sum_{i=1}^{\infty}\sum_{j=1}^{2}\overline{\psi}_{ij}(x)\overline{\psi}_{ij}(y)\rangle_H \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\langle g(y), \overline{\psi}_{ij}(y)\rangle_H \overline{\psi}_{ij}(x) \\ &= g(x).\end{aligned}$$

□

Theorem 4.6. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$ *and* $\mathbb{L}^{-1}$ *is existent, then the solution of Eq* (4.6) *satisfies the form*

$$\Phi(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} \alpha_{lk}^{ij} G_k(x_l, \Phi(x_l), \Phi'(x_l), \Phi''(x_l))\overline{\psi}_{ij}(x) \tag{4.9}$$

where

$$G(x,\Phi(x),\Phi'(x),\Phi''(x)) = g(x) - N(\Phi(x),\Phi'(x),\Phi''(x)) = (G_1, G_2)^T.$$

Proof. Since $\{\overline{\psi}_{ij}(x)\}_{(i,j)=(1,1)}^{(\infty,2)}$ is orthonormal system,$\Phi(x)$ is expressed as

$$\begin{aligned}
\Phi(x) &= \langle \Phi(y), R(x,y)\rangle_{H} \\
&= \langle \Phi(y), \sum_{i=1}^{\infty}\sum_{j=1}^{2}\overline{\psi}_{ij}(x)\overline{\psi}_{ij}(y)\rangle_{H} \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\langle \Phi(y), \overline{\psi}_{ij}(y)\rangle_{H}\overline{\psi}_{ij}(x) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\langle \Phi(y), \sum_{l=1}^{i}\sum_{k=1}^{j}\alpha_{lk}^{ij}\psi_{lk}(y)\rangle_{H}\overline{\psi}_{ij}(x) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\alpha_{lk}^{ij}\langle \Phi(y), \psi_{lk}(y)\rangle_{H}\overline{\psi}_{ij}(x) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\alpha_{lk}^{ij}\langle \Phi(y), \mathbb{L}^{*}r_{lk}(y)\rangle_{H}\overline{\psi}_{ij}(x) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\alpha_{lk}^{ij}\langle \mathbb{L}\Phi(y), r_{lk}(y)\rangle_{L}\overline{\psi}_{ij}(x) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\alpha_{lk}^{ij}G_{k}(x_{l}, \Phi(x_{l}), \Phi^{'}(x_{l}), \Phi^{''}(x_{l}))\overline{\psi}_{ij}(x),
\end{aligned}$$

where $r_{lk}(y) = R_{x_{l}}(y)e_{k}$.This completes the proof. □

Now, let

$$H_{N}[0,1] = Span\{\overline{\psi}_{11}, \overline{\psi}_{12}, \overline{\psi}_{21}, \overline{\psi}_{22}, \dots, \overline{\psi}_{N1}, \overline{\psi}_{N2}\}.$$

Define $H[0,1]-$orthogonal projection $\mathcal{T}_{N} : H[0,1] \to H_{N}[0,1]$ such that for $\Phi \in H[0,1]$,

$$\langle \mathcal{T}_{N}\Phi - \Phi, \zeta\rangle_{H} = 0, \qquad \forall \zeta \in H_{N}[0,1],$$

or equivalently,

$$\mathcal{T}_{N}\Phi = \sum_{i=1}^{N}\sum_{j=1}^{2}\langle \Phi, \overline{\psi}_{ij}\rangle\overline{\psi}_{ij}.$$

Then, we get the approximate solution as:

$$\Phi_{N}(x) = \sum_{i=1}^{N}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\alpha_{lk}^{ij}G_{k}(x_{l}, \mathcal{T}_{l-1}\Phi(x_{l}), (\mathcal{T}_{l-1}\Phi)^{'}(x_{l}), (\mathcal{T}_{l-1}\Phi)^{''}(x_{l}))\overline{\psi}_{ij}(x). \tag{4.10}$$

Here, $\mathcal{T}_{0}\Phi(x)$ is any fixed function in $H([0,1])$.

5. Convergence and error estimation

Theorem 5.1. *Assume that the problem* (4.6) *has a unique solution. If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$*, then* $\Phi_N(x)$ *in* (4.10) *is convergence to the* $\Phi(x)$ *and for any fixed* $\Phi_0(x) \in H([0,1])$*,* $\Phi_N(x)$ *is also represented by*

$$\Phi_N(x) = \sum_{i=1}^{N}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\alpha_{lk}^{ij}G_k(x_l,\Phi(x_l),\Phi^{'}(x_l),\Phi^{''}(x_l))\overline{\psi}_{ij}(x). \tag{5.1}$$

Proof. We have

$$\mathbb{L}\Phi_N(x) = \sum_{i=1}^{N}\sum_{j=1}^{2}\beta_{ij}\mathbb{L}\overline{\psi}_{ij}(x),$$

where

$$\beta_{ij} = \sum_{l=1}^{i}\sum_{k=1}^{j}\alpha_{lk}^{ij}G_k(x_l,\mathcal{T}_{l-1}\Phi(x_l),(\mathcal{T}_{l-1}\Phi)^{'}(x_l),(\mathcal{T}_{l-1}\Phi)^{''}(x_l)).$$

Then for $s \le N$ and $p \le 2$, we have

$$\begin{aligned}
(\mathbb{L}\Phi_N)_p(x_s) &= \sum_{i=1}^{N}\sum_{j=1}^{2}\beta_{ij}\langle\mathbb{L}\overline{\psi}_{ij}(x), r_{sp}(x)\rangle_H \\
&= \sum_{i=1}^{N}\sum_{j=1}^{2}\beta_{ij}\langle\overline{\psi}_{ij}(x), \mathbb{L}^{*}r_{sp}(x)\rangle_H \\
&= \sum_{i=1}^{N}\sum_{j=1}^{2}\beta_{ij}\langle\overline{\psi}_{ij}(x), \psi_{sp}(x)\rangle_H.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{s'=1}^{s}\sum_{p'=1}^{p}\beta_{s'p'}^{ij}(\mathbb{L}\Phi_N)_{p'}(x_{s'}) &= \sum_{i=1}^{N}\sum_{j=1}^{2}\beta_{ij}\Big\langle\overline{\psi}_{ij}(x), \sum_{s'=1}^{s}\sum_{p'=1}^{p}\beta_{s'p'}^{ij}\psi_{s'p'}(x)\Big\rangle_H \\
&= \sum_{i=1}^{N}\sum_{j=1}^{2}\beta_{ij}\langle\overline{\psi}_{ij}(x), \overline{\psi}_{s'p'}(x)\rangle_H \\
&= \beta_{sp}.
\end{aligned}$$

If $s = 1$, we get

$$(\mathbb{L}\Phi_N)_j(x_1) = G_j(x_1,\mathcal{T}_0\Phi(x_1),(\mathcal{T}_0\Phi)^{'}(x_1),(\mathcal{T}_0\Phi)^{''}(x_1)), \quad j = 1,2,$$

that is,

$$\mathbb{L}\Phi_N(x_1) = G(x_1,\mathcal{T}_0\Phi(x_1),(\mathcal{T}_0\Phi)^{'}(x_1),(\mathcal{T}_0\Phi)^{''}(x_1)).$$

For $s = 2$, we have

$$(\mathbb{L}\Phi_N)_j(x_2) = G_j(x_2,\mathcal{T}_1\Phi(x_2),(\mathcal{T}_1\Phi)^{'}(x_2),(\mathcal{T}_1\Phi)^{''}(x_2)), \quad j = 1,2,$$

that is,

$$\mathbb{L}\Phi_N(x_2) = G(x_2, \mathcal{T}_1\Phi(x_2), (\mathcal{T}_1\Phi)^{'}(x_2), (\mathcal{T}_1\Phi)^{''}(x_2)).$$

Hence it can be obtained by induction,

$$\mathbb{L}\Phi_N(x_n) = G(x_n, \mathcal{T}_{n-1}\Phi(x_n), (\mathcal{T}_{n-1}\Phi)^{'}(x_n), (\mathcal{T}_{n-1}\Phi)^{''}(x_n)).$$

Since $\{x_n\}_{n=1}^{\infty}$ is dense, for any $x \in [0, 1]$ there exists a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ such that $x_{n_i} \to x$, as $i \to \infty$. Then, we reach:

$$\begin{aligned}\lim_{i\to+\infty} \mathbb{L}\Phi_N(x_{n_i}) &= \lim_{i\to+\infty} G(x_{n_i}, \mathcal{T}_{n_i-1}\Phi(x_{n_i}), (\mathcal{T}_{n_i-1}\Phi)^{'}(x_{n_i}), (\mathcal{T}_{n_i-1}\Phi)^{''}(x_{n_i}))\\ &= G(x, \Phi(x), \Phi^{'}(x), \Phi^{''}(x))\\ &= \mathbb{L}\Phi(x).\end{aligned} \tag{5.2}$$

Moreover, according to (4.10) we have

$$\begin{aligned}&\lim_{s\to+\infty} \mathbb{L}\Phi_N(x_{n_s}) \qquad (5.3)\\ &= \lim_{s\to+\infty} \sum_{i=1}^{N}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} \alpha_{lk}^{ij} G_k(x_l, \mathcal{T}_{l-1}\Phi(x_l), (\mathcal{T}_{l-1}\Phi)^{'}(x_l), (\mathcal{T}_{l-1}\Phi)^{''}(x_l))\mathbb{L}\overline{\psi}_{ij}(x_{n_s})\\ &= \sum_{i=1}^{+\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} \alpha_{lk}^{ij} G_k(x_l, \mathcal{T}_{l-1}\Phi(x_l), (\mathcal{T}_{l-1}\Phi)^{'}(x_l), (\mathcal{T}_{l-1}\Phi)^{''}(x_l))\mathbb{L}\overline{\psi}_{ij}(x)\\ &= \lim_{N\to+\infty} \mathbb{L} \sum_{i=1}^{N}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} \alpha_{lk}^{ij} G_k(x_l, \mathcal{T}_{l-1}\Phi(x_l), (\mathcal{T}_{l-1}\Phi)^{'}(x_l), (\mathcal{T}_{l-1}\Phi)^{''}(x_l))\mathbb{L}\overline{\psi}_{ij}(x_{n_s})\\ &= \lim_{N\to+\infty} \mathbb{L}\Phi_N(x). \qquad (5.4)\end{aligned}$$

So, from Eqs (5.2) and (5.3), we conclude that

$$\lim_{N\to+\infty} \mathbb{L}\Phi_N(x) = \mathbb{L}\Phi(x). \tag{5.5}$$

Thus, we obtain

$$\lim_{N\to+\infty} \Phi_N(x) = \mathbb{L}^{-1} \lim_{N\to+\infty} \Big(\mathbb{L}\Phi_N(x)\Big) = \mathbb{L}^{-1}\Big(\Phi(x)\Big) = \Phi(x).$$

□

Theorem 5.2. *Let $\Phi_n(x) = (F_n(x), \Theta_n(x))^T$ be approximate solution that has obtained from the present method in the space $H[0, 1]$ and $\Phi(x) = (F(x), \Theta(x))^T$ be exact solution for the differential equation* (4.6) *with boundary conditions* (4.7)*. Also, assume that $x_n \to x$ $(n \to \infty)$, $\|\Phi_n\|_H$ is bounded and $G(t, \Phi(t), \Phi'(t), \Phi''(t))$ is continuous for $t \in [0, 1]$, then*

$$G(x_n, \Phi_{n-1}(x_n), \Phi'_{n-1}(x_n), \Phi''_{n-1}(x_n)) \to G(x, \Phi(x), \Phi'(x), \Phi''(x)),$$

as $n \to \infty$.

Proof. For any $x \in [0,1]$, using the boundedness of $\|\partial_x^i R_1(x,y)\|_{H_{w,R_1}}$ $(i = 0,1,2)$ and reproducing property of $R_1(x,y)$, we have

$$\begin{aligned}\|F_n^{(i)}(x) - F^{(i)}(x)\| =&\|(F_n(x) - F(x))^{(i)}\| = |\partial_x^i \langle F_n(y) - F(y), R_1(x,y)\rangle_{H_{w,R_1}}|\\ =&|\langle F_n(y) - F(y), \partial_x^i R_1(x,y)\rangle_{H_{w,R_1}}|\\ \le&\|F_n - F\|_{H_{w,R_1}} \|\partial_x^i R_1(x,y)\|_{H_{w,R_1}}\\ \le&\alpha_i \|F_n - F\|_{H_{w,R_1}} \to 0,\end{aligned} \tag{5.6}$$

where for $i = 0,1,2$, α_i are positive constants.

Similarly, for each $x \in [0,1]$ and $i = 0,1$, we get

$$\begin{aligned}\|\Theta_n^{(i)}(x) - \Theta^{(i)}(x)\| =&\|(\Theta_n(x) - \Theta(x))^{(i)}\| = |\partial_x^i \langle \Theta_n(y) - \Theta(y), R_2(x,y)\rangle_{H_{R_2}}|\\ =&|\langle \Theta_n(y) - \Theta(y), \partial_x^i R_2(x,y)\rangle_{H_{R_2}}|\\ \le&\|\Theta_n - \Theta\|_{H_{R_2}} \|\partial_x^i R_2(x,y)\|_{H_{R_2}}\\ \le&\beta_i \|\Theta_n - \Theta\|_{H_{R_2}} \to 0,\end{aligned} \tag{5.7}$$

where for $i = 0,1,2$, β_i are positive constants.

Furthermore, if $\Phi \in H[0,1]$, then $\Phi(x) = (F(x), \Theta(x))^T$ where $F(x) \in H_{w,R_1}[0,1]$ and $\Theta(x) \in H_{R_2}[0,1]$. Thus for $i = 0,1,2$, we have

$$\begin{aligned}\|\Phi_n^{(i)}(x) - \Phi^{(i)}(x)\| =& \sqrt{|F_n^{(i)}(x) - F^{(i)}(x)|^2 + |\Theta_n^{(i)}(x) - \Theta^{(i)}(x)|^2}\\ =& \sqrt{\alpha_i^2 \|F_n^{(i)}(x) - F^{(i)}(x)\|^2_{H_{w,R_1}} + \beta_i^2 \|\Theta_n^{(i)}(x) - \Theta^{(i)}(x)\|^2_{H_{R_2}}} \to 0.\end{aligned} \tag{5.8}$$

Note that, since $\Phi_n \in H[0,1]$, exist a constant c_1 such that

$$|\Phi'_{n-1}(x)| \le c_1, \quad \forall x \in [0,1].$$

Therefore

$$\begin{aligned}|\Phi_{n-1}(x_n) - \Phi(x)| =&|\Phi_{n-1}(x_n) - \Phi_{n-1}(x) + \Phi_{n-1}(x) - \Phi(x)|\\ =&|\Phi_{n-1}(x_n) - \Phi_{n-1}(x)| + |\Phi_{n-1}(x) - \Phi(x)|\\ =&|\Phi'_{n-1}(y_1)||x_n - x| + |\Phi_{n-1}(x) - \Phi(x)| \to 0, \quad \text{as} \quad n \to \infty,\end{aligned} \tag{5.9}$$

where y_1 lies between x_n and x. Now will show that $\Phi'_{n-1}(x_n) \to \Phi'(x)$. Since $\Phi_n(x) \in H[0,1]$, exist a constant c_2 such that $|\Phi''_{n-1}(x)| \le c_2$, so we get

$$\begin{aligned}|\Phi'_{n-1}(x_n) - \Phi'(x)| =&|\Phi'_{n-1}(x_n) - \Phi'_{n-1}(x) + \Phi'_{n-1}(x) - \Phi'(x)|\\ =&|\Phi'_{n-1}(x_n) - \Phi'_{n-1}(x)| + |\Phi'_{n-1}(x) - \Phi'(x)|\\ =&|\Phi''_{n-1}(y_2)||x_n - x| + |\Phi'_{n-1}(x) - \Phi'(x)| \to 0, \quad \text{as} \quad n \to \infty,\end{aligned} \tag{5.10}$$

where y_2 lies between x_n and x. Similarly, we can write

$$|\Phi''_{n-1}(x_n) - \Phi''(x)| \to 0, \quad \text{as} \quad n \to \infty.$$

Now, from the continuation of $G(t, \Phi(t), \Phi'(t), \Phi''(t))$, it is implies that

$$G(x_n, \Phi_{n-1}(x_n), \Phi'_{n-1}(x_n), \Phi''_{n-1}(x_n)) \to G(x, \Phi(x), \Phi'(x), \Phi''(x)), \quad \text{as} \quad n \to \infty.$$

□

6. Numerical experiment

In this section, some illustrative examples demonstrate the applicability, efficiency and utility of the proposed technique. The computations associated with the examples were performed using Maple16 on a personal computer.

Let us consider Eqs (1.1)–(1.4), using the shifted Legendre reproducing kernel Hilbert space method. We apply the technique on this problem with $N = 12$ and

$$x_i = -\frac{1}{2}cos(\frac{i\pi}{N}) + \frac{1}{2},\ \ i = 0, 1, 2, ..., N - 1.$$

Table 1 demonstrate the obtained solutions of $f''(x)$ and at $x = 1$ for various values of S, M, β and compares the results with homotopy analysis method (HAM) presented in [21]. Table 2 demonstrates the approximate solutions of velocity $\theta'(x)$ at $x = 1$ with $N = 12$ and $P_1 = M = P_2 = 1.0$, $\delta = 0.1$ for different values of S, β, γ and compares the result with the HAM presented in [21]. In [21], there is no analysis about the convergence or error estimate of results, whereas in the current work we discussed about the convergence of method and residual errors. Hence, we can claim from the error analysis that out obtained results are more accurate than [21]. For example, the result of $\theta'(1)$ with respect to $S = 4.00$ in [21] is 0.281319, but in the present techniqe we get 0.2880499297. It is evident that there is little difference between the obtained results, which the present method gives more accurate results.

Table 1. Values of $-f''(1)$ for different values of S, M and β.

S	M	β	**Hussain et al. [21]**	**Present**
1.00	1.00	0.00	3.180310	3.1803102750
		0.05	2.414897	2.4148967196
		0.10	1.945943	1.9459433694
		0.15	1.629328	1.6293275195
		0.20	–	1.4012474830
		0.50	–	0.7614007594
		1.00	–	0.4322969902
	0.00	0.10	1.928044	1.9280441735
	2.00		1.997081	1.9970814583
	3.00		2.074769	2.0747689044
2.00	1.00		1.994008	1.9940077672
3.00			2.038898	2.0388977767
0.10			–	1.8996168697
0.50			–	1.9205892930

Table 2. Results of $\theta'(1)$ for various values of S, β and γ with $P_1 = M = P_2 = 1.0$ and $\delta = 0.1$.

S	β	γ	**Hussain et al. [21]**	**Present**
1.00	0.10	0.00	0.415935	0.4159351994
		0.01	0.396709	0.3967094545
		0.05	0.326735	0.3267347576
		0.10	0.252278	0.2522783590
		0.50	–	-0.0766615072
	0.00	0.10	-0.687930	-0.6879304850
	0.01		-0.514271	-0.5142710439
	0.02		-0.368998	-0.3689980708
	0.03		-0.246254	-0.2462544541
	0.10		–	0.2522783590
	0.50		–	0.7301443477
	1.00		–	0.7892670209
2.00	0.10		0.263761	0.2637611894
3.00			0.273318	0.2733180923
4.00			0.281319	0.2880499297
0.10			–	0.2399056923
0.20			–	0.2413894057
0.50			–	0.2456708266

The effect of Hartmann number M on the radial velocity $f'(x)$ is exhibited in Figure 1. The radial velocity $f'(x)$, decreased for higher values of the Hartmann number on $0.24 \leq x \leq 0.76$. The influence of parameter β on $f'(x)$ is plotted in Figure 2 with $M = S = 1.0$. This Figure suggests that the $f'(x)$ show decreasing behavior with an increase in β. Figures 3 and 4 display the temperature profiles $\theta(x)$ for the various embedded parameters viz thermal slip parameter γ and Eckert number P_2 on interval $[0, 1]$. It is seen that when $\gamma = 0$, which corresponds to no thermal slip, the temperature of the fluid and that of the disks surfaces is the same, which in this case is 0 and 1 for lower and upper disks, respectively.

Since the exact solution of problems (1.1)–(1.4) is not known, we discuss the absolute residual error function which is a measure of how well the approximation satisfies the Eq (4.6) with $S = P_1 = M = P_2 = 1.0$ and $\beta = \gamma = \delta = 0.1$ as

$$Res(x) = |\mathbb{L}\Phi(x) + N(\Phi(x), \Phi^{'}(x), \Phi^{''}(x)) - g(x)|.$$

Note that the norm 2 of the residual function on the domain is

$$||Res||_2 = (\int_0^1 |Res(x)|^2 dx)^{\frac{1}{2}},$$

and it is employed in this paper to check the accuracy and the convergence of the proposed method. The absolute residual errors are plotted in Figure 5.

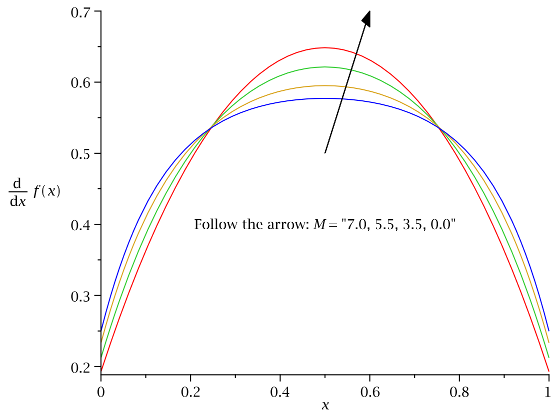

Figure 1. Influence of magnetic parameter M on the radial velocity $f'(x)$ for values of $S = 1.0$ and $\beta = 0.1$.

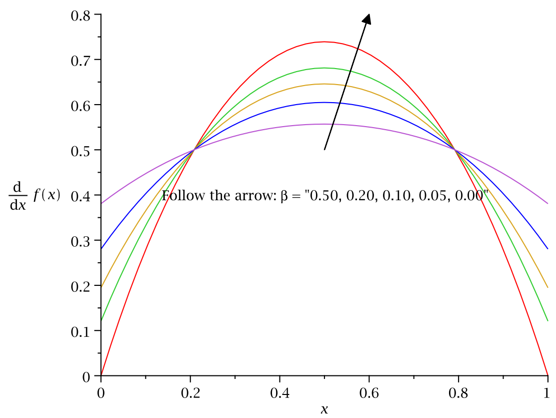

Figure 2. Influence of parameter β on the radial velocity $f'(x)$ for values of $S = 1.0$ and $M = 1.0$.

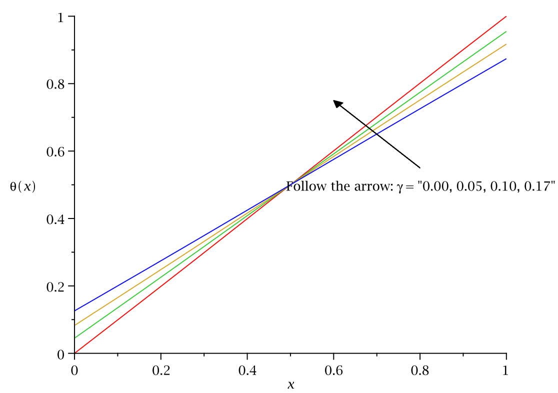

Figure 3. Influence of thermal slip parameter γ on the temperature profile $\theta(x)$ for values of $S = M = P_1 = 1.0, \beta = \delta = 0.1$ and $P_2 = 0.0$.

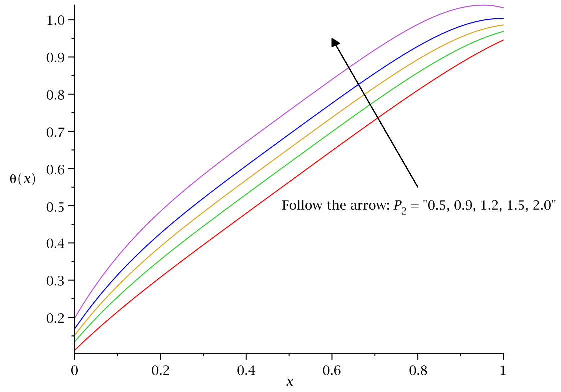

Figure 4. Influence of parameter P_2 on the temperature profile $\theta(x)$ for values of $P_1 = S = M = 1.0$ and $\beta = \gamma = \delta = 0.1$.

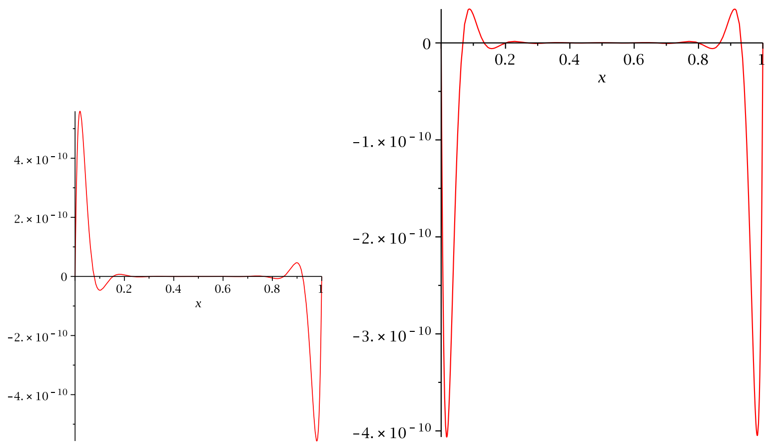

Figure 5. Residual errors $f'(x)$ and $\theta(x)$, with $S = P_1 = M = P_2 = 1.0$ and $\beta = \gamma = \delta = 0.1$, respectively (from left to right).

7. Conclusions and perspectives

In this paper, the shifted Legendre reproducing kernel method is employed to compute approximate solutions of a nonlinear system of ordinary differential equation. In this approach, a truncated series based on shifted Legendre reproducing kernel functions with easily computable components. The convergence analysis and error estimation of the approximate solution using the proposed method are investigated. The validity and applicability of the method is demonstrated by solving several numerical examples. The main advantage of the present method lies in the lower computational cost and high accuracy. System of differential equations appear in various branches of science and technology. Results of current study show that the shifted Legendre reproducing kernel method is a reliable technique for the physical models in the system of differential equations form. Moreover, this method could be developed for systems of differential equations with fractional order derivatives or system of integro-differential equations.

Acknowledgments

The authors wish to express their thanks to the referees for comments which improved the paper.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. F. Ahmad, A simple analytical solution for the steady flow of a third grade fluid in a porous half space, *Commun. Nonlinear Sci. Numer. Simul.*, **14** (2009), 2848–2852. https://doi.org/10.1016/j.cnsns.2008.09.029
2. A. Akgül, H. Ahmad, Reproducing kernel method for Fangzhu's oscillator for water collection from air, *Math. Methods Appl. Sci.*, 2020. https://doi.org/10.1002/mma.6853
3. O. A. Arqub, M. Al-Smadi, Numerical algorithm for solving two-point, second-order periodic boundary value problems for mixed integro-differential equations, *Appl. Math. Comput.*, **243** (2014), 911–922. https://doi.org/10.1016/j.amc.2014.06.063
4. M. Al-Smadi, O. A. Arqub, N. Shawagfeh, S. Momani, Numerical investigations for systems of second-order periodic boundary value problems using reproducing kernel method, *Appl. Math. Comput.*, **291** (2016), 137–148. https://doi.org/10.1016/j.amc.2016.06.002
5. X. Cheng, C. Zhong, Existence of positive solutions for a second order ordinary differential system, *J. Math. Anal. Appl.*, **312** (2005), 14–23. https://doi.org/10.1016/j.jmaa.2005.03.016
6. Y. M. Chu, U. Nazir, M. Sohail, M. M. Selim, J. R. Lee, Enhancement in thermal energy and solute particles using hybrid nanoparticles by engaging activation energy and chemical reaction over a parabolic surface via finite element approach, *Fractal Fract.*, **5** (2021), 119. https://doi.org/10.3390/fractalfract5030119
7. M. G. Cui, Y. Lin, *Nonlinear numerical analysis in reproducing kernel space*, Nova Science Publishers, 2009.
8. M. Dehghan, M. Lakestani, Numerical solution of nonlinear system of second-order boundary value problems using cubic Bspline scaling functions, *Int. J Comput. Math.*, **85** (2008), 1455–1461. https://doi.org/10.1080/00207160701534763
9. M. R. Foroutan, A. Ebadian, R. Asadi, A reproducing kernel Hilbert space method for solving the nonlinear three-point boundary value problems, *Int. J. Numer. Model.*, **32** (2019), e2573. https://doi.org/10.1002/jnm.2573
10. M. R. Foroutan, A. Ebadian, R. Asadi, Reproducing kernel method in Hilbert spaces for solving the linear and nonlinear four-point boundary value problems, *Int. J. Comput. Math.*, **95** (2018), 2128–2142. https://doi.org/10.1080/00207160.2017.1366464
11. M. R. Foroutan, A. Ebadian, S. Najafzadeh, Analysis of unsteady stagnation-point flow over a shrinking sheet and solving the equation with rational Chebyshev functions, *Math. Methods Appl. Sci.*, **40** (2017), 2610–2622. https://doi.org/10.1002/mma.4185

12. F. Z. Geng, Solving singular second order three-point boundary value problems using reproducing kernel Hilbert space method, *Appl. Math. Comput.*, **215** (2009), 2095–2102. https://doi.org/10.1016/j.amc.2009.08.002
13. F. Z. Geng, M. G. Cui, Solving singular nonlinear second-order periodic boundary value problems in the reproducing kernel space, *Appl. Math. Comput.*, **192** (2007), 389–398. https://doi.org/10.1016/j.amc.2007.03.016
14. S. N. Hajiseyedazizi, M. E. Samei, J. Alzabut, Y. M. Chu, On multi-step methods for singular fractional q-integro-differential equations, *Open Math.*, **19** (2021), 1378–1405. https://doi.org/10.1515/math-2021-0093
15. M. S. Hashemi, D. Baleanu, Numerical approximation of higher-order time-fractional telegraph equation by using a combination of a geometric approach and method of line, *J. Comput. Phys.*, **316** (2016), 10–20. https://doi.org/10.1016/j.jcp.2016.04.009
16. M. S. Hashemi, Constructing a new geometric numerical integration method to the nonlinear heat transfer equations, *Commun. Nonlinear Sci. Numer. Simul.*, **22** (2015), 990–1001. https://doi.org/10.1016/j.cnsns.2014.09.026
17. M. S. Hashemi, E. Darvishi, D. Baleanu, A geometric approach for solving the density-dependent diffusion Nagumo equation, *Adv. Differ. Equ.*, **2016** (2016), 89. https://doi.org/10.1186/s13662-016-0818-2
18. M. S. Hashemi, A novel simple algorithm for solving the magneto-hemodynamic flow in a semi-porous channel, *Eur. J. Mech. B/Fluids*, **65** (2017), 359–367. https://doi.org/10.1016/j.euromechflu.2017.05.008
19. M. S. Hashemi, Numerical solution to the telegraph equation via the geometric moving Kriging meshfree method, *Eur. Phys. J. Plus*, **134** (2019), 381. https://doi.org/10.1140/epjp/i2019-12741-8
20. Z. Y. He, A. Abbes, H. Jahanshahi, N. D. Alotaibi, Y. Wang, Fractional-order discrete-time SIR epidemic model with vaccination: Chaos and complexity, *Mathematics*, **10** (2022), 165. https://doi.org/10.3390/math10020165
21. A. Hussain, S. T. Mohyud-Din, T. A. Cheema, Analytical and numerical approaches to squeezing flow and heat transfer between two parallel disks with velocity slip and temperature jump, *Chin. Phys. Lett.*, **29** (2012), 114705.
22. M. A. Iqbal, Y. Wang, M. M. Miah, M. S. Osman, Study on date-Jimbo-Kashiwara-Miwa equation with conformable derivative dependent on time parameter to find the exact dynamic wave solutions, *Fractal Fract.*, **6** (2022), 4. https://doi.org/10.3390/fractalfract6010004
23. F. Jin, Z. S. Qian, Y. M. Chu, M. ur Rahman, On nonlinear evolution model for drinking behavior under Caputo-Fabrizio derivative, *J. Appl. Anal. Comput.*, **12** (2022), 790–806. https://doi.org/10.11948/20210357
24. S. C. S. Rao, M. Kumar, B-spline collocation method for nonlinear singularly-perturbed two-point boundary-value problems, *J. Optim. Theory Appl.*, **134** (2007), 91–105. https://doi.org/10.1007/s10957-007-9200-6

25. S. Rashid, S. Sultana, Y. Karaca, A. Khalid, Y. M. Chu, Some further extensions considering discrete proportional fractional operators, *Fractals*, **30** (2022), 2240026. https://doi.org/10.1142/S0218348X22400266

26. S. Rashid, E. I. Abouelmagd, A. Khalid, F. B. Farooq, Y. M. Chu, Some recent developments on dynamical $\hbar$-discrete fractional type inequalities in the frame of nonsingular and nonlocal kernels, *Fractals*, **30** (2022), 2240110 https://10.1142/S0218348X22401107

27. A. Saadatmandi, M. Dehghan, A. Eftekhari, Application of He's homotopy perturbation method for non-linear system of second-order boundary value problems, *Nonlinear Analy: Real World Appl.*, **10** (2009), 1912–1922. https://doi.org/10.1016/j.nonrwa.2008.02.032

28. A. Saadatmandi, J. A. D. Farsangi, Chebyshev finite difference method for a nonlinear system of second-order boundary value problems, *Appl. Math. Comput.*, **192** (2007), 586–591. https://doi.org/10.1016/j.amc.2007.02.148

29. S. Saitoh, Y. Sawano, *Theory of reproducing kernels and applications*, Springer, 2016. https://doi.org/10.1007/978-981-10-0530-5

30. J. Shen, T. Tang, *Spectral and high-order methods with applications*, Beijing: Science Press, 2006.

31. H. B. Thompson, C. Tisdell, Boundary value problems for systems of difference equations associated with systems of second-order ordinary differential equations, *Appl. Math. Lett.*, **15** (2002), 761–766. https://doi.org/10.1016/S0893-9659(02)00039-3

32. H. B. Thompson, C. Tisdell, Systems of difference equations associated with boundary value problems for second order systems of ordinary differential equations, *J. Math. Anal. Appl.*, **248** (2000), 333–347. https://doi.org/10.1006/jmaa.2000.6823

33. F. Z. Wang, M. N. Khan, I. Ahmad, H. Ahmad, H. Abu-Zinadah, Y. M. Chu, Numerical solution of traveling waves in chemical kinetics: Time-fractional fishers equations, *Fractals*, **30** (2022), 2240051. https://doi.org/10.1142/S0218348X22400515

34. L. H. Yang, Y. Lin, Reproducing kernel methods for solving linear initial-boundary-value problems, *Electron. J. Differ. Equ.*, **2008** (2008), 1–11.

35. C. P. Zhang, J. Niu, Y. Z. Lin, Numerical solutions for the three-point boundary value problem of nonlinear fractional differential equations, *Abstr. Appl. Anal.*, **2012** (2012), 360631. https://doi.org/10.1155/2012/360631

36. Z. H. Zhao, Y. Z. Lin, J. Niu, Convergence order of the reproducing kernel method for solving boundary value problems, *Math. Model. Anal.*, **21** (2016), 466–477 https://doi.org/10.3846/13926292.2016.1183240

37. T. H. Zhao, W. M. Qian, Y. M. Chu, On approximating the arc lemniscate functions, *Indian J. Pure Appl. Math.*, 2021. https://doi.org/10.1007/s13226-021-00016-9