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Research article

Solving a class of high-order fractional stochastic heat equations with fractional noise

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Abstract: This paper is concerned with a class of high-order fractional stochastic partial differential equations driven by fractional noise. We firstly prove the existence and uniqueness of the mild solution and then study the Hölder continuity of the solution with respect to space and time variables. In addition, we also prove the existence and Gaussian-type estimates for the density of the solution by using the techniques of Malliavin calculus.

Keywords: fractional differential operator; stochastic partial differential equation; fractional noise; Hölder continuity; Malliavin calculus
Mathematics Subject Classification: 60G35, 60H07, 60H15

1. Introduction

In this paper, we are interested in the following high-order fractional stochastic partial differential equations (SPDEs for short) driven by fractional noise

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \mathcal{D}_\delta^\alpha u(t,x) + H_0(t,x,u(t,x)) + \displaystyle\sum_{k=1}^{m} \frac{\partial^k H_k}{\partial x^k}(t,x,u(t,x)) + \dot{B}^H(t,x), \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

with $(t,x) \in [0,T]\times\mathbb{R}$ and $T > 0$, where $\mathcal{D}_\delta^\alpha$ is the fractional differential operator introduced in Debbi [8], and further studied by Debbi and Dozzi [7], Xie [21] recently. Moreover the coefficients $\{H_k, k = 1,2,\ldots,m\}$ with $m < [\alpha]$ which is an integer, are some measurable functions satisfying some conditions and $\dot{B}^H(t,x)$ denotes the fractional noise considered by Jiang *et al* [12], Hu *et al* [10] which will be explained in Section 2.

Partial differential equation plays a fundamental role in describing various phenomena, such as diffusion in a disorder or fractal medium, image processing or risk management. However, various models in the real world must take into account the uncertainty. Hence the investigation of SPDEs, obtained great attentions. And they have been successfully applied in different fields, such as population biology, quantum field, statistical physics, neurophysiology and so on. Gaussian noise are used widely to characterize some kinds of uncertainty in some models. The SPDEs driven by them are deeply studied until now, see Walsh [22], Dalang [6] , Chow [5], Debbi and Dozzi [7] for more details. In the meanwhile, there has been some recent interests in studying SPDEs driven by fractional noise (see Section 2 for the details). More works in the fields can also be found in Balan [1], Balan and Tudor [2], Bo *et al* [3],, Hu and Nualart [9], Hu, Nualart and Song [10], Hu, Lu and Nualart [11], Jiang *et al* [12], Liu and Yan [13], Liu and Tudor [14] and the references therein.

On the other hand, in recent years, there have been increased interest in fractional order calculus since the development of the regular integer order calculus. Then several kinds of fractional order integro-differential operators have been introduced, for example, Debbi and Dozzi [7], Podlubny [20] and etc. The fractional calculus, containing the fractional operators, such as the fractional Laplacian operator and the pseudo differential operator, has been widely used to system modelling and controller design concisely and precisely (for example, the indirect model reference adaptive control in [4], the output feedback control synthesis in [23] and so on.)

Motivated by the above results on SPDEs and fractional calculus, in this paper, we will study a class of high-order fractional SPDEs (1.1) which combines the fractional differential operator $\mathcal{D}_\delta^\alpha(\alpha > 1)$ and the fractional noise $\dot{B}^H$. It is known that the operator $\mathcal{D}_\delta^\alpha$ involved in SPDEs (1.1) extends the inverse of the generalized Riesz-Feller potential when $\alpha > 2$, the Riemann-Liouville type operator, the fractional Laplacian operator with $0 < \alpha \leq 2$ and a class of pseudo differential operator. Moreover the fractional SPDEs (1.1) includes the famous Ginzburg-Landau equations with or without conservation as examples. In fact, in Debbi [7], the author studied the nonlinear stochastic fractional partial differential equation containing $\mathcal{D}_\delta^\alpha(\alpha > 1)$ and Gaussian space-time white noise. While Xie [21] studied the similar class of stochastic fractional partial differential equation driven by impulsive noise, which is singular not only in time but also in space. However, until now, there was little work on high-order fractional SPDEs driven by fractional noise $\dot{B}^H$. Such equation can be viewed as a more flexible alternative to the equations driven by white noise, and it can be used to model the more complex physical phenomenon which is subject to random perturbations. Hence it is worth studying SPDEs (1.1) in this sense. Moreover the results obtained in this work generalizes the results in Debbi [7] to the setting of fractional noise and some results in Balan [1], Balan and Tudor [2], Bo *et al* [3] and Hu and Nualart [9] to the setting of fractional differential operator.

The rest of this paper is organized as follows. In Section 2, we briefly recall some properties of the fractional differential operator $\mathcal{D}_\delta^\alpha$, then introduce the definition of fractional noise and the Malliavin calculus with respect to it. In Section 3, we will show the existence and uniqueness for the solution to SPDE (1.1) under some assumptions. Hölder continuity of the solution $\{u(t,x),(t,x) \in [0,T] \times \mathbb{R}\}$ on both space and time parameters is obtained in Section 4. In Section 5, we prove that the law of the solution to (1.1) is absolutely continuous with respect to Lebesgue measure. Furthermore the Gaussian-type estimate for the density of such solution is also obtained by using the techniques of Malliavin calculus and the formulas developed by Nourdin and Viens [16].

2. Preliminaries

In this section, we will firstly recall some definitions and fundamental properties of the fractional differential operator $\mathcal{D}_\delta^\alpha$ and the fractional noise $\dot{B}^H$. We also recall the techniques of Malliavin calculus with respect to fractional noise.

2.1. Fractional differential operator $\mathcal{D}_\delta^\alpha$

From Debbi and Dozzi [7], we recall that the fractional differential operator $\mathcal{D}_\delta^\alpha$ is defined by

$$\mathcal{D}_\delta^\alpha f(x) := \mathcal{D}_\alpha f(x) = \mathcal{F}^{-1}\{-|\cdot|^\alpha e^{-i\delta\pi sgn(\cdot)/2}\mathcal{F}(f)\}(x),$$

where $\delta \leq \min\{\alpha - [\alpha]_2, 2 + [\alpha]_2 - \alpha\}$, $[\alpha]$ and $[\alpha]_2$ denote the largest integer and largest even integer less or equal to α, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier and inverse Fourier transformations respectively.

The operator $\mathcal{D}_\delta^\alpha$ is a non-selfadjoint, closed, densely defined operator on $L^2(\mathbb{R})$ and it is the infinitesimal generator of a semigroup which is in general not symmetric and not a contraction. This operator includes many important operators, such as fractional power of the Laplacian operator with $0 < \alpha \leq 2$ and $\delta = 0$, the Riemann-Liouville differential operator when $|\delta| = 2+[\alpha]_2-\alpha$ or $|\delta| = \alpha-[\alpha]_2$. Evidently, it is Laplacian itself when $\alpha = 2$ and $\delta = 0$.

The Green function $G_\alpha(t, x)$ associated to SPDE (1.1) on $[0, T] \times \mathbb{R}$ is the fundamental solution of the following Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}G_\alpha(t, x) = \mathcal{D}_\delta^\alpha G_\alpha(t, x), & (t, x) \in [0, T] \times \mathbb{R}, \\ G_\alpha(0, x) = \delta_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where δ_0 is the Dirac function. Using the Fourier transform, we see that $G_\alpha(t, x)$ is given by

$$G_\alpha(t, x) = \mathcal{F}^{-1}\left(e^{\delta\Psi_\alpha(\lambda)t}\right)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left\{-i\lambda x - t|\lambda|^\alpha e^{-i\delta\frac{\pi}{2}sgn(\lambda)}\right\} d\lambda. \tag{2.2}$$

which may be asymmetric in x (see for example, Debbi [8], Debbi and Dozzi [7] for some details). Moreover the Green function $G_\alpha(t, x)$ has some good properties such that it satisfies the Chapmann-Kolmogorov equation and is smooth in x for each fixed $t > 0$ and so on. Here, we will list some known fundamental properties for $G_\alpha(t, x)$ which will be used later on (see e.g. Debbi [8], Debbi and Dozzi [7] for their proofs).

Lemma 1. *1. The Green function $G_\alpha(t, x)$ is real but in general it is not symmetric relatively to x and it is not everywhere positive. Moreover $\int_{-\infty}^{+\infty} G_\alpha(t, x)dx = 1$.*

2. The scaling property holds: $G_\alpha(t, x) = t^{-\frac{1}{\alpha}}G_\alpha(1, t^{-\frac{1}{\alpha}}x)$. Moreover, for any $k > 0$

$$\frac{\partial^k G_\alpha}{\partial x^k}(t, x) = t^{-\frac{k+1}{\alpha}}\frac{\partial^k G_\alpha}{\partial y^k}(1, y)\Big|_{y=t^{-\frac{1}{\alpha}}x}. \tag{2.3}$$

3. There exists a constant C_α such that

$$\left|\frac{\partial^k G_\alpha}{\partial x^k}(1, x)\right| \leq C_\alpha\frac{1 + |x|^{\alpha+k-1}}{(1 + |x|^{\alpha+k})^2}, \quad k \in \mathbb{N}. \tag{2.4}$$

4. *Let* $\alpha \in (1, +\infty)/\mathbb{N}$*, for any fixed* $k \in \mathbb{N}$ *and each* $\beta \in \left(\frac{1}{\alpha+k+1}, \frac{\alpha+1}{k+1}\right)$*, we have*

$$\int_0^T \int_{\mathbb{R}} \left| \frac{\partial^k G_\alpha}{\partial x^k}(t, x) \right|^\beta dtdx < \infty.$$

5. *The Green function* $G_\alpha(t, x)$ *satisfies the semigroup property, or the Chapmann-Kolmogorov equation, i.e. for* $0 < s \leq t$

$$G_\alpha(t + s, x) = \int_{\mathbb{R}} G_\alpha(t, y) G_\alpha(s, x - y) dy. \tag{2.5}$$

2.2. Fractional noise

For some $T > 0$, a one-dimensional fractional Brownian motion $(W_t^H)_{t\in[0,T]}$ with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with covariance

$$E\left[W_t^H W_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right).$$

We recall the following fractional Brownian sheet considered in Jiang *et al* [12], Hu *et al* [10] and etc.

Definition 1. *The fractional Brownian sheet* $B^H = \{B^H(t, x), (t, x) \in [0, T] \times \mathbb{R}\}$ *with parameter* $H = (H_1, H_2)$ *for* $H_i \in (0, 1)$*,* $i = 1, 2$*, is a centered Gaussian process on some probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *with covariance*

$$\begin{aligned} R(t, s; x, y) &= E\left[B^H(t, x) B^H(s, y)\right] \\ &= \frac{1}{4}\left(t^{2H_1} + s^{2H_1} - |t - s|^{2H_1}\right)\left(|x|^{2H_2} + |y|^{2H_2} - |x - y|^{2H_2}\right), \end{aligned}$$

for all $s, t \in [0, T]$ *and* $x, y \in \mathbb{R}$*.*

We denote by $\mathcal{E}$ the set of step functions on $[0, T] \times \mathbb{R}$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle 1_{[0,t]\times[0,x]}, 1_{[0,s]\times[0,y]} \right\rangle_{\mathcal{H}} = R(t, s; x, y).$$

Thus the mapping $1_{[0,t]\times[0,x]} \mapsto B^H(1_{\{[0,t]\times[0,x]\}}) := B^H(t, x)$ is an isometry between $\mathcal{E}$ and the linear space span of $\left\{B^H(1_{\{[0,t]\times[0,x]\}}), (t, x) \in [0, T] \times \mathbb{R}\right\}$. Moreover, the mapping can be extended to an isometry from $\mathcal{H}$ to Gaussian space associated with B^H. This isometry will be denoted by $\varphi \mapsto B^H(\varphi)$ for $\varphi \in \mathcal{H}$. Therefore, we can regard $B^H(\varphi)$ as the stochastic integral with respect to B^H. In general, we use the notation

$$B^H(\varphi) = \int_0^T \int_{\mathbb{R}} \varphi(t, y) B^H(dt, dy), \quad \varphi \in \mathcal{H}.$$

Throughout this paper, we limit our consideration on the two-parameter fractional Brownian sheet with Hurst parameters $H_i \in (1/2, 1), i = 1, 2$. For any $0 \leq s < t \leq T$ and $x, y \in \mathbb{R}$ let

$$\Psi_H(t, s; x, y) = 4H_1H_2(2H_1 - 1)(2H_2 - 1)|t - s|^{2H_1-2}|x - y|^{2H_2-2}.$$

Furthermore, the following properties hold.

Proposition 1. *For $\varphi, \psi \in \mathcal{H}$, we have*

$$E\left[\int_0^T \int_{\mathbb{R}} \varphi(t,x) B^H(dt,dx)\right] = 0,$$

and

$$E\left[\int_0^T \int_{\mathbb{R}} \varphi(t,x) B^H(dt,dx) \int_0^T \int_{\mathbb{R}} \psi(t,x) B^H(dt,dx)\right]$$
$$= \int_{[0,T]^2} \int_{\mathbb{R}^2} \varphi(u,x) \Psi_H(u,v;x,y) \psi(v,y) dxdydudv.$$

Let us also recall an embedding theorem proved in Jiang *et al* [12].

Proposition 2. *If $H \in (1/2, 1)$ and $f, g \in L^{\frac{1}{H}}([a,b])$, then*

$$\int_a^b \int_a^b f(u)g(v)|u-v|^{2H-2} dudv \le C_H \|f\|_{L^{\frac{1}{H}}([a,b])} \|g\|_{L^{\frac{1}{H}}([a,b])},$$

where $C_H > 0$ is a constant depending only on H.

On the hand, since $B^H = \{B^H(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$ is Gaussian, we might develop the Malliavin calculus with respect to B^H in order to study the density properties for the solution of SPDE (1.1) (see e.g. Nualart [17]).

Let $\mathcal{S}$ be the class of smooth and cylindrical random variables of the form $F = f(B^H(\varphi_1), \ldots, B^H(\varphi_n))$, where $f \in C_b^\infty(\mathbb{R}^n)$ (i.e. the set of all functions with bounded derivatives of all orders) and $\varphi_i \in \mathcal{H}$ ($i = 1, \ldots, n$ and $n \in \mathbb{N}$). For each $F \in \mathcal{S}$, define the derivative $D_{t,x}F$ by

$$D_{t,x}F := \sum_{i=1}^n \frac{\partial f}{\partial x}(B^H(\varphi_1), \ldots, B^H(\varphi_n)) \varphi_i(t,x).$$

Let $\mathbb{D}^{1,2}$ be the completion of $\mathcal{S}$ under the norm

$$\|F\|_{1,2}^2 = E\left[|F|^2 + \|DF\|_{\mathcal{H}}^2\right].$$

Then $\mathbb{D}^{1,2}$ is the domain of the closed operator D on $L^2(\Omega)$ with the domain $\mathbb{D}_h$ being the closure of $\mathcal{S}$ under the norm

$$\|F\|_h^2 = E\left[|F|^2 + |D_h F|^2\right].$$

Let $\{h_n, n \ge 1\}$ be an orthonormal basis of $\mathcal{H}$. Then $F \in \mathbb{D}^{1,2}$ if and only if $F \in \mathbb{D}_{h_n}$ for each $n \in N$ and $\sum_{n=1}^\infty E|D_{h_n}F|^2 < \infty$. In this case, $D_h F = \langle DF, h \rangle_{\mathcal{H}}$. On the other hand, the divergence operator δ is the adjoint of the derivative operator D characterized by

$$E\langle DF, u \rangle_{\mathcal{H}} = E(F\delta(u)), \quad \text{for} \quad \text{any} \quad F \in \mathcal{S},$$

where $u \in L^2(\Omega, \mathcal{H})$. Then $Dom(\delta)$, the domain of δ, is the set of all functions $u \in L^2(\Omega, \mathcal{H})$ such that

$$E|\langle DF, u \rangle_{\mathcal{H}}| \le C(u) \|F\|_{L^2(\Omega)},$$

where $C(u)$ is some constant depending on u.

3. Existence of the solution

In this section, we will prove the existence and uniqueness of the solution to the high-order fractional SPDE (1.1). Let us first give the following notation of *mild solution* for SPDE (1.1) in the sense of Walsh [22].

Definition 2. *Let $T > 0$ be fixed. A random field $u = \{u(t, x), t, x \in [0, T] \times \mathbb{R}\}$ is a mild solution of SPDE* (1.1) *if*

$$\begin{aligned} u(t,x) = \int_{\mathbb{R}} G_\alpha(t, x-z) u_0(z) dz + \int_0^t \int_{\mathbb{R}} H_0(s, z, u(s, z)) G_\alpha(t-s, x-z) dz ds \\ + (-1)^k \int_0^t \int_{\mathbb{R}} \sum_{k=1}^m H_k(s, z, u(s, z)) \frac{\partial^k G_\alpha}{\partial z^k}(t-s, x-z) dz ds \\ + \int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-z) B^H(ds, dz), \end{aligned} \tag{3.1}$$

for any $(t, x) \in [0, T] \times \mathbb{R}$, where $G_\alpha(t, x)$ is the Green function solving Eq (2.1) *and $\frac{\partial^k G_\alpha}{\partial x^k}(t, x)$ is its partial derivative of order k with respect to the spatial variable.*

In order to explain our theorem, we present some conditions on the functions $\{H_k, k = 1, 2, \ldots, m\}$ for any $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$.

(C1) The growth conditions: For any $k = 0, 1, 2, \ldots, m$, there exists a positive constant K_T such that

$$|H_k(t, x, y)| \leq K_T(1 + |y|).$$

(C2) Lipschitz conditions: For any $k = 0, 1, 2, \ldots, m$, there exists a positive constant K_T such that

$$|H_k(t, x, y) - H_k(t, x, z)| \leq K_T |y - z|.$$

Then we can state the following main result in this section.

Theorem 1. *Under the conditions* **(C1)** *and* **(C2)**, *for $\alpha > 1$, $p \geq 2$, $m = [\alpha]$ and the assumptions that the initial condition u_0 is a measurable function and uniformly bounded on $\mathbb{R}$, then the SPDE* (1.1) *has a unique adapted mild solution which satisfies*

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} E[|u(t, x)|^p] < \infty.$$

Let us firstly give several useful lemmas.

Lemma 2. *For any $1/2 < H < 1$ and $t \in [0, T]$, there exists a positive constant K depending on α such that*

$$\int_{\mathbb{R}} |G_\alpha(t, x)|^{\frac{1}{H}} dx \leq K_\alpha t^{\frac{H-1}{\alpha H}}. \tag{3.2}$$

Proof. According to scaling property and estimates for the Green function $G_\alpha(t,x)$ given in Lemma 1, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} |G_\alpha(t,x)|^{\frac{1}{H}} dx &= t^{-\frac{1}{\alpha H}} \int_{\mathbb{R}} |G_\alpha(1, t^{-\frac{1}{\alpha}}x)|^{\frac{1}{H}} dx \\
&\le K_\alpha t^{\frac{1}{\alpha}-\frac{1}{\alpha H}} \int_{\mathbb{R}} \frac{1}{(1+|x|^{1+\alpha})^{\frac{1}{H}}} dx \\
&= K_\alpha t^{\frac{1}{\alpha}-\frac{1}{\alpha H}} \left(\int_0^1 \frac{1}{(1+x^{1+\alpha})^{\frac{1}{H}}} dx + \int_1^\infty \frac{1}{(1+x^{1+\alpha})^{\frac{1}{H}}} dx \right) \\
&\le K_{\alpha,H} t^{\frac{H-1}{\alpha H}}.
\end{aligned} \tag{3.3}$$

□

Lemma 3. *Let the sequence $\{u^{(n)}, n \ge 0\}$ be defined by*

$$u^{(0)}(t,x) = \int_{\mathbb{R}} G_\alpha(t, x-y) u_0(y) dy, \tag{3.4}$$

$$\begin{aligned}
u^{(n+1)}(t,x) =& u^{(0)}(t,x) + \int_0^t \int_{\mathbb{R}} H_0(s,y,u^{(n)}(s,y)) G_\alpha(t-s, x-y) dy ds \\
&+ \sum_{k=1}^m (-1)^k \int_0^t \int_{\mathbb{R}} H_k(s,y,u^{(n)}(s,y)) \frac{\partial^k G_\alpha}{\partial y^k}(t-s, x-y) dy ds \\
&+ \int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-y) B^H(ds, dy).
\end{aligned} \tag{3.5}$$

Then under conditions **(C1)**, **(C2)**, *for any $p \ge 2$ and $\alpha > 1$, the sequence $\{u^{(n)}(t,x), n \ge 1\}$ is well-defined and satisfies*

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} E\left|u^{(n)}(t,x)\right|^p < \infty. \tag{3.6}$$

Proof. We proceed by recurrence. Under the hypothesis that the initial condition $u_0(x)$ is $L^p(\Omega)$-bounded, it is easy to see that the function $u^{(0)}(t,x)$ and $u^{(1)}(t,x)$ exist and are also $L^p(\Omega)$-bounded.

Now let us prove (3.6). Note that, for each $n \in \mathbb{N}$ and $p \ge 2$, it follows that

$$\begin{aligned}
&E\left|u^{(n)}(t,x)\right|^p \\
&\le K_p \Bigg[E|u^0(t,x)|^p + \sum_{k=0}^m E\left| \int_0^t \int_{\mathbb{R}} H_k(s,y,u^{(n)}(s,y)) \frac{\partial^k G_\alpha}{\partial y^k}(t-s, x-y) dy ds \right|^p \\
&\qquad + E\left| \int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-y) B^H(ds, dy) \right|^p \Bigg] \\
&:= K_p \left[E|u^{(0)}(t,x)|^p + \sum_{k=0}^m A_{p,k}^{(n)}(t,x) + B_p^{(n)}(t,x) \right].
\end{aligned}$$

For the first term $E|u^{(0)}(t,x)|^p$, since the initial condition $u_0(x)$ is uniformly bounded, then we have $E|u^{(0)}(t,x)|^p \le ||u_0||_\infty$, where the notation $||u_0||_\infty$ is defined by $||u_0||_\infty := \sup_{x\in\mathbb{R}} |u_0(x)|$.

For the terms $A^{(n)}_{p,k}(t,x)$ with $k = 0, 1, \dots, m$, by using the Hölder inequality on the measure $\frac{\partial^k G_\alpha}{\partial y^k}(t - s, x - y)dy$ and the linear growth condition for the coefficients $H_k, k = 0, 1, \dots, n$, we have

$$\begin{aligned}
A^{(n)}_{p,k}(t,x) \le & \left(\int_0^t \int_{\mathbb{R}} \left|\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)\right| dyds\right)^{p-1} \\
& \cdot E\left[\int_0^t \int_{\mathbb{R}} \left|H_k(s,y,u^{(n)}(s,y))\right|^p \cdot \left|\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)\right| dyds\right] \\
\le & K_p E\left[\int_0^t \int_{\mathbb{R}} (1 + |u^{(n)}(s,y)|^p) \left|\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)\right| dyds\right] \\
\le & K_p \left[\int_0^t \sup_{y\in\mathbb{R}} E(1 + |u^{(n)}(s,y)|^p) \int_{\mathbb{R}} \left|\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)\right| dyds\right] \\
\le & K_{p,\alpha} \int_0^t \sup_{y\in\mathbb{R}} E(1 + |u^{(n)}(s,y)|^p)(t-s)^{-k/\alpha} ds,
\end{aligned}$$

where we have used the fact that, for some positive constant K

$$\int_{\mathbb{R}} \left|\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)\right| dy \le K(t-s)^{-k/\alpha}. \tag{3.7}$$

For all $(t, x) \in [0, T] \times \mathbb{R}$, since the stochastic integral $\int_0^t \int_{\mathbb{R}} G_\alpha(t - s, x - y)B^H(ds, dy)$ is Gaussian, then according to Lemma 1, we have

$$\begin{aligned}
& E\left|\int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y)B^H(ds,dy)\right|^p \\
& \le C_p \left(E\int_0^t \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G_\alpha(t-s,x-y)G_\alpha(t-r,x-z)\Psi_H(s,r;y,z)dydzdrds\right)^{p/2} \\
& \le C_{p,H_2} \left(\int_0^t \int_0^t ||G_\alpha(t-s,x-\cdot)||_{L^{1/H_2}(\mathbb{R})} ||G_\alpha(t-r,x-\cdot)||_{L^{1/H_2}(\mathbb{R})} |s-r|^{2H_1-2} drds\right)^{p/2} \\
& \le C_{p,H_1,H_2} \left(\int_0^t \left(||G_\alpha(t-s,x-\cdot)||_{L^{1/H_2}(\mathbb{R})}\right)^{1/H_1} ds\right)^{pH_1} \\
& \le C_{p,H_1,H_2} t^{\frac{p(\alpha H_1 + H_2 - 1)}{\alpha}}.
\end{aligned} \tag{3.8}$$

Then we can conclude the proof of this lemma by using Lemma 15 in Dalang [6]. □

Now let us give the details of the proof of Theorem 1.

Proof of Theorem 1. From (3.4) and (3.5), for each $n \in \mathbb{N}$, it follows that

$$\begin{aligned}
& E\left|u^{(n+1)}(t,x) - u^{(n)}(t,x)\right|^p \\
& = E\left|\sum_{k=0}^m (-1)^k \int_0^t \int_{\mathbb{R}} \left[H_k(s,y,u^{(n)}(s,y)) - H_k(s,y,u^{(n-1)}(s,y))\right] \frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)dyds\right|^p \\
& \le K_p \int_0^t \int_{\mathbb{R}} \sum_{k=0}^m E\left|H_k(s,y,u^{(n)}(s,y)) - H_k(s,y,u^{(n-1)}(s,y))\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)\right|^p dyds
\end{aligned}$$

$$:= K_p \sum_{k=0}^{m} A_{n,p}^{(k)}(t,x).$$

From Debbi and Dozzi [7], we know that

$$A_{n,p}^{(k)}(t,x) \le K_{\alpha,p} \int_0^t (t-s)^{-\frac{k}{\alpha}} \sup_{y\in\mathbb{R}} E|u^{(n)}(s,y) - u^{(n-1)}(s,y)|^p ds.$$

Hence,

$$\begin{aligned} E|u^{(n+1)}(s,y) - u^{(n)}(s,y)|^p &\le K \int_0^t \sum_{k=0}^{m} (t-s)^{-\frac{k}{\alpha}} \sup_{y\in\mathbb{R}} E|u^{(n)}(s,y) - u^{(n-1)}(s,y)|^p ds \\ &\le K \int_0^t \sup_{y\in\mathbb{R}} E|u^{(n)}(s,y) - u^{(n-1)}(s,y)|^p g_1(t-s) ds, \end{aligned} \tag{3.9}$$

where $g_1(t-s) = \sum_{k=0}^{m} (t-s)^{-\frac{k}{\alpha}}$ and $\int_0^T g_1(t)dt < \infty$ with $m < \alpha$.

Note that

$$\sup_{x\in\mathbb{R}} E|u^{(1)}(t,x) - u^{(0)}(t,x)|^p \le K_p \left(\sup_{x\in\mathbb{R}} E|u^0(t,x)|^p + \sup_{x\in\mathbb{R}} E|u^{(1)}(t,x)|^p \right) < \infty.$$

Hence

$$\sum_{n\ge 0} \sup_{(t,x)\in[0,T]\times\mathbb{R}} E\left|u^{(n+1)}(t,x) - u^{(n)}(t,x)\right|^p < \infty.$$

Hence, $\{u^{(n)}(t,x),(t,x) \in [0,T]\times\mathbb{R}\}_{n\ge 0}$ is a Cauchy sequence in $L^p(\Omega)$. Let

$$u(t,x) = \lim_{n\to\infty} u^{(n)}(t,x).$$

Then for each $(t,x) \in [0,T]\times\mathbb{R}$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} E|u(t,x)|^p < \infty. \tag{3.10}$$

Taking $n \to \infty$ in $L^p(\Omega)$ at both sides of (3.5). Then, it shows that $\{u(t,x),(t,x) \in [0,T]\times\mathbb{R}\}$ solves Eq (3.1).

Finally, we can prove the uniqueness of the mild solution to Eq (3.1) by a standard argument. So we omit the details. This completes the proof of this theorem. □

4. Hölder continuity for the solution

In this section, we will check the Hölder continuity of the solution $u = \{u(t,x) : (t,x) \in [0,T]\times\mathbb{R}\}$ of SPDE (1.1) on space and time variables, respectively.

Theorem 2. *Under the assumptions in Theorem 1, if we further assume that u_0 is β-Hölder continuous on $\mathbb{R}$ for $\beta \in (0,1)$, let u be the mild solution to SPDE (1.1), for $\theta \in \left[0, \min\left\{\alpha - [\alpha], \frac{\alpha-1}{2}\right\}\right)$ and $\mu \in \left[0, \min\left\{\frac{\beta}{\alpha}, \frac{(\alpha+1)H-1}{\alpha}, \frac{\alpha-m}{\alpha}\right\}\right)$, then we have*

1. *For fixed $x \in \mathbb{R}$, the process $u(t,x)$ is μ-Hölder continuous in t, P-a.s.,*

2. *For $\alpha < 3$ and for fixed $t \in [0,T]$, the process $u(t,x)$ is θ-Hölder continuous in x, P-a.s.*

Firstly we will give some useful estimates associated with the Green function $G_\alpha(t,x)$ which will be used frequently.

Lemma 4. *For $\theta \in [0, \min\{1, \frac{\alpha}{2} + H_2 - 1\})$ and $\mu \in \left[0, \frac{\frac{\alpha}{2}+H_2-1}{\alpha+1}\right)$, then for any $(t,x) \in [0,T] \times \mathbb{R}$, there exists a constant $K > 0$ such that*

$$E\left|\int_0^t \int_{\mathbb{R}} (G_\alpha(t-r, x-z) - G_\alpha(t-r, y-z)) B^H(dr, dz)\right|^2 \leq K|x-y|^{2\theta}, \tag{4.1}$$

$$E\left|\int_0^t \int_{\mathbb{R}} (G_\alpha(t-r, x-z) - G_\alpha(s-r, x-z)) B^H(dr, dz)\right|^2 \leq K|t-s|^{2\mu}, \tag{4.2}$$

$$E\left|\int_s^t \int_{\mathbb{R}} G_\alpha(t-r, x-z) B^H(dr, dz)\right|^2 \leq K|t-s|^{\frac{2(\alpha H_1 + H_2 - 1)}{\alpha}}. \tag{4.3}$$

Proof. We will divide into three steps to prove this lemma.

Step 1: for the space variable: For any $t \in [0,T]$, $x, y \in \mathbb{R}$ and $\theta \in [0,1)$,

$$\begin{aligned}
&E\left|\int_0^t \int_{\mathbb{R}} (G_\alpha(t-r, x-z) - G_\alpha(t-r, y-z)) B^H(dr, dz)\right|^2 \\
&= \|G_\alpha(t-*, x-\cdot) - G_\alpha(t-*, y-\cdot)\|_{\mathcal{H}}^2 \\
&= \left\| |G_\alpha(t-\cdot, x-\cdot) - G_\alpha(t-\cdot, y-\cdot)|^\theta \cdot |G_\alpha(t-\cdot, x-\cdot) - G_\alpha(t-\cdot, y-\cdot)|^{1-\theta} \right\|_{\mathcal{H}}^2 \\
&\leq C_\theta \left\| |G_\alpha(t-\cdot, x-\cdot) - G_\alpha(t-\cdot, y-\cdot)|^\theta \cdot |G_\alpha(t-\cdot, x-\cdot)|^{1-\theta} \right\|_{\mathcal{H}}^2 \\
&\qquad + C_\theta \left\| |G_\alpha(t-\cdot, x-\cdot) - G_\alpha(t-\cdot, y-\cdot)|^\theta \cdot |G_\alpha(t-\cdot, y-\cdot)|^{1-\theta} \right\|_{\mathcal{H}}^2 \\
&:= C_\theta(I_1 + I_2).
\end{aligned}$$

By mean-value theorem, for η between x and y, one can get that

$$\begin{aligned}
I_1 &= \left\| |x-y|^\theta \cdot \left|\frac{\partial}{\partial x} G_\alpha(t-\cdot, \eta-\cdot)\right|^\theta \cdot |G_\alpha(t-\cdot, x-\cdot)|^{1-\theta} \right\|_{\mathcal{H}}^2 \\
&= 4H_1H_2(2H_1-1)(2H_2-1)|x-y|^{2\theta} \int_0^t \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \left|\frac{\partial}{\partial x} G_\alpha(t-u_1, \eta-z_1)\right|^\theta |G_\alpha(t-u_1, x-z_1)|^{1-\theta} \\
&\quad \cdot \left|\frac{\partial}{\partial x} G_\alpha(t-u_2, \eta-z_2)\right|^\theta |G_\alpha(t-u_2, x-z_2)|^{1-\theta} |u_1-u_2|^{2H_1-2} |z_1-z_2|^{2H_2-2} dz_1 dz_2 du_1 du_2 \\
&\leq C|x-y|^{2\theta} \left(\int_0^t \left(\int_{\mathbb{R}} \left(\left|\frac{\partial}{\partial x} G_\alpha(t-u, \eta-z)\right|^\theta |G_\alpha(t-u, x-z)|^{1-\theta} \right)^{\frac{1}{H_2}} dz \right)^{\frac{H_2}{H_1}} du \right)^{2H_1},
\end{aligned}$$

where for the last inequality, we have used Proposition 1 twice. Then according to the scaling property and related estimates for $G_\alpha(t,x)$ given in Lemma 1, one gets

$$\int_{\mathbb{R}} \left|\frac{\partial}{\partial x} G_\alpha(t-u, \eta-z)\right|^{\frac{\theta}{H_2}} |G_\alpha(t-u, x-z)|^{\frac{1-\theta}{H_2}} dz$$

$$= (t-u)^{\frac{1}{\alpha}-\frac{1+\theta}{\alpha H_2}} \int_{\mathbb{R}} \left| \frac{\partial}{\partial x} G_\alpha(1, \widetilde{\eta} - z) \right|^{\frac{\theta}{H_2}} |G_\alpha(1,z)|^{\frac{1-\theta}{H_2}} dz$$
$$\leq C_{H_1,H_2,\theta} (t-u)^{\frac{1}{\alpha}-\frac{1+\theta}{\alpha H_2}}.$$

Therefore, if $1 + \frac{H_2-1-\theta}{\alpha H_1} > 0$, i.e. $\theta < \alpha H_1 + H_2 - 1$, then

$$I_1 \leq C_{T,H_1,H_2,\theta} |x-y|^{2\theta}.$$

Similarly, we can check

$$I_2 \leq C_{T,H_1,H_2,\theta} |x-y|^{2\theta}.$$

Hence the inequality (4.1) holds.

Step 2: for the time variable: As for the inequality (4.2), for any $t, s \in [0,T]$, $x \in \mathbb{R}$ and $\mu \in [0,1)$, one obtains

$$\begin{aligned}
&E\left| \int_0^t \int_{\mathbb{R}} (G_\alpha(t-r, x-z) - G_\alpha(s-r, x-z)) B^H(dr,dz) \right|^2 \\
&= \|G_\alpha(t-*, x-\cdot) - G_\alpha(s-*, x-\cdot)\|_{\mathcal{H}}^2 \\
&= \left\| |G_\alpha(t-*, x-\cdot) - G_\alpha(s-*, x-\cdot)|^\mu \cdot |G_\alpha(t-*, x-\cdot) - G_\alpha(s-*, x-\cdot)|^{1-\mu} \right\|_{\mathcal{H}}^2 \\
&\leq C_\mu \left\| |G_\alpha(t-*, x-\cdot) - G_\alpha(s-*, x-\cdot)|^\mu \cdot |G_\alpha(s-*, x-\cdot)|^{1-\mu} \right\|_{\mathcal{H}}^2 \\
&\qquad + C_\mu \left\| |G_\alpha(t-*, x-\cdot) - G_\alpha(s-*, x-\cdot)|^\mu \cdot |G_\alpha(s-*, x-\cdot)|^{1-\mu} \right\|_{\mathcal{H}}^2 \\
&:= C_\mu (II_1 + II_2).
\end{aligned}$$

By mean-value theorem, with Proposition 2, for ϱ between s and t, it holds that

$$\begin{aligned}
II_1 &= \left\| |t-s|^\mu \left| \frac{\partial}{\partial t} G_\alpha(\varrho - *, x-\cdot) \right|^\mu \cdot |G_\alpha(t-*, x-\cdot)|^{1-\mu} \right\|_{\mathcal{H}}^2 \\
&= 4H_1H_2(2H_1-1)(2H_2-1)|t-s|^{2\mu} \int_0^t \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\partial}{\partial x} G_\alpha(\varrho - u_1, x-z_1) \right|^\mu |G_\alpha(t-u_1, x-z_1)|^{1-\mu} \\
&\cdot \left| \frac{\partial}{\partial x} G_\alpha(\varrho - u_2, x-z_2) \right|^\mu |G_\alpha(t-u_2, x-z_2)|^{1-\mu} |u_1-u_2|^{2H_1-2} |z_1-z_2|^{2H_2-2} dz_1 dz_2 du_1 du_2 \\
&\leq C|t-s|^{2\mu} \left(\int_0^T \left(\int_{\mathbb{R}} \left(\left| \frac{\partial}{\partial t} G_\alpha(t-u, x-z) \right|^\mu |G_\alpha(t-u, x-z)|^{1-\mu} \right)^{\frac{1}{H_2}} dz \right)^{\frac{H_2}{H_1}} du \right)^{2H_1}.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{\mathbb{R}} \left| \frac{\partial}{\partial t} G_\alpha(\varrho - \cdot, x-\cdot) \right|^{\frac{\mu}{H_2}} |G_\alpha(t-u, x-z)|^{\frac{1-\mu}{H_2}} dz \\
&= C_\alpha \int_{\mathbb{R}} \Big((t-u)^{-\frac{\alpha\mu+1}{\alpha}} |G_\alpha(1, (\varrho-u)^{-\frac{1}{\alpha}}(x-z))|^\mu |G_\alpha(1, (t-u)^{-\frac{1}{\alpha}}(x-z))|^{1-\mu} \\
&\qquad + (t-u)^{-\frac{\alpha\mu+\mu+1}{\alpha}} |x-z|^\mu \left| \frac{\partial}{\partial x} G_\alpha(1, (t-u)^{-\frac{1}{\alpha}}(x-z)) \right|^\mu
\end{aligned}$$

$$\cdot|G_\alpha(1,(t-s)^{-\frac{1}{\alpha}}(x-z))|^{1-\mu}\Big)^{\frac{1}{H_2}}\,dz$$
$$\leq C_{\alpha,H_2}(t-u)^{-\frac{\alpha\mu+1}{\alpha H_2}+\frac{1}{\alpha}}+C_{\alpha,H_2}(t-u)^{-\frac{\alpha\mu+\mu+1}{H_2\alpha}+\frac{1}{\alpha}}.$$

Therefore, if $\frac{H_2-(\alpha+1)\mu-1}{H_1\alpha}+1>0$, i.e. $\mu<\frac{\alpha H_1+H_2-1}{\alpha+1}$, then for $\mu\in\Big[0,\frac{\alpha H_1+H_2-1}{\alpha+1}\Big)$,

$$II_1\leq C_{\alpha,H,T,\mu}|t-s|^{2\mu}.$$

Similarly,

$$II_2\leq C_{\alpha,H,T,\mu}|t-s|^{2\mu}.$$

So (4.2) holds.

Finally let us prove that (4.3) holds. Actually for any $s,t\in[0,T]$, according to (3.8), we can obtain (4.3) consequently. This completes the proof of the lemma.

□

Now we are in position to prove the Hölder continuity of the solution $\{u(t,x):(t,x)\in[0,T]\times\mathbb{R}\}$ on both space and time variables.

Proof of Theorem 2. Let $p\geq 2$, for $t,s\in[0,T]$ and $x,y\in\mathbb{R}$,

$$E|u(t,x)-u(s,y)|^p\leq K_p\left(E|u(t,x)-u(s,x)|^p+E|u(s,x)-u(s,y)|^p\right).$$

We proceed to prove this theorem in two steps.

Step 1: We firstly show the Hölder continuity in time variable t. In fact, for any $x\in\mathbb{R}$ and $t,s\in[0,T]$, one gets

$$\begin{aligned}E|u(t,x)-u(s,x)|^p\leq K_p\Bigg[&E\left|\int_{\mathbb{R}}G_\alpha(t,x-z)u_0(z)dz-\int_{\mathbb{R}}G_\alpha(s,x-z)u_0(z)dz\right|^p\\&+\sum_{k=0}^m E\Bigg|\int_0^t\int_{\mathbb{R}}\frac{\partial^k G_\alpha}{\partial z^k}(t-r,x-z)H_k(r,z,u(r,z))dzdr\\&\qquad-\int_0^s\int_{\mathbb{R}}\frac{\partial^k G_\alpha}{\partial z^k}(t-r,x-z)H_k(r,z,u(r,z))dzdr\Bigg|^p\\&+E\Bigg|\int_0^t\int_{\mathbb{R}}G_\alpha(t-r,x-z)B^H(dr,dz)\\&\qquad-\int_0^s\int_{\mathbb{R}}G_\alpha(s-r,x-z)B^H(dr,dz)\Bigg|^p\Bigg]\\:=A+\sum_{k=0}^m B_k+C.&\end{aligned}$$

Now we will estimate $A,\sum_{k=0}^m B_k$ and C respectively. Note that the initial value u_0 is $\beta-$ Hölder continuous with $0<\beta<1$, according to the semigroup property for $G_\alpha(t,x)$ stated in Lemma 1, one gets

$$E\left|\int_{\mathbb{R}}G_\alpha(t,x-z)u_0(z)dz-\int_{\mathbb{R}}G_\alpha(s,x-z)u_0(z)dz\right|^p$$

$$= E\left|\int_{\mathbb{R}}\int_{\mathbb{R}} G_\alpha(s,x-y)G_\alpha(t-s,y-z)u_0(z)dydz - \int_{\mathbb{R}} G_\alpha(s,x-z)u_0(z)dz\right|^p$$
$$= E\left|\int_{\mathbb{R}} G_\alpha(s,x-y)\int_{\mathbb{R}} G_\alpha(t-s,y-z)(u_0(z)-u_0(y))dzdy\right|^p$$
$$\le K\int_{\mathbb{R}} |G_\alpha(s,x-y)|\int_{\mathbb{R}} |G_\alpha(t-s,y-z)|\,|z-y|^{p\beta}dzdy$$
$$\le K|t-s|^{\frac{p\beta}{\alpha}}.$$

As for $\sum_{k=0}^m B_k$, we have that

$$\sum_{k=0}^m E\left|\int_0^t\int_{\mathbb{R}} \frac{\partial^k G_\alpha}{\partial z^k}(t-r,x-z)H_k(r,z,u(r,z))dzdr - \int_0^s\int_{\mathbb{R}} \frac{\partial^k G_\alpha}{\partial z^k}(t-r,x-z)H_k(r,z,u(r,z))dzdr\right|^p$$
$$\le K_p\left[\sum_{k=0}^m E\left|\int_0^s\int_{\mathbb{R}}\left(\frac{\partial^k G_\alpha}{\partial z^k}(t-r,x-z)-\frac{\partial^k G_\alpha}{\partial z^k}(s-r,x-z)\right)H_k(r,z,u(r,z))dzdr\right|^p + \sum_{k=0}^m E\left|\int_s^t\int_{\mathbb{R}} \frac{\partial^k G_\alpha}{\partial z^k}(t-r,x-z)H_k(r,z,u(r,z))dzdr\right|^p\right]$$
$$:= K_p\sum_{k=0}^m \left(B_{k,1}+B_{k,2}\right).$$

From the proof of Theorem 2 in Debbi and Dozzi [7], it holds that

$$B_{k,1}+B_{k,2}$$
$$\le K_p \sup_{(r,z)\in[0,T]\times\mathbb{R}} E\left(1+|u(r,z)|^p\right)\left|\int_0^s\int_{\mathbb{R}}\left(\frac{\partial^k G_\alpha}{\partial z^k}(t-r,x-z)-\frac{\partial^k G_\alpha}{\partial z^k}(s-r,x-z)\right)dzdr\right|^p + K_p \sup_{(r,z)\in[0,T]\times\mathbb{R}} E\left(1+|u(r,z)|^p\right)\left|\int_s^t\int_{\mathbb{R}} \frac{\partial^k G_\alpha}{\partial z^k}(t-r,x-z)dzdr\right|^p$$
$$\le K_p\left(\left|\int_0^s\int_{\mathbb{R}}\left(\frac{\partial^k G_\alpha}{\partial z^k}(t-r,x-z)-\frac{\partial^k G_\alpha}{\partial z^k}(s-r,x-z)\right)^2 dzdr\right|\right)^{\frac{p}{2}} + K_p\left|\int_s^t\int_{\mathbb{R}} \frac{\partial^k G_\alpha}{\partial z^k}(t-r,x-z)dzdr\right|^p$$
$$\le K|t-s|^{p\frac{\alpha-k}{\alpha}}.$$

Then

$$\sum_{k=0}^m B_k \le K_p\sum_{k=0}^m |t-s|^{p\frac{\alpha-k}{\alpha}} \le K|t-s|^{p\frac{\alpha-m}{\alpha}} \tag{4.4}$$

As for C, applying the estimates in Lemma 4, for any $p\ge 2$, it is easy to check that

$$C = E\left|\int_0^t\int_{\mathbb{R}} G_\alpha(t-r,x-z)B^H(dr,dz)\right.$$

$$
\begin{aligned}
&-\int_0^s \int_{\mathbb{R}} G_\alpha(s-r,x-z)B^H(dr,dz)\Big|^p \\
&\leq K_p\left(E\left|\int_0^s \int_{\mathbb{R}} (G_\alpha(t-r,x-z)-G_\alpha(s-r,x-z))B^H(dr,dz)\right|^p \right.\\
&\qquad \left. + E\left|\int_s^t \int_{\mathbb{R}} G_\alpha(t-r,x-z)B^H(dr,dz)\right|^p\right). \\
&:= K_p(C_1+C_2)
\end{aligned}
$$

Using (4.2), one gets that

$$
\begin{aligned}
C_1 &\leq K_p\left(E\left|\int_0^s \int_{\mathbb{R}} |G_\alpha(t-r,x-z)-G_\alpha(s-r,x-z)|B^H(dr,dz)\right|^2\right)^{\frac{p}{2}} \\
&\leq K_p\Bigg(\Big|\int_0^s \int_0^s \int_{\mathbb{R}} \int_{\mathbb{R}} |G_\alpha(t-r_1,x-z_1)-G_\alpha(s-r_1,x-z_1)| \\
&\qquad \cdot|G_\alpha(t-r_2,x-z_2)-G_\alpha(s-r_2,x-z_2)|\Psi_H(r_1,r_2;z_1,z_2)dz_1dz_2dr_1dr_2|\Bigg)^{\frac{p}{2}} \\
&\leq K_{p,\alpha}|t-s|^{p\mu}, \quad \mu\in\left[0,\frac{\alpha H_1+H_2-1}{\alpha+1}\right).
\end{aligned}
$$

Similarly, we can easily check

$$
C_2 \leq K_{p,\alpha}|t-s|^{p\frac{\alpha H_1+H_2-1}{\alpha}}.
$$

Hence, $\{u(t,x):(t,x)\in[0,T]\times\mathbb{R}\}$ is μ-Hölder continuous in t with $\mu\in\left[0,\min\left\{\frac{\beta}{\alpha},\frac{\alpha H_1+H_2-1}{\alpha+1},\frac{\alpha-m}{\alpha}\right\}\right)$.

Step 2: Now we show the Hölder continuous of $\{u(t,x):(t,x)\in[0,T]\times\mathbb{R}\}$ in space variable. Let $\alpha<3$ For any fixed $t\in[0,T]$, $x,y\in\mathbb{R}$ and $p\geq 2$, we have

$$
\begin{aligned}
&E|u(t,x)-u(t,y)|^p \\
&\leq K_p\Bigg(E\left|\int_{\mathbb{R}} G_\alpha(t,x-z)u_0(z)dz-\int_D G_\alpha(t,y-z)u_0(z)dz\right|^p \\
&\quad +\sum_{k=0}^m E\left|\int_0^t \int_{\mathbb{R}}\left(\frac{\partial^k G_\alpha}{\partial z^k}(t-r,x-z)-\frac{\partial^k G_\alpha}{\partial z^k}(t-r,y-z)\right)H_k(r,z,u(r,z))dzdr\right|^p \\
&\quad +E\left|\int_0^t \int_{\mathbb{R}} |G_\alpha(t-r,x-z)-G_\alpha(t-r,y-z)|B^H(dr,dz)\right|^p\Bigg) \\
&:= D+\sum_{k=0}^m E_k+F.
\end{aligned}
$$

Now we estimate $D,\sum_{k=0}^m E_k$ and F respectively. For the $D,\sum_{k=0}^m E_k$, from Debbi and Dozzi [7], we know that

$$
D+E_k\leq K_{p,\alpha}|x-y|^{p\frac{\alpha-1}{2}}, \quad \text{with} \quad k\leq m<[\alpha], \tag{4.5}
$$

and

$$
D+E_k\leq K_{p,\alpha}|x-y|^{p(\alpha-[\alpha])}, \quad \text{with} \quad k=[\alpha]. \tag{4.6}
$$

Using (4.1), one obtains

$$\begin{aligned} F &= K_p \sup_{(r,z)\in[0,T]\times\mathbb{R}} (1 + E|u(r,z)|^p) \|G_\alpha(t-r, x-z) - G_\alpha(t-r, y-z)\|_{\mathcal{H}}^p \\ &\le K_{p,\alpha,H}|x-y|^{p\theta}, \quad 0 < \theta < \alpha/2 + H_2 - 1. \end{aligned}$$

Hence $\{u(t,x) : (t,x) \in [0,T] \times \mathbb{R}\}$ is θ-Hölder continuous in x with $\theta \in \Big[0, \min\{\alpha - [\alpha], \frac{\alpha-1}{2}, \alpha/2 + H_2 - 1\}\Big)$. This completes the proof of the theorem.

□

5. Density estimates for the solution

In this section, we will prove the absolute continuity of the law of the mild solution $u = \{u(t,x); t \in [0,T], x \in \mathbb{R}\}$ to SPDE (1.1) at any fixed point $(t,x) \in [0,T] \times \mathbb{R}$. We will also establish the lower and upper Gaussian-type bounds for the density of the solution. We firstly prove $u(t,x) \in \mathbb{D}^{1,2}$ and then derive the expression of $Du(t,x)$.

Lemma 5. *Under the assumptions in Theorem 1 and $H_k \in C_b^1([0,T] \times \mathbb{R} \times \mathbb{R})$, $k = 0, 1, \cdots, m$. Then the solution of Eq* (3.1) *belongs to $\mathbb{D}^{1,2}$ and its Malliavin derivative satisfies*

$$\begin{aligned} D_{v,z}u(t,x) =& G_\alpha(t-v, x-z) \\ &+ \sum_{k=0}^{m} \int_v^t \int_{\mathbb{R}} (-1)^k \frac{\partial^k G_\alpha}{\partial y^k}(t-s, x-y) H_k'(s,y,u(s,y)) D_{v,z}u(s,y)dyds, \end{aligned} \tag{5.1}$$

for all $v \le t$ and $z \in \mathbb{R}$. If $v > t$, then $D_{v,z}u(t,x) = 0$.

Proof. Recall the sequence $\{u^{(n)}(t,x), n \ge 0\}$ defined by (3.4) and (3.5) in Lemma 3. Now we will prove $u^{(n)}(t,x) \in \mathbb{D}^{1,2}$ by induction.

Since the coefficients H_k with $k = 0, 1, \ldots, m$ are Lipschitz, by a standard argument, one can show that the sequence $\{u^{(n)}(t,x), n \ge 1\}$ converges to u in $L^p(\Omega)(p \ge 2)$ uniformly for $(t,x) \in [0,T] \times \mathbb{R}$ as $n \to \infty$. Then an argument similar to Zhang and Zheng [24] shows that for each $n \in \mathbb{N}$ and $h \in \mathcal{H}$, the sequence $u^{(n)}(t,x) \in \mathbb{D}_h$ and it satisfies that

$$\begin{aligned} D_h u^{(n+1)}(t,x) &= \langle G_\alpha(t-\cdot, x-\cdot), h\rangle_{\mathcal{H}} \\ &+ \sum_{k=0}^{m} (-1)^k \int_0^t \int_{\mathbb{R}} \frac{\partial^k G_\alpha}{\partial z^k}(t-r, x-z) H_k'(r,z,u(r,z)) D_h u^{(n)}(r,z)dzdr. \end{aligned} \tag{5.2}$$

Following the similar lines of Zhang and Zheng [24] to sequences (3.4) and (3.5), we can conclude that there exists a random process $u_h(t,x)$ such that $D_h u^{(n)}(t,x) \to u_h(t,x)$ in $L^p(\Omega)$ uniformly in $(t,x) \in [0,T] \times \mathbb{R}$ as $n \to \infty$ and $u_h(t,x)$ satisfies the following

$$\begin{aligned} u_h(t,x) &= \langle G_\alpha(t-\cdot, x-\cdot), h\rangle_{\mathcal{H}} \\ &+ \sum_{k=0}^{m} (-1)^k \int_0^t \int_{\mathbb{R}} \frac{\partial^k G_\alpha}{\partial z^k}(t-r, x-z) H_k'(r,z,u(r,z)) u_h(r,z)dzdr. \end{aligned} \tag{5.3}$$

Hence, from the closeness of the operator D_h it follows that $u(t,x) \in \mathbb{D}_h$, $D_h u(t,x) = u_h(t,x)$ and

$$D_h u(t,x) = \langle G_\alpha(t-\cdot, x-\cdot), h\rangle_{\mathcal{H}} + \sum_{k=0}^{m}(-1)^k \int_0^t \int_{\mathbb{R}} \frac{\partial^k G_\alpha}{\partial z^k}(t-r, x-z) H_k'(r,z,u(r,z)) D_h u(r,z) dz dr. \tag{5.4}$$

Next we proceed to show that $u(t,x) \in \mathbb{D}^{1,2}$. Recall that $\{h_n, n \geq 1\}$ is the orthonormal basis of $\mathcal{H}$ introduced in Section 2. For each n, by (5.3), we obtain

$$\begin{aligned} E|D_{h_n} u(t,x)|^2 &= E\,\bigg|\langle G_\alpha(t-\cdot, x-\cdot), h\rangle_{\mathcal{H}} \\ &\quad + \sum_{k=0}^{m}(-1)^k \int_0^t \int_{\mathbb{R}} \frac{\partial^k G_\alpha}{\partial z^k}(t-r, x-z) H_k'(r,z,u(r,z)) D_{h_n} u(r,z) dz dr\bigg|^2 \\ &\leq K \sum_{k=0}^{m} E\left|\int_0^t \int_{\mathbb{R}} \frac{\partial^k G_\alpha}{\partial z^k}(t-r, x-z) H_k'(r,z,u(r,z)) D_{h_n} u(r,z) dz dr\right|^2 \\ &\quad + K|\langle G_\alpha(t-\cdot, x-\cdot), h\rangle_{\mathcal{H}}|^2, \end{aligned} \tag{5.5}$$

with $K > 0$ a constant whose value may change from line to line.

$$U_N(t) = \sup_{x\in\mathbb{R}} E \sum_{n=1}^{N} |D_{h_n} u(t,x)|^2. \tag{5.6}$$

By (5.5), Hölder inequality and Corollary 2 in Debbi and Dozzi [7], for $p = q = 2$, we have

$$\begin{aligned} U_N(t) &\leq K \sum_{k=0}^{m} \int_0^t (t-s)^{-\frac{k}{\alpha}} U_N(s) ds + K\|G_\alpha(t-\cdot, x-\cdot)\|_{\mathcal{H}}^2 \\ &\leq K + K \int_0^t \sum_{k=0}^{m} (t-s)^{-\frac{k}{\alpha}} U_N(s) ds \\ &\leq K\left(1 + \int_0^t \sum_{k=0}^{m} (t-s)^{-\frac{k}{\alpha}} ds + K \int_0^t \sum_{k=0}^{m} (t-s)^{-\frac{k}{\alpha}} ds \int_0^s \sum_{k=0}^{m} (s-r)^{-\frac{k}{\alpha}} U_N(r) dr\right) \\ &\leq K + K \int_0^t \sum_{k=0}^{m} (s-r)^{-\frac{k}{\alpha}} U_N(r) \int_r^t \sum_{k=0}^{m} (t-s)^{-\frac{k}{\alpha}} ds dr \\ &\leq K + K \int_0^t \sum_{k=0}^{m} (t-s)^{-\frac{k+m-\alpha}{\alpha}} U_N(s) ds \\ &\leq K + K \int_0^t \sum_{k=0}^{m} (t-s)^{-\frac{k+m-\alpha}{\alpha}} ds + K \int_0^t \sum_{k=0}^{m} (t-s)^{-\frac{k+m-\alpha}{\alpha}} \int_0^s \sum_{k=0}^{m} (s-r)^{-\frac{k+m-\alpha}{\alpha}} U_N(r) dr ds \\ &\leq K\left(g_1(t) + \int_0^t g_2(t,s) U_N(s) ds\right), \end{aligned} \tag{5.7}$$

where g_1 and g_2 are two bounded functions defined on $[0, T]$. Then the Gronwall's lemma yields that

$$U_N(t) \leq K e^{KT^{\frac{2\alpha-2m}{\alpha}}},$$

where K is independent of m. Let $N \to \infty$, then we get

$$\sup_{x\in\mathbb{R}} E\sum_{n=1}^{\infty}|D_{h_n}u(t,x)|^2 < \infty,$$

which implies $u(t,x) \in \mathbb{D}^{1,2}$.

Since $u(t,x)$ is $\mathcal{F}_t$-adapted, there exists a measurable function $D_{v,z}u(t,x) \in \mathcal{H}$ such that $D_{v,z}u(t,x) = 0$ for $v > t$ and for all $h \in \mathcal{H}$,

$$D_h u(t,x) = \langle Du(t,x), h\rangle_{\mathcal{H}}. \tag{5.8}$$

It follows from (5.4), (5.8) and Fubini's theorem that

$$\begin{aligned}\langle Du(t,x), h\rangle_{\mathcal{H}} &= \langle G_\alpha(t-\cdot, x-\cdot), h\rangle_{\mathcal{H}} \\ &+ \sum_{k=0}^{m}(-1)^k\int_0^t\int_{\mathbb{R}}\frac{\partial^k G_\alpha}{\partial z^k}(t-r,x-z)H_k'(r,z,u(r,z))\langle Du(r,z),h\rangle_{\mathcal{H}}dzdr.\end{aligned} \tag{5.9}$$

This completes the proof of this lemma.

□

Then we have the following main results in this section.

Theorem 3. *Under the assumptions in Theorem 1 and $H_k \in C_b^1([0,T]\times\mathbb{R}\times\mathbb{R})$, $k = 0,1,\ldots,m$, let u be the solution to* (1.1)*. Then for fixed $(t,x) \in [0,T]\times\mathbb{R}$, the law of $u(t,x)$ is absolutely continuous with respect to Lebesgue measure. Moreover, with $\alpha > 1$, the density of random variable $u(t,x)$ satisfies the following: for almost every $z \in \mathbb{R}$,*

$$\frac{E|u(t,x)-\tau|}{C_2\sigma(t)}\exp\left\{-\frac{(z-\tau)^2}{C_1\sigma(t)}\right\} \le p(z) \le \frac{E|u(t,x)-\tau|}{C_1\sigma(t)}\exp\left\{-\frac{(z-\tau)^2}{C_2\sigma(t)}\right\}. \tag{5.10}$$

where $\tau = Eu(t,x)$, $\sigma(t) = t^{\frac{2(\alpha H_1+H_2-1)}{\alpha}}$ and C_1, C_2 are positive constants depending on $\|H_k'\|_\infty, \delta, \alpha, T$.

Theorem 3 will be a consequence of Theorem 3.1 in [16] and the following Proposition 3. We use the notation $F = u(t,x) - Eu(t,x)$ and we remind that we will need to find almost sure lower and upper bounds for the random variable $g_F(F)$, where

$$\begin{aligned}g_F(F) &= \int_0^\infty e^{-\sigma}E\left[E'\left(\left\langle DF,\widetilde{DF}\right\rangle_{\mathcal{H}}\right)|F\right]d\sigma \\ &= \int_0^\infty e^{-\sigma}E\left[E'\left(\left\langle Du(t,x),\widetilde{Du(t,x)}\right\rangle_{\mathcal{H}}\right)|F\right]d\sigma\end{aligned} \tag{5.11}$$

where $\widetilde{DF} = (DF)(e^{-\sigma}\omega + \sqrt{1-e^{-2\sigma}}\omega')$.

Proposition 3. *Fix $T > 0$ and assume that the functions H_k (with $k = 0,1,\ldots,m$) is of $C_b^1([0,T]\times\mathbb{R}\times\mathbb{R})$ and has a bounded derivative. Then, with $\alpha > 1$, there exist two positive constants C_1 and C_2 such that*

$$C_1 t^{\frac{2(\alpha H_1+H_2-1)}{\alpha}} \le g_F(F) \le C_2 t^{\frac{2(\alpha H_1+H_2-1)}{\alpha}}, \tag{5.12}$$

for all $t \in [0,T]$.

In order to prove Proposition 3, we will need the following technical lemmas

Lemma 6. *Let* $\frac{1}{2} < H_1, H_2 < 1$*, for any* s, t *such that* $0 \leq s < t \leq T$ *and* $2m + 1 < 2\alpha$*, then there exists some constant* $K > 0$ *such that for* $0 < \varepsilon < t$

$$\sup_{(r,y)\in[t-\varepsilon,t]\times\mathbb{R}} E\left(\int_{t-\varepsilon}^{t}\int_{\mathbb{R}} |D_{v,z}u(r,y)|^2 dzdv\right) \leq K\varepsilon^{1-\frac{1}{\alpha}}, \tag{5.13}$$

and

$$\sup_{\sigma\in\mathbb{R}} \sup_{(r,y)\in[t-\varepsilon,t]\times\mathbb{R}} E\left(\int_{t-\varepsilon}^{t}\int_{\mathbb{R}} E'\left(|D_{v,z}\widetilde{u(r},y)|^2\right| dzdv\right) \leq K\varepsilon^{1-\frac{1}{\alpha}}. \tag{5.14}$$

Proof. For $s \in [t-\varepsilon, t]$, put

$$L_s(t) = \sup_{x\in\mathbb{R}} E\left(\int_{s}^{t}\int_{\mathbb{R}} |D_{v,z}u(t,x)|^2 dzdv\right).$$

Then by (5.1) and the properties for $G_\alpha(t,x)$ in Lemma 1, for $0 < t_0 < t$, we have

$$\begin{aligned} &E\left(\int_{t_0}^{t}\int_{\mathbb{R}} |D_{v,z}u(t,x)|^2 dzdv\right) \\ &\leq KE\left[\int_{t_0}^{t}\int_{\mathbb{R}}\left(\int_{v}^{t}\int_{\mathbb{R}}\sum_{k=0}^{m}\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)H_k'(s,y,u(s,y))D_{v,z}u(s,y)dyds\right)^2 dzdv\right] \\ &\quad + K\int_{t_0}^{t}\int_{\mathbb{R}} G_\alpha^2(t-v,x-z)dzdv. \end{aligned} \tag{5.15}$$

From the estimates about the Green function $G_\alpha(t,x)$ in Lemma 1, we obtain

$$\begin{aligned} \int_{s}^{t}\int_{\mathbb{R}} G_\alpha^2(t-v,x-z)dzdv &= \int_{s}^{t}(t-v)^{-\frac{2}{\alpha}}\int_{\mathbb{R}} G_\alpha^2(1,(t-v)^{-\frac{1}{\alpha}}(x-z)dzdv \\ &= \int_{s}^{t}(t-v)^{-\frac{1}{\alpha}}\int_{\mathbb{R}} G_\alpha^2(1,z)dzdv \\ &\leq K_\alpha(t-s)^{1-\frac{1}{\alpha}}, \end{aligned} \tag{5.16}$$

where we have used the fact that

$$\int_{\mathbb{R}} G_\alpha^2(1,z)dz \leq K_\alpha \int_{\mathbb{R}} \frac{1}{(1+|z|^\alpha)^2}dz = 2K_\alpha\left(\int_0^1 dz + \int_1^\infty z^{-2\alpha}dz\right) < K_\alpha.$$

Furthermore by formulas of change of variables and estimates on the Green function $G_\alpha(t,x)$ in

Lemma 1, we have

$$
\begin{aligned}
&E\left[\int_{t_0}^{t}\int_{\mathbb{R}}|D_{v,z}u(t,x)|^2dzdv\right]\\
&\le K\int_{t_0}^{t}\int_{\mathbb{R}}\sum_{k=0}^{m}\left(\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)\right)^2 dydsE\left(\int_{t_0}^{s}\int_{\mathbb{R}}(D_{v,z}u(s,y))^2dzdv\right)\\
&\qquad + K(t-s)^{1-\frac{1}{\alpha}} \qquad (5.17)\\
&\le K\int_{t_0}^{t}\sum_{k=0}^{m}(t-s)^{-\frac{2k}{\alpha}}L_{t_0}(s)ds\int_{\mathbb{R}}\frac{(1+|x-y|^{\alpha+k-1})^2}{(1+|x-y|^{\alpha+k})^4}dy + K(t-s)^{1-\frac{1}{\alpha}}\\
&\le K\int_{t_0}^{t}(t-s)^{-\frac{2m}{\alpha}}L_{t_0}(s)ds + K(t-s)^{1-\frac{1}{\alpha}}
\end{aligned}
$$

Hence

$$
\begin{aligned}
L_{t_0}(t) &\le K\int_{t_0}^{t}(t-s)^{-\frac{2m}{\alpha}}L_{t_0}(s)ds + K(t-t_0)^{1-\frac{1}{\alpha}}\\
&\le K\int_{t_0}^{t}(t-s)^{-\frac{2m}{\alpha}}\left(K\int_{t_0}^{s}(s-v)^{-\frac{2m}{\alpha}}L_{t_0}(v)dvds + K(s-t_0)^{1-\frac{1}{\alpha}}\right)ds + K(t-t_0)^{1-\frac{1}{\alpha}}\\
&\le K\int_{t_0}^{t}L_{t_0}(v)dv\int_{v}^{t}(t-s)^{-\frac{2m}{\alpha}}(s-v)^{-\frac{2m}{\alpha}}ds \qquad (5.18)\\
&\qquad + K\int_{t_0}^{t}(t-s)^{-\frac{2m}{\alpha}}(s-t_0)^{1-\frac{1}{\alpha}}ds + K(t-t_0)^{1-\frac{1}{\alpha}}\\
&\le K(t-t_0)^{1-\frac{1}{\alpha}} + K\int_{t_0}^{t}L_{t_0}(v)dv.
\end{aligned}
$$

It follows from Gronwall's lemma that

$$L_{t_0}(t) \le K(t-t_0)^{1-\frac{1}{\alpha}}. \qquad (5.19)$$

Thus the proof of this lemma is completed.

□

In order to prove Proposition 3, we will also need the following lemma, whose proof is similar to that of the above Lemma 6, Lemma 4.6 in Nualart and Quer-Sardanyons [18] or Lemma 5 in Nualart and Quer-Sardanyons [19]. So we omit the details here.

Lemma 7. *For $\varrho \in (0,1]$, then there exists some positive constant C such that depending on $\|H_k'\|_\infty$, σ such that*

$$\sup_{(r,y)\in[(1-\varrho)t,t]\times\mathbb{R}} E[\|Du(r,y)\|^2_{\mathcal{H}([(1-\varrho)t,t]\times\mathbb{R})}|F] \le C(\varrho t)^{\frac{2(\alpha H_1+H_2-1)}{\alpha}}, \qquad (5.20)$$

and

$$\sup_{\sigma\ge 1}\sup_{(r,y)\in[(1-\varrho)t,t]\times\mathbb{R}} E\left[E'\left(\|\widetilde{Du(r,y)}\|^2_{\mathcal{H}([(1-\varrho)t,t]\times\mathbb{R})}\right)|F\right] \le C(\varrho t)^{\frac{2(\alpha H_1+H_2-1)}{\alpha}}, \quad \text{a.s.} \qquad (5.21)$$

with $\mathcal{H}([(1-\varrho)t,t]\times\mathbb{R})$ denotes the Hilbert space $\mathcal{H}$ associated with B^H over the rectangle $[(1-\varrho)t,t]\times\mathbb{R}$.

Proof of Proposition 3. We first recall that the Malliavin derivative of $u(t,x), (t,x) \in [0,T]\times\mathbb{R}$, satisfies $D_{v,z}u(s,y) \geq 0$, for all $(v,z) \in [0,T]\times\mathbb{R}$, a.s. This is because the Malliavin derivative solves the linear equation (5.1) and the Green function $G_\alpha(t-s, x-y)$ is nonnegative. This fact is standard and used in several works (see, among others, the proof of Proposition 3.2 in Nualart and Quer-Sardanyons [18]). We can follow the similar arguments in the Section 5 in [15] to prove this assertion. Here we leave the details to the interested readers.

Let us deal with the proof of (5.12) in two steps. Our methods used here is essentially due to Nualart and Quer-Sardanyons [18], [19] that is mainly because the noise in this work is also additive.

Step 1. <u>The lower bound.</u> Fix $\varrho \in (0,1]$ and let us first derive the lower bound of (5.12). Since the Malliavin derivative of $u(t,x)$ is non-negative, formula (5.11) yields

$$g_F(F) \geq \int_0^\infty e^{-\sigma} E\left[E'\left(\left\langle Du(t,x), \widetilde{Du(t,x)}\right\rangle_{\mathcal{H}([(1-\varrho)t,t]\times\mathbb{R})}\right)|F\right] d\sigma. \tag{5.22}$$

By (5.1), we can decompose the right-hand side of the above (5.22) in a sum of four terms.

$$\Theta_0(t,x;\varrho) = \|G_\alpha(t-*;x-\cdot)\|^2_{\mathcal{H}([(1-\varrho)t,t]\times\mathbb{R})}, \tag{5.23}$$

$$\begin{aligned}\Theta_1(t,x;\varrho) = \sum_{k=0}^m E\bigg[\int_{(1-\varrho)t}^t \int_{\mathbb{R}} \frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y) H_k'(s,y,u(s,y)) \\ \cdot \langle G_\alpha(t-\cdot;x,\cdot), Du(s,y)\rangle_{\mathcal{H}([(1-\varrho)t,t]\times\mathbb{R})}\, dyds|F\bigg],\end{aligned} \tag{5.24}$$

$$\begin{aligned}\Theta_2(t,x;\varrho) = \sum_{k=0}^m \int_0^\infty e^{-\sigma} E\bigg[E'\bigg(\int_{(1-\varrho)t}^t \int_{\mathbb{R}} \frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y) H_k'(s,y,u(s,y)) \\ \cdot \left\langle G_\alpha(t-\cdot,x-\cdot), \widetilde{Du(s,y)}\right\rangle_{\mathcal{H}([(1-\varrho)t,t]\times\mathbb{R})} dyds|F\bigg)\bigg] d\sigma,\end{aligned} \tag{5.25}$$

$$\begin{aligned}&\Theta_3(t,x;\varrho) \\ &= \sum_{k=0}^m \int_0^{+\infty} e^{-\sigma} E\bigg[E' \int_{(1-\varrho)t}^t \int_{\mathbb{R}} \int_{(1-\varrho)t}^t \int_{\mathbb{R}} \frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y) H_k'(s,y,u(s,y)) \\ &\quad \cdot \frac{\partial^k G_\alpha}{\partial y^k}(t-r,x-z) H_k'(r,z,u(r,z)) \left\langle Du(s,y), \widetilde{Du(r,z)}\right\rangle_{\mathcal{H}([(1-\varrho)t,t]\times\mathbb{R})} drdsdydz|F\bigg)\bigg] d\sigma.\end{aligned} \tag{5.26}$$

Firstly we notice that, by the estimates for $G_\alpha(t,x)$ in Lemma 1, one can easily get that

$$\Theta_0(t,x;\varrho) \geq k_1(\varrho t)^{\frac{2(\alpha H_1 + H_2 - 1)}{\alpha}}.$$

Thus we can write

$$g_F(F) \geq C(\varrho t)^{\frac{2(\alpha H_1 + H_2 - 1)}{\alpha}} - |\Theta_1(t,x;\varrho) + \Theta_2(t,x;\varrho) + \Theta_3(t,x;\varrho)|. \tag{5.27}$$

so that we will need to obtain the upper bounds for the terms $\Theta_i(t,x;\varrho), i = 1,2,3$. We apply Fubini's theorem, the boundedness of H_k' with $k = 0,1,\ldots,m$, the estimate (4.3) in Lemma 4 and the bound

(5.20) in Lemma 7. So we have the following estimate

$$
\begin{aligned}
&|\Theta_1(t,x;\varrho)| \\
&\leq C\|G_\alpha(t-*;x,\cdot)\|_{\mathcal{H}([(1-\varrho)t,t]\times\mathbb{R})}\sup_{k=0,1,\ldots m}\|H_k'\|_\infty \\
&\quad\cdot\sum_{k=0}^{m}\left[\int_{(1-\varrho)t}^{t}\int_{\mathbb{R}}\left|\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)\right|E\|Du(s,y)\|_{\mathcal{H}([(1-\varrho)t,t]\times\mathbb{R})}dyds|F\right], \\
&\leq C\|G_\alpha(t-*,x-\cdot)\|_{\mathcal{H}([(1-\varrho)t,t]\times\mathbb{R})}\sup_{k=0,1,\ldots m}\|H_k'\|_\infty\sup_{(s,y)\in[(1-\varrho)t,t]\times\mathbb{R}}E\|Du(s,y)\|_{\mathcal{H}([(1-\varrho)t,t]\times\mathbb{R})} \\
&\quad\cdot\sum_{k=0}^{m}\int_{(1-\varrho)t}^{t}\int_{\mathbb{R}}\left|\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)\right|dyds \\
&\leq C(\varrho t)^{\frac{2(\alpha H_1+H_2-1)}{\alpha}+1-\frac{1}{\alpha}+1-\frac{m}{\alpha}}.
\end{aligned}
\tag{5.28}
$$

In order to get an upper bound for $|\Theta_2(t,x;\varrho)|$, one can proceed using exactly the same arguments as for $|\Theta_1(t,x;\varrho)|$, but apply (5.21) in Lemma 7 instead of (5.20) in Lemma 7. Hence one obtains

$$
|\Theta_2(t,x;\varrho)| \leq C(\varrho t)^{\frac{2(\alpha H_1+H_2-1)}{\alpha}+1-\frac{1}{\alpha}+1-\frac{m}{\alpha}}, \quad \text{a.s.}, \tag{5.29}
$$

Let us finally estimate $|\Theta_3(t,x;\varrho)|$. For this, we apply Fubini's theorem, the fact H_k' with $k=0,1,\ldots,m$ is bounded, the Cauchy-Schwartz inequality, and we finally invoke Lemma 7

$$
\begin{aligned}
|\Theta_3(t,x;\varrho)| \leq C&\sup_{k=0,1,\ldots,m}\|H_k'\|_\infty\sum_{k=0}^{m}\int_0^{+\infty}e^{-\sigma}\left[\int_{(1-\varrho)t}^{t}\int_{\mathbb{R}}\int_{(1-\varrho)t}^{t}\int_{\mathbb{R}}\right. \\
&\cdot\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)\frac{\partial^k G_\alpha}{\partial y^k}(t-\bar{s},x-\bar{y}) \\
&\cdot\left.\left(E\left[\|Du(s,y)\|_{\mathcal{H}([(1-\varrho)t,t]\times\mathbb{R})}E'\left(\left\|\widetilde{Du(\bar{s},\bar{y})}\right\|_{\mathcal{H}([(1-\varrho)t,t]\times\mathbb{R})}\right)|F\right]\right)dydsd\bar{y}d\bar{s}\right]d\sigma,
\end{aligned}
\tag{5.30}
$$

At this point, we apply Cauchy-Schwartz inequality with respect to the conditional expectation with respect to F. One can use the bound (5.21) in Lemma 7 and obtain

$$
|\Theta_3(t,x;\varrho)| \leq C(\varrho t)^{\frac{2(\alpha H_1+H_2-1)}{\alpha}+2-\frac{m}{\alpha}}, \quad \text{a.s.}. \tag{5.31}
$$

Eventually, plugging the bounds (5.28), (5.29), (5.31) in (5.27), we have

$$
\begin{aligned}
g_F(F) &\geq k_1(\varrho t)^{\frac{2(\alpha H_1+H_2-1)}{\alpha}} - C\left(2(\varrho t)^{\frac{2(\alpha H_1+H_2-1)}{\alpha}+1-\frac{1}{\alpha}+1-\frac{m}{\alpha}} + (\varrho t)^{\frac{2(\alpha H_1+H_2-1)}{\alpha}+2-\frac{m}{\alpha}}\right) \\
&\geq (\varrho t)^{\frac{2(\alpha H_1+H_2-1)}{\alpha}}\left[k_1 - C\left(2(\varrho T)^{1-\frac{1}{\alpha}+1-\frac{m}{\alpha}} + (\varrho T)^{2-\frac{m}{\alpha}}\right)\right].
\end{aligned}
\tag{5.32}
$$

Hence if we assume that $\varrho < 1 \wedge \frac{1}{T}$, it only remains to choose ϱ sufficiently small such that the quantity $k_1 - C\left(2(\varrho T)^{1-\frac{1}{\alpha}+1-\frac{m}{\alpha}} + (\varrho T)^{2-\frac{m}{\alpha}}\right)$ is strictly positive, then we can write

$$
g_F(F) \geq C_1(\varrho t)^{\frac{2(\alpha H_1+H_2-1)}{\alpha}}.
$$

Thus, the lower bound in (5.12) has been proved.

Step 2: The upper bound. The upper bound in (5.12) is almost an immediate consequence of the computations which we have just performed for the lower bound. More precisely, according to $g_F(F)$ and the considerations in the first part of the proof of the upper bound for the density, we have the following

$$g_F(F) \le \sum_{i=0}^{3} \Theta_i(t,x;1),$$

where we notice that we have substituted δ by 1 in $\Theta_i(t,x;\delta), i = 0,1,2,3$. We have already seen that

$$\Theta_i(t,x;1) \le Ct^{\frac{2(\alpha H_1+H_2-1)}{\alpha}+2-\frac{1}{\alpha}-\frac{m}{\alpha}}, \quad \text{a.s.}, \quad i = 1,2,$$

and

$$\Theta_3(t,x;1) \le Ct^{\frac{2(\alpha H_1+H_2-1)}{\alpha}+2-\frac{m}{\alpha}}, \quad \text{a.s.}$$

So we just need to bound $\Theta_0(t,x;1)$, which follows directly from (3.8). Thus

$$\begin{aligned} g_F(F) &\le C\left(t^{\frac{2(\alpha H_1+H_2-1)}{\alpha}} + t^{\frac{2(\alpha H_1+H_2-1)}{\alpha}+2-\frac{1}{\alpha}-\frac{m}{\alpha}} + t^{\frac{2(\alpha H_1+H_2-1)}{\alpha}+2-\frac{m}{\alpha}}\right) \\ &\le Ct^{\frac{2(\alpha H_1+H_2-1)}{\alpha}}\left(2 + T^{1-\frac{1}{\alpha}-\frac{m}{\alpha}} + T^{2-\frac{m}{\alpha}}\right), \quad a.s. \end{aligned}$$

for some constant $C > 0$. Therefore

$$g_F(F) \le C_2 t^{\frac{2(\alpha H_1+H_2-1)}{\alpha}},$$

for all $\alpha > 1$ and $m < \alpha$, where the positive constant C_2 depends on T, m and α. Therefore we conclude the proof.

□

Proof of Theorem 3. We will divide into two steps to prove this theorem. Firstly in order to prove the existence of the density for solution to SPDE (1.1), by Theorem 2.1.3 in Nualart [17], we only need to check that

$$\|Du(t,x)\|_{\mathcal{H}} > 0, \quad \text{a.s.}$$

Note that (see for example, Liu and Tudor [14])

$$\|Du(t,x)\|_{\mathcal{H}} > 0, \text{a.s.} \quad \text{i.f.f} \quad \|Du(t,x)\|_{L^2([0,t]\times\mathbb{R})} > 0.$$

Hence we only need to prove that

$$\int_0^t \int_{\mathbb{R}} |D_{r,z}u(t,x)|^2 dzdr > 0, \quad \text{a.s.} \tag{5.33}$$

Recall (5.1), note that

$$\begin{aligned} \int_0^t \int_D |D_{r,z}u(t,x)|^2 dzdr &\ge \int_{t-\varepsilon}^t \int_{\mathbb{R}} |D_{v,z}u(t,x)|^2 dzdv \\ &\ge \frac{1}{2} I_1(t,x,\varepsilon) - I_2(t,x,\varepsilon), \end{aligned} \tag{5.34}$$

where

$$I_1(t,x,\varepsilon) = \int_{t-\varepsilon}^{t}\int_{\mathbb{R}} G_\alpha^2(t-v,x-z)dzdv,$$

$$I_2(t,x,\varepsilon) = \int_{t-\varepsilon}^{t}\int_{\mathbb{R}}\left|\int_{v}^{t}\int_{\mathbb{R}}\sum_{k=0}^{m}\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)H_k'(s,y,u(s,y))D_{v,z}u(s,y)dyds\right|^2 dzdv,$$

Set

$$T(v,z) = \int_{v}^{t}\int_{\mathbb{R}}\sum_{k=0}^{m}\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)H_k'(s,y,u(s,y))D_{v,z}u(s,y)dyds.$$

Then for $H_k \in C_b^1([0,T]\times\mathbb{R}\times\mathbb{R}), k = 0,1,\ldots,m$ and $0<\varepsilon<t$,

$$\begin{aligned}
E\left(I_2(t,x,\varepsilon)\right) &= E\left(\int_{t-\varepsilon}^{t}\int_{\mathbb{R}}|T(v,z)|^2 dzdv\right)\\
&\le KE\left[\int_{t-\varepsilon}^{t}\int_{\mathbb{R}}\left(\int_{v}^{t}\int_{\mathbb{R}}\sum_{k=0}^{m}\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)D_{v,z}u(s,y)dyds\right)^2 dzdv\right]\\
&\le KE\left[\int_{t-\varepsilon}^{t}\int_{\mathbb{R}}dzdv\int_{v}^{t}\int_{\mathbb{R}}\left(\sum_{k=0}^{m}\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)\right)^2(D_{v,z}u(s,y))^2 dyds\right]\\
&\le K\int_{t-\varepsilon}^{t}\int_{\mathbb{R}}dyds\left(\sum_{k=0}^{m}\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)\right)^2 E\left[\int_{t-\varepsilon}^{s}\int_{\mathbb{R}}(D_{v,z}u(s,y))^2 dzdv\right]\\
&\le K\int_{t-\varepsilon}^{t}\int_{\mathbb{R}}dyds\left(\sum_{k=0}^{m}\frac{\partial^k G_\alpha}{\partial y^k}(t-s,x-y)\right)^2 L_{t-\varepsilon}(s)\\
&\le K\varepsilon^{1-\frac{1}{\alpha}}\sum_{k=0}^{m}\int_{t-\varepsilon}^{t}(t-s)^{-\frac{2k}{\alpha}}ds\int_{\mathbb{R}}\frac{(1+|x-y|^{\alpha+k-1})^2}{(1+|x-y|^{\alpha+k})^4}dy\\
&\le K\varepsilon^{2-\frac{1}{\alpha}-\frac{2m}{\alpha}},\quad \text{with}\quad 2m+1<2\alpha.
\end{aligned}\tag{5.35}$$

which is similar to that of (5.17).

From (5.16), we know there exists some constant $K>0$ such that

$$I_1(t,x,\varepsilon) \le K\varepsilon^{1-\frac{1}{\alpha}}.\tag{5.36}$$

Then for each $\varepsilon_0>0$, according to (5.34), (5.35) and (5.36)

$$\begin{aligned}
P\left(\int_0^T\int_{\mathbb{R}}|D_{r,z}u(t,x)|^2 dzdr>0\right) &\ge \sup_{\varepsilon\in(0,\varepsilon_0]} P\left(\frac{1}{2}I_1(t,x,\varepsilon)-I_2(t,x,\varepsilon)>0\right)\\
&\ge \sup_{\varepsilon\in(0,\varepsilon_0]} P\left(I_2(t,x,\varepsilon)\le C\varepsilon^{1-\frac{1}{\alpha}}\right)\\
&\ge 1-\inf_{\varepsilon\in(0,\varepsilon_0]}\left\{\frac{1}{K\varepsilon^{1-\frac{1}{\alpha}}}E(I_2(t,x,\varepsilon))\right\}\\
&\ge 1-\inf_{\varepsilon\in(0,\varepsilon_0]}\frac{1}{K}\varepsilon^{1-\frac{2m}{\alpha}} = 1.
\end{aligned}\tag{5.37}$$

This concludes the proof of the existence of the density for solution to SPDE (1.1).

Secondly, for any fixed $(t,x) \in [0,T] \times \mathbb{R}$, we know that the random variable $F = u(t,x) - E(u(t,x))$ is centered belongs to $\mathbb{D}^{1,2}$ and by (5.12), it holds that $0 < C_1 t^{\frac{2(\alpha H_1 + H_2 - 1)}{\alpha}} \leq g_F(F)$, for all $t \in [0,T]$. We then apply Theorem 3.1 and Corollary 3.3 in Nourdin and Viens [16], and obtain that the probability density $\rho : \mathbb{R} \mapsto \mathbb{R}$ of the random variable F is given by

$$\rho(z) = \frac{E|u(t,x) - E(u(t,x))|}{2g_F(z)} \exp\left\{-\int_0^z \frac{y}{g_F(y)} dy\right\},$$

for almost every $z \in \mathbb{R}$. Then, the density p of the random variable $u(t,x)$ satisfies

$$p(z) = \frac{E|u(t,x) - E(u(t,x))|}{2g_F(z - E(u(t,x)))} \exp\left\{-\int_0^{z - E(u(t,x))} \frac{y}{g_F(y)} dy\right\}. \tag{5.38}$$

In order to conclude the proof, we only need to use the bounds obtained in Proposition 3 into (5.38).

□

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' contributions

The authors declare that this study was accomplished in collaboration with the same responsibility. All authors read and approved the final manuscript.

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