AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(6): 10596–10608.
DOI:10.3934/math.2022591
Received: 13 January 2022
Revised: 23 February 2022
Accepted: 09 March 2022
Published: 29 March 2022

Research article

On the distribution of k-full lattice points in $\mathbb{Z}^2$

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Abstract: Let $\mathbb{Z}^2$ be the two-dimensional integer lattice. For an integer $k \geq 2$, we say a non-zero lattice point in $\mathbb{Z}^2$ is k-full if the greatest common divisor of its coordinates is a k-full number. In this paper, we first prove that the density of k-full lattice points in $\mathbb{Z}^2$ is $c_k = \prod_p (1 - p^{-2} + p^{-2k})$, where the product runs over all primes. Then we show that the density of k-full lattice points on a path of an α-random walk in $\mathbb{Z}^2$ is almost surely c_k, which is independent on α.

Keywords: k-full lattice points; k-full number; density; random walk; two-dimensional integer lattice
Mathematics Subject Classification: 60G50, 11H06, 11N37

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1. Introduction

Let $k \geq 2$ be a fixed integer. In $\mathbb{Z}$, we say an integer n with $|n| > 1$ is a k-full number if for any prime $p \mid n$ we have that $p^k \mid n$. Integers ± 1 are also considered to be k-full numbers. Particularly, 2-full numbers are said to be square-full. For $x \geq 2$, let $N_k(x)$ be the number of k-full numbers not exceeding x. Erdős and Szekeres [5] showed that

$$N_k(x) = \sum_{i=k}^{2k-1} c_{i,k} x^{\frac{1}{i}} + O(x^{\theta_k + \varepsilon}),$$

holds for $\theta_k \leq 1/(k+1)$ and any $\varepsilon > 0$. Here $c_{i,k}$ are constants, which can be explicitly computed. This result has been improved by many other authors. For example, see Bateman and Grosswald [1] and Krătzel [8, 9].

In the two-dimensional lattice $\mathbb{Z}^2$, we say a non-zero lattice point (m, n) is k-full if and only if $\gcd(m, n)$ is a k-full number, where $\gcd(*, *)$ is the greatest common divisor function. Particularly, 2-full lattice points in $\mathbb{Z}^2$ are said to be square-full. For example, lattice points $(2, 3)$ and $(12, 20)$ are square-full, but point $(12, 21)$ is not.

k-full lattice points are natural analogues of k-free lattice points. We say an non-zero integer n is a k-free number if it is not divisible by any k-th ($k \geq 1$) power of primes. A non-zero lattice point (m, n) in $\mathbb{Z}^2$ is said to be k-free if $\gcd(m, n)$ is a k-free number. From [10], we see that the density of k-free lattice points in $\mathbb{Z}^2$ is $1/\zeta(2k)$. We refer to [2, 6] for more work on k-free lattice points from different aspects.

Our first result gives the density of k-full lattice points in $\mathbb{Z}^2$.

Theorem 1.1. *For $k \geq 2$, let $S_k(x)$ be the number of k-full lattice points in the square area* $[1, x] \times [1, x]$. *Then for $x \geq 2$ we have that*

$$S_k(x) = c_k x^2 + O(x \log^2 x),$$

where

$$c_k = \prod_p \Big(1 - \frac{1}{p^2} + \frac{1}{p^{2k}}\Big) \tag{1.1}$$

with the product running over all primes and the implied O-constant does not depend on k.

In particular, for $k = 2$, by Theorem 1.1 and the Euler product of $\zeta(s)$, which is the Riemann zeta function, we obtain that the density of square-full lattice points in $\mathbb{Z}^2$ is

$$c_2 = \prod_p \Big(1 - \frac{1}{p^2} + \frac{1}{p^4}\Big) = \zeta(4)\zeta(6)\zeta^{-1}(2)\zeta^{-1}(12) \approx 0.66922.$$

We also investigate k-full lattice points in $\{0, 1, 2, \cdots\}^2$ from the viewpoint of random walks. For $0 < \alpha < 1$, an α-random walk is defined by

$$P_{i+1} = P_i + \begin{cases} (1, 0), & \text{with probability } \alpha, \\ (0, 1), & \text{with probability } 1 - \alpha \end{cases}$$

for $i = 0, 1, 2, \cdots$, where $P_i = (x_i, y_i)$ is the coordinate of the α-random walker at the i-th step and $P_0 = (0, 0)$. In 2019, Cilleruelo, Fernández and Fernández [3] considered visible lattice points in α-random walks in $\mathbb{Z}^2$. They proved that (see Theorem A, [3]) the density of visible lattice points on a path of an α-random walker is almost surely $1/\zeta(2)$.

Our second result gives the density of k-full lattice points on a path of an α-random walker. Before stating the result, we introduce some notations first. For an α-random walk, define a sequence of random variables $\{W_i\}_{i \in \mathbb{N}}$ by

$$W_i = \begin{cases} 1, & P_i \text{ is } k\text{-full}, \\ 0, & \text{otherwise}. \end{cases}$$

For any $n \geq 1$, define a random variable $\overline{S}_{k,\alpha}(n)$ by

$$\overline{S}_{k,\alpha}(n) = \frac{W_1 + W_2 + \cdots + W_n}{n},$$

then $\overline{S}_{k,\alpha}(n)$ indicates the proportion of k-full lattice points in the first n steps of an α-random walker.

Theorem 1.2. *For any $\alpha \in (0,1)$, we have that*

$$\lim_{n\to+\infty} \overline{S}_{k,\alpha}(n) = c_k$$

almost surely, where c_k is the same as in Theorem 1.1.

Note that the density c_k in Theorem 1.2 is independent on α and coincides with the density of k-full lattice points in $\mathbb{Z}^2$.

Notations. As usual, for real functions f and g, we use the expressions $f = O(g)$ and $f \ll g$ to mean $|f| \le Cg$ for a constant $C > 0$. When this constant C depends on some parameter α, we write $f \ll_\alpha g$ and $f = O_\alpha(g)$. We use $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ to denote the sets of all real numbers, integers and positive integers, respectively. Moreover, we use $\mathbb{P}$, $\mathbb{E}$ and $\mathbb{V}$ to denote taking probability, expectation and variance, respectively. The symbol $\prod_p$ always means taking product over all primes.

2. Preliminaries

In the present section, we apply elementary methods to give some preliminary results with the aim of proving our Theorems.

2.1. *Divisor functions*

We give some bounds for sums involving divisor functions, which would be used later. For $l \ge 2$, let

$$\tau_l(n) := \sum_{n=d_1 d_2\cdots d_l} 1$$

be the l-dimensional divisor function. Particularly, we always write $\tau(n) = \tau_2(n)$. By (1.80) in [7], we have that

$$\sum_{1\le i\le n} \tau_3(i) \ll n\log^2 n \tag{2.1}$$

and

$$\sum_{1\le i\le n} \tau_3^2(i) \ll n\log^8 n \tag{2.2}$$

for $n \ge 2$. By bound (2.1) and partial summation, we have the following lemma.

Lemma 2.1. *For any integer $n \ge 2$, we have that*

$$\sum_{1\le i<j\le n} \frac{\tau_3(i)\tau_3(j)}{\sqrt{i}} = O(n^{3/2}\log^4 n) \quad \text{and} \quad \sum_{1\le i<j\le n} \frac{\tau_3(j)}{\sqrt{j-i}} = O(n^{3/2}\log^2 n).$$

Proof. To prove the first equality, we write that

$$\sum_{1\le i<j\le n} \frac{\tau_3(i)\tau_3(j)}{\sqrt{i}} = \sum_{1<j\le n} \tau_3(j) \sum_{1\le i<j} \frac{\tau_3(i)}{\sqrt{i}}.$$

Applying partial summation to the sum over i, we obtain that

$$\sum_{1\le i<j\le n}\frac{\tau_3(j)\tau_3(i)}{\sqrt{i}} \ll n^{1/2}\log^2 n\sum_{1<j\le n}\tau_3(j) \ll n^{3/2}\log^4 n,$$

where we have used (2.1).

To prove the second equality, we write that

$$\sum_{1\le i<j\le n}\frac{\tau_3(j)}{\sqrt{j-i}} = \sum_{1<j\le n}\tau_3(j)\sum_{1\le i<j}\frac{1}{\sqrt{j-i}}.$$

Note that $\sum_{1\le i<j}(j-i)^{-1/2} \ll \sqrt{j}$, then we have that

$$\sum_{1\le i<j\le n}\frac{\tau_3(j)}{\sqrt{j-i}} \ll \sqrt{n}\sum_{1<j\le n}\tau_3(j) \ll n^{3/2}\log^2 n,$$

where we have used (2.1) again. □

2.2. Two arithmetic functions

Denote the characteristic function of k-full numbers by

$$h_k(n) = \begin{cases} 1, & n \text{ is } k\text{−full}, \\ 0, & \text{otherwise}. \end{cases}$$

It is obvious that $h_k(n)$ is multiplicative and

$$h_k(p^m) = \begin{cases} 1, & m \ge k \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

for any prime power p^m. For $k \ge 2$, define

$$g_k(n) := \sum_{rd=n}\mu(r)h_k(d), \tag{2.4}$$

where μ is the Möbius function. Obviously, for $n \ge 1$ we have that

$$|g_k(n)| \le \tau(n). \tag{2.5}$$

Note that $g_k(n)$ is multiplicative and by (2.3), we have that

$$g_k(p^m) = \begin{cases} -1, & m = 1, \\ 1, & m = k, \\ 0, & \text{otherwise} \end{cases}$$

for any prime power p^m. It follows that

$$\sum_{n=1}^{\infty}\frac{g_k(n)}{n^2} = \prod_p\Big(1-\frac{1}{p^2}+\frac{1}{p^{2k}}\Big), \tag{2.6}$$

where the symbol $\prod_p$ means taking product over all primes.

Lemma 2.2. *For fixed integer $k \geq 2$ and any $x \geq 2$, we have that*

$$\sum_{n\leq x} \frac{g_k(n)}{n^2} = \prod_p \Big(1 - \frac{1}{p^2} + \frac{1}{p^{2k}}\Big) + O(x^{-1}\log x),$$

where $\prod_p$ means taking product over all primes.

Proof. Extending the range of the sum over n, we have that

$$\sum_{n\leq x} \frac{g_k(n)}{n^2} = \sum_{n=1}^{\infty} \frac{g_k(n)}{n^2} - \sum_{n>x} \frac{g_k(n)}{n^2}.$$

Using bound (2.5), we have that

$$\sum_{n>x} \frac{g_k(n)}{n^2} \ll \sum_{n>x} \frac{\tau(n)}{n^2} \ll x^{-1}\log x.$$

where we have used the asymptotic formula (see (1.75) in [7])

$$\sum_{n\leq x} \tau(n) = x\log x + O(x) \tag{2.7}$$

and partial summation. Hence we have that

$$\sum_{n\leq x} \frac{g_k(n)}{n^2} = \sum_{n=1}^{\infty} \frac{g_k(n)}{n^2} + O(x^{-1}\log x).$$

This together with (2.6) gives our desired result. □

For $k \geq 2$, define

$$f_k(n) = \sum_{rd|n} \frac{\mu(r)h_k(d)}{rd}. \tag{2.8}$$

Obviously, for $n \geq 1$ we have that

$$|f_k(n)| \leq \tau_3(n). \tag{2.9}$$

Lemma 2.3. *For fixed integer $k \geq 2$ and any $x \geq 2$, we have that*

$$\sum_{1\leq n\leq x} f_k(n) = x\prod_p \Big(1 - \frac{1}{p^2} + \frac{1}{p^{2k}}\Big) + O(\log^2 x).$$

Proof. In (2.8), let $rd = w$, then we have that

$$f_k(n) = \sum_{w|n} \frac{1}{w} \sum_{rd=w} \mu(r)h_k(d) = \sum_{w|n} \frac{g_k(w)}{w}. \tag{2.10}$$

It follows that

$$\sum_{1\le n\le x} f_k(n) = \sum_{1\le n\le x}\sum_{w|n}\frac{g_k(w)}{w} = \sum_{w\le x}\frac{g_k(w)}{w}\sum_{\substack{1\le n\le x\\ n\equiv 0 \bmod w}} 1,$$

where we have changed the order of summations. Further, we have that

$$\sum_{1\le n\le x} f_k(n) = \sum_{w\le x}\frac{g_k(w)}{w}\Big(\frac{x}{w}+O(1)\Big) = x\sum_{w\le x}\frac{g_k(w)}{w^2} + O\Big(\sum_{w\le x}\frac{|g_k(w)|}{w}\Big).$$

Extending the range of the sum over w, we obtain

$$\sum_{1\le n\le x} f_k(n) = x\sum_{w=1}^{\infty}\frac{g_k(w)}{w^2} + O\Big(x\sum_{w>x}\frac{|g_k(w)|}{w^2}\Big) + O\Big(\sum_{w\le x}\frac{|g_k(w)|}{w}\Big).$$

Using (2.5), (2.7) and partial summation to estimate the O-terms, we obtain

$$\sum_{1\le n\le x} f_k(n) = x\sum_{w=1}^{\infty}\frac{g_k(w)}{w^2} + O(\log^2 x).$$

This together with (2.6) gives our desired result. □

2.3. Tools from probability and number theory

We need the following second moment method from probability.

Lemma 2.4 (Lemma 2.5, [3]). *For a sequence of uniformly bounded random variables $(W_i)_{i\ge1}$, let $\overline{S}_n = (W_1 + \cdots + W_n)/n$. If the expectation $\mathbb{E}(\overline{S}_n)$ and the variance $\mathbb{V}(\overline{S}_n)$ of $\overline{S}_n$ satisfy*

$$\lim_{n\to\infty}\mathbb{E}(\overline{S}_n) = \mu$$

and

$$\mathbb{V}(\overline{S}_n) \ll_\delta n^{-\delta}$$

for some constant $\delta > 0$ and any $n \ge 1$, then we have that

$$\lim_{n\to\infty}\overline{S}_n = \mu$$

almost surely.

We also need the following number theoretical result.

Lemma 2.5 (Lemma 2.1, [3]). *For any $0 < \alpha < 1$ and integers $n \ge 1$, $1 \le d \le n$ and $r \in \{0, 1, \dots, d-1\}$, there holds*

$$\sum_{l\equiv r \bmod d}\binom{n}{l}\alpha^l(1-\alpha)^{n-l} = \frac{1}{d} + O_\alpha\Big(\frac{1}{\sqrt{n}}\Big),$$

where the implied constant depends on α.

For brevity, we denote

$$C_\alpha(n,s) := \binom{n}{s}\alpha^s(1-\alpha)^{n-s}. \tag{2.11}$$

For integer $k \geq 2$, let

$$\mathcal{P}_{k,\alpha}(a,b,n) := \sum_{\substack{0\leq m\leq n \\ \gcd(m+a,b) \text{ is } k-\text{full}}} C_\alpha(n,m), \tag{2.12}$$

where a, b, n are integers with $b \neq 0$ and $n \geq 1$. Then we have the following result.

Lemma 2.6. *For $0 < \alpha < 1$ and any integers a, b, n with $b \neq 0$, $n \geq 1$, we have that*

$$\mathcal{P}_{k,\alpha}(a,b,n) = f_k(b) + O_\alpha\Big(\frac{\tau_3(b)}{\sqrt{n}}\Big),$$

where f_k is defined by (2.8) *and the implied O-constant depends only on α.*

Proof. By (2.12), we have that

$$\mathcal{P}_{k,\alpha}(a,b,n) = \sum_{d|b} h_k(d) \sum_{\substack{0\leq m\leq n \\ \gcd(m+a,b)=d}} C_\alpha(n,m). \tag{2.13}$$

For simplicity, let

$$\mathcal{F} = \mathcal{F}_\alpha(n,a,b,d) := \sum_{\substack{0\leq m\leq n \\ \gcd(m+a,b)=d}} C_\alpha(n,m).$$

For $d \mid b$, we have that

$$\mathcal{F} = \sum_{\substack{0\leq m\leq n, d|(m+a) \\ \gcd((m+a)/d, b/d)=1}} C_\alpha(n,m).$$

Using the formula

$$\sum_{r|n}\mu(r) = \begin{cases} 1, & n=1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.14}$$

and changing the order of summations, for $d \mid b$, we obtain that

$$\begin{aligned}\mathcal{F} &= \sum_{\substack{0\leq m\leq n \\ d|(m+a)}} C_\alpha(n,m) \sum_{r|\gcd((m+a)/d, b/d)} \mu(r) \\ &= \sum_{rd|b}\mu(r) \sum_{\substack{0\leq m\leq n \\ m\equiv -a \bmod rd}} C_\alpha(n,m),\end{aligned}$$

where μ is the Möbius function. Moreover, we write

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2, \tag{2.15}$$

where

$$\mathcal{F}_1 = \sum_{\substack{rd \le n \\ rd|b}} \mu(r) \sum_{\substack{0 \le m \le n \\ m \equiv -a \bmod rd}} C_\alpha(n,m) \quad \text{and} \quad \mathcal{F}_2 = \sum_{\substack{rd > n \\ rd|b}} \mu(r) \sum_{\substack{0 \le m \le n \\ m \equiv -a \bmod rd}} C_\alpha(n,m).$$

We first consider the sum $\mathcal{F}_1$, applying Lemma 2.5 to the sum over m, we obtain that

$$\mathcal{F}_1 = \sum_{\substack{rd \le n \\ rd|b}} \frac{\mu(r)}{rd} + O_\alpha\Big(\frac{1}{\sqrt{n}} \sum_{\substack{rd \le n \\ rd|b}} 1\Big) \tag{2.16}$$

For $\mathcal{F}_2$, since $rd > n$, then the sum over m in $\mathcal{F}_2$ consists of at most one term. Estimating this term by the local central limit theorem (see Theorem 3.5.2, [4])

$$\max_{0 \le l \le n} \binom{n}{l} \alpha^l (1-\alpha)^{n-l} = O_\alpha\Big(\frac{1}{\sqrt{n}}\Big),$$

we obtain that

$$\mathcal{F}_2 \ll \frac{1}{\sqrt{n}} \sum_{\substack{rd > n \\ rd|b}} 1 \tag{2.17}$$

By (2.15)-(2.17), we have that

$$\mathcal{F} = \sum_{\substack{rd \le n \\ rd|b}} \frac{\mu(r)}{rd} + O_\alpha\Big(\frac{1}{\sqrt{n}} \sum_{\substack{rd \le n \\ rd|b}} 1\Big) + O_\alpha\Big(\frac{1}{\sqrt{n}} \sum_{\substack{rd > n \\ rd|b}} 1\Big) = \sum_{\substack{rd \le n \\ rd|b}} \frac{\mu(r)}{rd} + O_\alpha\Big(\frac{\tau(b/d)}{\sqrt{n}}\Big).$$

Extending the range of the sums over r and d, we have that

$$\mathcal{F} = \sum_{rd|b} \frac{\mu(r)}{rd} - \sum_{\substack{rd > n \\ rd|b}} \frac{\mu(r)}{rd} + O_\alpha\Big(\frac{\tau(b/d)}{\sqrt{n}}\Big) = \sum_{rd|b} \frac{\mu(r)}{rd} + O_\alpha\Big(\frac{\tau(b/d)}{\sqrt{n}}\Big), \tag{2.18}$$

where we have used

$$\sum_{\substack{rd > n \\ rd|b}} \frac{\mu(r)}{rd} \ll \frac{\tau(b/d)}{n} \ll \frac{\tau(b/d)}{\sqrt{n}}.$$

Inserting (2.18) into (2.13), we have that

$$\mathcal{P}_{k,\alpha}(a,b,n) = \sum_{d|b} h_k(d) \Big(\sum_{rd|b} \frac{\mu(r)}{rd} + O_\alpha\Big(\frac{\tau(b/d)}{\sqrt{n}}\Big)\Big).$$

The contribution of the O-term to $\mathcal{P}_{k,\alpha}$ is

$$\ll_\alpha \frac{1}{\sqrt{n}} \sum_{d|b} \tau(b/d) = \frac{\tau_3(b)}{\sqrt{n}}.$$

Hence, we have that

$$\mathcal{P}_{k,\alpha}(a,b,n) = f_k(b) + O_\alpha\Big(\frac{\tau_3(b)}{\sqrt{n}}\Big),$$

where $f_k(b)$ is given by (2.8). This completes our proof. □

3. Proof of Theorem 1.1

Given Lemma 2.2, the proof of the theorem is straightforward. By the definition of the k-full lattice points, we have that

$$S_k(x) = \sum_{\substack{m,n\le x\\ \gcd(m,n) \text{ is } k-\text{full}}} 1.$$

It follows that

$$S_k(x) = \sum_{d\le x} h_k(d)\mathcal{A}_d(x), \tag{3.1}$$

where

$$\mathcal{A}_d(x) := \sum_{\substack{m,n\le x\\ \gcd(m,n)=d}} 1.$$

By the definition of $\mathcal{A}_d(x)$ and applying the formula (2.14), we have that

$$\mathcal{A}_d(x) = \sum_{\substack{m,n\le x; d|m,d|n\\ \gcd(m/d,n/d)=1}} 1 = \sum_{\substack{m,n\le x\\ d|m,d|n}} \sum_{r|\gcd(m/d,n/d)} \mu(r),$$

where μ is the Möbius function. Changing the order of summations, we obtain

$$\mathcal{A}_d(x) = \sum_{\substack{m,n\le x\\ d|m,d|n}} \sum_{\substack{r|(m/d)\\ r|(n/d)}} \mu(r) = \sum_{r\le \frac{x}{d}} \mu(r) \sum_{\substack{m,n\le x\\ m\equiv \bmod 0(rd)\\ n\equiv \bmod 0(rd)}} 1.$$

It follows that

$$\begin{aligned} \mathcal{A}_d(x) &= \sum_{r\le x/d} \mu(r)\Big(\frac{x}{rd} + O(1)\Big)^2 \\ &= x^2 \sum_{r\le x/d} \frac{\mu(r)}{(rd)^2} + O\Big(\frac{x}{d}\sum_{r\le x/d}\frac{1}{r}\Big). \end{aligned} \tag{3.2}$$

Inserting (3.2) into (3.1), we have that

$$\begin{aligned} S_k(x) &= x^2 \sum_{d\le x} h_k(d) \sum_{r\le \frac{x}{d}} \frac{\mu(r)}{(rd)^2} + O\Big(\sum_{d\le x}\frac{x}{d}\sum_{r\le x/d}\frac{1}{r}\Big) \\ &= x^2 \sum_{rd\le x} \frac{\mu(r)h_k(d)}{(rd)^2} + O(x\log^2 x), \end{aligned}$$

where we have used partial summation to estimate the O-term. Let $rd = n$, we have that

$$\begin{aligned} S_k(x) &= x^2 \sum_{n\le x} \frac{1}{n^2} \sum_{rd=n} \mu(r)h_k(d) + O(x\log^2 x) \\ &= x^2 \sum_{n\le x} \frac{g_k(n)}{n^2} + O(x\log^2 x), \end{aligned} \tag{3.3}$$

where $g_k(n)$ is defined by (2.4). From Lemma 2.2 and (3.3), we obtain our desired result.

4. Proof of Theorem 1.2

We first compute the expectation $\mathbb{E}(\overline{S}_n)$ and prove the following result.

Proposition 4.1. *Let* $0 < \alpha < 1$ *and integer* $k \geq 2$ *be fixed. Then for* $n \geq 2$*, we have that*

$$\mathbb{E}(\overline{S}_n) = \prod_p \Big(1 - \frac{1}{p^2} + \frac{1}{p^{2k}}\Big) + O_\alpha(n^{-1/2}\log^2 n),$$

where the product runs over all primes and the implied O-constant depends only on α.

Proof. By the linearity of expectation and the definition of W_i, we note that

$$\mathbb{E}(\overline{S}_n) = \frac{1}{n}\sum_{1\leq i\leq n}\mathbb{E}(W_i) = \frac{1}{n}\sum_{1\leq i\leq n}\mathbb{P}(P_i \text{ is } k-\text{full}), \tag{4.1}$$

where $P_i = (x_i, y_i)$ is the coordinate of the α-random walker at the i-th step. Observe that $x_i + y_i = i$, thus we can write $P_i = (l, i-l)$ for some $l = 0, 1, \ldots, i$. The probability that $P_i = (l, i-l)$ is

$$\mathbb{P}(P_i = (l, i-l)) = C_\alpha(i, l),$$

where the function $C_\alpha(i, l)$ is given by (2.11). Since a lattice point P_i is k-full if and only if $\gcd(l, i-l) = \gcd(l, i) = d$, where d is k-full, we infer that

$$\mathbb{P}(P_i \text{ is } k-\text{full}) = \sum_{\substack{0\leq l\leq i\\ \gcd(l,i) \text{ is } k-\text{full}}} C_\alpha(i, l)$$

for any $1 \leq i \leq n$. Applying Lemma 2.6 to the sum over l, we have that

$$\mathbb{P}(P_i \text{ is } k-\text{full}) = f_k(i) + O_\alpha\Big(\frac{\tau_3(i)}{\sqrt{i}}\Big), \tag{4.2}$$

where f_k is given by (2.8). Hence from (4.1) and (4.2), we obtain

$$\mathbb{E}(\overline{S}_n) = \frac{1}{n}\sum_{1\leq i\leq n} f_k(i) + O_\alpha\Big(\frac{1}{n}\sum_{1\leq i\leq n}\frac{\tau_3(i)}{\sqrt{i}}\Big).$$

Using bound (2.1) and partial summation to estimate the O-term, we obtain

$$\mathbb{E}(\overline{S}_n) = \frac{1}{n}\sum_{1\leq i\leq n} f_k(i) + O_\alpha(n^{-1/2}\log^2 n). \tag{4.3}$$

This together with Lemma 2.3 yields Proposition 4.1. □

Now we estimate the variance of $\overline{S}_n$.

Proposition 4.2. *Let* $0 < \alpha < 1$ *and integer* $k \geq 2$ *be fixed. Then for* $n \geq 2$*, we have that*

$$\mathbb{V}(\overline{S}_n) = O_\alpha(n^{-1/2}\log^4 n),$$

where the implied O-constant depends only on α.

Proof. By the definition of $\overline{S}_n$, we have that

$$\mathbb{V}(\overline{S}_n) = \frac{1}{n^2}\sum_{1\le i\le n}\mathbb{E}(W_i^2) + \frac{2}{n^2}\sum_{1\le i<j\le n}\mathbb{E}(W_iW_j) - \frac{1}{n^2}\mathbb{E}^2\Big(\sum_{1\le i\le n}W_i\Big). \tag{4.4}$$

Firstly, by the definition of W_i, we have that

$$\sum_{1\le i\le n}\mathbb{E}(W_i^2) = \sum_{1\le i\le n}\mathbb{E}(W_i) = \sum_{1\le i\le n}\mathbb{P}(W_i) = O(n). \tag{4.5}$$

Secondly, for the third term on the right hand side of (4.4), by the definition of $\overline{S}_n$ and (4.3), we obtain

$$\mathbb{E}^2\Big(\sum_{1\le i\le n}W_i\Big) = \Big(\sum_{1\le i\le n}f_k(i)\Big)^2 + O(n^{3/2}\log^4 n), \tag{4.6}$$

where we have used (2.9) and (2.1) to obtain the O-term in the above.

Thirdly, we deal with the second term on the right hand side of (4.4). For $1 \le i < j \le n$, let P_i and P_j be the coordinates of the i-th and j-th steps of a path of the α-random walk, respectively. Here, we remark that P_j depends on P_i. By the definition of W_i, we have that

$$\mathbb{E}(W_iW_j) = \mathbb{P}(P_i, P_j \text{ are both } k-\text{full}).$$

Note that $P_i = (l, i-l)$ for some $0 \le l \le i$, then we have that $P_j = (l+m, j-l-m)$ for some $0 \le m \le j-i$. The probability that P_i and P_j are both k-full is

$$\sum_{\substack{0\le l\le i\\ \gcd(l,i-l)\text{ is }k-\text{full}}}\ \sum_{\substack{0\le m\le j-i\\ \gcd(l+m,j-l-m)\text{ is }k-\text{full}}}\mathbb{P}(P_i = (l, i-l), P_j = (l+m, j-l-m)).$$

Note that

$$\mathbb{P}(P_i = (l, i-l), P_j = (l+m, j-l-m)) = C_\alpha(i,l)C_\alpha(j-i,m),$$

$\gcd(l, i-l) = \gcd(l, i)$ and $\gcd(l+m, j-l-m) = \gcd(l+m, j)$. Then we have that

$$\mathbb{E}(W_iW_j) = \sum_{\substack{0\le l\le i\\ \gcd(l,i)\text{ is }k-\text{full}}} C_\alpha(i,l)\mathcal{P}_{k,\alpha}(l, j, j-i), \tag{4.7}$$

where $\mathcal{P}_{k,\alpha}$ is given by (2.12). For $\mathcal{P}_{k,\alpha}(l, j, j-i)$, applying Lemma 2.6, we obtain that

$$\mathcal{P}_{k,\alpha}(l, j, j-i) = f_k(j) + O_\alpha\Big(\frac{\tau_3(j)}{\sqrt{j-i}}\Big), \tag{4.8}$$

where $f_k(j)$ is given by (2.8). Inserting (4.8) into (4.7), we obtain that

$$\mathbb{E}(W_iW_j) = \sum_{\substack{0\le l\le i\\ \gcd(l,i)\text{ is }k-\text{full}}} C_\alpha(i,l)\Big(f_k(j) + O_\alpha\Big(\frac{\tau_3(j)}{\sqrt{j-i}}\Big)\Big).$$

By the binomial theorem, the contribution of the O-term to $\mathbb{E}(W_iW_j)$ is $O_\alpha(\tau_3(j)/\sqrt{j-i})$. Hence, we have that

$$\mathbb{E}(W_iW_j) = f_k(j)\sum_{\substack{0\le l\le i\\ \gcd(l,i)\text{ is }k-\text{full}}} C_\alpha(i,l) + O_\alpha\Big(\frac{\tau_3(j)}{\sqrt{j-i}}\Big).$$

Applying Lemma 2.6 again to the sum over l, we obtain that

$$\begin{aligned}\mathbb{E}(W_iW_j) =& f_k(j)\Big(f_k(i) + O_\alpha\Big(\frac{\tau_3(i)}{\sqrt{i}}\Big)\Big) + O_\alpha\Big(\frac{\tau_3(j)}{\sqrt{j-i}}\Big)\\ =& f_k(i)f_k(j) + O_\alpha\Big(\frac{\tau_3(j)\tau_3(i)}{\sqrt{i}}\Big) + O_\alpha\Big(\frac{\tau_3(j)}{\sqrt{j-i}}\Big),\end{aligned}$$

where we have used bound (2.9). Summing over $1 \le i < j \le n$ and using Lemma 2.1 to estimate the contribution of the above O-terms, we obtain that

$$\sum_{1\le i<j\le n}\mathbb{E}(W_iW_j) = \sum_{1\le i<j\le n} f_k(i)f_k(j) + O_\alpha(n^{3/2}\log^4 n).$$

Note that

$$2\sum_{1\le i<j\le n} f_k(i)f_k(j) = \Big(\sum_{1\le i\le n} f_k(i)\Big)^2 - \sum_{1\le i\le n} f_k^2(i) = \Big(\sum_{1\le i\le n} f_k(i)\Big)^2 + O(n\log^8 n), \tag{4.9}$$

where we have used bound (2.2) and

$$\sum_{1\le i\le n} f_k^2(i) \ll \sum_{1\le i\le n}\tau_3^2(i) \ll n\log^8 n.$$

Then we have that

$$2\sum_{1\le i<j\le n}\mathbb{E}(W_iW_j) = \Big(\sum_{1\le i\le n} f_k(i)\Big)^2 + O_\alpha(n^{3/2}\log^4 n). \tag{4.10}$$

Now Proposition 4.2 follows from inserting (4.5), (4.10) and (4.6) into (4.4). □

Combining Propositions 4.1, 4.2 with Lemma 2.4, we obtain Theorem 1.2.

Acknowledgment

This work is supported by Shandong Provincial Natural Science Foundation (Grant No. ZR2019BA028). The author appreciates Professor Kui Liu's helpful disscussion. The author also thanks the anonymous referees for valuable suggestions.

Conflict of interest

The author declares that there is no conflict of interest in this paper.

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