AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(6): 10564–10581.
DOI:10.3934/math.2022589
Received: 08 November 2021
Revised: 30 January 2022
Accepted: 06 March 2022
Published: 29 March 2022

Research article

Positive solutions for a system of Hadamard fractional $(\varrho_1, \varrho_2, \varrho_3)$-Laplacian operator with a parameter in the boundary

Ahmed Hussein Msmali*

Department of Mathematics, College of Science, Jazan University, Jazan, Saudi Arabia

* **Correspondence:** Email: amsmali@jazanu.edu.sa.

Abstract: In this paper, we are gratified to explore existence of positive solutions for a tripled nonlinear Hadamard fractional differential system with $(\varrho_1, \varrho_2, \varrho_3)$-Laplacian operator in terms of the parameter $(\sigma_1, \sigma_2, \sigma_3)$ are obtained, by applying Avery-Henderson and Leggett-Williams fixed point theorems. As an application, an example is given to illustrate the effectiveness of the main result.

Keywords: Hadamard fractional system; positive solutions; boundary value problems; p-Laplacian; fixed point theorems
Mathematics Subject Classification: 34A08, 34B15, 34B18, 34B27

---

1. Introduction

In this paper, we study the tripled nonlinear Hadamard fractional differential system with $(\varrho_1, \varrho_2, \varrho_3)$-Laplacian operators

$$D_{1^+}^{\beta_1}(\phi_{\varrho_1}(D_{1^+}^{\alpha_1}u(t))) = f_1(t, u(t), v(t), w(t)), \ t \in (1, e), \tag{1.1}$$

$$D_{1^+}^{\beta_2}(\phi_{\varrho_2}(D_{1^+}^{\alpha_2}v(t))) = f_2(t, u(t), v(t), w(t)), \ t \in (1, e), \tag{1.2}$$

$$D_{1^+}^{\beta_3}(\phi_{\varrho_3}(D_{1^+}^{\alpha_3}w(t))) = f_3(t, u(t), v(t), w(t)), \ t \in (1, e), \tag{1.3}$$

subject to the boundary conditions

$$\begin{aligned} &D_{1^+}^{\alpha_1}u(1) = 0, \quad D_{1^+}^{\delta_1}(\phi_{\varrho_1}(D_{1^+}^{\alpha_1}u(e))) = 0, \\ &u(1) = u'(1) = 0, \ \ \text{ß}_1 \ D_{1^+}^{\gamma_1}u(e) = \kappa_1 \ D_{1^+}^{\gamma_1}u(\eta) + \sigma_1; \end{aligned} \tag{1.4}$$

$$\begin{aligned} &D_{1^+}^{\alpha_2}v(1) = 0, \quad D_{1^+}^{\delta_2}(\phi_{\varrho_2}(D_{1^+}^{\alpha_2}v(e))) = 0, \\ &v(1) = v'(1) = 0, \ \ \text{ß}_2 \ D_{1^+}^{\gamma_2}v(e) = \kappa_2 \ D_{1^+}^{\gamma_2}v(\eta) + \sigma_2; \end{aligned} \tag{1.5}$$

$$\begin{aligned} &D_{1^+}^{\alpha_3}w(1) = 0, \quad D_{1^+}^{\delta_3}(\phi_{\varrho_3}(D_{1^+}^{\alpha_3}w(e))) = 0, \\ &w(1) = w'(1) = 0, \ \ \text{ß}_3 \ D_{1^+}^{\gamma_3}w(e) = \kappa_3 \ D_{1^+}^{\gamma_3}w(\eta) + \sigma_3; \end{aligned} \tag{1.6}$$

where $\alpha_i \in (2,3], \beta_i \in (1,2], \gamma_i \in (0,1]$, $\delta_i \in (0,1]$, $D^{\mho}_{1^+}$ denotes the Hadamard fractional derivative of order $\mho$ for ($\mho = \alpha_i, \beta_i, \delta_i, \gamma_i, i = 1,2,3$), $\alpha_i - \gamma_i - 1 > 0$, $\eta \in (1, e)$, $\sigma_i > 0$ is a parameter, $\varrho_i > 1$, $\phi_{\varrho_i}(s) = |s|^{\varrho_i - 2}s$, $\phi^{-1}_{\varrho_i} = \phi_{\varphi_i}$, $\frac{1}{\varrho_i} + \frac{1}{\varphi_i} = 1$ and $f_i \in C([1,e] \times R^3_+ \to R_+)$, for $i = 1,2,3$.

The positive solutions of boundary vale problems connected with system of fractional differential equations were deliberate by many novelist [9–11, 29] and enlarge to p-Laplacian subject to various boundary conditions [13, 21–26, 30–32, 38]. Our results generalize from the papers [14, 19, 33], when a parameter is involved in the boundary conditions. Later, many scholars study the Hadamard fractional differential equation or system [3–5, 15, 16, 34, 41, 42] and these results are further extended to p-Laplacian operator [17, 39, 40]. Newly researchers are focus on the the theory of tripled system of Hadamard fractional differential equations associated with p-Laplacian operator [35, 36]. For numerous supplication of the fractional calculus in diverse scientific and many engineering fields, the readers may turn to the reference books [1, 12, 18, 27, 28] and the the papers [2, 7, 8, 37].

We assemble the following postulate all over:

(A1) The functions $f_1, f_2, f_3 : [1, e] \times R^3_+ \to R_+$ are continuous.
(A2) $\text{ß}_i, \kappa_i$ are positive constants such that $\text{ß}_i > \kappa_i (\log \eta)^{\alpha_i - \gamma_i - 1}, \forall i = 1,2,3$.
(A3) $\Im_1, \Im_2, \Im_3, \Re_1, \Re_2, \Re_3$ are positive constants such that
$\frac{1}{\Im_1} + \frac{1}{\Im_2} + \frac{1}{\Im_3} + \frac{1}{\Re_1} + \frac{1}{\Re_2} + \frac{1}{\Re_3} \le 1$.

The rest of the article is categorize in the following manner. In Section 2, we come up with some definitions and lemmas that provide us with some useful details with respect to the behavior of solution of the problem (1.1)–(1.6), then we build the Green function and bounds for the homogeneous problem corresponding to (1.1)–(1.6). In Section 3, we get going a measure for the existence of positive solutions for the problem (1.1)–(1.6) by applying varies fixed point theorems in a Banach spaces. Finally, as an application, an example to exhibit our results is given.

2. Preliminaries

In this section, we will come up with some definitions and lemmas that will be worn in the proof of used by us main results.

Definition 2.1. *[18] The Hadamard fractional derivative of order $\alpha > 0$ of a function $f : [1, \infty) \to R$ is given by*

$$D^{\alpha}_{1^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(t\frac{d}{dt}\Big)^n \int_1^t \Big(\log\frac{t}{s}\Big)^{n-\alpha-1} f(s)\frac{ds}{s}, \quad n-1 < \alpha < n,$$

where $n = [\alpha] + 1$, $[\alpha]$ represent the integer part of the real number α and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2. *[18] The Hadamard fractional integral of order $\alpha > 0$ is given by*

$$I^q f(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} f(s)\frac{ds}{s}, \ \ \alpha > 0,$$

provided that integral exists.

In what follows, we work out the Green function kindred with (1.1) and (1.4), we let $\phi_{\varrho_1}(D^{\alpha_1}_{1^+} u(t)) = -\varpi(t)$, for $t \in [1, e]$. Then, from (1.1) and (1.4) we obtain

$$\begin{cases} -D^{\beta_1}_{1^+}\varpi(t) = f_1(t, u(t), v(t), w(t)), \ t \in (1, e), \\ \varpi(1) = 0; \ \ D^{\delta_1}_{1^+}\varpi(e) = 0. \end{cases} \tag{2.1}$$

Lemma 2.1. *The boundary value problem* (2.1) *grip the form*

$$\varpi(t) = \int_1^e H_1(t,s) f_1(s,u(s),v(s),w(s))\frac{ds}{s},$$

where

$$H_1(t,s) = \frac{1}{\Gamma(\beta_1)}\begin{cases} (\log t)^{\beta_1-1}(1-\log s)^{\beta_1-\delta_1-1} - \left(\log \frac{t}{s}\right)^{\beta_1-1}, & 1 \le s \le t \le e, \\ (\log t)^{\beta_1-1}(1-\log s)^{\beta_1-\delta_1-1}, & 1 \le t \le s \le e. \end{cases} \tag{2.2}$$

Proof. We utilize proposal in Lemma 2 of [41]. For some $d_i \in R (i = 1, 2)$, we have

$$\varpi(t) = d_1(\log t)^{\beta_1-1} + d_2(\log t)^{\beta_1-2} - \frac{1}{\Gamma(\beta_1)}\int_1^t \left(\log \frac{t}{s}\right)^{\beta_1-1} f_1(s,u(s),v(s),w(s))\frac{ds}{s}.$$

From the condition $\varpi(1) = 0$, we have $d_2 = 0$. Hence,

$$\varpi(t) = d_1(\log t)^{\beta_1-1} - \frac{1}{\Gamma(\beta_1)}\int_1^t \left(\log \frac{t}{s}\right)^{\beta_1-1} f_1(s,u(s),v(s),w(s))\frac{ds}{s}$$

and

$$D_{1^+}^{\delta_1}(\varpi(t)) = d_1\frac{\Gamma(\beta_1)}{\Gamma(\beta_1-\delta_1)}(\log t)^{\beta_1-\delta_1-1} - \frac{1}{\Gamma(\beta_1-\delta_1)}\int_1^t \left(\log \frac{t}{s}\right)^{\beta_1-\delta_1-1} f_1(s,u(s),v(s),w(s))\frac{ds}{s}.$$

Consequently, $D_{1^+}^{\delta_1}(\varpi(e)) = 0$, implies that

$$d_1 = \frac{1}{\Gamma(\beta_1)}\int_1^e (1-\log s)^{\beta_1-\delta_1-1} f_1(s,u(s),v(s),w(s))\frac{ds}{s}.$$

Therefore,

$$\begin{aligned}\varpi(t) &= \frac{1}{\Gamma(\beta_1)}\int_1^e (\log t)^{\beta_1-1}(1-\log s)^{\beta_1-\delta_1-1} f_1(s,u(s),v(s),w(s))\frac{ds}{s} \\ &\quad - \frac{1}{\Gamma(\beta_1)}\int_1^t \left(\log \frac{t}{s}\right)^{\beta_1-1} f_1(s,u(s),v(s),w(s))\frac{ds}{s} \\ &= \int_1^e H_1(t,s) f_1(s,u(s),v(s),w(s))\frac{ds}{s}.\end{aligned}$$

Note that $\phi_{\varrho_1}(D_{1^+}^{\alpha_1}u(t)) = -\varpi(t)$, then $\phi_{\varrho_1}(-D_{1^+}^{\alpha_1}u(t)) = \varpi(t)$ and $-D_{1^+}^{\alpha_1}u(t) = \phi_{\varphi_1}(\varpi(t))$, where φ_1 is a constant with $\varrho_1^{-1} + \varphi_1^{-1} = 1$. Then, from (1.1) and (1.4), we have

$$\begin{cases} -D_{1^+}^{\alpha_1}u(t) = \phi_{\varphi_1}(\varpi(t)), \ t \in (1,e), \\ u(1) = u'(1) = 0; \ \ ß_1 D_{1^+}^{\gamma_1}u(e) = \kappa_1 D_{1^+}^{\gamma_1}u(\eta) + \sigma_1. \end{cases} \tag{2.3}$$

□

Lemma 2.2. *The boundary value problem* (2.3) *is equivalent to the integral equation*

$$u(t) = \int_1^e G_1(t,s)\phi_{\varphi_1}(\varpi(s))\frac{ds}{s} + \frac{\sigma_1\Gamma(\alpha_1-\gamma_1)(\log t)^{\alpha_1-1}}{\Delta_1},$$

where

$$G_1(t,s)=\begin{cases} \underset{t\in[1,\eta]}{G_1(t,s)} = \begin{cases} G_{11}(t,s), & 1\le t\le s\le \eta < \mathrm{e},\\ G_{12}(t,s), & 1\le s\le \min\{t,\eta\} < \mathrm{e},\end{cases}\\ \\ \underset{t\in[\eta,e]}{G_1(t,s)} = \begin{cases} G_{13}(t,s), & 1\le \max\{t,\eta\}\le \mathrm{s}\le \mathrm{e},\\ G_{14}(t,s), & 1<\eta\le s\le t\le \mathrm{e},\end{cases}\end{cases} \tag{2.4}$$

$$G_{11}(t,s)=\frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}(1-\log s)^{\alpha_1-\gamma_1-1}-\kappa_1(\log t)^{\alpha_1-1}\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-\gamma_1-1}\Big],$$
$$G_{12}(t,s)=\frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}(1-\log s)^{\alpha_1-\gamma_1-1}-\kappa_1(\log t)^{\alpha_1-1}\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-\gamma_1-1}-\Lambda_1\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}\Big],$$
$$G_{13}(t,s)=\frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}(1-\log s)^{\alpha_1-\gamma_1-1}\Big],$$
$$G_{14}(t,s)=\frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}(1-\log s)^{\alpha_1-\gamma_1-1}-\Lambda_1\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}\Big],$$

Here $\Delta_1=\Gamma(\alpha_1)\Lambda_1\neq 0$; $\Lambda_1=ß_1-\kappa_1(\log\eta)^{\alpha_1-\gamma_1-1}$.

Proof. We went along with the idea in Lemma 2.1. For some $c_i\in R(i=1,2,3)$,

$$u(t)=c_1(\log t)^{\alpha_1-1}+c_2(\log t)^{\alpha_1-2}+c_3(\log t)^{\alpha_1-3}-\frac{1}{\Gamma(\alpha_1)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}\phi_{\varphi_1}(\varpi(s))\frac{ds}{s}.$$

From the boundary condition $u(1)=0, u'(1)=0$, we have $c_2=c_3=0$. Hence

$$u(t)=c_1(\log t)^{\alpha_1-1}-\frac{1}{\Gamma(\alpha_1)}\int_1^t(\log\frac{t}{s})^{\alpha_1-1}\phi_{\varphi_1}(\varpi(s))\frac{ds}{s}$$

and

$$D_{1^+}^{\gamma_1}(u(t))=c_1\frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1-\gamma_1)}(\log t)^{\alpha_1-\gamma_1-1}-\frac{1}{\Gamma(\alpha_1-\gamma_1)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha_1-\gamma_1-1}\phi_{\varphi_1}(\varpi(s))\frac{ds}{s}.$$

Consequently $ß_1D_{1^+}^{\gamma_1}u(e)=\kappa_1D_{1^+}^{\gamma_1}u(\eta)+\sigma_1$ implies that

$$c_1=\frac{1}{\Delta_1}\Big[ß_1\int_1^e(1-\log s)^{\alpha_1-\gamma_1-1}\phi_{\varphi_1}(\varpi(s))\frac{ds}{s}-\kappa_1\int_1^\eta\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-\gamma_1-1}\phi_{\varphi_1}(\varpi(s))\frac{ds}{s}\Big]+\frac{\sigma_1\Gamma(\alpha_1-\gamma_1)}{\Delta_1}.$$

As a result,

$$\begin{aligned}u(t)&=\frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}\int_1^e(1-\log s)^{\alpha_1-\gamma_1-1}\phi_{\varphi_1}(\varpi(s))\frac{ds}{s}\\&\quad-\kappa_1(\log t)^{\alpha_1-1}\int_1^\eta\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-\gamma_1-1}\phi_{\varphi_1}(\varpi(s))\frac{ds}{s}\\&\quad-\Lambda_1\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}\phi_{\varphi_1}(\varpi(s))\frac{ds}{s}\Big]+\frac{\sigma_1\Gamma(\alpha_1-\gamma_1)(\log t)^{\alpha_1-1}}{\Delta_1}\\&=\int_1^e G_1(t,s)\phi_{\varphi_1}(\varpi(s))\frac{ds}{s}+\frac{\sigma_1\Gamma(\alpha_1-\gamma_1)(\log t)^{\alpha_1-1}}{\Delta_1}.\end{aligned}$$

□

Note $\varpi(t) = \int_1^e H_1(t,s)f_1(s,u(s),v(s),w(s))\frac{ds}{s}$, $\quad t \in [1,e]$, and we have that (1.1) and (1.4) is identical to the integral equation

$$u(t) = \int_1^e G_1(t,s)\phi_{\varphi_1}\Big(\int_1^e H_1(t,\tau)f_1(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\Big)\frac{ds}{s} + \frac{\sigma_1\Gamma(\alpha_1-\gamma_1)(\log t)^{\alpha_1-1}}{\Delta_1}.$$

Lemma 2.3. *Suppose that* (*A*2) *hold. Then the Green function* $G_1(t,s)$ *specified in* (2.4) *is nonnegative, for all* $t, s \in [1,e]$.

Proof. Consider the Green's function $G_1(t,s)$ given by (2.4).
Let $1 \le t \le s \le \eta \le e$, then

$$\begin{aligned}
G_{11}(t,s) &= \frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}(1-\log s)^{\alpha_1-\gamma_1-1} - \kappa_1(\log t)^{\alpha_1-1}\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-\gamma_1-1}\Big]\\
&= \frac{(\log t)^{\alpha_1-1}}{\Delta_1}\Big[ß_1(1-\log s)^{\alpha_1-\gamma_1-1} - \kappa_1\Big(1-\frac{\log s}{\log\eta}\Big)^{\alpha_1-\gamma_1-1}(\log\eta)^{\alpha_1-\gamma_1-1}\Big]\\
&\ge \frac{(\log t)^{\alpha_1-1}}{\Delta_1}\Big[ß_1 - \kappa_1(\log\eta)^{\alpha_1-\gamma_1-1}\Big](1-\log s)^{\alpha_1-\gamma_1-1}\\
&= \frac{(\log t)^{\alpha_1-1}}{\Delta_1}\Big[\Lambda_1(1-\log s)^{\alpha_1-\gamma_1-1}\Big] \ge 0.
\end{aligned}$$

Let $1 \le s \le \min\{t,\eta\} \le e$, we have

$$\begin{aligned}
G_{12}(t,s) &= \frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}(1-\log s)^{\alpha_1-\gamma_1-1} - \kappa_1(\log t)^{\alpha_1-1}\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-\gamma_1-1} - \Lambda_1\Big(\ln\frac{t}{s}\Big)^{\alpha_1-1}\Big]\\
&= \frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}(1-\log s)^{\alpha_1-\gamma_1-1} - \kappa_1(\log t)^{\alpha_1-1}\Big(1-\frac{\log s}{\log\eta}\Big)^{\alpha_1-\gamma_1-1}(\log\eta)^{\alpha_1-\gamma_1-1}\\
&\qquad - \Lambda_1\Big(1-\frac{\log s}{\log t}\Big)^{\alpha_1-1}(\log t)^{\alpha_1-1}\Big]\\
&\ge \frac{(\log t)^{\alpha_1-1}}{\Delta_1}\Big[[ß_1 - \kappa_1(\log\eta)^{\alpha_1-\gamma_1-1}](1-\log s)^{\alpha_1-\gamma_1-1} - \Lambda_1(1-\log s)^{\alpha_1-1}\Big]\\
&= \frac{\Lambda_1(\log t)^{\alpha_1-1}}{\Delta_1}\Big[(1-\log s)^{-\gamma_1} - 1\Big](1-\log s)^{\alpha_1-1} \ge 0.
\end{aligned}$$

Let $1 \le \max\{t,\eta\} \le s \le e$, we have

$$G_{13}(t,s) = \frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}(1-\log s)^{\alpha_1-\gamma_1-1}\Big] \ge 0.$$

Let $1 < \eta \le s \le t \le e$, we have

$$\begin{aligned}
G_{14}(t,s) &= \frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}(1-\log s)^{\alpha_1-\gamma_1-1} - \Lambda_1\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}\Big]\\
&= \frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}(1-\log s)^{\alpha_1-\gamma_1-1} - \Lambda_1\Big(1-\frac{\log s}{\log t}\Big)^{\alpha_1-1}(\log t)^{\alpha_1-1}\Big]\\
&\ge \frac{(\log t)^{\alpha_1-1}}{\Delta_1}\Big[ß_1(1-\log s)^{-\gamma_1} - \Lambda_1\Big](1-\log s)^{\alpha_1-1} \ge 0,
\end{aligned}$$

which implies $G_1(t,s) \ge 0$. Hence the nonnegative is proved. □

Lemma 2.4. *Let $\Delta_1 > 0$. Then the Green function $G_1(t,s)$ is given by* (2.4) *satisfies the following inequalities:*

(i) $G_1(t,s) \leq G_1(e,s)$, *for all* $(t,s) \in [1,e] \times [1,e]$,
(ii) $G_1(t,s) \geq (\frac{1}{4})^{\alpha_1 - 1} G_1(e,s)$, *for all* $(t,s) \in I \times (1,e)$, *where* $I = [e^{1/4}, e^{3/4}]$.

Proof. Consider the Green's function $G_1(t,s)$ given by (2.4).

(i) Let $1 \leq t \leq s \leq \eta \leq e$, then

$$\begin{aligned}
\frac{\partial G_{11}(t,s)}{\partial t} &= \frac{(\alpha_1 - 1)}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-2}(1-\log s)^{\alpha_1-\gamma_1-1} - \kappa_1(\log t)^{\alpha_1-2}\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-\gamma_1-1}\Big] \\
&\geq \frac{(\alpha_1-1)(\log t)^{\alpha_1-2}}{\Delta_1}\Big[ß_1 - \kappa_1(\log\eta)^{\alpha_1-\gamma_1-1}\Big](1-\log s)^{\alpha_1-\gamma_1-1} \\
&= \frac{(\alpha_1-1)(\log t)^{\alpha_1-2}}{\Delta_1}\Lambda_1(1-\log s)^{\alpha_1-\gamma_1-1} \geq 0.
\end{aligned}$$

Let $1 \leq s \leq \min\{t,\eta\} \leq e$, we have

$$\begin{aligned}
\frac{\partial G_{12}(t,s)}{\partial t} &= \frac{(\alpha_1 - 1)}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-2}(1-\log s)^{\alpha_1-\gamma_1-1} - \kappa_1(\log t)^{\alpha_1-2}\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-\gamma_1-1} - \Lambda_1\Big(\ln\frac{t}{s}\Big)^{\alpha_1-2}\Big] \\
&\geq \frac{(\alpha_1-1)(\log t)^{\alpha_1-2}}{\Delta_1}\Big[\Lambda_1(1-\log s)^{\alpha_1-\gamma_1-1} - \Lambda_1(1-\log s)^{\alpha_1-2}\Big] \\
&> \frac{\Lambda_1(\alpha_1-1)(\log t)^{\alpha_1-2}}{\Delta_1}\Big[(1-\log s)^{-\gamma_1} - 1\Big](1-\log s)^{\alpha_1-1} \\
&= \frac{\Lambda_1(\alpha_1-1)(\log t)^{\alpha_1-2}}{\Delta_1}\Big[\gamma_1(\log s) + O(\log s)^2\Big](1-\log s)^{\alpha_1-1} \geq 0.
\end{aligned}$$

Let $1 \leq \max\{t,\eta\} \leq s \leq e$, we have

$$\frac{\partial G_{13}(t,s)}{\partial t} = \frac{(\alpha_1 - 1)}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-2}(1-\log s)^{\alpha_1-\gamma_1-1}\Big] \geq 0.$$

Let $1 < \eta \leq s \leq t \leq e$, we have

$$\begin{aligned}
\frac{\partial G_{14}(t,s)}{\partial t} &= \frac{(\alpha_1 - 1)}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-2}(1-\log s)^{\alpha_1-\gamma_1-1} - \Lambda_1\Big(\log\frac{t}{s}\Big)^{\alpha_1-2}\Big] \\
&\geq \frac{(\alpha_1-1)(\log t)^{\alpha_1-2}}{\Delta_1}\Big[ß_1(1-\log s)^{\alpha_1-\gamma_1-1} - \Lambda_1(1-\log s)^{\alpha_1-2}\Big] \\
&> \frac{(\alpha_1-1)(\log t)^{\alpha_1-2}}{\Delta_1}\Big[ß_1(1-\log s)^{-\gamma_1} - \Lambda_1\Big](1-\log s)^{\alpha_1-1} \\
&= \frac{(\alpha_1-1)(\log t)^{\alpha_1-2}}{\Delta_1}\Big[ß_1\gamma_1(\log s) + O(\log s)^2 + \kappa_1(\log\eta)^{\alpha_1-\gamma_1-1}\Big](1-\log s)^{\alpha_1-1} \geq 0.
\end{aligned}$$

Therefore, $G_1(t,s)$ is increasing with respect to t, which implies that $G_1(t,s) \leq G_1(e,s)$. Hence, inequality (i) is proved.

(ii) Let $1 \le t \le s \le \eta \le e$ and $t \in I$. Then

$$\begin{aligned} G_{11}(t,s) &= \frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}(1-\log s)^{\alpha_1-\gamma_1-1} - \kappa_1(\log t)^{\alpha_1-1}\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-\gamma_1-1}\Big] \\ &= \frac{(\log t)^{\alpha_1-1}}{\Delta_1}\Big[ß_1(1-\log s)^{\alpha_1-\gamma_1-1} - \kappa_1\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-\gamma_1-1}\Big] \\ &\ge \Big(\frac{1}{4}\Big)^{\alpha_1-1} G_{11}(e,s). \end{aligned}$$

Let $1 \le s \le \min\{t,\eta\} \le e$ and $t \in I$. Then

$$\begin{aligned} G_{12}(t,s) &= \frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}(1-\log s)^{\alpha_1-\gamma_1-1} - \kappa_1(\log t)^{\alpha_1-1}\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-\gamma_1-1} - \Lambda_1\Big(\ln\frac{t}{s}\Big)^{\alpha_1-1}\Big] \\ &\ge \frac{(\log t)^{\alpha_1-1}}{\Delta_1}\Big[ß_1(1-\log s)^{\alpha_1-\gamma_1-1} - \kappa_1\Big(\log\frac{\eta}{s}\Big)^{\alpha_1-\gamma_1-1} - \Lambda_1(1-\log s)^{\alpha_1-1}\Big] \\ &\ge \Big(\frac{1}{4}\Big)^{\alpha_1-1} G_{12}(e,s). \end{aligned}$$

Let $1 \le \max\{t,\eta\} \le s \le e$ and $t \in I$. Then

$$G_{13}(t,s) = \frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}(1-\log s)^{\alpha_1-\gamma_1-1}\Big] = (\log t)^{\alpha_1-1}G_{13}(e,s) \ge \Big(\frac{1}{4}\Big)^{\alpha_1-1} G_{13}(e,s).$$

Let $1 < \eta \le s \le t \le e$ and $t \in I$. Then

$$\begin{aligned} G_{14}(t,s) &= \frac{1}{\Delta_1}\Big[ß_1(\log t)^{\alpha_1-1}(1-\log s)^{\alpha_1-\gamma_1-1} - \Lambda_1\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}\Big] \\ &\ge \frac{(\log t)^{\alpha_1-1}}{\Delta_1}\Big[ß_1(1-\log s)^{\alpha_1-\gamma_1-1} - \Lambda_1(1-\log s)^{\alpha_1-1}\Big] \\ &\ge \Big(\frac{1}{4}\Big)^{\alpha_1-1} G_{14}(e,s). \end{aligned}$$

Therefore, $G_1(t,s) \ge \left(\frac{1}{4}\right)^{\alpha_1-1} G_1(e,s)$. Hence, the inequality (ii) is proved. □

Lemma 2.5. *Suppose* $(A1)$ *and* $(A2)$ *holds. Then the Green function* $H_1(t,s)$ *is given by* (2.2) *satisfies the following inequalities:*

(i) $0 \le H_1(t,s) \le H_1(s,s)$, *for all* $(t,s) \in [1,e]\times[1,e]$,
(ii) $H_1(t,s) \ge \varsigma_1(s)H_1(s,s)$, *for all* $(t,s) \in I\times(1,e)$, *where* $I = [e^{1/4}, e^{3/4}]$, *and for* $\xi \in I$,

$$\varsigma_1(s) = \begin{cases} \dfrac{(\frac{3}{4})^{\beta_1-1}(1-\log s)^{\beta_1-\delta_1-1} - (\frac{3}{4}-\log s)^{\beta_1-1}}{(\log s)^{\beta_1-1}(1-\log s)^{\beta_1-\delta_1-1}}, & s \in (1,\xi], \\ \dfrac{1}{4^{\beta_1-1}(\log s)^{\beta_1-1}}, & s \in [\xi, e). \end{cases}$$

Remark. In a similar manner, the results of the Green's functions $G_i(t,s)$, $G_i(t,s)$; i=2, 3 for the homogeneous BVP corresponding to the Hadamard fractional differential Eqs (1.2), (1.5) and (1.3), (1.6) are obtained.

Consider the following conditions:

(i) $G_i(t,s) \geq \aleph G_i(e,s)$, for all $(t,s) \in [1,e] \times [1,e]$ i=1,2,3,
(ii) $H_i(t,s) \geq \varsigma(s) H_i(e,s)$, for all $(t,s) \in I \times (1,e)$, i=1,2,3,

where $I = [e^{1/4}, e^{3/4}], \aleph = \min\left\{(\frac{1}{4})^{\alpha_1-1}, (\frac{1}{4})^{\alpha_2-1}, (\frac{1}{4})^{\alpha_3-1}\right\}, \varsigma(s) = \min\{\varsigma_1(s), \varsigma_2(s), \varsigma_3(s)\}$.

Next, let us recall some notations about cones and two fixed point theorems. For more details, we refer the reader to [6, 20].

Let B be a real Banach space. A closed convex set P in B is called a cone if the following conditions are satisfied:

(i) if $x \in P$, then $\lambda x \in P$ for any $\lambda \geq 0$;
(ii) if $x \in P$ and $-x \in P$ then $x = 0$.

A non-negative continuous functional ψ is said to be a concave on P if ψ is continuous and

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y), \quad x, y \in P, \quad t \in [0,1].$$

Letting a, b, c be three positive constants, and ψ be a nonnegative continuous functional on P, we denote

$$P_a = \{y \in P : \|y\| < a\}, \qquad \overline{P_a} = \{y \in P : \|y\| \leq a\},$$
$$P(\psi, a) = \{x \in P : \psi(x) < a\}, \qquad \overline{P(\psi, a)} = \{x \in P : \psi(x) \leq a\},$$
$$\partial P(\psi, a) = \{x \in P : \psi(x) = a\}, \qquad P(\psi, b, c) = \{y \in P : b \leq \psi(y), \ \|y\| \leq c\}.$$

Theorem 2.6. *[6] Let P be a cone in a real Banach space B, φ and ψ be two increasing, non-negative, and continuous functionals on P, and θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that for some $r > 0$ and $\gamma > 0$*

$$\psi(x) \leq \theta(x) \leq \varphi(x), \quad \|x\| \leq \gamma\psi(x), \quad x \in \overline{P(\psi, r)}.$$

Moreover, suppose that there exist a completely continuous operator $T : \overline{P(\psi, r)} \to P$ and $0 < p < q < r$ such that

$$\theta(\lambda x) \leq \lambda\theta(x), \ \ 0 \leq \lambda \leq 1, \ \ x \in \partial P(\theta, q),$$

and

(i) $\psi(Tx) > r$ for all $x \in \partial P(\psi, r)$,
(ii) $\theta(Tx) < q$ for all $x \in \partial P(\theta, q)$,
(iii) $P(\varphi, p) \neq \emptyset$ and $\varphi(Tx) > p$ for all $x \in \partial P(\psi, p)$,

then T has at least two fixed points x_1, x_2 belonging to $\overline{P(\psi, r)}$ such that $p < \varphi(x_1)$, $\theta(x_1) < q$ and $q < \theta(x_2)$, $\psi(x_2) < r$.

Theorem 2.7. *(Leggett-Williams [20]) Let $F : \overline{P_d} \to \overline{P_d}$ be a completely continuous operator and let ψ be a nonnegative continuous concave functional on P such that $\psi(x) \leq \|x\|$ for all $x \in \overline{P_d}$. Suppose that there exist $0 < a < b < c \leq d$ such that*

(i) $\{x \in P(\psi, b, c) \mid \psi(x) > b\} \neq \emptyset$ and $\psi(Fx) > b$ for $x \in P(\psi, b, c)$;
(ii) $\|Fx\| < a$ for $\|x\| \leq a$;
(iii) $\psi(Fx) > b$ for $x \in P(\psi, b, d)$ with $\|Fx\| > c$.

Then F has at least three fixed points x_1, x_2 and x_3 in $\overline{P_d}$ satisfying $\|x_1\| < a, b < \psi(x_2), a < \|x_3\|$ and $\psi(x_3) < b$.

3. Main result

In this section, let $E = C[1,e]$, then E is a Banach space with the norm $\|u\| = \max_{t\in[1,e]} |u(t)|$. Let $B = E\times E\times E$, then B is a Banach space with the norm $\|(u,v,w)\| = \|u\|+\|v\|+\|w\|$. We define a cone $P\subset B$ by

$$P = \Big\{(u,v,w)\in B,\ u(t)\geq 0,\ v(t)\geq 0,\ w(t)\geq 0,\ \forall t\in[1,e]$$

and

$$\min_{t\in I}[u(t)+v(t)+w(t)] \geq \aleph\|(u,v,w)\|\Big\},$$

where $I = [e^{1/4}, e^{3/4}]$, $\aleph = \min\Big\{(\frac{1}{4})^{\alpha_1-1}, (\frac{1}{4})^{\alpha_2-1}, (\frac{1}{4})^{\alpha_3-1}\Big\}$.

It is well know that the triple system of Hadamard fractional order BVP (1.1)–(1.6) is equivalent to

$$\begin{cases} u(t) &= \int_1^e G_1(t,s)\phi_{\varphi_1}\big(\int_1^e H_1(t,\tau)f_1(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\big)\frac{ds}{s} + \frac{\sigma_1\Gamma(\alpha_1-\gamma_1)(\log t)^{\alpha_1-1}}{\Delta_1},\\ v(t) &= \int_1^e G_2(t,s)\phi_{\varphi_2}\big(\int_1^e H_2(t,\tau)f_2(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\big)\frac{ds}{s} + \frac{\sigma_2\Gamma(\alpha_2-\gamma_2)(\log t)^{\alpha_2-1}}{\Delta_2},\\ w(t) &= \int_1^e G_3(t,s)\phi_{\varphi_3}\big(\int_1^e H_3(t,\tau)f_3(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\big)\frac{ds}{s} + \frac{\sigma_3\Gamma(\alpha_3-\gamma_3)(\log t)^{\alpha_3-1}}{\Delta_3}.\end{cases}$$

We define the operators $F_1, F_2, F_3 : P\to E$ and $F : P\to B$ by

$$F(u,v,w) = (F_1(u,v,w), F_2(u,v,w), F_3(u,v,w)) \tag{3.1}$$

with

$$\begin{cases} F_1(u,v,w)(t) &= \int_1^e G_1(t,s)\phi_{\varphi_1}\big(\int_1^e H_1(t,\tau)f_1(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\big)\frac{ds}{s}\\ &\quad + \frac{\sigma_1\Gamma(\alpha_1-\gamma_1)(\log t)^{\alpha_1-1}}{\Delta_1},\ t\in[1,e],\\ F_2(u,v,w)(t) &= \int_1^e G_2(t,s)\phi_{\varphi_2}\big(\int_1^e H_2(t,\tau)f_2(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\big)\frac{ds}{s}\\ &\quad + \frac{\sigma_2\Gamma(\alpha_2-\gamma_2)(\log t)^{\alpha_2-1}}{\Delta_2},\ t\in[1,e],\\ F_3(u,v,w)(t) &= \int_1^e G_3(t,s)\phi_{\varphi_3}\big(\int_1^e H_3(t,\tau)f_3(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\big)\frac{ds}{s}\\ &\quad + \frac{\sigma_3\Gamma(\alpha_3-\gamma_3)(\log t)^{\alpha_3-1}}{\Delta_3},\ t\in[1,e].\end{cases}$$

If $(u,v,w)\in P$ is a fixed point of operator F, then (u,v,w) is a solution of problem (1.1)–(1.6). So, we will investigate the existence of fixed points of operator F.

Lemma 3.1. *$F : P\to P$ is completely continuous.*

Proof. By using standard arguments, we can easily show that, the operator F is completely continuous and we need only to prove $F(P)\subset P$. Let $(u,v,w)\in P$, by Lemma 2.4, we have

$$\|F_1(u,v,w)\| \leq \int_1^e G_1(e,s)\phi_{\varphi_1}\Big(\int_1^e H_1(t,\tau)f_1(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\Big)\frac{ds}{s} + \frac{\sigma_1\Gamma(\alpha_1-\gamma_1)}{\Delta_1},$$
$$\|F_2(u,v,w)\| \leq \int_1^e G_2(e,s)\phi_{\varphi_2}\Big(\int_1^e H_2(t,\tau)f_2(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\Big)\frac{ds}{s} + \frac{\sigma_2\Gamma(\alpha_2-\gamma_2)}{\Delta_2},$$
$$\|F_3(u,v,w)\| \leq \int_1^e G_3(e,s)\phi_{\varphi_3}\Big(\int_1^e H_3(t,\tau)f_3(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\Big)\frac{ds}{s} + \frac{\sigma_3\Gamma(\alpha_3-\gamma_3)}{\Delta_3},$$

and

$$\begin{aligned}
\min_{t\in I} F_1(u,v,w)(t) = \min_{t\in I}\Big[\int_1^e G_1(t,s)\phi_{\varphi_1}\Big(\int_1^e H_1(t,\tau)f_1(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\Big)\frac{ds}{s} \\
+ \frac{\sigma_1\Gamma(\alpha_1-\gamma_1)(\log t)^{\alpha_1-1}}{\Delta_1}\Big], \\
\geq \Big(\frac{1}{4}\Big)^{\alpha_1-1}\Big[\int_1^e G_1(e,s)\phi_{\varphi_1}\Big(\int_1^e H_1(t,\tau)f_1(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\Big)\frac{ds}{s} + \frac{\sigma_1\Gamma(\alpha_1-\gamma_1)}{\Delta_1}\Big] \\
\geq \aleph\Big[\int_1^e G_1(e,s)\phi_{\varphi_1}\Big(\int_1^e H_1(t,\tau)f_1(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\Big)\frac{ds}{s} + \frac{\sigma_1\Gamma(\alpha_1-\gamma_1)}{\Delta_1}\Big] \\
\geq \aleph\|F_1(u,v,w)\|.
\end{aligned}$$

Similarly, $\min_{t\in I} F_2(u,v,w)(t) \geq \aleph\|F_2(u,v,w)\|$ and $\min_{t\in I} F_3(u,v,w)(t) \geq \aleph\|F_3(u,v,w)\|$. Therefore

$$\begin{aligned}
&\min_{t\in I}\{F_1(u,v,w)(t) + F_2(u,v,w)(t) + F_3(u,v,w)(t)\} \\
&\quad \geq \aleph\|F_1(u,v,w)\| + \aleph\|F_2(u,v,w)\| + \aleph\|F_3(u,v,w)\| \\
&\quad = \aleph\|F_1(u,v,w), F_2(u,v,w), F_3(u,v,w)\| \\
&\quad = \aleph\|F(u,v,w)\|.
\end{aligned}$$

Hence, we get $F(P) \subset P$. By using standard arguments involving the Arzela-Ascoli theorem, we can easily show that F_1, F_2 and F_3 are completely continuous, and then F is a completely continuous operator from P to P. □

We use the following notation for our convenience:

$$\begin{aligned}
M = \max\Big\{\Big[\int_{s\in I} G_1(e,s)\phi_{\varphi_1}\Big(\int_{\tau\in I}\varsigma(\tau)H_1(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s}\Big]^{-1}, \Big[\int_{s\in I} G_2(e,s)\phi_{\varphi_2}\Big(\int_{\tau\in I}\varsigma(\tau)H_2(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s}\Big]^{-1}, \\
\Big[\int_{s\in I} G_3(e,s)\phi_{\varphi_3}\Big(\int_{\tau\in I}\varsigma(\tau)H_3(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s}\Big]^{-1}\Big\}. \\
L = \min\Big\{\Big[\int_1^e G_1(e,s)\phi_{\varphi_1}\Big(\int_1^e H_1(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s}\Big]^{-1}, \Big[\int_1^e G_2(e,s)\phi_{\varphi_2}\Big(\int_1^e H_2(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s}\Big]^{-1}, \\
\Big[\int_1^e G_3(e,s)\phi_{\varphi_3}\Big(\int_1^e H_3(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s}\Big]^{-1}\Big\}.
\end{aligned}$$

Now let the non-negative, continuous functionals ψ, θ and φ be defined on the cone P by

$$\begin{aligned}
\psi(u,v,w) &= \min_{t\in I}\{u(t)+v(t)+w(t)\}, \\
\theta(u,v,w) &= \max_{t\in I}\{u(t)+v(t)+w(t)\}, \\
\varphi(u,v,w) &= \max_{t\in[1,e]}\{u(t)+v(t)+w(t)\},
\end{aligned} \tag{3.2}$$

and let $P(\psi, r) = \{(u,v,w) \in P : \psi(u,v,w) < r\}$.

Theorem 3.2. *Assume that* (A1)–(A3) *hold. Suppose there exist positive real numbers* $0 < p < q < r$ *such that* $0 < \sigma_i < \frac{q\Delta_i}{\Re_i\Gamma(\alpha_i-\gamma_i)} \leq \frac{r\Delta_i}{\Re_i\Gamma(\alpha_i-\gamma_i)}$ *such that* $f_i (i = 1,2,3)$ *satisfies the following conditions:*

(D1) $f_i(t,u,v,w) > \phi_{\varrho_i}(\frac{rM}{3\aleph})$ *for all* $t \in I$,$(u,v,w) \in [r, \frac{r}{\aleph}] \times [r, \frac{r}{\aleph}] \times [r, \frac{r}{\aleph}]$,
(D2) $f_i(t,u,v,w) < \phi_{\varrho_i}(\frac{qL}{\Im_i})$ *for all* $t \in [1,e]$,$(u,v,w) \in [0, \frac{q}{\aleph}] \times [0, \frac{q}{\aleph}] \times [0, \frac{q}{\aleph}]$,
(D3) $f_i(t,u,v,w) > \phi_{\varrho_i}(\frac{pM}{3\aleph})$ *for all* $t \in I$,$(u,v,w) \in [\aleph p, p] \times [\aleph p, p] \times [\aleph p, p]$.

Then the system (1.1)–(1.6) *has at least two positive solutions* (u_1, v_1, w_1) *and* (u_2, v_2, w_2) *such that*

$$p < \varphi(u_1, v_1, w_1) \ \text{ with } \ \theta(u_1, v_1, w_1) < q,$$

$$q < \theta(u_2, v_2, w_2) \ \text{ with } \ \psi(u_2, v_2, w_2) < r.$$

Proof. Due to Lemma 3.1, we have operator $F : P \to P$ is completely continuous. From (3.2), for each $(u,v,w) \in P$, we have $\psi(u,v,w) \le \theta(u,v,w) \le \varphi(u,v,w)$, and

$$\|(u,v,w)\| \le \frac{1}{\aleph} \min_{t\in I}\{u(t) + v(t) + w(t)\} = \frac{1}{\aleph}\psi(u,v,w).$$

For all $(u,v,w) \in P$, $\lambda \in [0,1]$ we have

$$\theta(\lambda u, \lambda v, \lambda w) = \max_{t\in I}\{\lambda u(t) + \lambda v(t) + \lambda w(t)\} = \lambda \max_{t\in I}\{u(t) + v(t) + w(t)\} = \lambda\theta(u,v,w).$$

It is clear that $\theta(0,0,0) = 0$. Now we show that the remaining conditions of Theorem 2.6 are satisfied. Firstly, we shall show that condition (*i*) of Theorem 2.6 is satisfied. Since $(u,v,w) \in \partial P(\psi, r)$, we have

$$\min_{t\in I}\{u(t) + v(t) + w(t)\} = r, r \le \|u\| + \|v\| + \|w\| \le \frac{r}{\aleph}.$$

From $(D1)$, we have

$$\begin{aligned}
\psi(F(u,v,w)) &= \min_{t\in I} F(u,v,w)(t) \\
&= \min_{t\in I} \sum_{i=1}^{3} \Big[\int_1^e G_i(t,s)\phi_{\varphi_i}\Big(\int_1^e H_i(t,\tau) f_i(\tau, u(\tau), v(\tau), w(\tau)) \frac{d\tau}{\tau} \Big) \frac{ds}{s} \\
&\qquad + \frac{\sigma_i \Gamma(\alpha_i - \gamma_i)(\log t)^{\alpha_i - 1}}{\Delta_i} \Big] \\
&\ge \aleph \sum_{i=1}^{3} \Big[\int_{s\in I} G_i(e,s)\phi_{\varphi_i}\Big(\int_{\tau\in I} \varsigma(\tau) H_i(\tau,\tau) f_i(\tau, u(\tau), v(\tau), w(\tau)) \frac{d\tau}{\tau} \Big) \frac{ds}{s} + \frac{\sigma_i \Gamma(\alpha_i - \gamma_i)}{\Delta_i} \Big] \qquad (3.3) \\
&> \frac{rM}{3\aleph} \int_{s\in I} \aleph G_1(e,s) \phi_{\varphi_1}\Big(\int_{\tau\in I} \varsigma(\tau) H_1(\tau,\tau) \frac{d\tau}{\tau} \Big) \frac{ds}{s} + \frac{rM}{3\aleph} \int_{s\in I} \aleph G_2(e,s) \phi_{\varphi_2}\Big(\int_{\tau\in I} \varsigma(\tau) H_2(\tau,\tau) \frac{d\tau}{\tau} \Big) \frac{ds}{s} \\
&\quad + \frac{rM}{3\aleph} \int_{s\in I} \aleph G_3(e,s) \phi_{\varphi_3}\Big(\int_{\tau\in I} \varsigma(\tau) H_3(\tau,\tau) \frac{d\tau}{\tau} \Big) \frac{ds}{s} \\
&\ge \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r.
\end{aligned}$$

Now, we shall show that condition (*ii*) of Theorem 2.6 is satisfied. Since $(u,v,w) \in \partial P(\theta, q)$, we have $0 \leq u(t) + v(t) + w(t) \leq \|u\| + \|v\| + \|w\| \leq \frac{q}{\aleph}$ for $t \in [1, e]$. From $(D2)$, we have

$$\begin{aligned}
\theta(F(u,v,w)) &= \max_{t\in I} F(u,v,w)(t) \\
&= \max_{t\in I} \sum_{i=1}^{3} \Big[\int_1^e G_i(t,s)\phi_{\varphi_i}\Big(\int_1^e H_i(t,\tau) f_i(\tau, u(\tau), v(\tau), w(\tau)) \frac{d\tau}{\tau}\Big)\frac{ds}{s} \\
&\qquad + \frac{\sigma_i \Gamma(\alpha_i - \gamma_i)(\log t)^{\alpha_i - 1}}{\Delta_i}\Big] \\
&\leq \sum_{i=1}^{3} \int_1^e G_i(e,s)\phi_{\varphi_i}\Big(\int_1^e H_i(\tau,\tau) f_i(\tau, u(\tau), v(\tau), w(\tau)) \frac{d\tau}{\tau}\Big)\frac{ds}{s} + \frac{\sigma_i \Gamma(\alpha_i - \gamma_i)}{\Delta_i} \\
&< \frac{qL}{\mathfrak{I}_1} \int_1^e G_1(e,s)\phi_{\varphi_1}\Big(\int_1^e H_1(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s} + \frac{qL}{\mathfrak{I}_2} \int_1^e G_2(e,s)\phi_{\varphi_2}\Big(\int_1^e H_2(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s} \\
&\quad + \frac{qL}{\mathfrak{I}_3} \int_1^e G_3(e,s)\phi_{\varphi_3}\Big(\int_1^e H_3(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s} + \frac{q}{\mathfrak{R}_1} + \frac{q}{\mathfrak{R}_2} + \frac{q}{\mathfrak{R}_3} \\
&= q\Big[\frac{1}{\mathfrak{I}_1} + \frac{1}{\mathfrak{I}_2} + \frac{1}{\mathfrak{I}_3} + \frac{1}{\mathfrak{R}_1} + \frac{1}{\mathfrak{R}_2} + \frac{1}{\mathfrak{R}_3}\Big] \leq q.
\end{aligned}$$

Finally using hypothesis $(D3)$, we shall show that condition (iii) of Theorem 2.6 is satisfied. Since $(0,0,0) \in P$ and $p > 0$, $P(\varphi, p) \neq \emptyset$. Since $(u,v,w) \in \partial P(\varphi, p)$, $\aleph p \leq u(t) + v(t) + w(t) \leq \|u\| + \|v\| + \|w\| = p$ for $t \in I$. From $(D3)$, we have $\varphi(F(u,v,w)) = \max_{t\in[1,e]} F(u,v,w) \geq p$. The process of proof is same as (3.3), so we omit it. Therefore, the hypothesis of Theorem 2.6 have been satisfied. Thus, the operator $F(u,v,w)$ has at least two positive solutions (u_1, v_1, w_1) and (u_2, v_2, w_2) such that

$$p < \varphi(u_1, v_1, w_1) \text{ with } \theta(u_1, v_1, w_1) < q,$$

$$q < \theta(u_2, v_2, w_2) \text{ with } \psi(u_2, v_2, w_2) < r.$$

Hence, the system (1.1)–(1.6) has at least two positive solutions (u_1, v_1, w_1) and (u_2, v_2, w_2). □

Theorem 3.3. *Assume that* $(A1)$–$(A3)$ *hold. Suppose that there exist* $0 < a < b < \aleph d$ *and* $0 < \sigma_i < \frac{a\Delta_i}{\mathfrak{R}_i \Gamma(\alpha_i - \gamma_i)} \leq \frac{d\Delta_i}{\mathfrak{R}_i \Gamma(\alpha_i - \gamma_i)}$ *such that* $f_i (i = 1, 2, 3)$ *satisfies the following conditions:*

$(D4)$ $f_i(t,u,v,w) < \phi_{\varrho_i}(\frac{dL}{\mathfrak{I}_i})$, *for all* $t \in [1,e]$, $(u,v,w) \in [0,d] \times [0,d] \times [0,d]$,
$(D5)$ $f_i(t,u,v,w) > \phi_{\varrho_i}(\frac{bM}{3\aleph})$, *for all* $t \in I$, $(u,v,w) \in [b, \frac{b}{\aleph}] \times [b, \frac{b}{\aleph}] \times [b, \frac{b}{\aleph}]$,
$(D6)$ $f_i(t,u,v,w) < \phi_{\varrho_i}(\frac{aL}{\mathfrak{I}_i})$, *for all* $t \in [1,e]$, $(u,v,w) \in [0,a] \times [0,a] \times [0,a]$.

Then the system (1.1)–(1.6) *has at least three positive solution* $(u_1, v_1, w_1), (u_2, v_2, w_2)$ *and* (u_3, v_3, w_3) *with* $\varphi(u_1, v_1, w_1) < a$, $b < \psi(u_2, v_2, w_2) < \varphi(u_2, v_2, w_2) < d$, $a < \varphi(u_3, v_3, w_3) < d$, *with* $\psi(u_3, v_3, w_3) < b$.

Proof. Firstly, if $(u,v,w) \in \overline{P_d}$, then we may assert that $F : \overline{P_d} \to \overline{P_d}$ is a completely continuous operator. To see this, suppose $(u,v,w) \in \overline{P_d}$, then $\|(u,v,w)\| \leq d$. It follows from Lemma 2.4, Lemma 2.5 and $(D4)$, that

$$\begin{aligned}
\|F(u,v,w)\| &= \max_{t\in[1,e]}\{F_1(u,v,w)(t)+F_2(u,v,w)(t)+F_3(u,v,w)(t)\}\\
&= \max_{t\in[1,e]}\sum_{i=1}^{3}\Big[\int_1^e G_i(t,s)\phi_{\varphi_i}\Big(\int_1^e H_i(t,\tau)f_i(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\Big)\frac{ds}{s}\\
&\qquad\qquad + \frac{\sigma_i\Gamma(\alpha_i-\gamma_i)(\log t)^{\alpha_i-1}}{\Delta_i}\Big]\\
&\le \sum_{i=1}^{3}\int_1^e G_i(e,s)\phi_{\varphi_i}\Big(\int_1^e H_i(\tau,\tau)f_i(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\Big)\frac{ds}{s}+\frac{\sigma_i\Gamma(\alpha_i-\gamma_i)}{\Delta_i}\\
&< \frac{dL}{\mathfrak{I}_1}\int_1^e G_1(e,s)\phi_{\varphi_1}\Big(\int_1^e H_1(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s}+\frac{dL}{\mathfrak{I}_2}\int_1^e G_2(e,s)\phi_{\varphi_2}\Big(\int_1^e H_2(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s}\\
&\quad+\frac{dL}{\mathfrak{I}_3}\int_1^e G_3(e,s)\phi_{\varphi_3}\Big(\int_1^e H_3(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s}+\frac{d}{\mathfrak{R}_1}+\frac{d}{\mathfrak{R}_2}+\frac{d}{\mathfrak{R}_3}\\
&= d\Big[\frac{1}{\mathfrak{I}_1}+\frac{1}{\mathfrak{I}_2}+\frac{1}{\mathfrak{I}_3}+\frac{1}{\mathfrak{R}_1}+\frac{1}{\mathfrak{R}_2}+\frac{1}{\mathfrak{R}_3}\Big]\le d
\end{aligned}$$

Therefore, $F:\overline{P_d}\to\overline{P_d}$. This together with Lemma 3.1 implies that $F:\overline{P_d}\to\overline{P_d}$ is a completely continuous operator. In the similarly way, if $(u,v,w)\in\overline{P_a}$, then from $(D6)$ yields $\|F(u,v,w)\|<a$. This shows that condition (ii) of Theorem 2.7 is fulfilled.

Now, we let $u(t)+v(t)+w(t)=\frac{b}{\aleph}$ for $t\in[1,e]$. It is easy to verify that $u(t)+v(t)+w(t)=\frac{b}{\aleph}\in P(\psi,b,\frac{b}{\aleph})$ and $\psi(u,v,w)=\frac{b}{\aleph}>b$, and so $\{(u,v,w)\in P(\psi,b,\frac{b}{\aleph});\psi(u,v,w)>b\}\neq\emptyset$. Thus, for all $(u,v,w)\in P(\psi,b,\frac{b}{\aleph})$, we have that $b\le u(t)+v(t)+w(t)\le\frac{b}{\aleph}$ for $t\in I$ and $F(u,v,w)\in P$, from $(D5)$, we have

$$\begin{aligned}
\psi(F(u,v,w)(t)) &= \min_{t\in I}\{F_1(u,v,w)(t)+F_2(u,v,w)(t)+F_3(u,v,w)(t)\}\\
&= \min_{t\in I}\sum_{i=1}^{3}\Big[\int_1^e G_i(t,s)\phi_{\varphi_i}\Big(\int_1^e H_i(t,\tau)f_i(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\Big)\frac{ds}{s}\\
&\qquad\qquad + \frac{\sigma_i\Gamma(\alpha_i-\gamma_i)(\log t)^{\alpha_i-1}}{\Delta_i}\Big]\\
&\ge \aleph\sum_{i=1}^{3}\Big[\int_{s\in I} G_i(e,s)\phi_{\varphi_i}\Big(\int_1^e H_i(t,\tau)f_i(\tau,u(\tau),v(\tau),w(\tau))\frac{d\tau}{\tau}\Big)\frac{ds}{s}+\frac{\sigma_i\Gamma(\alpha_i-\gamma_i)}{\Delta_i}\Big]\\
&> \frac{bM}{3\aleph}\int_{s\in I}\aleph G_1(e,s)\phi_{\varphi_1}\Big(\int_{\tau\in I}\varsigma(\tau)H_1(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s}\\
&\quad+\frac{bM}{3\aleph}\int_{s\in I}\aleph G_2(e,s)\phi_{\varphi_2}\Big(\int_{\tau\in I}\varsigma(\tau)H_2(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s}\\
&\quad+\frac{bM}{3\aleph}\int_{s\in I}\aleph G_3(e,s)\phi_{\varphi_3}\Big(\int_{\tau\in I}\varsigma(\tau)H_3(\tau,\tau)\frac{d\tau}{\tau}\Big)\frac{ds}{s}\\
&\ge \frac{b}{3}+\frac{b}{3}+\frac{b}{3}=b.
\end{aligned}$$

Hence the condition (i) of Theorem 2.7 is verified. Next, we prove that (iii) of Theorem 2.7 is satisfied. By Lemma 3.1, we have $\min_{t\in I} |F_1(u,v,w)(t) + F_2(u,v,w)(t) + F_3(u,v,w)(t)| > \aleph\|F(u,v,w)\| > d$ for $(u,v,w) \in P(\psi,b,d)$ with $\|F(u,v,w)\| > \frac{b}{\aleph}$. To sum up, all the conditions of Theorem 2.7 are satisfied, then there exist three positive solutions $(u_1,v_1,w_1),(u_2,v_2,w_2)$ and (u_3,v_3,w_3) with $\varphi(u_1,v_1,w_1) < a,\ b < \psi(u_2,v_2,w_2) < \varphi(u_2,v_2,w_2) < d,\ \ a < \varphi(u_3,v_3,w_3) < d$, with $\psi(u_3,v_3,w_3) < b$. □

Example 3.1. *Let* α_1=2.2, α_2=2.5, α_3=2.8, β_1=1.2, β_2=1.5, β_3=1.8, γ_1=0.2, γ_2=0.5, $\gamma_3 = 0.8$, $\delta_1 = 0.2$, $\delta_2 = 0.5$, $\delta_3 = 0.8$, ß$_1 = 4$, ß$_2 = 5$, ß$_3 = 6$, $\kappa_1 = 3$, $\kappa_2 = 4$, $\kappa_3 = 5$, $\varrho_1 = \varrho_2 = \varrho_3 = 2$.

We consider the system of Hadamard fractional differential equations:

$$\begin{cases} D_{1^+}^{1.2}(\phi_2(D_{1^+}^{2.2}u(t))) = f_1(t,u(t),v(t),w(t)),\ t \in (1,e), \\ D_{1^+}^{1.5}(\phi_2(D_{1^+}^{2.5}v(t))) = f_2(t,u(t),v(t),w(t)),\ t \in (1,e), \\ D_{1^+}^{1.8}(\phi_2(D_{1^+}^{2.8}w(t))) = f_3(t,u(t),v(t),w(t)),\ t \in (1,e), \end{cases} \tag{3.4}$$

subject to the boundary conditions

$$\begin{cases} D_{1^+}^{2.2}u(1) = 0, \quad D_{1^+}^{0.2}(\phi_{\varrho_1}(D_{1^+}^{2.2}u(e))) = 0, \\ u(1) = u'(1) = 0,\ \ 4\, D_{1^+}^{0.2}u(e) = 3\, D_{1^+}^{0.2}u(\eta) + \sigma_1; \\ D_{1^+}^{2.5}v(1) = 0, \quad D_{1^+}^{0.5}(\phi_{\varrho_2}(D_{1^+}^{2.5}v(e))) = 0, \\ v(1) = v'(1) = 0,\ \ 5\, D_{1^+}^{0.5}v(e) = 4\, D_{1^+}^{0.5}v(\eta) + \sigma_2; \\ D_{1^+}^{2.8}w(1) = 0, \quad D_{1^+}^{0.8}(\phi_{\varrho_3}(D_{1^+}^{2.8}w(e))) = 0, \\ w(1) = w'(1) = 0,\ \ 6\, D_{1^+}^{0.8}w(e) = 5\, D_{1^+}^{0.8}w(\eta) + \sigma_3; \end{cases} \tag{3.5}$$

where $\sigma_1,\ \sigma_2,\ \sigma_3$ *are parameters. We have* $\aleph$=min{0.189465, 0.125, 0.082469} = 0.082469, $\Delta_1 = 3.804563 > 0$, $\Delta_2 = 5.677232 > 0$, $\Delta_3 = 8.530643 > 0$, *so assumption* ($A2$) *is satisfied. Beside we deduce* M = max{30.90692, 20.39956, 22.28683} = 30.90692 *and* L = min{17.4452, 80.5, 6.621083} = 6.621083.

We consider the functions

$$f_1(t,u,v,w) = \begin{cases} 4, & 0 \le u+v+w \le 4, \\ 4 + 625(u+v+w-4), & 4 < u+v+w \le 5, \\ (e^{-(u+v+w)} + 1)\sin t + 625, & u+v+w > 5, \end{cases}$$

$$f_2(t,u,v,w) = \begin{cases} (\sin(u+v+w) + t)\log t + \frac{1}{5}, & 0 \le u+v+w \le 4, \\ 2e^{-t} + 630[(u+v+w) - 4], & 4 < u+v+w \le 5, \\ (e^{-(u+v+w)} + 100)\frac{1}{2}\log t + 630, & u+v+w > 5, \end{cases}$$

$$f_3(t,u,v,w) = \begin{cases} \frac{1}{2}(3-t)4^{-(u+v+w-1)}, & 0 \le u+v+w \le 4, \\ 7(e^{(u+v+w)-54}) - \log t, & 4 < u+v+w \le 5, \\ 587(e^{-t}+1) - (\frac{1}{2}\log(u+v+w) - 1), & u+v+w > 5. \end{cases}$$

Choosing $a = 4,\ b = 5,\ d = 727.55,\ \frac{1}{\Im_1} = \frac{1}{\Im_2} = \frac{1}{\Im_3} = \frac{1}{\Re_1} = \frac{1}{\Re_2} = \frac{1}{\Re_3} = \frac{1}{6}$ *then* $0 < a < b < \aleph d$ *and* $f_i (i = 1,2,3)$ *fulfilling the following conditions:*

($D4$) $f_i(t,u,v,w) < 802.8615 = \phi_{\varrho_i}(\frac{dL}{\Im_i}),\ t \in [1,e], |u| + |v| + |w| \in [0, 727.55]$,

$(D5)$ $f_i(t,u,v,w) > 624.6024 = \phi_{\varrho_i}(\frac{bM}{3\aleph})$, $t \in [e^{1/4}, e^{3/4}]$, $|u| + |v| + |w| \in [5, 60.6289]$,
$(D6)$ $f_i(t,u,v,w) < 4.414055 = \phi_{\varrho_i}(\frac{aL}{\mho_i})$, $t \in [1,e]$,$|u| + |v| + |w| \in [0,4]$.

Thus, all conditions of Theorem 3.3 are fulfilled. Hence, for $\sigma_1 \leq 461.335$*,* $\sigma_2 \leq 688.4117$*,* $\sigma_3 \leq 1034.412$*, the system of* (3.4) *and* (3.5) *has at least three positive solutions.*

4. Conclusions

By using Avery-Henderson and Leggett-Williams fixed point theorems under suitable conditions, we have presented the existence of multiple positive solutions for the system of Hadamard fractional differential equation with $(\varrho_1, \varrho_2, \varrho_3)$-Laplacian operator in terms of the parameter $(\sigma_1, \sigma_2, \sigma_3)$. Finally, we have given an example to demonstrate our result.

Conflict of interest

The authors declare that they have no competing interests.

References

1. T. Atanackovic, S. Pilipovic, B. Stankovic, D. Zorica, *Fractional calculus with applications in mechanics*, New York: Wiley, 2014. http://dx.doi.org/10.1002/9781118577530
2. A. Arafa, S. Rida, M. Khalil, Fractional modeling dynamics of HIV and CD4$^+$ T-cells during primary infection, *Nonlinear Biomed. Phys.*, **6** (2012), 1. http://dx.doi.org/10.1186/1753-4631-6-1
3. S. Aljoudi, B. Ahmad, J. Nieto, A. Alsaedi, On coupled Hadamard type sequential fractional differential equations with variable coefficients and nonlocal integral boundary conditions, *Filomat*, **31** (2017), 6041–6049. http://dx.doi.org/10.2298/FIL1719041A
4. S. Aljoudi, B. Ahmad, J. Nieto, A. Alsaedi, A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions, *Chaos Soliton. Fract.*, **91** (2016), 39–46. http://dx.doi.org/10.1016/j.chaos.2016.05.005
5. B. Ahmad, S. Ntouyas, A. Alsaedi, A. Albideewi, A study of a coupled system of Hadamard fractional differential equations with nonlocal coupled initial-multipoint conditions, *Adv. Differ. Equ.*, **2021** (2021), 33. http://dx.doi.org/10.1186/s13662-020-03198-4
6. R. Avery, J. Henderson, Two positive fixed points of nonlinear operators on ordered Banach spaces, *Communications on Applied Nonlinear Analysis*, **8** (2001), 27–36.
7. Y. Ding, H. Ye, A fractional-order differential equation model of HIV infection of CD4$^+$ T-cells, *Math. Comput. Model.*, **50** (2009), 386–392. http://dx.doi.org/10.1016/j.mcm.2009.04.019
8. R. Garra, E. Orsingher, F. Polito, A note on Hadamard fractional differential equations with varying coefficients and their applications in probability, *Mathematics*, **6** (2018), 4. http://dx.doi.org/10.3390/math6010004
9. J. Henderson, R. Luca, A. Tudorache, Existence and nonexistence of positive solutions for coupled Riemann-Liouville fractional boundary value problems, *Discrete Dyn. Nat. Soc.*, **2016** (2016), 2823971. http://dx.doi.org/10.1155/2016/2823971

10. J. Henderson, R. Luca, *Boundary value problems for systems of differential, difference and fractional equations: positive solutions*, Amsterdam: Elsevier, 2016.

11. J. Henderson, R. Luca, Systems of Riemann-Liouville fractional equations with multi-point boundary conditions, *Appl. Math. Comput.*, **309** (2017), 303–323. http://dx.doi.org/10.1016/j.amc.2017.03.044

12. R. Hilfer, *Applications of fractional calculus in physics*, New Jersey: World Scientific, 2000. http://dx.doi.org/10.1142/3779

13. W. Han, J. Jiang, Existence and mutiplicity of positive solutions for a system of nonlinear fractional multi-point boundary velua problems with p-Laplacian operator, *J. Appl. Anal. Comput.*, **11** (2021), 351–366. http://dx.doi.org/10.11948/20200021

14. Z. Han, H. Lu, S. Sun, Positive solutions to boundary-value problems of p-Laplacian fractional differential equations with a parameter in the boundary, *Electron. J. Differ. Equ.*, **2012** (2012), 1–14.

15. H. Huang, W. Liu, Positive solutions for a class of nonlinear Hadamard fractional differential equations with a parameter, *Adv. Differ. Equ.*, **2018** (2018), 96. http://dx.doi.org/10.1186/s13662-018-1551-9

16. S. Hamani, W. Benhamida, J. Henderson, Boundary value problems for Caputo-Hadamard fractional differential equations, *Advances in the Theory of Nonlinear Analysis and its Application*, **2** (2018), 138–145. http://dx.doi.org/10.31197/atnaa.419517

17. J. Jiang, D. O'Regan, J. Xu, Y. Cui, Positive solutions for a Hadamard fractional p-Laplacian three-point boundary value problem, *Mathematics*, **7** (2019), 439. http://dx.doi.org/10.3390/math7050439

18. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006. http://dx.doi.org/10.1016/S0304-0208(06)80001-0

19. L. Kong, D. Piao, L. Wang, Positive solutions for third boundary value problems with p-Laplacian, *Results Math.*, **55** (2009), 111–128. http://dx.doi.org/10.1007/s00025-009-0383-z

20. R. Leggett, L. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.*, **28** (1979), 673–688. http://dx.doi.org/10.1512/iumj.1979.28.28046

21. Z. Lv, J. Liu, J. Xu, Multiple positive solutions for a system of Caputo fractional p-Laplacian boundary value problems, *Complexity*, **2020** (2020), 6723791. http://dx.doi.org/10.1155/2020/6723791

22. R. Luca, Positive solutions for a system of fractional differential equations with p-Laplacian operator and multi-point boundary conditions, *Nonlinear Anal.-Model.*, **23** (2018), 771–801. http://dx.doi.org/10.15388/NA.2018.5.8

23. R. Luca, On a system of fractional boundary value problems with p-Laplacian operator, *Dynam. Syst. Appl.*, **28** (2019), 691–713. http://dx.doi.org/10.12732/dsa.v28i3.10

24. Y. Li, Multiple positive solutions for nonlinear mixed fractional differential equation with p-Laplacian operator, *Adv. Differ. Equ.*, **2019** (2019), 112. http://dx.doi.org/10.1186/s13662-019-2041-4

25. D. Li, Y. Liu, C. Wang, Multiple positive solutions for fractional three-point boundary value problem with p-Laplacian operator, *Math. Probl. Eng.*, **2020** (2020), 2327580. http://dx.doi.org/10.1155/2020/2327580
26. Y. Liu, D. Xie, C. Bai, D. Yang, Multiple positive solutions for a coupled system of fractional multi-point BVP with p-Laplacian operator, *Adv. Differ. Equ.*, **2017** (2017), 168. http://dx.doi.org/10.1186/s13662-017-1221-3
27. K. Oldham, J. Spanier, *The fractional calculus: theory and applications of differentiation and integration to arbitrary order*, New York: Academic Press, 1974.
28. Y. Povstenko, *Fractional thermoelasticity*, New York: Springer, 2015. http://dx.doi.org/10.1007/978-3-319-15335-3
29. T. Qi, Y. Liu, Y. Cui, Existence of solutions for a class of coupled fractional differential systems with nonlocal boundary conditions, *J. Funct. Space.*, **2017** (2017), 6703860. http://dx.doi.org/10.1155/2017/6703860
30. S. Rao, Multiplicity of positive solutions for fractional differential equation with p-Laplacian boundary value problem, *Int. J. Differ. Equ.*, **2016** (2016), 6906049. http://dx.doi.org/10.1155/2016/6906049
31. S. Rao, M. Zico, Positive solutions for a coupled system of nonlinear semipositone fractional boundary value problems, *Int. J. Differ. Equ.*, **2019** (2019), 2893857. http://dx.doi.org/10.1155/2019/2893857
32. S. Rao, Multiple positive solutions for coupled system of p-Laplacian fractional order three-point boundary value problems, *Rocky Mountain J. Math.*, **49** (2019), 609–626. http://dx.doi.org/10.1216/RMJ-2019-49-2-609
33. V. Raju, B. Krushna, On a coupled sytem of fractional order differential equations with Riemann-Liouville type boundary conditions, *J. Nonlinear Funct. Anal.*, **2018** (2018), 25. http://dx.doi.org/10.23952/jnfa.2018.25
34. S. Rao, A. Msmali, M. Singh, A. Ahmadini, Existence and uniqueness for a system of Caputo-Hadamard fractional differential equations with multi-point boundary conditions, *J. Funct. Space.*, **2020** (2020), 8821471. http://dx.doi.org/10.1155/2020/8821471
35. S. Rao, A. Ahmadini, Multiple positive solutions for a system of (p_1, p_2, p_3)-Laplacian Hadamard fractional order BVP with parameters, *Adv. Differ. Equ.*, **2021** (2021), 436. http://dx.doi.org/10.1186/s13662-021-03591-7
36. S. Rao, M. Singh, M. Meetei, Multiplicity of positive solutions for Hadamard fractional differential equations with p-Laplacian operator, *Bound. Value Probl.*, **2020** (2020), 43. http://dx.doi.org/10.1186/s13661-020-01341-4
37. R. Saxena, R. Garra, E. Orsingher, Analytical solution of space-time fractional telegraph-type equations involving Hilfer and Hadamard derivatives, *Integr. Transf. Spec. F.*, **27** (2016), 30–42. http://dx.doi.org/10.1080/10652469.2015.1092142
38. A. Tudorache, R. Luca, Positive solutions for a system of Riemann-Liouville fractional boundary value problems with p-Laplacian operators, *Adv. Differ. Equ.*, **2020** (2020), 292. http://dx.doi.org/10.1186/s13662-020-02750-6

39. G. Wang, T. Wang, On a nonlinear Hadamard type fractional differential equation with p-Laplacian operator and strip condition, *J. Nonlinear Sci. Appl.*, **9**, (2016), 5073–5081. http://dx.doi.org/10.22436/jnsa.009.07.10

40. J. Xu, J. Jiang, D. O'Regan, Positive solutions for a class of p-Laplacian Hadamard fractional order three point boundary value problems, *Mathematics*, **8** (2020), 308. http://dx.doi.org/10.3390/math8030308

41. W. Yang, Positive solutions for singular coupled integral boundary value problems of nonlinear Hadamard fractional differential equations, *J. Nonlinear Sci. Appl.*, **8** (2015), 110–129. http://dx.doi.org/10.22436/jnsa.008.02.04

42. C. Zhai, W. Wang, Solution for a system of Hadamard fractional differential equations with integral conditions, *Numer. Func. Anal. Opt.*, **41** (2020), 209–229. http://dx.doi.org/10.1080/01630563.2019.1620771

AIMS Press

© 2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)