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Research article

Super warped products with a semi-symmetric non-metric connection

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Abstract: In this paper, we define a semi-symmetric non-metric connection on super Riemannian manifolds. And we compute the curvature tensor and the Ricci tensor of a semi-symmetric non-metric connection on super warped product spaces. Next, we introduce two kinds of super warped product spaces with a semi-symmetric non-metric connection and give the conditions that two super warped product spaces with a semi-symmetric non-metric connection are the Einstein super spaces with a semi-symmetric non-metric connection.

Keywords: semi-symmetric non-metric connection; the curvature tensor; Ricci tensor; super warped product space; the Einstein super space
Mathematics Subject Classification: 53C40, 53C42

1. Introduction

The (singly) warped product $B \times_h F$ of two pseudo-Riemannian manifolds (B, g_B) and (F, g_F) with a smooth function $h : B \to (0, \infty)$ is the product manifold $B \times F$ with the metric tensor $g = g_B \oplus h^2 g_F$. Here, (B, g_B) is called the base manifold, (F, g_F) is called as the fiber manifold and h is called as the warping function. Generalized Robertson-Walker space-times and standard static space-times are two well-known warped product spaces. The concept of warped products was first introduced by Bishop and ONeil (see [4]) to construct examples of Riemannian manifolds with negative curvature. In Riemannian geometry, warped product manifolds and their generic forms have been used to construct new examples with interesting curvature properties since then. In [7], F. Dobarro and E. Dozo had studied from the viewpoint of partial differential equations and variational methods, the problem of showing when a Riemannian metric of constant scalar curvature can be produced on a product manifolds by a warped product construction. In [8], Ehrlich, Jung and Kim got explicit solutions to warping function to have a constant scalar curvature for generalized Robertson-Walker space-times. In [3], explicit solutions were also obtained for the warping function to make the space-time as Einstein when the fiber is also Einstein.

N. S. Agashe and M. R. Chafle introduced the notion of a semi-symmetric non-metric connection and studied some of its properties and submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection [1, 2]. In [13, 14], Sular and Özgur studied warped product manifolds with a semi-symmetric metric connection and a semi-symmetric non-metric connection, they computed curvature of semi-symmetric metric connection and semi-symmetric non-metric connection and considered Einstein warped product manifolds with a semi-symmetric metric connection and a semi-symmetric non-metric connection. In [16], Wang studied the Einstein multiply warped products with a semi-symmetric metric connection and the multiply warped products with a semi-symmetric metric connection with constant scalar curvature.

On the other hand, in [5], the definition of super warped product spaces was given. Einstein warped products were studied in [9]. In [10], several new super warped product spaces were given and the authors also studied the Einstein equations with cosmological constant in these new super warped product spaces. In [17], Wang studied super warped product spaces with a semi-symmetric metric connection. Our motivation is to study super warped product spaces with a semi-symmetric non-metric connection.

In Section 2, we state some definitions of super manifolds and super Riemannian metrics. We also define a semi-symmetric non-metric connection on super Riemannian manifolds and prove that there is a unique semi-symmetric non-metric connection on super Riemannian manifolds which is non-metric and has the semi-symmetric torsion. In Section 3, we compute the curvature tensor and the Ricci tensor of a semi-symmetric non-metric connection on super warped product spaces. In Section 4, we introduce two kinds of super warped product spaces with a semi-symmetric non-metric connection and give the conditions that two super warped product spaces with a semi-symmetric non-metric connection are the Einstein super spaces with a semi-symmetric non-metric connection.

2. A semi-symmetric non-metric connection on super Riemannian manifolds

In this section, we give some definitions about Riemannian supergeometry.

Definition 2.1. *(Definition 1 in [5]) A locally $\mathbb{Z}_2$-ringed space is a pair $S := (|S|, O_S)$ where $|S|$ is a second-countable Hausdorff space, and a O_S is a sheaf of $\mathbb{Z}_2$-graded $\mathbb{Z}_2$-commutative associative unital $\mathbb{R}$-algebras, such that the stalks $O_{S,p}$, $p \in |S|$ are local rings.*

In this context, $\mathbb{Z}_2$-commutative means that any two sections $s, t \in O_S(|U|)$, $|U| \subset |S|$ open, of homogeneous degree $|s| \in \mathbb{Z}_2$ and $|t| \in \mathbb{Z}_2$ commute up to the sign rule $st = (-1)^{|s||t|}ts$. $\mathbb{Z}_2$-ring space $U^{m|n} := (U, C^\infty_{U^m} \otimes \wedge\mathbb{R}^n)$, is called standard superdomain where $C^\infty_{U^m}$ is the sheaf of smooth functions on U and $\wedge\mathbb{R}^n$ is the exterior algebra of $\mathbb{R}^n$. We can employ (natural) coordinates $x^I := (x^a, \xi^A)$ on any $\mathbb{Z}_2$-domain, where x^a form a coordinate system on U and the ξ^A are formal coordinates.

Definition 2.2. *(Notation and preliminary concepts in [6]) A supermanifold of dimension $m|n$ is a super ringed space $M = (|M|, O_M)$ that is locally isomorphic to $\mathbb{R}^{m|n}$ and $|M|$ is a second countable and Hausdorff topological space.*

The tangent sheaf $\mathcal{T}M$ of a $\mathbb{Z}_2$-manifold M is defined as the sheaf of derivations of sections of the structure sheaf, i.e., $\mathcal{T}M(|U|) := \mathrm{Der}(O_M(|U|))$, for arbitrary open set $|U| \subset |M|$. Naturally, this is a sheaf of locally free O_M-modules. Global sections of the tangent sheaf are referred to as *vector* *fields*. We denote the $\mathcal{O}_M(|M|)$-module of vector fields as $\mathrm{Vect}(M)$. The dual of the tangent sheaf is the *cotangent sheaf*, which we denote as $\mathcal{T}^*M$. This is also a sheaf of locally free $\mathcal{O}_M$-modules. Global section of the cotangent sheaf we will refer to as *one-forms* and we denote the $\mathcal{O}_M(|M|)$-module of one-forms as $\Omega^1(M)$.

Definition 2.3. *(Definition 4 in [5]) A Riemannian metric on a $\mathbb{Z}_2$-manifold M is a $\mathbb{Z}_2$-homogeneous, $\mathbb{Z}_2$-symmetric, non-degenerate, $\mathcal{O}_M$-linear morphisms of sheaves $\langle -, - \rangle_g : \mathcal{T}M \otimes \mathcal{T}M \to \mathcal{O}_M$. A $\mathbb{Z}_2$-manifold equipped with a Riemannian metric is referred to as a Riemannian $\mathbb{Z}_2$-manifold.*

We will insist that the Riemannian metric is homogeneous with respect to the $\mathbb{Z}_2$-degree, and we will denote the degree of the metric as $|g| \in \mathbb{Z}_2$. Explicitly, a Riemannian metric has the following properties:
(1) $|\langle X, Y \rangle_g| = |X| + |Y| + |g|$,
(2) $\langle X, Y \rangle_g = (-1)^{|X||Y|} \langle Y, X \rangle_g$,
(3) If $\langle X, Y \rangle_g = 0$ for all $Y \in Vect(M)$, then $X = 0$,
(4) $\langle fX + Y, Z \rangle_g = f \langle X, Z \rangle_g + \langle Y, Z \rangle_g$,
for arbitrary (homogeneous) $X, Y, Z \in \mathrm{Vect}(M)$ and $f \in C^\infty(M)$. We will say that a Riemannian metric is even if and only if it has degree zero. Similarly, we will say that a Riemannian metric is odd if and only if it has degree one. Any Riemannian metric we consider will be either even or odd as we will only be considering homogeneous metrics.

Now we recall the definition of the warped product of Riemannian $\mathbb{Z}_2$-manifolds. For details, see the Section 2.3 in [5]. Let $M_1 \times M_2$ be the product of two $\mathbb{Z}_2$-manifolds M_1 and M_2. Let $(M_i, g_i)(i = 1, 2)$ be Riemannian $\mathbb{Z}_2$-manifolds whose Riemannian metric are of the same $\mathbb{Z}_2$-degree. Let $\mu \in C^\infty(M_1)$ be a degree 0 invertible global functions that is strictly positive, i.e., $\varepsilon_{M_1}(\mu)$ a strictly positive function on $|M_1|$ where ε is simply "throwing away" the formal coordinates. Then the warped product is defined as

$$M_1 \times_\mu M_2 := (M_1 \times M_2, g := \pi_1^* g_1 + (\pi_1^* \mu) \pi_2^* g_2),$$

where $\pi_i : M_1 \times M_2 \to M_i \ (i = 1, 2)$ is the projection. By Proposition 4 in [5], the warped product $M_1 \times_\mu M_2$ is a Riemannian $\mathbb{Z}_2$-manifold.

Definition 2.4. *(Definition 9 in [5]) An affine connection on a $\mathbb{Z}_2$-manifold is a $\mathbb{Z}_2$-degree preserving map*

$$\nabla : \ \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M); \ \ (X, Y) \mapsto \nabla_X Y,$$

which satisfies the following
1) Bi-linearity

$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z; \ \ \nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z,$$

2) $C^\infty(M)$-linearrity in the first argument

$$\nabla_{fX} Y = f \nabla_X Y,$$

3) The Leibniz rule

$$\nabla_X (fY) = X(f)Y + (-1)^{|X||f|} f \nabla_X Y,$$

for all homogeneous $X, Y, Z \in \mathrm{Vect}(M)$ and $f \in C^\infty(M)$.

Definition 2.5. *(Definition 10 in [5]) The torsion tensor of an affine connection* $T_\nabla : \ \mathrm{Vect}(M) \otimes_{C^\infty(M)} \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ *is defined as*

$$T_\nabla(X, Y) := \nabla_X Y - (-1)^{|X||Y|}\nabla_Y X - [X, Y],$$

for any (homogeneous) $X, Y \in \mathrm{Vect}(M)$*. An affine connection is said to be symmetric if the torsion vanishes.*

Definition 2.6. *(Definition 11 in [5]) An affine connection on a Riemannian* $\mathbb{Z}_2$*-manifold* (M, g) *is said to be metric compatible if and only if*

$$X \langle Y, Z\rangle_g = \langle \nabla_X Y, Z\rangle_g + (-1)^{|X||Y|} \langle Y, \nabla_X Z\rangle_g ,$$

for any $X, Y, Z \in \mathrm{Vect}(M)$.

Theorem 2.7. *(Theorem 1 in [5]) There is a unique symmetric (torsionless) and metric compatible affine connection* ∇^L *on a Riemannian* $\mathbb{Z}_2$*-manifold* (M, g) *which satisfies the Koszul formula*

$$\begin{aligned} 2\left\langle \nabla^L_X Y, Z\right\rangle_g &= X \langle Y, Z\rangle_g + \langle [X, Y], Z\rangle_g \\ &+ (-1)^{|X|(|Y|+|Z|)}(Y \langle Z, X\rangle_g - \langle [Y, Z], X\rangle_g) \\ &- (-1)^{|Z|(|X|+|Y|)}(Z \langle X, Y\rangle_g - \langle [Z, X], Y\rangle_g), \end{aligned} \tag{2.1}$$

for all homogeneous $X, Y, Z \in \mathrm{Vect}(M)$.

Definition 2.8. *(Definition 13 in [5]) The Riemannian curvature tensor of an affine connection*

$$R_\nabla : \ \mathrm{Vect}(M) \otimes_{C^\infty(M)} \mathrm{Vect}(M) \otimes_{C^\infty(M)} \mathrm{Vect}(M) \to \mathrm{Vect}(M)$$

is defined as

$$R_\nabla(X, Y)Z = \nabla_X\nabla_Y - (-1)^{|X||Y|}\nabla_Y\nabla_X - \nabla_{[X,Y]}Z, \tag{2.2}$$

for all X, Y *and* $Z \in \mathrm{Vect}(M)$.

Directly from the definition it is clear that

$$R_\nabla(X, Y)Z = -(-1)^{|X||Y|}R_\nabla(Y, X)Z, \tag{2.3}$$

for all X, Y and $Z \in \mathrm{Vect}(M)$.

Definition 2.9. *(Definition 14 in [5]) The Ricci curvature tensor of an affine connection is the symmetric rank-*2 *covariant tensor defined as*

$$Ric_\nabla(X, Y) := (-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|X|+|Y|)}\frac{1}{2}\left[R_\nabla(\partial_{x^I}, X)Y + (-1)^{|X||Y|}R_\nabla(\partial_{x^I}, Y)X\right]^I, \tag{2.4}$$

where $X, Y \in \mathrm{Vect}(M)$ *and* $[\ \ ]^I$ *denotes the coefficient of* ∂_{x^I} *and* ∂_{x^I} *is the natural frame of* $\mathcal{T}M$.

Definition 2.10. *(Definition 16 in [5]) Let* $f \in C^\infty(M)$ *be an arbitrary function on a Riemannian* $\mathbb{Z}_2$*-manifold* (M, g)*. The gradient of* f *is the unique vector field* $\mathrm{grad}_g f$ *such that*

$$X(f) = (-1)^{|f||g|}\left\langle X, \mathrm{grad}_g f\right\rangle_g , \tag{2.5}$$

for all $X \in \mathrm{Vect}(M)$.

Definition 2.11. *(Definition 17 in [5]) Let* (M, g) *be a Riemannian* $\mathbb{Z}_2$*-manifold and let* ∇^L *be the associated Levi-Civita connection. The covariant divergence is the map* $\mathrm{Div}_L : \mathrm{Vect}(M) \to C^\infty(M)$*, given by*

$$\mathrm{Div}_L(X) = (-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|X|)}(\nabla_{\partial_{x^I}} X)^I, \tag{2.6}$$

for any arbitrary $X \in \mathrm{Vect}(M)$.

Definition 2.12. *(Definition 18 in [5]) Let* (M, g) *be a Riemannian* $\mathbb{Z}_2$*-manifold and let* ∇^L *be the associated Levi-Civita connection. The connection Laplacian (acting on functions) is the differential operator of* $\mathbb{Z}_2$*-degree* $|g|$ *defined as*

$$\triangle_g(f) = \mathrm{Div}_L(\mathrm{grad}_g f), \tag{2.7}$$

for any and all $f \in C^\infty(M)$.

Definition 2.13. *Let* (M, g) *be a Riemannian* $\mathbb{Z}_2$*-manifold and* $P \in \mathrm{Vect}(M)$ *which satisfied* $|g| + |P| = 0$ *and we define a semi-symmetric non-metric connection* $\widehat{\nabla}$ *on* (M, g)

$$\widehat{\nabla}_X Y = \nabla^L_X Y + X \cdot g(Y, P) = \nabla^L_X Y + (-1)^{|X||Y|} g(Y, P) X, \tag{2.8}$$

for any homogenous $X, Y \in \mathrm{Vect}(M)$ *and where* $X \cdot f = (-1)^{|X||f|} f X$ *for* $f \in C^\infty(M)$.

Obviously, we have $\widehat{\nabla}_{X+Y} Z = \widehat{\nabla}_X Z + \widehat{\nabla}_Y Z$; $\widehat{\nabla}_X(Y + Z) = \widehat{\nabla}_X Y + \widehat{\nabla}_X Z$, for any homogenous $X, Y, Z \in \mathrm{Vect}(M)$. We can verify that $\widehat{\nabla}_X Y$ satisfies the Definition 2.4, then $\widehat{\nabla}_X Y$ is an affine connection. By Definition 2.5, we get

$$T_{\widehat{\nabla}}(X, Y) = X \cdot g(Y, P) - (-1)^{|X||Y|} Y \cdot g(X, P). \tag{2.9}$$

Then, we call that $\widehat{\nabla}_X Y$ is a semi-symmetric connection. By Definition 2.6 and Definition 2.13, we get

$$\begin{aligned}
&\left\langle \widehat{\nabla}_X Y, Z \right\rangle_g + (-1)^{|X||Y|} \left\langle Y, \widehat{\nabla}_X Z \right\rangle_g \\
&= X \langle Y, Z \rangle_g + \langle X \cdot g(Y, P), Z \rangle_g + (-1)^{|X||Y|} \langle Y, X \cdot g(Z, P) \rangle_g \\
&= X \langle Y, Z \rangle_g + (-1)^{|Y||X|} g(Y, P) g(X, Z) + (-1)^{|X||Y|}(-1)^{|Z|(|X|+|Y|)} g(Z, P) g(Y, X).
\end{aligned} \tag{2.10}$$

So $\widehat{\nabla}$ doesn't preserve the metric.

Theorem 2.14. *There is a unique non-metric compatible affine connection* $\widehat{\nabla}$ *on a Riemannian* $\mathbb{Z}_2$*-manifold* (M, g) *which satisfies (2.9) and (2.10).*

Proof. By (2.9), we know that a semi-symmetric non-metric connection $\widehat{\nabla}$ satisfies the conditions in Theorem 2.14, then we only need to prove the uniqueness. Let ∇^* be the other connection which satisfies (2.9) and (2.10). And let $\nabla^*_X Y = \nabla^L_X Y + B(X, Y)$, then

$$B(fX, Y) = f B(X, Y), \quad B(X, fY) = (-1)^{|f||X|} B(X, Y). \tag{2.11}$$

By ∇^L preserving the metric and (2.10), we get

$$\begin{aligned}
&g(\nabla^*_X Y, Z) + (-1)^{|X||Y|} g(Y, \nabla^*_X Z) \\
&= g(\nabla^L_X Y, Z) + g(B(X, Y), Z) + (-1)^{|X||Y|} g(Y, \nabla^L_X Z) + (-1)^{|X||Y|} g(Y, B(X, Z)) \\
&= X \langle Y, Z \rangle_g + (-1)^{|Y||X|} g(Y, P) g(X, Z) + (-1)^{|X||Y|}(-1)^{|Z|(|X|+|Y|)} g(Z, P) g(Y, X).
\end{aligned} \tag{2.12}$$

So

$$g(B(X,Y),Z) + (-1)^{|X||Y|}g(Y,B(X,Z))$$
$$= (-1)^{|Y||X|}g(Y,P)g(X,Z) + (-1)^{|X||Y|}(-1)^{|Z|(|X|+|Y|)}g(Z,P)g(Y,X). \tag{2.13}$$

By ∇^L having no torsion, we have

$$\begin{aligned} T_{\nabla^*}(X,Y) &= \nabla^*_X Y - (-1)^{|X||Y|}\nabla^*_Y X - [X,Y] \\ &= \nabla^L_X Y + B(X,Y) - (-1)^{|X||Y|}\nabla^L_Y X - (-1)^{|X||Y|}B(Y,X) - [X,Y] \\ &= B(X,Y) - (-1)^{|X||Y|}B(Y,X). \end{aligned} \tag{2.14}$$

By (2.13) and (2.14) and $|B| = 0$, we have

$$g(T_{\nabla^*}(X,Y),Z) + (-1)^{|Z|(|X|+|Y|)}g(T_{\nabla^*}(Z,X),Y) + (-1)^{|X||Y|}(-1)^{|Z|(|X|+|Y|)}g(T_{\nabla^*}(Z,Y),X)$$
$$= 2g(B(X,Y),Z) - 2(-1)^{|X||Y|}(-1)^{|Z|(|X|+|Y|)}g(Z,P)g(Y,X). \tag{2.15}$$

By (2.9) and (2.15), we get

$$2g(B(X,Y),Z) = 2g(X\cdot g(Y,P),Z),$$

then $B(X,Y) = X\cdot g(Y,P)$. So $\nabla^* = \widehat{\nabla}$, we get the proof of uniqueness. □

Proposition 2.15. *The following equality holds*

$$\begin{aligned} R_{\widehat{\nabla}}(X,Y)Z = R^L(X,Y)Z &+ (-1)^{(|X|+|Y|)|Z|}[g(Z,\nabla^L_X P)Y - (-1)^{|X||Y|}g(Z,\nabla^L_Y P)X] \\ &+ (-1)^{(|X|+|Y|)|Z|}\pi(Z)[(-1)^{|X||Y|}\pi(Y)X - \pi(X)Y], \end{aligned} \tag{2.16}$$

where π is a one form defined by $\pi(Z) := g(Z,P)$ and $|\pi| = 0$.

Proof. By Definition 2.8 and Definition 2.13, we have

$$\begin{aligned} R_{\widehat{\nabla}}(X,Y)Z = &\nabla^L_X\nabla^L_Y Z + (-1)^{|Y||Z|}\nabla^L_X(\pi(Z)Y) - (-1)^{|X||Y|}[\nabla^L_Y\nabla^L_X Z + (-1)^{|X||Z|}\nabla^L_Y(\pi(Z)X)] \\ &+ (-1)^{|X|(|Y|+|Z|)}\pi(\nabla^L_Y Z)X + (-1)^{|X|(|Y|+|Z|)}(-1)^{|Y||Z|}\pi(Z)\pi(Y)X \\ &- (-1)^{|X||Y|}(-1)^{|Y|(|X|+|Z|)}[\pi(\nabla^L_X Z)Y + (-1)^{|X||Z|}\pi(Z)\pi(X)Y] - \nabla^L_{[X,Y]}Z \\ &- (-1)^{(|X|+|Y|)|Z|}\pi(Z)[X,Y], \end{aligned} \tag{2.17}$$

since ∇^L preserving metric, we have

$$\nabla^L_X(\pi(Z)Y) = \pi(\nabla^L_X Z)Y + (-1)^{|X||Z|}g(Z,\nabla^L_X P)Y + (-1)^{|X||Z|}\pi(Z)\nabla^L_X Y. \tag{2.18}$$

Then, by (2.17) and (2.18), we can get Proposition 2.15. □

3. Super warped products with a semi-symmetric non-metric connection

Let $(M = M_1 \times_\mu M_2, g_\mu = \pi_1^* g_1 + \pi_1^*(\mu)\pi_2^* g_2)$ be the super warped product with $|g| = |g_1| = |g_2|$ and $|\mu| = 0$. For simplicity, we assume that $\mu = h^2$ with $|h| = 0$. Let $\nabla^{L,\mu}$ be the Levi-Civita connection on (M, g_μ) and ∇^{L,M_1} (resp. ∇^{L,M_2}) be the Levi-Civita connection on (M_1, g_1) (resp. (M_2, g_2)).

Lemma 3.1. *(Lemma 3.1 in [17]) For $X, Y, Z \in \mathrm{Vect}(M_1)$ and $U, W, V \in \mathrm{Vect}(M_2)$, we have*

$$
\begin{aligned}
&(1)\ \nabla^{L,\mu}_X Y = \nabla^{L,M_1}_X Y,\ \ (2)\ \nabla^{L,\mu}_X U = \frac{X(h)}{h}U,\\
&(3)\ \nabla^{L,\mu}_U X = (-1)^{|U||X|}\frac{X(h)}{h}U,\ \ (4)\ \nabla^{L,\mu}_U W = -hg_2(U,W)\mathrm{grad}_{g_1}h + \nabla^{L,M_2}_U W.
\end{aligned}
\tag{3.1}
$$

Let $R^{L,\mu}$ denote the curvature tensor of the Levi-Civita connection on (M, g_μ) and let R^{L,M_1} (resp. R^{L,M_2}) be the curvature tensor of the Levi-Civita connection on (M_1, g_1) (resp. (M_2, g_2)). Let $H^h_{M_1}(X, Y) := XY(h) - \nabla^{L,M_1}_X Y(h)$, then $H^h_{M_1}(fX, Y) = fH^h_{M_1}(X, Y)$ and $H^h_{M_1}(X, fY) = (-1)^{|f||X|} fH^h_{M_1}(X, Y)$. $H^h_{M_1}$ is a $(0, 2)$ tensor.

Proposition 3.2. *(Proposition 3.2 in [17]) For $X, Y, Z \in \mathrm{Vect}(M_1)$ and $U, V, W \in \mathrm{Vect}(M_2)$, we have*

$$
\begin{aligned}
&(1)\ R^{L,\mu}(X,Y)Z = R^{L,M_1}(X,Y)Z,\ \ (2)\ R^{L,\mu}(V,X)Y = -(-1)^{|V|(|X|+|Y|)}\frac{H^h_{M_1}(X,Y)}{h}V,\\
&(3)\ R^{L,\mu}(X,Y)V = 0,\ \ (4)\ R^{L,\mu}(V,W)X = 0,\\
&(5)\ R^{L,\mu}(X,V)W = -(-1)^{|X|(|V|+|W|+|g|)}\frac{g_\mu(V,W)}{h}\nabla^{L,M_1}_X(\mathrm{grad}_{g_1}h),\\
&(6)\ R^{L,\mu}(V,W)U = R^{L,M_2}(V,W)U - (-1)^{|V|(|W|+|U|)}g_2(W,U)(\mathrm{grad}_{g_1}h)(h)V\\
&+ (-1)^{|W||U|}g_2(V,U)(\mathrm{grad}_{g_1}h)(h)W.
\end{aligned}
\tag{3.2}
$$

For $\overline{X}, \overline{Y}, P \in \mathrm{Vect}(M)$, we define

$$
\widehat{\nabla}^{\mu}_{\overline{X}}\overline{Y} = \nabla^{L,\mu}_{\overline{X}}\overline{Y} + \overline{X}\cdot g_\mu(\overline{Y}, P).
\tag{3.3}
$$

For $X, Y, P \in \mathrm{Vect}(M_1)$, we define

$$
\widehat{\nabla}^{M_1}_X Y = \nabla^{L,M_1}_X Y + X\cdot g_1(Y,P).
\tag{3.4}
$$

By Lemma 3.1, (3.3) and (3.4), we have

Lemma 3.3. *For $X, Y, P \in \mathrm{Vect}(M_1)$ and $U, W \in \mathrm{Vect}(M_2)$ and $\pi(X) = g_1(X, P)$, we have*

$$
\begin{aligned}
&(1)\ \widehat{\nabla}^{\mu}_X Y = \widehat{\nabla}^{M_1}_X Y,\ \ (2)\ \widehat{\nabla}^{\mu}_X U = \frac{X(h)}{h}U,\\
&(3)\ \widehat{\nabla}^{\mu}_U X = (-1)^{|U||X|}[\frac{X(h)}{h} + \pi(X)]U,\\
&(4)\ \widehat{\nabla}^{\mu}_U W = -hg_2(U,W)\mathrm{grad}_{g_1}h + \nabla^{L,N}_U W.
\end{aligned}
\tag{3.5}
$$

Lemma 3.4. *For $X, Y \in \mathrm{Vect}(M_1)$ and $U, W, P \in \mathrm{Vect}(M_2)$, we have*

$$
\begin{aligned}
&(1)\ \widehat{\nabla}^{\mu}_X Y = \nabla^{L,M_1}_X Y - g_1(X,Y)P,\ \ (2)\ \widehat{\nabla}^{\mu}_X U = \frac{X(h)}{h}U + X\cdot g_\mu(U,P),\\
&(3)\ \widehat{\nabla}^{\mu}_U X = (-1)^{|U||X|}\frac{X(h)}{h}U,\\
&(4)\ \widehat{\nabla}^{\mu}_U W = -hg_2(U,W)\mathrm{grad}_{g_1}h + \nabla^{L,M_2}_U W + U\cdot g_\mu(W,P).
\end{aligned}
\tag{3.6}
$$

By Proposition 2.15, Lemmas 3.3 and 3.4, we get the following propositions.

Proposition 3.5. *For $X, Y, Z, P \in \mathrm{Vect}(M_1)$ and $U, V, W \in \mathrm{Vect}(M_2)$, we have*

$$(1)\ R_{\widehat{\nabla}^{\mu}}(X, Y)Z = R_{\widehat{\nabla}^{M_1}}(X, Y)Z,$$

$$(2)\ R_{\widehat{\nabla}^{\mu}}(V, X)Y = -(-1)^{|V|(|X|+|Y|)}\left[\frac{H^h_{M_1}(X, Y)}{h} + (-1)^{|X||Y|}g_1(Y, \nabla^{L,M_1}_X P) - \pi(X)\pi(Y)\right]V,$$

$$(3)\ R_{\widehat{\nabla}^{\mu}}(X, Y)V = 0,\quad (4)\ R_{\widehat{\nabla}^{\mu}}(V, W)X = 0,$$

$$(5)\ R_{\widehat{\nabla}^{\mu}}(X, V)W = -(-1)^{|X|(|V|+|W|+|g|)}g_{\mu}(V, W)\left[\frac{\nabla^{L,M_1}_X(\mathrm{grad}_{g_1} h)}{h} + (-1)^{(|X|+|P|)|g|}\frac{P(h)}{h}X\right],$$

when $|g| = |P| = 0$, then

$$R_{\widehat{\nabla}^{\mu}}(X, V)W = -(-1)^{|X|(|V|+|W|)}g_{\mu}(V, W)\left[\frac{\nabla^{L,M_1}_X(\mathrm{grad}_{g_1} h)}{h} + \frac{P(h)}{h}X\right],$$

$$\begin{aligned}(6)\ R_{\widehat{\nabla}^{\mu}}(U, V)W = R^{L,M_2}(U, V)W &+ \left[(-1)^{|g|(|W|+|g|)}\frac{(\mathrm{grad}_{g_1} h)(h)}{h^2} + (-1)^{|P|(|W|+|g|)}\frac{P(h)}{h}\right]\\ &\cdot\left[(-1)^{|V||W|}(-1)^{|P||U|}g_{\mu}(U, W)V - (-1)^{|U|(|V|+|W|)}(-1)^{|P||V|}g_{\mu}(V, W)U\right],\end{aligned}$$

when $|g| = |P| = 0$, then

$$\begin{aligned}R_{\widehat{\nabla}^{\mu}}(U, V)W = R^{L,M_2}(U, V)W &+ \left[\frac{(\mathrm{grad}_{g_1} h)(h)}{h^2} + \frac{P(h)}{h}\right]\\ &\cdot\left[(-1)^{|V||W|}g_{\mu}(U, W)V - (-1)^{|U|(|V|+|W|)}g_{\mu}(V, W)U\right].\end{aligned}\tag{3.7}$$

Proof. (1) By Lemma 3.1 and Definition 2.8, we get

$$\begin{aligned}R_{\widehat{\nabla}^{\mu}}(X, Y)Z &= \widehat{\nabla}^{\mu}_X\widehat{\nabla}^{\mu}_Y - (-1)^{|X||Y|}\widehat{\nabla}^{\mu}_Y\widehat{\nabla}^{\mu}_X - \widehat{\nabla}^{\mu}_{[X,Y]}Z,\\ &= \widehat{\nabla}^{M_1}_X\widehat{\nabla}^{M_1}_Y - (-1)^{|X||Y|}\widehat{\nabla}^{M_1}_Y\widehat{\nabla}^{M_1}_X - \widehat{\nabla}^{M_1}_{[X,Y]}Z,\\ &= R_{\widehat{\nabla}^{M_1}}(X, Y)Z,\end{aligned}\tag{3.8}$$

so (1) holds.

(2) By Lemma 3.1 and Proposition 2.15, we have

$$\begin{aligned}R_{\widehat{\nabla}^{\mu}}(V, X)Y &= R^{L,\mu}(V, X)Y + (-1)^{(|V|+|X|)|Y|}[g_{\mu}(Y, \nabla^{L,\mu}_V P)X - (-1)^{|X||V|}g_{\mu}(Y, \nabla^{L,\mu}_X P)V]\\ &\quad + (-1)^{(|X|+|V|)|Y|}\pi(Y)[(-1)^{|X||V|}\pi(X)V - \pi(V)X],\\ &= R^{L,\mu}(V, X)Y + (-1)^{(|V|+|X|)|Y|}(-1)^{|X||V|}[\pi(Y)\pi(X)V - g_1(Y, \nabla^{L,M_1}_X P)V],\\ &= -(-1)^{|V|(|X|+|Y|)}\left[\frac{H^h_{M_1}(X, Y)}{h} + (-1)^{|X||Y|}g_1(Y, \nabla^{L,M_1}_X P) - \pi(X)\pi(Y)\right]V,\end{aligned}\tag{3.9}$$

so we get (2).

(3) By Lemma 3.1 and Proposition 2.15, we have

$$\begin{aligned}R_{\widehat{\nabla}^{\mu}}(X,Y)V &= R^{L,\mu}(X,Y)V + (-1)^{(|X|+|Y|)|V|}[g_{\mu}(V,\nabla^{L,\mu}_{X}P)Y - (-1)^{|X||Y|}g_{\mu}(V,\nabla^{L,\mu}_{Y}P)X]\\ &+ (-1)^{(|X|+|Y|)|V|}\pi(V)[(-1)^{|X||Y|}\pi(Y)X - \pi(X)Y]\\ &= 0,\end{aligned} \tag{3.10}$$

then we get (3).

(4) Similar to (3), we get $R_{\widehat{\nabla}^{\mu}}(V,W)X = 0$.

(5) By Lemma 3.1 and Proposition 2.15, we have

$$\begin{aligned}R_{\widehat{\nabla}^{\mu}}(X,V)W &= R^{L,\mu}(X,V)W + (-1)^{(|V|+|X|)|W|}[g_{\mu}(W,\nabla^{L,\mu}_{X}P)V - (-1)^{|X||V|}g_{\mu}(W,\nabla^{L,\mu}_{V}P)X]\\ &+ (-1)^{(|X|+|V|)|W|}\pi(W)[(-1)^{|X||V|}\pi(V)X - \pi(X)V],\\ &= -(-1)^{(|V|+|W|+|g|)|X|}\frac{g_{\mu}(V,W)}{h}\nabla^{L,M_1}_{X}(grad_{g_1}h)\\ &- (-1)^{|W|(|X|+|V|)}(-1)^{|X||V|}g_{\mu}(W,(-1)^{|P||V|}\frac{P(h)}{h}V)X,\end{aligned} \tag{3.11}$$

and by

$$\begin{aligned}&(-1)^{|W|(|X|+|V|)}(-1)^{|X||V|}g_{\mu}(W,(-1)^{|P||V|}\frac{P(h)}{h}V)X\\ &= (-1)^{|X|(|W|+|V|+|g|)}(-1)^{(|X|+|P|)|g|}g_{\mu}(V,W)\frac{P(h)}{h}X,\end{aligned} \tag{3.12}$$

so we have

$$R_{\widehat{\nabla}^{\mu}}(X,V)W = -(-1)^{|X|(|V|+|W|+|g|)}g_{\mu}(V,W)\left[\frac{\nabla^{L,M_1}_{X}(\mathrm{grad}_{g_1}h)}{h} + (-1)^{(|X|+|P|)|g|}\frac{P(h)}{h}X\right]. \tag{3.13}$$

Obviously, we can get

$$\begin{aligned}&\text{when } |g| = |P| = 0, \text{ then}\\ &R_{\widehat{\nabla}^{\mu}}(X,V)W = -(-1)^{|X|(|V|+|W|)}g_{\mu}(V,W)\left[\frac{\nabla^{L,M_1}_{X}(\mathrm{grad}_{g_1}h)}{h} + \frac{P(h)}{h}X\right].\end{aligned} \tag{3.14}$$

(6) By Lemma 3.1 and Proposition 2.15, we have

$$\begin{aligned}R_{\widehat{\nabla}^{\mu}}(U,V)W &= R^{L,\mu}(U,V)W + (-1)^{(|V|+|U|)|W|}[g_{\mu}(W,\nabla^{L,\mu}_{U}P)V - (-1)^{|U||V|}g_{\mu}(W,\nabla^{L,\mu}_{V}P)U]\\ &+ (-1)^{(|V|+|U|)|W|}\pi(W)[(-1)^{|U||V|}\pi(V)U - \pi(U)V]\\ &= R^{L,M_2}(U,V)W - (-1)^{(|V|+|W|)|U|}g_2(V,W)(grad_{g_1}h)(h)U + (-1)^{|W||V|}g_2(U,W)\\ &(grad_{g_1}h)(h)V + (-1)^{(|V|+|U|)|W|}(-1)^{|U||P|}(-1)^{|W||P|}\frac{P(h)}{h}g_{\mu}(W,U)V\\ &- (-1)^{(|V|+|U|)|W|}(-1)^{|V||U|}(-1)^{|V||P|}(-1)^{|W||P|}\frac{P(h)}{h}g_{\mu}(W,V)U,\end{aligned} \tag{3.15}$$

by

$$
\begin{aligned}
&-(-1)^{(|V|+|W|)|U|}g_2(V,W)(grad_{g_1}h)(h)U+(-1)^{|W||V|}g_2(U,W)(grad_{g_1}h)(h)V\\
&=(-1)^{|g_1|(|W|+|g_2|)}(grad_{g_1}h)(h)[-(-1)^{(|V|+|W|)|U|}(-1)^{|V||g_1|}g_2(V,W)U+(-1)^{|W||U|}(-1)^{|U||g_1|}g_2(U,W)V],
\end{aligned}
\tag{3.16}
$$

and

$$
\begin{aligned}
&(-1)^{(|V|+|U|)|W|}(-1)^{|U||P|}(-1)^{|W||P|}\frac{P(h)}{h}g_\mu(W,U)V-(-1)^{(|V|+|U|)|W|}\\
&(-1)^{|V||U|}(-1)^{|V||P|}(-1)^{|W||P|}\frac{P(h)}{h}g_\mu(W,V)U\\
&=(-1)^{|P|(|W|+|g|)}\frac{P(h)}{h}[(-1)^{|V||W|}(-1)^{|P||U|}g_\mu(U,W)V-(-1)^{(|W|+|V|)|U|}(-1)^{|P||V|}g_\mu(V,W)U],
\end{aligned}
\tag{3.17}
$$

so we have

$$
\begin{aligned}
R_{\widehat{\nabla}^\mu}(U,V)W=R^{L,M_2}(U,V)W&+\left[(-1)^{|g|(|W|+|g|)}\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}+(-1)^{|P|(|W|+|g|)}\frac{P(h)}{h}\right]\\
&\cdot\left[(-1)^{|V||W|}(-1)^{|P||U|}g_\mu(U,W)V-(-1)^{|U|(|V|+|W|)}(-1)^{|P||V|}g_\mu(V,W)U\right].
\end{aligned}
\tag{3.18}
$$

Obviously, we can get

$$
\begin{aligned}
&\text{when }\ |g|=|P|=0,\ \text{ then}\\
&R_{\widehat{\nabla}^\mu}(U,V)W\\
&=R^{L,M_2}(U,V)W+\left[\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}+\frac{P(h)}{h}\right]\cdot\left[(-1)^{|V||W|}g_\mu(U,W)V-(-1)^{|U|(|V|+|W|)}g_\mu(V,W)U\right].
\end{aligned}
\tag{3.19}
$$

□

Proposition 3.6. *For $X,Y,Z\in\mathrm{Vect}(M_1)$ and $U,V,W,P\in\mathrm{Vect}(M_2)$, we have*

$$
\begin{aligned}
&(1)\ R_{\widehat{\nabla}^\mu}(X,Y)Z=R^{L,M_1}(X,Y)Z,\\
&(2)\ R_{\widehat{\nabla}^\mu}(V,X)Y=-(-1)^{|V|(|X|+|Y|)}\frac{H^h_{M_1}(X,Y)}{h}V-(-1)^{|X||Y|}hg_2(V,P)g_1(Y,\mathrm{grad}_{g_1}h)X\\
&\qquad-(-1)^{|g(X,Y)||V|}g_1(X,Y)[\nabla^{L,M_2}_V P-hg_2(V,P)\mathrm{grad}_{g_1}h],\\
&(3)\ R_{\widehat{\nabla}^\mu}(X,Y)V=(-1)^{(|X|+|Y|)|V|}\pi(V)[\frac{X(h)}{h}Y-(-1)^{|X||Y|}\frac{Y(h)}{h}X],\\
&(4)\ R_{\widehat{\nabla}^\mu}(V,W)X=-(-1)^{|X||W|}hg_2(V,P)g_1(X,\mathrm{grad}_{g_1}h)W+(-1)^{(|V|+|X|)|W|}hg_2(W,P)g_1(X,\mathrm{grad}_{g_1}h)V,\\
&(5)\ R_{\widehat{\nabla}^\mu}(X,V)W=-(-1)^{|X|(|V|+|W|+|g|)}\frac{g_\mu(V,W)}{h}\nabla^{L,M_1}_X(\mathrm{grad}_{g_1}h)+(-1)^{(|X|+|V|)|W|}[(-1)^{|X||W|}\frac{X(h)}{h}\\
&\qquad g_\mu(W,P)V-(-1)^{|X||V|}g_\mu(W,\nabla^{L,M_2}_V P)X]+(-1)^{(|X|+|V|)|W|}(-1)^{|X||V|}\pi(W)\pi(V)X,\\
&(6)\ R_{\widehat{\nabla}^\mu}(U,V)W=R^{L,M_2}(U,V)W-(-1)^{|U|(|V|+|W|)}g_2(V,W)(\mathrm{grad}_{g_1}h)(h)U+(-1)^{|V||W|}g_2(U,W)\\
&\qquad(\mathrm{grad}_{g_1}h)(h)V+(-1)^{(|U|+|V|)|W|}[g_\mu(W,\nabla^{L,M_2}_U P)V-(-1)^{|U||V|}g_\mu(W,\nabla^{L,M_2}_V P)U]\\
&\qquad+(-1)^{(|U|+|V|)|W|}\pi(W)[(-1)^{|U||V|}\pi(V)U-\pi(U)V].
\end{aligned}
\tag{3.20}
$$

In the following, we compute the Ricci tensor of M. Let M_1 (resp. M_2) have the (p, m) (resp. (q, n)) dimension. Let $\partial_{x^I} = \{\partial_{x^a}, \partial_{\xi^A}\}$ (resp. $\partial_{y^J} = \{\partial_{y^b}, \partial_{\eta^B}\}$) denote the natural tangent frames on M_1 (resp. M_2). Let $\mathrm{Ric}^{L,\mu}$ (resp. Ric^{L,M_1}, Ric^{L,M_2}) denote the Ricci tensor of (M, g_μ) (resp. (M_1, g_1), (M_2, g_2)). Then by (2.4), (2.7) and (3.2), we have

Proposition 3.7. *The following equalities holds*

$$
\begin{aligned}
&(1)\ \mathrm{Ric}^{L,\mu}(\partial_{x^I}, \partial_{x^K}) = \mathrm{Ric}^{L,M_1}(\partial_{x^I}, \partial_{x^K}) - \frac{(q-n)}{h} H^h_{M_1}(\partial_{x^I}, \partial_{x^K}),\\
&(2)\ \mathrm{Ric}^{L,\mu}(\partial_{x^I}, \partial_{y^J}) = \mathrm{Ric}^{L,\mu}(\partial_{y^J}, \partial_{x^I}) = 0,\\
&(3)\ \mathrm{Ric}^{L,\mu}(\partial_{y^L}, \partial_{y^J}) = \mathrm{Ric}^{L,M_2}(\partial_{y^L}, \partial_{y^J}) - g_\mu(\partial_{y^L}, \partial_{y^J}) \cdot [\frac{\triangle^L_{g_1}(h)}{h} + (q-n-1)\frac{(\mathrm{grad}_{g_1} h)(h)}{h^2}].
\end{aligned}
\tag{3.21}
$$

Let $\mathrm{Ric}^{\widehat{\nabla}^\mu}$ (resp. $\mathrm{Ric}^{\widehat{\nabla}^{M_1}}$) denote the Ricci tensor of $(M, \widehat{\nabla}^\mu, g_\mu)$ (resp. $(M_1, \widehat{\nabla}^{M_1}, g_1)$. Then by Proposition 3.5, (2.4) and (2.6), we have

Proposition 3.8. *The following equalities holds*

$$
\begin{aligned}
(1)\ \mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_{x^I}, \partial_{x^K}) = \mathrm{Ric}^{\widehat{\nabla}^{M_1}}(\partial_{x^I}, \partial_{x^K}) - (q-n)\Big[&\frac{H^h_{M_1}(\partial_{x^I}, \partial_{x^K})}{h} - \pi(\partial_{x^I})\pi(\partial_{x^K})\\
&+ \frac{(-1)^{|\partial_{x^I}||\partial_{x^K}|} g_1(\partial_{x^K}, \nabla^{L,M}_{\partial_{x^I}} P)}{2} + \frac{g_1(\partial_{x^I}, \nabla^{L,M_1}_{\partial_{x^K}} P)}{2}\Big],
\end{aligned}
\tag{3.22}
$$

$$
(2)\ \mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_{x^I}, \partial_{y^J}) = \mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_{y^J}, \partial_{x^I}) = 0,
$$

when $|g| = |P| = 0$, then

$$
\begin{aligned}
(3)\ \mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_{y^L}, \partial_{y^J}) = \mathrm{Ric}^{L,M_2}(\partial_{y^L}, \partial_{y^J}) - g_\mu(\partial_{y^L}, \partial_{y^J})[&\frac{\triangle^L_{g_1}(h)}{h} + (q-n-1)\frac{(\mathrm{grad}_{g_1} h)(h)}{h^2}\\
&+ (q-n-1+p-m)\frac{P(h)}{h}].
\end{aligned}
\tag{3.23}
$$

Proof. (1) By Definition 2.9, we have

$$
\begin{aligned}
\mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_{x^I}, \partial_{x^K}) &= \sum_L (-1)^{|\partial_{x^L}|(|\partial_{x^L}|+|\partial_{x^I}|+|\partial_{x^K}|)} \frac{1}{2}[R_{\widehat{\nabla}^\mu}(\partial_{x^L}, \partial_{x^I})\partial_{x^K} + (-1)^{|\partial_{x^I}||\partial_{x^K}|} R_{\widehat{\nabla}^\mu}(\partial_{x^L}, \partial_{x^K})\partial_{x^I}]^L\\
&+ \sum_J (-1)^{|\partial_{y^J}|(|\partial_{y^J}|+|\partial_{x^I}|+|\partial_{x^K}|)} \frac{1}{2}[R_{\widehat{\nabla}^\mu}(\partial_{y^J}, \partial_{x^I})\partial_{x^K} + (-1)^{|\partial_{x^I}||\partial_{x^K}|} R_{\widehat{\nabla}^\mu}(\partial_{y^J}, \partial_{x^K})\partial_{x^I}]^J\\
&= \sum_L (-1)^{|\partial_{x^L}|(|\partial_{x^L}|+|\partial_{x^I}|+|\partial_{x^K}|)} \frac{1}{2}[R_{\widehat{\nabla}^{M_1}}(\partial_{x^L}, \partial_{x^I})\partial_{x^K} + (-1)^{|\partial_{x^I}||\partial_{x^K}|} R_{\widehat{\nabla}^{M_1}}(\partial_{x^L}, \partial_{x^K})\partial_{x^I}]^L\\
&+ \sum_J (-1)^{|\partial_{y^J}|(|\partial_{y^J}|+|\partial_{x^I}|+|\partial_{x^K}|)} \frac{1}{2}[R_{\widehat{\nabla}^\mu}(\partial_{y^J}, \partial_{x^I})\partial_{x^K} + (-1)^{|\partial_{x^I}||\partial_{x^K}|} R_{\widehat{\nabla}^\mu}(\partial_{y^J}, \partial_{x^K})\partial_{x^I}]^J\\
&= \mathrm{Ric}^{\widehat{\nabla}^{M_1}}(\partial_{x^I}, \partial_{x^K}) - \sum_J (-1)^{|\partial_{y^J}||\partial_{y^J}|} \frac{1}{2}\Big[\frac{H^h_{M_1}(\partial_{x^I}, \partial_{x^K})}{h} + (-1)^{|\partial_{x^I}||\partial_{x^K}|}\frac{H^h_{M_1}(\partial_{x^K}, \partial_{x^I})}{h}\\
&+ (-1)^{|\partial_{x^I}||\partial_{x^K}|} g_1(\partial_{x^K}, \widehat{\nabla}^{L,M_1}_{\partial_{x^L}} P) + g_1(\partial_{x^L}, \widehat{\nabla}^{L,M_1}_{\partial_{x^K}} P) - 2\pi(\partial_{x^L})\pi(\partial_{x^K})\Big]
\end{aligned}
$$

$$= \mathrm{Ric}^{\widehat{\nabla}^{M_1}}(\partial_{x^I}, \partial_{x^K}) - (q-n)\Big[\frac{H^h_{M_1}(\partial_{x^I}, \partial_{x^K})}{h} + \frac{(-1)^{|\partial_{x^I}||\partial_{x^K}|} g_1(\partial_{x^K}, \nabla^{L,M_1}_{\partial_{x^I}} P)}{2}$$
$$+ \frac{g_1(\partial_{x^I}, \nabla^{L,M_1}_{\partial_{x^K}} P)}{2} - \pi(\partial_{x^I})\pi(\partial_{x^K})\Big], \tag{3.24}$$

so (1) holds.
(2) Similar to (2) in Propsition 3.7, we get

$$\mathrm{Ric}^{\widehat{\nabla}^{\mu}}(\partial_{x^I}, \partial_{y^J}) = \mathrm{Ric}^{\widehat{\nabla}^{\mu}}(\partial_{y^J}, \partial_{x^I}) = 0. \tag{3.25}$$

(3) By Definition 2.9, we have

$$\mathrm{Ric}^{\widehat{\nabla}^{\mu}}(\partial_{y^L}, \partial_{y^J}) = \sum_I (-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|\partial_{y^L}|+|\partial_{y^J}|)} \frac{1}{2}[R_{\widehat{\nabla}^{\mu}}(\partial_{x^I}, \partial_{y^L})\partial_{y^J} + (-1)^{|\partial_{y^L}||\partial_{y^J}|} R_{\widehat{\nabla}^{\mu}}(\partial_{x^I}, \partial_{y^J})\partial_{y^L}]^I$$
$$+ \sum_K (-1)^{|\partial_{y^K}|(|\partial_{y^K}|+|\partial_{y^L}|+|\partial_{y^J}|)} \frac{1}{2}[R_{\widehat{\nabla}^{\mu}}(\partial_{y^K}, \partial_{y^L})\partial_{y^J} + (-1)^{|\partial_{y^L}||\partial_{y^J}|} R_{\widehat{\nabla}^{\mu}}(\partial_{y^K}, \partial_{y^J})\partial_{y^L}]^K$$
$$= \Delta_1 + \Delta_2, \tag{3.26}$$

where

$$\Delta_1 := \sum_I (-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|\partial_{y^L}|+|\partial_{y^J}|)} \frac{1}{2}[R_{\widehat{\nabla}^{\mu}}(\partial_{x^I}, \partial_{y^L})\partial_{y^J} + (-1)^{|\partial_{y^L}||\partial_{y^J}|} R_{\widehat{\nabla}^{\mu}}(\partial_{x^I}, \partial_{y^J})\partial_{y^L}]^I,$$
$$\Delta_2 := \sum_K (-1)^{|\partial_{y^K}|(|\partial_{y^K}|+|\partial_{y^L}|+|\partial_{y^J}|)} \frac{1}{2}[R_{\widehat{\nabla}^{\mu}}(\partial_{y^K}, \partial_{y^L})\partial_{y^J} + (-1)^{|\partial_{y^L}||\partial_{y^J}|} R_{\widehat{\nabla}^{\mu}}(\partial_{y^K}, \partial_{y^J})\partial_{y^L}]^K, \tag{3.27}$$

by Propsition 3.2, we have

$$R_{\widehat{\nabla}^{\mu}}(\partial_{x^I}, \partial_{y^L})\partial_{y^J} = -(-1)^{|\partial_{x^I}|(|\partial_{y^L}|+|g|+|\partial_{y^J}|)} g_\mu(\partial_{y^L}, \partial_{y^J})\Big[\frac{\nabla^{L,M_1}_{\partial_{x^I}}(grad_{g_1} h)}{h} + (-1)^{(|\partial_{x^I}|+|P|)|g|}\frac{P(h)}{h}\partial_{x^I}\Big], \tag{3.28}$$

then, we get

$$\Delta_1 = -g_\mu(\partial_{y^L}, \partial_{y^J})[\frac{\triangle^L_{g_1}(h)}{h} + \sum_I (-1)^{|\partial_{x^I}||\partial_{x^I}|}(-1)^{|P||g|}\frac{P(h)}{h}], \tag{3.29}$$

$$\Delta_2 = \sum_K (-1)^{|\partial_{y^K}|(|\partial_{y^K}|+|\partial_{y^L}|+|\partial_{y^J}|)} \frac{1}{2}\Big\{R^{L,M_2}(\partial_{y^K}, \partial_{y^L})\partial_{y^J} + \frac{(grad_{g_1} h)(h)}{h^2}\Big[(-1)^{|\partial_{y^L}||\partial_{y^J}|} g_\mu(\partial_{y^K}, \partial_{y^J})\delta^K_L$$
$$- (-1)^{|\partial_{y^K}|(|\partial_{y^L}|+|\partial_{y^J}|)} g_\mu(\partial_{y^L}, \partial_{y^J})\Big]\Big\},$$
$$= Ric^{L,M_2}(\partial_{y^L}, \partial_{y^J}) - g_\mu(\partial_{y^L}, \partial_{y^J})\Big[\frac{\triangle^L_g(h)}{h} + (p-m)P(h) + (q-n-1)\frac{grad_{g_1} h}{h}\Big], \tag{3.30}$$

when $|g| = |P| = 0$, then

$$\Delta_1 + \Delta_2 = \mathrm{Ric}^{L,M_2}(\partial_{y^L}, \partial_{y^J}) - g_\mu(\partial_{y^L}, \partial_{y^J})[\frac{\triangle^L_{g_1}(h)}{h} + (q-n-1)\frac{(\mathrm{grad}_{g_1} h)(h)}{h^2} + (q-n-1+p-m)\frac{P(h)}{h}], \tag{3.31}$$

so (3) holds. □

4. Special super warped products with a semi-symmetric non-metric connection

In this section, we construct an Einstein super warped product with a semi-symmetric non-metric connection. Let $(M_2^{(q,n)}, g_2)$ be a super Riemannian manifold and $\mathbb{R}^{(1,0)}$ be the real line. We consider the super Riemannian manifold $M = \mathbb{R}^{(1,0)} \times_\mu M_2^{(q,n)}$ and $g_\mu = -dt \otimes dt + h^2 g_2$, where $h(t)$ and $\mu(t) = h(t)^2$ be non-zero functions for $t \in \mathbb{R}$ and $|g_2| = 0$.

Let $P = \partial_t$, then by Definition 2.8 and Definition 2.9, we get $R_{\widehat{\nabla}^{\mathbb{R}}}(\partial_t, \partial_t)\partial_t = 0$ and $\mathrm{Ric}^{\widehat{\nabla}^{\mathbb{R}}}(\partial_t, \partial_t) = 0$. By computations, we have $H^h_{M_1}(\partial_t, \partial_t) = h''$, $\mathrm{grad}_{g_1}(h) = -h'\partial_t$ and $\triangle^L_{g_1}(h) = -h''$. By Propsition 3.8, we have

Proposition 4.1. *The following equalities holds*

$$\text{(1)}\ \mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_t, \partial_t) = -(q-n)(\frac{h''}{h} - 1),$$

$$\text{(2)}\ \mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_t, \partial_{y^J}) = \mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_{y^J}, \partial_t) = 0,$$

$$\text{(3)}\ \mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_{y^L}, \partial_{y^J}) = \mathrm{Ric}^{L,M_2}(\partial_{y^L}, \partial_{y^J}) - g_\mu(\partial_{y^L}, \partial_{y^J}) \cdot [-\frac{h''}{h} - (q-n-1)\frac{(h')^2}{h^2} + (q-n)\frac{h'}{h}]. \tag{4.1}$$

Proof. (1) By (1) in Propsition 3.8, we have

$$\begin{aligned}\mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_t, \partial_t) &= \mathrm{Ric}^{\widehat{\nabla}^R}(\partial_t, \partial_t) - (q-n)\Big[\frac{H^h_{M_1}(\partial_t, \partial_t)}{h} - \pi(\partial_t)\pi(\partial_t) \\ &+ \frac{(-1)^{|\partial_t||\partial_t|}g_1(\partial_t, \nabla^{L,R}_{\partial_t}\partial_t)}{2} + \frac{g_1(\partial_t, \nabla^{L,R}_{\partial_t}\partial_t)}{2}\Big] \\ &= -(q-n)(\frac{h''}{h} - 1).\end{aligned} \tag{4.2}$$

(2) By (2) in Propsition 3.8, we have

$$\mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_t, \partial_{y^J}) = \mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_{y^J}, \partial_t) = 0. \tag{4.3}$$

(3) By (3) in Propsition 3.8, $(\mathrm{grad}_{g_1} h)(h) = -h'\partial_t$ and $\triangle^L_{g_1}(h) = -h''$, we have

$$\begin{aligned}\mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_{y^L}, \partial_{y^J}) &= \mathrm{Ric}^{L,M_2}(\partial_{y^L}, \partial_{y^J}) - g_\mu(\partial_{y^L}, \partial_{y^J})[\frac{\triangle^L_{g_1}(h)}{h} + (q-n-1)\frac{(\mathrm{grad}_{g_1} h)(h)}{h^2} \\ &+ (q-n-1+p-m)\frac{P(h)}{h}] \\ &= \mathrm{Ric}^{L,M_2}(\partial_{y^L}, \partial_{y^J}) - g_\mu(\partial_{y^L}, \partial_{y^J})[\frac{-h''}{h} - (q-n-1)\frac{h'^2}{h^2} + (q-n)\frac{h'}{h}].\end{aligned} \tag{4.4}$$

□

Definition 4.2. *We call that* $(M, g_\mu, \widehat{\nabla}^\mu)$ *is Einstein if* $\mathrm{Ric}^{\widehat{\nabla}^\mu}(\overline{X}, \overline{Y}) = \lambda g_\mu(\overline{X}, \overline{Y})$, *for* $\overline{X}, \overline{Y} \in \mathrm{Vect}(M)$ *and a constant* λ.

As in the ordinary warped product case (see Theorem 15 in [16]), by (4.1) and Definition 4.2, we have the following theorems.

Theorem 4.3. *Let $M = \mathbb{R}^{(1,0)} \times_\mu M_2^{(q,n)}$ and $g_\mu = -dt \otimes dt + h^2 g_2$ and $P = \partial_t$. Then $(M, g_\mu, \widehat{\nabla}^\mu)$ is Einstein with the Einstein constant λ if and only if the following conditions are satisfied*
(1) $(M_2^{(q,n)}, \nabla^{L,M_2})$ is Einstein with the Einstein constant c_0.
(2)

$$(q-n)(\frac{h''}{h} - 1) = \lambda. \tag{4.5}$$

(3)

$$\lambda h^2 - h''h - (q-n-1)(h')^2 + (q-n)hh' = c_0. \tag{4.6}$$

Proof. (1) By (3) in Propsition 4.1, we have

$$\mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_{y^L}, \partial_{y^J}) = \mathrm{Ric}^{L,M_2}(\partial_{y^L}, \partial_{y^J}) - g_\mu(\partial_{y^L}, \partial_{y^J}) \cdot [-\frac{h''}{h} - (q-n-1)\frac{(h')^2}{h^2} + (q-n)\frac{h'}{h}], \tag{4.7}$$

then

$$\begin{aligned}
\mathrm{Ric}^{L,M_2}(\partial_{y^L}, \partial_{y^J}) &= \mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_{y^L}, \partial_{y^J}) + g_\mu(\partial_{y^L}, \partial_{y^J}) \cdot [-\frac{h''}{h} - (q-n-1)\frac{(h')^2}{h^2} + (q-n)\frac{h'}{h}] \\
&= \lambda g_\mu(\partial_{y^L}, \partial_{y^J}) + h^2 g_2(\partial_{y^L}, \partial_{y^J}) \cdot [-\frac{h''}{h} - (q-n-1)\frac{(h')^2}{h^2} + (q-n)\frac{h'}{h}] \\
&= \lambda h^2 g_2(\partial_{y^L}, \partial_{y^J}) + h^2 g_2(\partial_{y^L}, \partial_{y^J}) \cdot [-\frac{h''}{h} - (q-n-1)\frac{(h')^2}{h^2} + (q-n)\frac{h'}{h}] \\
&= h^2 \left(\lambda - \frac{h''}{h} - (q-n-1)\frac{(h')^2}{h^2} + (q-n)\frac{h'}{h}\right) g_2(\partial_{y^L}, \partial_{y^J}) \\
&= L(t) g_2(\partial_{y^L}, \partial_{y^J}),
\end{aligned} \tag{4.8}$$

by two sides of the Eq (4.8) act simultaneously on ∂_t and $g_2(\partial_{y^L}, \partial_{y^J}) \neq 0$, we have $L(t) = c_0$, so $\mathrm{Ric}^{L,M_2}(\partial_{y^L}, \partial_{y^J}) = c_0 g_2(\partial_{y^L}, \partial_{y^J})$, therefore (1) holds.
(2) By (1) in Propsition 4.1 and Definition 4.2, we have

$$\mathrm{Ric}^{\widehat{\nabla}^\mu}(\partial_t, \partial_t) = \lambda g_\mu(\partial_t, \partial_t) = -\lambda = -(q-n)(\frac{h''}{h} - 1), \tag{4.9}$$

then we get $\lambda = (q-n)(\frac{h''}{h} - 1)$.
(3) By (1), we get $\lambda h^2 - h''h - (q-n-1)(h')^2 + (q-n)hh' = c_0$. □

By Theorem 4.3, similar to the ordinary warped product case (see Theorem 3.1 in [15]), we have

Theorem 4.4. *Let $M = \mathbb{R}^{(1,0)} \times_\mu M_2^{(q,n)}$ and $g_\mu = -dt \otimes dt + h^2 g_2$ and $P = \partial_t$, when $q - n = 1$, then (M, g_μ, ∇^μ) is Einstein with the Einstein constant $-\lambda_0$ if and only if the following conditions are satisfied*
(1) $(M_2^{(q,n)}, \nabla^{L,M_2})$ is Einstein with the Einstein constant $c_0 = hh' - h^2$.
(2-1) $\lambda_0 < 1$, $f(t) = c_1 e^{\sqrt{1-\lambda_0}t} + c_2 e^{-\sqrt{1-\lambda_0}t}$,
(2-2) $\lambda_0 = 1$, $f(t) = c_1 + c_2 t$,
(2-3) $\lambda_0 > 1$, $f(t) = c_1 \cos\left(\sqrt{\lambda_0 - 1}t\right) + c_2 \sin\left(\sqrt{\lambda_0 - 1}t\right)$,

Proof. (1) Let $\lambda = -\lambda_0$ and $c_0 = -\lambda_N$, then

$$\lambda_N - hh'' - (q-n-1)h'^2 - \lambda_0 h^2 + (q-n)hh' = 0, \tag{4.10}$$

when $q-n=1$, then

$$\lambda_N - hh'' - \lambda_0 h^2 + hh' = 0. \tag{4.11}$$

By $(q-n)(\frac{h''}{h}-1) = -\lambda_0$, we have $\lambda_N = h^2 - hh'$ and $h'' = (1-\lambda_0)h$, so $(M_2^{(q,n)}, \nabla^{L,M_2})$ is Einstein with the Einstein constant $c_0 = hh' - h^2$.
(2) By $h'' = (1-\lambda_0)h$, we have characteristic equation $\mu^2 - (1-\lambda_0) = 0$, then $\Delta = 4(1-\lambda_0)$, so (2) holds. □

Proposition 4.5. *Let $M = \mathbb{R}^{(1,0)} \times_\mu M_2^{(q,n)}$ and $g_\mu = -dt\otimes dt + h^2 g_2$ and $P = \partial_t$, when $q-n=0$, then $(M, g_\mu, \widehat{\nabla}^\mu)$ is Einstein with the Einstein constant $-\lambda_0$ if and only if the following conditions are satisfied*
(1) $(M_2^{(q,n)}, \nabla^{L,M_2})$ is Einstein with the Einstein constant c_0.
(2) $\lambda_0 = 0$,
(3) $c_0 + hh'' - h'^2 = 0$.

Proof. When $q-n=0$, we have $\lambda_0 = 0$ and $\lambda_N - hh'' + h'^2 = 0$, then we get Propsition 4.5. □

Theorem 4.6. *Let $M = \mathbb{R}^{(1,0)} \times_\mu M_2^{(q,n)}$ and $g_\mu = -dt\otimes dt + h^2 g_2$ and $P = \partial_t$, when $q-n \neq 0, 1$, then $(M, g_\mu, \widehat{\nabla}^\mu)$ is Einstein with the Einstein constant $-\lambda_0$ if and only if $(M_2^{(q,n)}, \nabla^{L,M_2})$ is Einstein with the Einstein constant c_0 and one of the following conditions holds*
(1) $\lambda_0 = \lambda_N = 0, h = c_1 e^t$;
(2) $\lambda_0 = q-n, h = c_1 = \sqrt{\frac{\lambda_N}{q-n}}$.

Proof. Let $\lambda = -\lambda_0$ and $c_0 = -\lambda_N$, then by (4.5), we have $h'' = (1-\frac{\lambda_0}{q-n})h$. By (4.6), we get

$$\lambda_N - (q-n-1)h'^2 + (q-n)hh' - (1+\lambda_0 - \frac{\lambda_0}{q-n})h^2 = 0, \tag{4.12}$$

when $q-n \neq 0, 1$, we get

$$\frac{\lambda_N}{1-q+n} + h'^2 + \frac{(q-n)}{1-q+n}hh' + (\frac{\lambda_0}{q-n} - \frac{1}{1-q+n})h^2 = 0. \tag{4.13}$$

Let $q-n = l$, $\frac{\lambda_0}{q-n} = \frac{\lambda_0}{l} = d_0$, $\frac{\lambda_N}{1-q+n} = \frac{\lambda_N}{1-L} = \overline{d_0}$,
Case (a). When $d_0 < 1$, let $a_0 = \sqrt{1-d_0}$, $b_0 = -\sqrt{1-d_0}$, then $a_0 + b_0 = 0$, $a_0 b_0 = d_0 - 1$ and $h = c_1 e^{a_0 t} + c_2 e^{b_0 t}$, by (4.13), we have

$$\begin{aligned}&\overline{d_0} + c_1^2(a_0^2 + a_0 b_0 + 1 - \frac{1}{1-l} + \frac{l}{1-l}a_0)e^{2a_0 t} + c_2^2(b_0^2 + a_0 b_0 + 1 - \frac{1}{1-l} + \frac{l}{1-l}b_0)e^{2b_0 t}\\ &+ c_1 c_2(4a_0 b_0 - \frac{l}{1-l}) = 0,\end{aligned} \tag{4.14}$$

then

$$\begin{aligned}&\overline{d_0} + c_1 c_2(4a_0 b_0 - \frac{l}{1-l}) = 0,\\ &c_1^2(a_0^2 + a_0 b_0 + 1 - \frac{1}{1-l} + \frac{l}{1-l}a_0) = 0,\\ &c_2^2(b_0^2 + a_0 b_0 + 1 - \frac{1}{1-l} + \frac{l}{1-l}b_0) = 0.\end{aligned} \tag{4.15}$$

Case (a-1). When $c_1 = 0,\ \ c_2 \neq 0$, we get $a_0 = -1,\ \ b_0 = 1$, then this is a contradiction.
Case (a-2). When $c_1 \neq 0,\ \ c_2 = 0$, we get $a_0 = 1,\ \ b_0 = -1,\ \ \lambda_N = 0,\ \ \lambda_0 = 0,\ \ h = c_1 e^t$.
Case (a-3). When $c_1 \neq 0,\ \ c_2 \neq 0$, then there is no solution.
Case (b). When $d_0 = 1$, then $h = c_1 + c_2 t$, by (4.13), we have

$$\overline{d_0} + c_1^2(d_0 - \frac{1}{1-l}) + c_2^2[1 + (d_0 - \frac{1}{1-l})t^2 + \frac{l}{1-l}t] + c_1c_2[2(d_0 - \frac{1}{1-l})t + \frac{l}{1-l}] = 0. \tag{4.16}$$

Case (b-1). When $c_1 = 0,\ \ c_2 \neq 0$, then $\overline{d_0} + c_2^2[1 + (d_0 - \frac{1}{1-l})t^2 + \frac{l}{1-l}t] = 0$, we get $c_2 = 0$, this is a contradiction.
Case (b-2). When $c_1 \neq 0,\ \ c_2 = 0$, then $\overline{d_0} + c_1^2(d_0 - \frac{1}{1-l}) = 0$, we get $h = c_1 = \frac{\lambda_N}{l}$.
Case (b-3). When $c_1 \neq 0,\ \ c_2 \neq 0$, then $c_2^2(d_0 - \frac{1}{1-l}) = 0,\ \ c_2^2\frac{l}{1-l} + 2c_1c_2(d_0 - \frac{1}{1-l}) = 0$, we get $c_2 = 0$, so this is a contradiction.
Case (c). When $d_0 > 1$, let $h_0 = \sqrt{d_0 - 1}$, then $h = c_1 cosh_0 t + c_2 sinh_0 t$, by (4.13), we have

$$\begin{aligned}&\overline{d_0} + (sinh_0t)^2[c_1^2h_0^2 + c_2(d_0 - \frac{1}{1-l}) - c_1c_2\frac{l}{1-l}h_0] + (cosh_0t)^2[c_2^2h_0^2 + c_1(d_0 - \frac{1}{1-l}) + c_1c_2\frac{l}{1-l}h_0]\\ &+ cosh_0tsinh_0t[-2c_1c_2h_0^2 + 2c_1c_2(d_0 - \frac{l}{1-l}) - c_1^2h_0\frac{1}{1-l} + c_2^2h_0\frac{1}{1-l}] = 0,\end{aligned} \tag{4.17}$$

then

$$\begin{aligned}&\overline{d_0} + c_1^2h_0^2 + c_2(d_0 - \frac{1}{1-l}) - c_1c_2\frac{l}{1-l}h_0 = 0,\\ &\overline{d_0} + c_2^2h_0^2 + c_1(d_0 - \frac{1}{1-l}) + c_1c_2\frac{l}{1-l}h_0 = 0,\\ &-2c_1c_2h_0^2 + 2c_1c_2(d_0 - \frac{l}{1-l}) - c_1^2h_0\frac{1}{1-l} + c_2^2h_0\frac{1}{1-l} = 0.\end{aligned} \tag{4.18}$$

By (4.18), we can get $c_1 = c_2 = 0$, so this is a contradiction. □

Nextly, we give another example. Let $M_1 = \mathbb{R}^{(1,2)}$ with coordinates (t, ξ, η) and $|t| = 0,\ \ |\xi| = |\eta| = 1$. We give a metric $g_1 = -dt \otimes dt + d\xi \otimes d\eta - d\eta \otimes d\xi$ on M_1, i.e.,

$$g_1(\partial_t, \partial_t) = -1,\ \ g_1(\partial_\xi, \partial_\eta) = -1,\ \ g_1(\partial_\eta, \partial_\xi) = 1,\ \ g_1(\partial_{x^I}, \partial_{x^K}) = 0, \tag{4.19}$$

for the other pair $(\partial_{x^I}, \partial_{x^K})$. Let $\widetilde{M} = \mathbb{R}^{(1,2)} \times_\mu M_2^{(q,n)}$ and $g_\mu = g_1 + h(t)^2 g_2$ and $P = \partial_t$. By Proposition 7 in [5], we have the Christoffel symbols $\Gamma^L_{JI} = 0$, then

$$\nabla^{L,g_1}_{\partial_{x^J}}\partial_{x^K} = 0,\ \ R^{L,g_1}(X,Y)Z = 0,\ \ \mathrm{Ric}^{L,g_1}(X,Y) = 0. \tag{4.20}$$

We have

$$H^h_{M_1}(\partial_t, \partial_t) = h'',\ \ H^h_M(\partial_{x^J}, \partial_{x^K}) = 0,\ \ \text{for the other pair } (\partial_{x^I}, \partial_{x^K}). \tag{4.21}$$

$$\mathrm{grad}_{g_1}(h) = -h'\partial_t,\ \ \triangle^L_{g_1}(h) = -h''. \tag{4.22}$$

By Proposition 3.7 and the Einstein condition, we have

Theorem 4.7. *Let $\widetilde{M} = \mathbb{R}^{(1,2)} \times_\mu M_2^{(q,n)}$ and $g_\mu = g_1 + h^2 g_2$ and $P = \partial_t$. Then $(\widetilde{M}, g_\mu, \nabla^{L,\mu})$ is Einstein with the Einstein constant λ if and only if one of the following conditions is satisfied*
(1) $\lambda = 0$, $q = n$, $(M_2^{(q,n)}, \nabla^{L,M_2})$ is Einstein with the Einstein constant $-c_0$ and $hh'' - h'^2 = c_0$.
(2) $\lambda = 0$, $q - n - 1 = 0$, $(M_2^{(q,n)}, \nabla^{L,M_2})$ is Einstein with the Einstein constant 0 and $h = c_1 t + c_2$ where c_1, c_2 are constant.
(3) $\lambda = 0$, $q - n - 1 \neq 0, -1$, $(M_2^{(q,n)}, \nabla^{L,M_2})$ is Einstein with the Einstein constant $-c_0$ and $h = \pm\sqrt{\frac{c_0}{q-n-1}}t + c_2$, $\frac{c_0}{q-n-1} \geq 0$.

Proof. By (1) in Propsition 3.7, we have

$$\mathrm{Ric}^{L,\mu}(\partial_{x^I}, \partial_{x^K}) = \mathrm{Ric}^{L,M_1}(\partial_{x^I}, \partial_{x^K}) - \frac{(q-n)}{h} H^h_{M_1}(\partial_{x^I}, \partial_{x^K}), \tag{4.23}$$

then

$$\lambda g_\mu(\partial_{x^I}, \partial_{x^K}) = -\frac{(q-n)}{h} H^h_{M_1}(\partial_{x^I}, \partial_{x^K}), \tag{4.24}$$

so we get $\lambda = 0$ and $q = n$ or $h'' = 0$.
By $\lambda = 0$, then we have

$$\mathrm{Ric}^{L,\mu}(\partial_{x^I}, \partial_{y^J}) = \mathrm{Ric}^{L,\mu}(\partial_{y^J}, \partial_{x^I}) = 0. \tag{4.25}$$

By (3) in Propsition 3.7 and (4.25), we have

$$\mathrm{Ric}^{L,\mu}(\partial_{y^I}, \partial_{y^J}) = g_2(\partial_{y^I}, \partial_{y^J})[-hh'' - (q-n-1)h'^2]. \tag{4.26}$$

Then we get:
(Case-a). When $\lambda = 0, q = n$, by $hh'' + (q-n-1)h'^2 = c_0$, we get $(M_2^{(q,n)}, \nabla^{L,M_2})$ is Einstein with the Einstein constant $-c_0$ and $hh'' - h'^2 = c_0$.
(Case-b). When $\lambda = 0, h'' = 0$, by $hh'' + (q-n-1)h'^2 = c_0$, we have $(q-n-1)h'^2 = c_0$.
(Case-b-1). When $q - n - 1 = 0$, $(M_2^{(q,n)}, \nabla^{L,M_2})$ is Einstein with the Einstein constant 0 and $h = c_1 t + c_2$ where c_1, c_2 are constant.
(Case-b-2). When $q \neq n$, $q - n - 1 \neq 0$, $(M_2^{(q,n)}, \nabla^{L,M_2})$ is Einstein with the Einstein constant $-c_0$ and $h = \pm\sqrt{\frac{c_0}{q-n-1}}t + c_2$, $\frac{c_0}{q-n-1} \geq 0$. □

By (2.16) and (4.20), we can get

$$\begin{aligned}
&R^{\widehat{\nabla}^{\mathbb{R}^{(1,2)}}}(\partial_t, \partial_\xi)\partial_t = -\partial_\xi, \quad R^{\widehat{\nabla}^{\mathbb{R}^{(1,2)}}}(\partial_t, \partial_\eta)\partial_t = -\partial_\eta, \\
&R^{\widehat{\nabla}^{\mathbb{R}^{(1,2)}}}(\partial_\xi, \partial_t)\partial_t = \partial_\xi, \quad R^{\widehat{\nabla}^{\mathbb{R}^{(1,2)}}}(\partial_\eta, \partial_t)\partial_t = \partial_\eta, \\
&R^{\widehat{\nabla}^{\mathbb{R}^{(1,2)}}}(\partial_{x^J}, \partial_{x^K})\partial_{x^L} = 0,
\end{aligned} \tag{4.27}$$

for other pairs $(\partial_{x^J}, \partial_{x^K}, \partial_{x^L})$. By (2.4) and (4.27), we have

$$\mathrm{Ric}^{\widehat{\nabla}^{\mathbb{R}^{(1,2)}}}(\partial_t, \partial_t) = 2, \quad \mathrm{Ric}^{\widehat{\nabla}^{\mathbb{R}^{(1,2)}}}(\partial_{x^J}, \partial_{x^L}) = 0, \tag{4.28}$$

for other pairs $(\partial_{x^J}, \partial_{x^L})$.

If $(\widetilde{M}, g_\mu, \widehat{\nabla}^\mu)$ is Einstein with the Einstein constant λ, by Propsition 2.15 and (4.28), we have

$$\lambda = 0, \;\; 2 - (q-n)(\frac{h''}{h} - 1) = -\lambda. \tag{4.29}$$

Solving (4.29), we get

$$h = c_1 e^{\sqrt{1+\frac{2}{q-n}}t} + c_2 e^{-\sqrt{1+\frac{2}{q-n}}t}. \tag{4.30}$$

By (3.22) (3) and the Einstein condition, we get $(M_2^{(q,n)}, \nabla^{L,M_2})$ is Einstein with the Einstein constant c_0 and

$$\lambda h^2 - h''h - (q-n-1)(h')^2 + (q-n-2)hh' = c_0. \tag{4.31}$$

Then we have the following theorem

Theorem 4.8. *Let $\widetilde{M} = \mathbb{R}^{(1,2)} \times_\mu M_2^{(q,n)}$ and $g_\mu = g_1 + h^2 g_2$ and $P = \partial_t$. Then $(\widetilde{M}, g_\mu, \widehat{\nabla}^\mu)$ is Einstein with the Einstein constant λ if and only if $(M_2^{(q,n)}, \nabla^{L,M_2})$ is Einstein with the Einstein constant $c_0 = 0$ and $\lambda = 0$, $h = c^*$, $q - n + 2 = 0$.*

Proof. Let $k = 1 + \frac{2}{q-n}$, by $\lambda = 0$, (4.30) and (4.31), we have

$$\begin{aligned}&[-(q-n-1)k^2c_1^2 + (q-n-2)kc_1^2]e^{2kt} + [-(q-n-1)k^2c_2^2 + (q-n-2)kc_2^2]e^{-2kt} - c_1k^2e^{kt} - c_2k^2e^{-kt}\\ &= c_0 - 2c_1c_2(q-n-1)k^2.\end{aligned} \tag{4.32}$$

Let $b_1 = -(q-n-1)k^2c_1^2 + (q-n-2)kc_1^2$, $b_2 = -(q-n-1)k^2c_2^2 + (q-n-2)kc_2^2$, $b_3 = -c_1k^2$, $b_4 = -c_2k^2$, $b_5 = c_0 - 2c_1c_2(q-n-1)k^2$, we get

$$b_1e^{2kt} + b_2e^{-2kt} + b_3e^{kt} + b_4e^{-kt} = b_5. \tag{4.33}$$

When $k \neq 0$, we have

$$\begin{cases} b_1 + b_2 + b_3 + b_4 = b_5 \\ 2kb_1 - 2kb_2 + kb_3 - kb_4 = 0 \\ 4k^2b_1 + 4k^2b_2 + k^2b_3 + k^2b_4 = 0 \\ 8k^3b_1 - 8k^3b_2 + k^3b_3 - k^3b_4 = 0 \\ 16k^4b_1 + 416k^4b_2 + k^4b_3 + k^4b_4 = 0, \end{cases} \tag{4.34}$$

by (4.34), we get $b_1 = b_2 = b_3 = b_4 = b_5 = 0$, then $c_1 = c_2 = 0$, so this is a contradiction.
When $k = 0$, we get $q - n + 2 = 0$, $c_0 = 0$ and $h = c_1 + c_2 = c^*$. □

5. Conclusions

For Riemannian supergeometry, we give some definitions about a semi-symmetric non-metric connection. Then by computations, we get the curvature tensor $R_{\widehat{\nabla}^{\mathbb{R}}}$ and the Ricci tensor $\mathrm{Ric}^{\widehat{\nabla}^\mu}$ of a semi-symmetric non-metric connection on super warped product spaces respectively. We find that they are different from Riemannian geometry. Next, we construct an Einstein super warped product with a semi-symmetric non-metric connection and give another example. The main results of this paper are Theorems 4.3–4.8, which are the conditions that two super warped product spaces with a semi-symmetric non-metric connection are the Einstein super spaces with a semi-symmetric non-metric connection. Moreover, some properties of a semi-symmetric non-metric connection on super warped product spaces are discussed in this paper.

In the future, we can do more research on super Riemannian manifolds.

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Conflict of interest

The authors declare no conflict of interest.

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