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Research article

Some novel mathematical results on the existence and uniqueness of generalized Caputo-type initial value problems with delay

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Abstract: In this article, we propose some novel results on the existence and uniqueness of generalized Caputo-type initial value problems with delay by using fixed point theory. The characteristics of space of continuous and measurable functions are the main basis of our results. The proposed results are very useful to prove the existence of a unique solution for the various types of fractional-order systems defined under the generalized Caputo fractional derivative consisting of delay terms.

Keywords: initial value problem; generalised Caputo fractional derivative; delay; existence; uniqueness
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1. Introduction

Nowadays, fractional-order operators [1–3] are one of the effective tools to analyze real-world problems. A number of scientific problems have been explored by using fractional operators. *Nabi et al.* in [4] used the Caputo operator to find the solution of a Covid-19 (a deathly epidemic) model. In [5], authors have analyzed a Caputo-type fractional-order HIV model. In [6], authors have derived a novel, four-dimensional memristor-based chaotic circuit model, by using the Atangana-Baleanu derivative. *Vellappandi et al.* in [7] have derived an optimal control problem for the mosaic epidemic in the sense of the Caputo operator. Virus transmission in the butterfly population was studied in ref. [8] by using a generalized Caputo derivative. A fruitful application of fractional order operators to do the mathematical modeling of plankton-oxygen dynamics can be seen from a ref. [9]. Recently, *Erturk et al.* in [10] have defined a new lagrangian in the fractional sense to define the motion of a beam on the nanowire. In [11], a generalized Caputo derivative was used to solve a psychological problem. Also, a number of numerical schemes have come to the literature to solve the fractional initial value problems

(FIVP). In [12], authors have defined a generalized form of the Predictor-Corrector (P-C) method to simulate fractional-order systems. *Kumar et al.* in [13] have explored a very short and effective method to solve FIVP.

Delayed fractional differential equations (DFDEs) have gotten a lot of interest recently because of their applicability in mathematical modeling of real-world issues where the fractional rate of change is influenced by genetic factors. A recent study on a delay-type mathematical model for defining the oncolytic virotherapy can be seen from a ref. [14]. *Odibat et al.* in [15] have modified the P-C method to solve generalized Caputo type delay problems. Proving the existence of a unique solution for the DFDEs is always a challenging task because of their complexity. A number of studies have been given by the researchers to fulfill this research gap. In [16], *Abbas* was derived the existence of a solution for the DFDEs but had not mentioned the uniqueness part. Similarly, without discussing uniqueness, authors in [17] had derived the global solution existence on a finite time interval. Authors in [18] have explored and derived the uniqueness of the global solution to the DFDEs with the help of generalized Gronwall inequality. However, their results contain a flaw that has been specified in the study [19]. Authors in [19] had given some theorems on the existence and uniqueness of solution for initial values problems for DFDE by using Caputo derivative without requiring the Lipschitz property of the considered function with respect to the delay variable, but for the non-delay variable (see e.g. [20–23] and references therein). Some other significant mathematical studies in the sense of fractional derivatives can be done from the ref. [24–28].

In this paper, we will derive a theorem for the DFDE in the sense of generalized Caputo derivative by using the previously published results of ref. [29]. The main motivation behind the proposal of this study is to extend the Caputo-type results of existence and uniqueness of initial value problems with delay, for the generalized Caputo-type fractional derivative sense. The main difference between the Caputo and generalized Caputo fractional derivatives is the presence of an extra parameter (say ρ) along with the fractional order (say γ) which makes the generalized Caputo derivative an advanced version of the Caputo derivative. Nowadays, the generalized Caputo derivative is being used to model a number of real-world problems (see ref. [8,11,13,30]). To date, there are no significant results to prove the existence of a unique solution for the DFDEs in the generalized Caputo sense. Because of it, many fractional-order systems have been numerically simulated without proving the solution's existence (see ref. [15]). This research paper will definitely fill the research gap of the DFDEs literature.

The given study is formulated in a number of sections. In section 2 we recall some preliminaries. In section 3, we derive our main results by proving the existence of a unique solution in the sense of a considered fractional operator with the help of some important lemma and results. In the end, we conclude the novelty of our findings.

2. Preliminaries

Firstly, we remind some necessary definitions and results.

Definition 1. *[29] Consider* $[b, c](-\infty \leq b < c \leq \infty)$ *be a finite or infinite interval on the real axis* $R = (-\infty, \infty), k \in N \cup 0$ *and* $1 \leq p \leq \infty$*. We express the usual Lebesgue space* $L_p[b, c](1 \leq p \leq \infty)$ *and continuous space* $C^k[b, c]$ *by*

$$L_p[b, c] \;:=\; \left\{ y : [b, c] \to R;\ \Lambda \textit{ is measurable in [b,c] and } \int_b^c |y(t)|^p dt < \infty \right\},$$

$$\begin{aligned} L_\infty[b,c] &:= \{y:[b,c]\to R;\ \Lambda \textit{ is measurable in [b,c] and bounded essentially in [b,c]}\},\\ C^k[b,c] &:= \{y:[b,c]\to R;\ \Lambda \textit{ has a continuous kth derivative in [b,c] }\},\\ C[b,c] &:= C^0[b,c]. \end{aligned}$$

Also, $L_p[b,c](1 \le p \le \infty)$ *denotes the set of Lebesgue real-valued measurable functions* y *on* Ω *those follow* $\|y\|_p < \infty$*, where*

$$\|y\|_p = \left(\int_b^c |y(t)|^p dt\right)^{1/p} \quad (1 \le p \le \infty)$$

and

$$\|y\|_\infty = \sup_{b\le t\le c} |y(t)|.$$

Definition 2. *[29] For* $n \in \mathbb{N}$*,* $AC^n[b,c]$ *denotes the space of real-valued functions* $y(t)$ *having continuous derivatives up to the order* $n-1$ *in* $[b,c](-\infty < b < c < \infty)$*, i.e. the functions* y *for which a function* $u \in L_1[b,c]$ *exists almost everywhere such that*

$$AC^n[b,c] \ := \ \left\{y:[b,c]\to R; y^{(n-1)}(t) = y^{(n-1)}(b) + \int_b^t u(\vartheta)d\vartheta\right\},$$

where $u = y^{(n)}$*.*

Definition 3. *[2] Gamma function is defined by*

$$\Gamma(z) = \int\limits_0^\infty t^{z-1}e^{-t}dt, \tag{2.1}$$

where z *be any complex number s.t* $Re(z) > 0$*.*

Definition 4. *[2] The one-parametrized Mittag-Leffler function is defined by*
$E_\gamma(\omega) = \sum_{n=0}^{\infty} \frac{\omega^n}{\Gamma(\gamma n+1)},\ \ \gamma > 0,\ \omega \in \mathbb{C}.$

Definition 5. *[2] The Riemann-Liouville (RL) fractional integral of a function* $f: R^+ \to R$ *is given by*
$J^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\vartheta)^{\gamma-1} f(\vartheta)d\vartheta, \quad \gamma > 0,$
$J^0 f(t) = f(t).$

Definition 6. *[2] The Caputo fractional derivative of* $f \in AC^m[0,c]$ $(c \in R^+)$ *is defined by*

$${}_0^C D_t^\gamma f(t) = \begin{cases} \frac{d^m f(t)}{dt^m}, & \gamma = m \in \mathbb{N} \\ \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-\vartheta)^{m-\gamma-1} f^{(m)}(\vartheta)\, d\vartheta, & m-1 < \gamma < m\ , m \in \mathbb{N}. \end{cases} \tag{2.2}$$

Definition 7. *[31] The generalized fractional integral,* ${}^R I_{c_+}^{\gamma,\rho}$*, of order* $\gamma > 0$ *is defined by*

$$({}^R I_{c_+}^{\gamma,\rho} f)(t) = \frac{\rho^{1-\gamma}}{\Gamma(\gamma)} \int_c^t (t^\rho - \vartheta^\rho)^{\gamma-1} \vartheta^{\rho-1} f(\vartheta)d\vartheta, \ \ t > c, \tag{2.3}$$

where $c \ge 0,\ \rho > 0,$ *and* $m-1 < \gamma \le m$*.*

Definition 8. *[31] The generalized R-L fractional derivative, ${}^{R}_{0}D^{\gamma,\rho}_{t}$, of order $\gamma > 0$ is defined by*

$$({}^{R}_{0}D^{\gamma,\rho}_{t} f)(t) = \frac{\rho^{\gamma-m+1}}{\Gamma(m-\gamma)} \frac{d^m}{dt^m} \int_c^t \vartheta^{\rho-1}(t^\rho - \vartheta^\rho)^{m-\gamma-1} f(\vartheta) d\vartheta, \quad t > c, \tag{2.4}$$

where $c \geq 0,\ \rho > 0,$ and $m - 1 < \gamma \leq m$.

Definition 9. *[32] The generalized Caputo fractional derivative, ${}^{C}_{0}D^{\gamma,\rho}_{t}$, of order $\gamma > 0$ is defined by*

$$({}^{C}_{0}D^{\gamma,\rho}_{t} f)(t) = \frac{\rho^{\gamma-m+1}}{\Gamma(m-\gamma)} \int_c^t \vartheta^{\rho-1}(t^\rho - \vartheta^\rho)^{m-\gamma-1} f^{(m)}(\vartheta) d\vartheta, \quad t > c, \tag{2.5}$$

where $\rho > 0,\ c \geq 0,$ and $m - 1 < \gamma \leq m$.

3. Existence and uniqueness analysis

After getting the direction from the aforementioned works [16–19, 29], we consider the initial value problem (IVP) of DFDEs in the space of continuous and measurable functions, and establish some sufficient cases for the existence of a unique solution in the sense of generalized Caputo derivative for the following form

$${}^{C}_{0}D^{\gamma,\rho}_{t} f(t) = \Lambda(t, f_t), \tag{3.1}$$

with the initial conditions

$$D^k f(t) = \phi^{(k)}(t) \text{ on } [-r, 0], k = 0, 1, \cdots m-1, D^k := \frac{d^k}{dt^k}, \tag{3.2}$$

where $t \in [0, c], \rho > 0, \gamma > 0, m = [\gamma] + 1$, and $[\gamma]$ denotes the integer part of γ. ${}^{C}D^{\gamma,\rho}$ is the generalized Caputo derivative operator, $\Lambda : [0, c] \times B \to R$ is a continuous function following necessary assumed conditions that will be mentioned later. $f_t \in B$ where B is a phase space. For the function f given on $[-r, c]$ and f_t, $t \in [0, c]$, the element of B defined by $f_t(\theta) = f(t + \theta), \theta \in [-r, 0]$. Particularly, $f_0 = \phi(t) \in B$.

Lemma 1. *[29] If $f \in C((-\infty, c], R^n)$ then f_t is a continuous function of t in $[0, c]$.*

Proof. Because f is continuous in $(-\infty, c]$ then for any large $0 < r < \infty$, it is uniformly continuous in $[-r, c]$. Then for every $\epsilon > 0$, there exists $\rho > 0$ such that $|f(t) - f(\vartheta)| < \epsilon$ if $|t - \vartheta| < \rho$. Hence, for t, ϑ in $[0, c]$ and $|t - \vartheta| < \rho$, we have $|f(t + \theta) - f(\vartheta + \theta)| < \epsilon$ for any $\theta \in [-r, 0]$. □

Lemma 2. *Let us define $k : [0, c] \times [0, c] \to R$ by*

$$k(t, \vartheta) = \begin{cases} \vartheta^{\rho-1}(t^\rho - \vartheta^\rho)^{\gamma-1} & \text{if } 0 \leq \vartheta < t \leq c, \\ 0 & \text{if } 0 \leq t \leq \vartheta \leq c. \end{cases} \tag{3.3}$$

and let $\gamma \in R^+$. There exist numbers $0 = c_0 < c_1 < c_2 < \cdots < c_n = c$ such that $\forall\ i \in \{0, 1, \cdots n-1\}$ and $t \in [c_i, c_{i+1}]$ we have

$$\frac{L\rho^{1-\gamma}}{\Gamma(\gamma)} \int_{c_i}^{\min\{t, c_{i+1}\}} |k(t, \vartheta)| d\vartheta < 1,$$

where $L \in R^+$ is some constant.

Proof. Here we consider the three different cases: $\gamma = 1, \gamma > 1$ and $0 < \gamma < 1$. In the case $\gamma = 1$, the result of the Lemma is clearly correct. When $\gamma > 1$, k is continuous and then bounded on the compact set $\{(t, \vartheta) : 0 \leq \vartheta \leq t \leq c\}$. Assume $c_i := ic/n$, i.e. the interval $[0, c]$ is splitted into n equal sections, we can find that

$$\frac{L\rho^{1-\gamma}}{\Gamma(\gamma)} \int_{c_i}^{\min\{t,c_{i+1}\}} |k(t,\vartheta)| d\vartheta \leq \frac{L\rho^{1-\gamma}}{\Gamma(\gamma)} \int_{c_i}^{c_{i+1}} |k(t,\vartheta)| d\vartheta \leq \frac{L\rho^{1-\gamma}}{\Gamma(\gamma)} \|k\|_\infty (c_{i+1} - c_i) = \frac{L\rho^{1-\gamma}\|k\|_\infty c}{\Gamma(\gamma) n},$$

here the Chebyshev norm of k is utilized over the above given set. Choosing

$$n := \left[\frac{L\rho^{1-\gamma}\|k\|_\infty c}{\Gamma(\gamma)}\right] + 1,$$

we get the required results easily. Now for $0 < \gamma < 1$, we go throw a little different way. Let $c_i := ic/n$, for $t \leq c_{i+1}$, we have

$$\begin{aligned}
\frac{L\rho^{1-\gamma}}{\Gamma(\gamma)} \int_{c_i}^{\min\{t,c_{i+1}\}} |k(t,\vartheta)| d\vartheta &= \frac{L\rho^{1-\gamma}}{\Gamma(\gamma)} \int_{c_i}^{t} \vartheta^{\rho-1} (t^\rho - \vartheta^\rho)^{\gamma-1} d\vartheta = \frac{L\rho^{-\gamma}}{\Gamma(\gamma+1)} (t^\rho - c_i^\rho)^\gamma \\
&\leq \frac{L\rho^{-\gamma}}{\Gamma(\gamma+1)} (c_{i+1}^\rho - c_i^\rho)^\gamma = \frac{L\rho^{-\gamma}}{\Gamma(\gamma+1)} (c/n)^{\rho\gamma}.
\end{aligned}$$

By choosing

$$n := \left[c \left(\frac{L\rho^{-\gamma}}{\Gamma(\gamma+1)}\right)^{\frac{1}{\rho\gamma}}\right] + 1,$$

the required result is received. Therewith, for $t \geq c_{i+1}$, we may compute

$$\begin{aligned}
\Psi(t) := L \int_{c_i}^{\min\{t,c_{i+1}\}} |p(t,\vartheta)| d\vartheta = \frac{L\rho^{1-\gamma}}{\Gamma(\gamma)} \int_{c_i}^{c_{i+1}} \vartheta^{\rho-1} (t^\rho - \vartheta^\rho)^{\gamma-1} d\vartheta \\
= \frac{L\rho^{-\gamma}}{\Gamma(\gamma+1)} [(t^\rho - c_i^\rho)^\gamma - (t^\rho - c_{i+1}^\rho)^\gamma.
\end{aligned}$$

Noticing that

$$\Psi'(t) = \frac{L\rho^{1-\gamma}}{\Gamma(\gamma)} [(t^\rho - c_i^\rho)^{\gamma-1} t^{\rho-1} - (t^\rho - c_{i+1}^\rho)^{\gamma-1} t^{\rho-1}] < 0,$$

since $\gamma < 1$, we can find that $\Psi(t) \leq \Psi(c_{i+1}) < 1$ for $t \geq c_{i+1}$. So, by collecting the above given outputs, our proof is finished. □

Remark 1. *In the form of above mentioned parameters, we may take*

$$n := \max\left\{\left[\frac{L\rho^{1-\gamma}\|k\|_\infty c}{\Gamma(\gamma)}\right] + 1, \left[c\left(\frac{L\rho^{-\gamma}}{\Gamma(\gamma+1)}\right)^{\frac{1}{\rho\gamma}}\right] + 1\right\}$$

accordingly

$$\sigma := \max\left\{\frac{L\rho^{1-\gamma}\|k\|_\infty c}{\Gamma(\gamma) n}, \frac{L\rho^{-\gamma}}{\Gamma(\gamma+1)} (c/n)^{\rho\gamma}\right\} < 1,$$

this implies the explored result of Lemma 2.

Lemma 3. *Let $f(t) \in AC^n[0,c]$ and the function $\Lambda : [0,c] \times B \to R$ is a continuous function. Then $f \in [-r,c]$ becomes the solution to the IVP* (3.1)-(3.2) *if and only if it becomes the solution of delay Volterra integral equation*

$$\begin{cases} f(t) = \sum_{k=0}^{m-1} \phi^{(k)}(0)\frac{t^k}{k!} + \dfrac{\rho^{1-\gamma}}{\Gamma(\gamma)} \int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}\Lambda(\vartheta, f_\vartheta)d\vartheta, & \forall t \in [0,c] \\ f(t) = \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{3.4}$$

Theorem 1. *Choose $\gamma \in R^+, m = [\gamma] + 1$, and $f(t) \in AC^m[0,c]$. Consider the set $G : [0,c] \times B$ and the continuous function $\Lambda : G \to R$ which satisfies the Lipschitz constraint for the second variable with a Lipschitz constant $L \in R^+$ which is free from t, f_1, and f_2. Then there exists a unique solution $f(t) \in C[-r,c]$ of the IVP* (3.1) *and* (3.2).

Proof. Firstly, we split the interval $[0,c]$ into distinct sub-interval by using a row of real numbers $0 = c_0 < c_1 < c_2 < \cdots < c_n = c$. Now first we will derive the results for the interval $[c_0, c_1]$. In this regard, we take the following sequence of functions:

$$f^0(t) := \Omega(t) := \sum_{k=0}^{m-1} \phi^{(k)}(0)\frac{t^k}{k!}$$

and

$$f^a(t) := \Omega(t) + \frac{\rho^{1-\gamma}}{\Gamma(\gamma)} \int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}\Lambda(\vartheta, f_\vartheta^{a-1})d\vartheta, \quad a = 1, 2, \cdots \tag{3.5}$$

Now to prove the continuity of the functions $f^i(t)$ those satisfy the Eq (3.5) on $[c_0, c_1]$, we may apply the mathematical induction. Let us take the case $a = 1$, that is

$$f^1(t) := \Omega(t) + \frac{\rho^{1-\gamma}}{\Gamma(\gamma)} \int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}\Lambda(\vartheta, f_\vartheta)d\vartheta. \tag{3.6}$$

By Lemma (1) and hypothesis, this is straight to observe that $\Lambda(t, f_t)$ is a continuous function in $[0,c]$, and then $\Lambda(t, f_t) \in L_1[0,c]$ with respect to t. Firstly, we check the continuity of $f^1(t)$ on $[c_0, c_1]$.
We begin by noting that, for $c_0 \le t_1 \le t_2 \le c_1$,

$$|f^1(t_1) - f^1(t_2)| = \frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\left| \int_0^{t_1} \vartheta^{\rho-1}(t_1^\rho - \vartheta^\rho)^{\gamma-1}\Lambda(\vartheta, f_\vartheta)d\vartheta - \int_0^{t_2} \vartheta^{\rho-1}(t_2^\rho - \vartheta^\rho)^{\gamma-1}\Lambda(\vartheta, f_\vartheta)d\vartheta\right|$$

$$= \frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\left| \int_0^{t_1} [(t_1^\rho - \vartheta^\rho)^{\gamma-1} - (t_2^\rho - \vartheta^\rho)^{\gamma-1}]\vartheta^{\rho-1}\Lambda(\vartheta, f_\vartheta)d\vartheta + \int_{t_1}^{t_2} \vartheta^{\rho-1}(t_2^\rho - \vartheta^\rho)^{\gamma-1}\Lambda(\vartheta, f_\vartheta)d\vartheta\right|$$

$$\le \frac{M\rho^{1-\gamma}}{\Gamma(\gamma)}\left(\int_0^{t_1} |(t_1^\rho - \vartheta^\rho)^{\gamma-1} - (t_2^\rho - \vartheta^\rho)^{\gamma-1}|\vartheta^{\rho-1}d\vartheta + \int_{t_1}^{t_2} \vartheta^{\rho-1}(t_2^\rho - \vartheta^\rho)^{\gamma-1}d\vartheta\right).$$

The right-side second integral of the last equation gives the value $\frac{1}{\rho\gamma}(t_2^\rho - \vartheta^\rho)^\gamma$. Now for the first integral, we take two cases $\gamma < 1$, $\gamma = 1$, respectively. For $\gamma = 1$, the integral gives zero value. In the case $\gamma < 1$, we have $(t_1^\rho - \vartheta^\rho)^{\gamma-1} \ge (t_2^\rho - \vartheta^\rho)^{\gamma-1}$. Thus,

$$\int_0^{t_1} |(t_1^\rho - \vartheta^\rho)^{\gamma-1} - (t_2^\rho - \vartheta^\rho)^{\gamma-1}|\vartheta^{\rho-1}d\vartheta = \int_0^{t_1} [(t_1^\rho - \vartheta^\rho)^{\gamma-1} - (t_2^\rho - \vartheta^\rho)^{\gamma-1}]\vartheta^{\rho-1}d\vartheta$$

$$= \frac{1}{\rho\gamma}(t_1^\rho\gamma - t_2^\rho\gamma) + \frac{1}{\rho\gamma}(t_2^\rho - t_1^\rho)^\gamma$$
$$\leq \frac{1}{\rho\gamma}(t_2^\rho - t_1^\rho)^\gamma.$$

Combining these results, we have

$$|f^1(t_1) - f^1(t_2)| \leq \frac{2M}{\rho^\gamma\Gamma(\gamma+1)}(t_2^\rho - t_1^\rho)^\gamma \tag{3.7}$$

if $\gamma \leq 1$. In either case, the right-side portion of (3.7) converges to 0 as $t_2 \to t_1$, which justifies the continuity of f^1, since $\Omega(t)$ itself is continuous.
Now for the induction step $a-1 \to a$, following the similar way as in the case $a = 1$, we can investigate the continuity of $f^a(t)$ on $[c_0, c_1]$. Moreover, if we define

$$\phi^a(t) := f^a(t) - f^{a-1}(t), \quad a = 1, 2, \cdots$$

and $\phi^0(t) := \Omega(t) = f^0(t)$, then functions $\phi^a(t)$ are all continuous on $[c_0, c_1]$. For $a = 0, 1, \cdots$, it is explicit that

$$f^a(t) = \sum_{l=0}^{a} \phi^l(t).$$

Moreover, for $a = 2, 3, \cdots$, we have

$$\phi^a(t) = \frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}[\Lambda(\vartheta, f_\vartheta^{a-1}) - \Lambda(\vartheta, f_\vartheta^{a-2})]d\vartheta. \tag{3.8}$$

Using the Lipschitz constraint on Λ and the condition $f^a(t) = \phi(t)$ for $[-r, 0]$. If $t \in [c+0, c_1]$, then from (3.8), Lemma 2 and Remark 1, we can explore a fixed number $\sigma \in (0, 1)$ such that

$$\begin{aligned}
|\phi^a(t)| &\leq \frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}|\Lambda(\vartheta, f_\vartheta^{a-1}) - \Lambda(\vartheta, f_\vartheta^{a-2})|d\vartheta \\
&\leq \frac{L\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}|f_\vartheta^{a-1} - f_\vartheta^{a-2}|d\vartheta \\
&\leq \frac{L\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}\max_{\vartheta+\theta\in[-r,t]}|f^{a-1}(\vartheta+\theta) - f^{a-2}(\vartheta+\theta)|d\vartheta \\
&\leq \frac{L\rho^{1-\gamma}}{\Gamma(\gamma)}\max_{\vartheta+\theta\in[-r,t]}|f^{a-1}(\vartheta+\theta) - f^{a-2}(\vartheta+\theta)|\int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}d\vartheta \\
&= \max_{\vartheta+\theta\in[-r,t]}|\phi^{a-1}(\vartheta)|\frac{L\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}d\vartheta \leq \sigma \max_{\vartheta\in[c_0,c_1]}|f^{a-1}(\vartheta) - f^{a-2}(\vartheta)| \\
&= \sigma \max_{\vartheta\in[c_0,c_1]}|\phi^{a-1}(\vartheta)|
\end{aligned}$$

for $a = 2, 3, \cdots$, which implies that $\max_{t\in[c_0,c_1]}|\phi^a(t)| \leq \sigma^{a-1}\max_{t\in[c_0,c_1]}|\phi^1(t)|$. So, we explored a convergent majorant of the series $\sum_{a=o}^{\infty}\phi^a$ on $[c_0, c_1]$, and hence the series converges uniformly. As f^a is the $a-th$ partial sum of the series, it satisfies the uniform convergence of the sequence $\{f^a\}_{a=1}^{\infty}$ in

$[c_0, c_1]$, and the limits concur. We specify the limit of that sequence by f. As we explored above that f^a is continuous in $[c_0, c_1]$ $\forall$ a. Hence, form the definition of uniform convergence, $f \in C[c_0, c_1]$.

Now we target to prove that the function f solves the Voltera equation (3.4), and thus the IVP (3.1) and (3.2). In this regard, we identify that the uniformly convergence of the sequence $\{f^a\}_{a=1}^{\infty}$ and the Lipschitz constraint of Λ imply $|\Lambda(t, f(t+\theta)) - \Lambda(t, f^a(t+\theta))| \leq L|f(t) - f^a(t)| \to 0$ uniformly for $t \in [c_0, c_1]$. Another way, the sequence $\{\Lambda(\cdot, f^a_{\cdot})\}_{a=1}^{\infty}$ uniformly converges against $\Lambda(\cdot, f^a_{\cdot})$. So we may interchange the fractional integration and limit operations. Which gives

$$\begin{aligned} f(t) &= \lim_{a\to\infty} f^a(t) = \lim_{a\to\infty}\left(\Omega(t) + \frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}\Lambda(\vartheta, f_\vartheta^{a-1})d\vartheta\right) \\ &= \Omega(t) + \frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}\lim_{a\to\infty}\Lambda(\vartheta, f_\vartheta^{a-1})d\vartheta \\ &= \sum_{k=0}^{m-1}\phi^{(k)}(0)\frac{t^k}{k!} + \frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}\Lambda(\vartheta, f_\vartheta)d\vartheta, \end{aligned}$$

which is the demanded relation (3.4).

Next, for the interval $[c_0, c_1]$, it is pending to prove the uniqueness of the solution. We may suppose that $\widetilde{f}$ is another solution of (3.4), then it gives that

$$\begin{aligned} |f(t) - \widetilde{f}(t)| &\leq \frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}|\Lambda(\vartheta, f_\vartheta) - \Lambda(\vartheta, \widetilde{f}_\vartheta)|d\vartheta \\ &= \frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}\max_{\vartheta+\theta\in[-r,t]}|f(\vartheta+\theta) - \widetilde{f}(\vartheta+\theta)|d\vartheta \\ &\leq \sigma \max_{\vartheta\in[c_0,c_1]}|f(\vartheta) - \widetilde{f}(\vartheta)| \end{aligned}$$

for some $\sigma \in (0, 1)$ by Lemma (2) and Remark (1). Since the above given condition uniformly holds for all $t \in [c_0, c_1]$, we deduce

$$\max_{t\in[c_0,c_1]}|f(t) - \widetilde{f}(t)| \leq \sigma \max_{t\in[c_0,c_1]}|f(t) - \widetilde{f}(t)|,$$

which implies the necessary uniqueness narration $f \equiv \widetilde{f}$.

Now our demand is to prove the same results from the first sub-interval $[c_0, c_1]$ to rest off the sub-intervals $[c_{i-1}, c_i](i = 2, 3, \cdots n)$. We will perform it by using the results which are derived for the interval $[c_0, c_1]$. So by assuming that the claim exists on $[c_{i-1}, c_i]$ for some i, we will show its existence on $[c_{i-1}, c_i]$, if $i < n$. It means we have to show that in this interval, the Eq (3.4) exists a unique continuous solution. For this, we rewrite the (3.4) in the form

$$f(t) = \Omega_i(t) + \frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_{c_i}^t (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}\Lambda(\vartheta, f_\vartheta)d\vartheta,$$

with

$$\Omega_i(t) = \sum_{k=0}^{m-1}\phi^{(k)}(0)\frac{t^k}{k!} + \frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^{c_i} (t^\rho - \vartheta^\rho)^{\gamma-1}\vartheta^{\rho-1}\Lambda(\vartheta, f_\vartheta)d\vartheta.$$

Here the main point is that the Ω_i is a known function because its definition contains only proposed data and the values of the solution f on $[0, c_i]$ which has previously been estimated. Also, as investigated that Ω_i is continuous on $[c_i, c_{i+1}]$. Reminding the descriptions of the functions f^a and ϕ^a, we derive

$$\begin{aligned} f^{0(i)}(t) &:= \Omega_i(t), \\ f^{a(i)}(t) &:= \Omega_i(t) + \frac{\rho^{1-\gamma}}{\Gamma(\gamma)} \int_{c_i}^{t} (t^\rho - \vartheta^\rho)^{\gamma-1} \vartheta^{\rho-1} \Lambda(\vartheta, f_\vartheta^{a-1(i)}) d\vartheta, \quad a = 1, 2, \cdots, \\ \phi^{a(i)}(t) &:= f^{a(i)}(t) - f^{a-1(i)}(t), \quad a = 1, 2, \cdots, \end{aligned}$$

and

$$\phi^{0(i)}(t) \ := \ \Omega_i(t) = f^{0(i)}(t).$$

Based on the previous investigations, all these functions $f^{a(i)}(t)$ and $\phi^{a(i)}(t)$ are also continuous on $[c_i, c_{i+1}]$ for $a = 0, 1, \cdots$, and this is straightforward that

$$f^{a(i)}(t) \ = \ \sum_{l=0}^{a} \phi^{l(i)}(t).$$

As given above, a convergent majorant for $\sum_{l=0}^{\infty} \phi^{l(i)}$ can be investigated which gives the existence of uniform limit $f := \lim_{a\to\infty} f^{a(i)}$ on $[c_i, c_{i+1}]$. As, $f(t)$ has been presented in a piecewise form on $[0, c_i]$ and $[c_i, c_{i+1}]$. So, at the point c_i, we may get the limit of $f(t)$. From

$$\lim_{t\to c_i+0} \int_{c_i}^{t} (t^\rho - \vartheta^\rho)^{\gamma-1} \vartheta^{\rho-1} d\vartheta = \lim_{t\to c_i+0} \left[\frac{-(t^\rho - \vartheta^\rho)^\gamma}{\rho\gamma} \right]_{c_i}^{t} = \lim_{t\to c_i+0} \frac{(t^\rho - c_i^\rho)^\gamma}{\rho\gamma} = 0,$$

which implies □

$$\begin{aligned} \lim_{t\to c_i+0} \int_{c_i}^{t} (t^\rho - \vartheta^\rho)^{\gamma-1} \vartheta^{\rho-1} \Lambda(\vartheta, f_\vartheta) d\vartheta &= \Lambda(t^*, f_t^*) \lim_{t\to c_i+0} \int_{c_i}^{t} (t^\rho - \vartheta^\rho)^{\gamma-1} \vartheta^{\rho-1} d\vartheta \\ &= \Lambda(t^*, f_t^*) \lim_{t\to c_i+0} \frac{(t^\rho - c_i^\rho)^\gamma}{\rho\gamma} = 0 \end{aligned}$$

for some $t^* \in [c_i, t]$. Thus

$$\begin{aligned} \lim_{t\to c_i-0} f(t) &= \lim_{t\to c_i-0} \left(\sum_{k=0}^{m-1} \phi^{(k)}(0) \frac{t^k}{k!} + \frac{\rho^{1-\gamma}}{\Gamma(\gamma)} \int_0^t (t^\rho - \vartheta^\rho)^{\gamma-1} \vartheta^{\rho-1} \Lambda(\vartheta, f_\vartheta) d\vartheta \right) \\ &= \sum_{k=0}^{m-1} \phi^{(k)}(0) \frac{c_i^k}{k!} + \frac{\rho^{1-\gamma}}{\Gamma(\gamma)} \int_0^{c_i} (c_i^\rho - \vartheta^\rho)^{\gamma-1} \vartheta^{\rho-1} \Lambda(\vartheta, f_\vartheta) d\vartheta = \Omega_i(c_i) = f(c_i) \end{aligned}$$

and

$$\lim_{t\to c_i+0} f(t) = \lim_{t\to c_i+0} \left(\Omega_i(t) + \frac{\rho^{1-\gamma}}{\Gamma(\gamma)} \int_{c_i}^{t} (t^\rho - \vartheta^\rho)^{\gamma-1} \vartheta^{\rho-1} \Lambda(\vartheta, f_\vartheta) d\vartheta \right) = \Omega_i(c_i).$$

Therefore, $f(t)$ is continuous at the point c_i, this gives that there exists a unique function $f(t) \in C[-r, c]$ which solves the Eqn. 3.4. Hence our proof is finished.

Remark 2. *The above-given results of the proposed theorem can be easily applied to the various delay-type fractional-order systems in the form of generalized-Caputo type initial value problems to justify their existence of a unique solution. For example, the authors in ref. [15] had solved some important delay-type problems in the sense of generalized-Caputo fractional derivatives by proposing a modified form of the Predictor-Corrector method. In their study, they did not discuss anything about the existence of a unique solution to the proposed problems. Now by using Theorem 1 of the proposed study, the existence and uniqueness results for the examples 6.1, 6.2, 6.3, and 6.4 of ref. [15] can easily be derived (because all examples satisfy the statement of the proposed Theorem 1). Moreover, there are a number of delay-type epidemics and ecological models available in the literature which can be easily derived in the generalized Caputo-type fractional-order sense by using the above-mentioned results.*

4. Conclusions

In this article, we have proposed a novel theorem for the existence of a unique solution for the generalized Caputo-type initial value problems with delay by using fixed point results. The proposed theorem fills the research gap of existence and uniqueness for the considered fractional derivative with delay. The given results provide a novel mathematical foundation for the research on the asymptotic nature of solutions to DFDEs. In the future, the given results can be applied to various kinds of delay-type dynamical systems to check their solution existence in the generalized Caputo sense.

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Conflict of interest

This work does not have any conflicts of interest.

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