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Research article

Some integral inequalities in interval fractional calculus for left and right coordinated interval-valued functions

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Abstract: Integral inequalities play a crucial role in both theoretical and applied mathematics. Because of the relevance of these notions, we have discussed a new class of introduced generalized convex function called as coordinated left and right convex interval-valued function (coordinated LR-convex IVF) using the pseudo-order relation ($\leq_p$). On interval space, this order relation is defined. First, a pseudo-order relation is used to show Hermite-Hadamard type inequality (HH type inequality) for coordinated LR-convex IVF. Second for coordinated LR-convex IVF, Some HH type inequalities are also derived for the product of two coordinated LR-convex IVFs. Furthermore, we have demonstrated that our conclusions cover a broad range of new and well-known inequalities for coordinated LR-convex IVFs and their variant forms as special instances which are defined by Zhao et al. and Budak et al. Finally, we have shown that the inclusion relation "$\supseteq$" confidents to the pseudo-order relation "$\leq_p$" for coordinated LR-convex IVFs. The concepts and methodologies presented in this study might serve as a springboard for additional research in this field, as well as a tool for investigating probability and optimization research, among other things.

Keywords: coordinated left and right convex interval-valued functions; double interval Riemann-Liouville-type integrals; Hermite-Hadamard type inequalities
Mathematical Subject Classification: Primary 26A33, 26A51, 26D07, 26D10; Secondary 26D15, 26D20

1. Introduction

In convex function theory, the classical Hermite-Hadamard inequality is one of the most well-known inequalities with geometrical interpretation, and it has a wide range of applications, see [1,2].
Let $\mathfrak{S}: K \to \mathbb{R}^+$ be a convex function on a convex set K and $\rho, \varsigma \in K$ with $\rho \neq \varsigma$. Then,

$$\mathfrak{S}\left(\frac{\rho+\varsigma}{2}\right) \leq \frac{1}{\varsigma-\rho} \int_\rho^\varsigma \mathfrak{S}(\varpi)d\varpi \leq \frac{\mathfrak{S}(\rho) + \mathfrak{S}(\varsigma)}{2}. \tag{1}$$

In [3], Fejér looked at the key extensions of HH-inequality which is known as Hermite-Hadamard-Fejér inequality (HH-Fejér inequality).
Let $\mathfrak{S}: K \to \mathbb{R}^+$ be a convex function on a convex set K and $\rho, \varsigma \in K$ with $\rho \neq \varsigma$. Then,

$$\mathfrak{S}\left(\frac{\rho+\varsigma}{2}\right) \leq \frac{1}{\int_\rho^\varsigma \mathfrak{D}(\varpi)d\varpi} \int_\rho^\varsigma \mathfrak{S}(\varpi)\mathfrak{D}(\varpi)d\varpi \leq \frac{\mathfrak{S}(\rho)+ \mathfrak{S}(\varsigma)}{2}\int_\rho^\varsigma \mathfrak{D}(\varpi)d\varpi. \tag{2}$$

If $\mathfrak{D}(\varpi) = 1$, then we obtain (1) from (2). We should remark that Hermite-Hadamard inequality is a refinement of the idea of convexity, and it can be simply deduced from Jensen's inequality. In recent years, the Hermite-Hadamard inequality for convex functions has gotten a lot of attention, and there have been a lot of improvements and generalizations examined. Sarikaya [4] proved the Hadamard type inequality for coordinated convex functions such that

Let $\mathfrak{G}: \Delta \to \mathbb{R}^+$ be a coordinate convex function on $\Delta = [\varsigma, \rho] \times [\mu, \nu]$. If $\mathfrak{G}$ is double fractional integrable, then following inequalities hold:

$$\mathfrak{G}\left(\frac{\mu+\nu}{2}, \frac{\varsigma+\rho}{2}\right) \leq \frac{\Gamma(\alpha+1)}{4(\nu-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}\left(\nu, \frac{\varsigma+\rho}{2}\right) + \mathcal{J}_{\nu^-}^{\alpha}\mathfrak{G}\left(\mu, \frac{\varsigma+\rho}{2}\right)\right] + \frac{\Gamma(\beta+1)}{4(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\varsigma^+}^{\beta}\,\mathfrak{G}\left(\frac{\mu+\nu}{2}, \rho\right) + \mathcal{J}_{\rho^-}^{\beta}\,\mathfrak{G}\left(\frac{\mu+\nu}{2}, \varsigma\right)\right]$$

$$\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^\alpha(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\mu^+,\varsigma^+}^{\alpha,\beta}\,\mathfrak{G}(\nu,\rho) + \mathcal{J}_{\mu^+,\rho^-}^{\alpha,\beta}\,\mathfrak{G}(\nu,\varsigma) + \mathcal{J}_{\nu^-,\varsigma^+}^{\alpha,\beta}\,\mathfrak{G}(\mu,\rho) + \mathcal{J}_{\nu^-,\rho^-}^{\alpha,\beta}\,\mathfrak{G}(\mu,\varsigma)\right]$$

$$\leq \frac{\Gamma(\alpha+1)}{8(\nu-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\varsigma)\mathfrak{G}\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\rho) + \mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\varsigma) + \mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\rho)\right]$$

$$+ \frac{\Gamma(\beta+1)}{4(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\varsigma^+}^{\beta}\,\mathfrak{G}(\mu,\rho)\widetilde{+}\mathcal{J}_{\rho^-}^{\beta}\,\mathfrak{G}(\nu,\varsigma) + \mathcal{J}_{\varsigma^+}^{\beta}\,\mathfrak{G}(\mu,\rho) + \mathcal{J}_{\rho^-}^{\beta}\,\mathfrak{G}(\nu,\varsigma)\right]$$

$$\leq \frac{\mathfrak{G}(\mu,\varsigma)+\mathfrak{G}(\nu,\varsigma)+\mathfrak{G}(\mu,\rho)+\mathfrak{G}(\nu,\rho)}{4}. \tag{3}$$

If $\alpha = 1$, then we obtain the following Dragomir inequality [5] on coordinates:

$$\mathfrak{G}\left(\frac{\mu+\nu}{2}, \frac{\varsigma+\rho}{2}\right)$$

$$\leq \frac{1}{2}\left[\frac{1}{\nu-\mu}\int_\mu^\nu \mathfrak{G}\left(x, \frac{\varsigma+\rho}{2}\right)dx + \frac{1}{\rho-\varsigma}\int_\varsigma^\rho \mathfrak{G}\left(\frac{\mu+\nu}{2}, y\right)dy\right] \leq \frac{1}{(\nu-\mu)(\rho-\varsigma)}\int_\mu^\nu\int_\varsigma^\rho \mathfrak{G}(x,y)dydx$$

$$\leq \frac{1}{4(\nu-\mu)}\left[\int_{\mu}^{\nu} \mathfrak{G}(x,\varsigma)\,dx + \int_{\mu}^{\nu} \mathfrak{G}(x,\rho)dx\right] + \frac{1}{4(\rho-\varsigma)}\left[\int_{\varsigma}^{\rho} \mathfrak{G}(\mu,y)dy + \int_{\varsigma}^{\rho} \mathfrak{G}(\nu,y)dy\right]$$

$$\leq \frac{\mathfrak{G}(\mu,\varsigma)+\mathfrak{G}(\nu,\varsigma)+\mathfrak{G}(\mu,\rho)+\mathfrak{G}(\nu,\rho)}{4}. \tag{4}$$

For more details related to inequalities, see [6–9] and reference therein.

Interval analysis, on the other hand, is a well-known example of set-valued analysis, which is the study of sets in the context of mathematical analysis and general topology. It was created as a way of dealing with the interval uncertainty that can be found in many mathematical or computer models of deterministic real-world phenomena. Archimede's method, which is used to calculate the circumference of a circle, is an old example of an interval enclosure. Moore [10], who is credited with being the first user of intervals in computational mathematics, published the first book on interval analysis in 1966. Following the publication of his book, a number of scientists began to research the theory and applications of interval arithmetic. Interval analysis is now a helpful technique in a variety of fields that are interested in ambiguous data because of its applicability. Computer graphics, experimental and computational physics, error analysis, robotics, and many more fields have applications.

Furthermore, in recent years, numerous major inequalities (Hermite-Hadamard, Ostrowski and others) have been addressed for interval-valued functions. Chalco-Cano et al. used the Hukuhara derivative for interval-valued functions to construct Ostrowski type inequalities for interval-valued functions in [11–14]. For interval-valued functions, Román-Flores et al. developed Minkowski and Beckenbach's inequality in [15]. For fuzzy interval-valued function, Khan et al. [16–18] derived some new versions of Hermite-Hadamard type inequalities and proved their validity with the help of non-trivial examples. Moreover, Khan et al. [19,20] discussed some novel types of Hermite-Hadamard type inequalities in fuzzy-interval fractional calculus and proved that many classical versions are special cases of these inequalities. Recently, Khan et al. [21] introduced the new class of convexity in fuzzy-interval calculus which is known as coordinated convex fuzzy-interval-valued functions and with the support of these classes, some Hermite-Hadamard type inequalities are obtained via newly defined fuzzy-interval double integrals. We encourage readers to [22–54] for other related results.

The following is an overview of the paper's structure. Section 2 recalls some preliminary notions and definitions. Moreover, some properties of introduced coordinated *LR*-convex IVF are also discussed. Section 3 presents some Hermite-Hadamard type inequalities for coordinated *LR*-convex IVF. With the help of this class, some fractional integral inequalities are also derived for the coordinated *LR*-convex IVF and for the product of two coordinated *LR*-convex IVFs. The fourth section, Conclusions and Future Work, brings us to a close.

2. Preliminaries and known results

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_I$ be the space of all closed and bounded intervals of $\mathbb{R}$, such that $\mathfrak{U} \in \mathbb{R}_I$ is defined by

$$\mathfrak{U} = [\mathfrak{U}_*, \mathfrak{U}^*] = \{y \in \mathbb{R} |\, \mathfrak{U}_* \leq y \leq \mathfrak{U}^*\}, \qquad (\mathfrak{U}_*, \mathfrak{U}^* \in \mathbb{R}). \tag{5}$$

If $\mathfrak{U}_* = \mathfrak{U}^*$, then $\mathfrak{U}$ is said to be degenerate. If $\mathfrak{U}_* \geq 0$, then $[\mathfrak{U}_*, \mathfrak{U}^*]$ is called positive interval. The set of all positive interval is denoted by $\mathbb{R}_I^+$ and defined as $\mathbb{R}_I^+ = \{[\mathfrak{U}_*, \mathfrak{U}^*]: [\mathfrak{U}_*, \mathfrak{U}^*] \in$

$\mathbb{R}_I$ and $\mathfrak{U}_* \geq 0\}$.
Let $\varrho \in \mathbb{R}$ and $\varrho\mathfrak{U}$ be defined by

$$\varrho.\mathfrak{U} = \begin{cases} [\varrho\mathfrak{U}_*, \varrho\mathfrak{U}^*] \text{ if } \varrho > 0, \\ \{0\} \qquad \text{if } \varrho = 0, \\ [\varrho\mathfrak{U}^*, \varrho\mathfrak{U}_*] \text{ if } \varrho < 0. \end{cases} \tag{6}$$

Then, the Minkowski difference $\mathfrak{D} - \mathfrak{U}$, addition $\mathfrak{U} + \mathfrak{D}$ and $\mathfrak{U} \times \mathfrak{D}$ for $\mathfrak{U}, \mathfrak{D} \in \mathbb{R}_I$ are defined by

$$\begin{aligned} [\mathfrak{D}_*, \mathfrak{D}^*] - [\mathfrak{U}_*, \mathfrak{U}^*] &= [\mathfrak{D}_* - \mathfrak{U}_*,\ \mathfrak{D}^* - \mathfrak{U}^*], \\ [\mathfrak{D}_*, \mathfrak{D}^*] + [\mathfrak{U}_*, \mathfrak{U}^*] &= [\mathfrak{D}_* + \mathfrak{U}_*,\ \mathfrak{D}^* + \mathfrak{U}^*], \end{aligned} \tag{7}$$

and

$$[\mathfrak{D}_*, \mathfrak{D}^*] \times [\mathfrak{U}_*, \mathfrak{U}^*] = [min\{\mathfrak{D}_*\mathfrak{U}_*, \mathfrak{D}^*\mathfrak{U}_*, \mathfrak{D}_*\mathfrak{U}^*, \mathfrak{D}^*\mathfrak{U}^*\}, max\{\mathfrak{D}_*\mathfrak{U}_*, \mathfrak{D}^*\mathfrak{U}_*, \mathfrak{D}_*\mathfrak{U}^*, \mathfrak{D}^*\mathfrak{U}^*\}].$$

The inclusion " $\supseteq$ " means that
$\mathfrak{U} \supseteq \mathfrak{D}$ if and only if, $[\mathfrak{U}_*, \mathfrak{U}^*] \supseteq [\mathfrak{D}_*, \mathfrak{D}^*]$, and if and only if

$$\mathfrak{U}_* \leq \mathfrak{D}_*,\ \mathfrak{D}^* \leq \mathfrak{U}^*. \tag{8}$$

Remark 1. [36] (i) The relation " $\leq_p$ " is defined on $\mathbb{R}_I$ by

$$[\mathfrak{D}_*, \mathfrak{D}^*] \leq_p [\mathfrak{U}_*, \mathfrak{U}^*] \text{ if and only if } \mathfrak{D}_* \leq \mathfrak{U}_*,\ \mathfrak{D}^* \leq \mathfrak{U}^*, \tag{9}$$

for all $[\mathfrak{D}_*, \mathfrak{D}^*]$, $[\mathfrak{U}_*, \mathfrak{U}^*] \in \mathbb{R}_I$, and it is a pseudo order relation. The relation $[\mathfrak{D}_*, \mathfrak{D}^*] \leq_p [\mathfrak{U}_*, \mathfrak{U}^*]$ coincident to $[\mathfrak{D}_*, \mathfrak{D}^*] \leq [\mathfrak{U}_*, \mathfrak{U}^*]$ on $\mathbb{R}_I$ when it is " $\leq_p$ "
(ii) It can be easily seen that " $\leq_p$ " looks like "left and right" on the real line $\mathbb{R}$, so we call " $\leq_p$ " is "left and right" (or "LR" order, in short).
For $[\mathfrak{D}_*, \mathfrak{D}^*]$, $[\mathfrak{U}_*, \mathfrak{U}^*] \in \mathbb{R}_I$, the Hausdorff-Pompeiu distance between intervals $[\mathfrak{D}_*, \mathfrak{D}^*]$ and $[\mathfrak{U}_*, \mathfrak{U}^*]$ is defined by

$$d([\mathfrak{D}_*, \mathfrak{D}^*], [\mathfrak{U}_*, \mathfrak{U}^*]) = max\{|\mathfrak{D}_* - \mathfrak{U}_*|, |\mathfrak{D}^* - \mathfrak{U}^*|\}. \tag{10}$$

It is familiar fact that $(\mathbb{R}_I, d)$ is a complete metric space.
Theorem 1. [10] If $\mathfrak{G}: [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}_I$ is an *I-V-F* given by (x) $[\mathfrak{G}_*(x), \mathfrak{G}^*(x)]$, then $\mathfrak{G}$ is Riemann integrable over $[\mu, \nu]$ if and only if, $\mathfrak{G}_*$ and $\mathfrak{G}^*$ both are Riemann integrable over $[\mu, \nu]$ such that

$$(IR)\int_\mu^\nu \mathfrak{G}(x)dx = \left[(R)\int_\mu^\nu \mathfrak{G}_*(x)dx, (R)\int_\mu^\nu \mathfrak{G}^*(x)dx\right]. \tag{11}$$

The collection of all Riemann integrable real valued functions and Riemann integrable *I-V-F* is denoted by $\mathcal{R}_{[\mu,\nu]}$ and $\mathfrak{T}\mathcal{R}_{[\mu,\nu]}$, respectively.

Definition 1. [31, 33] Let $\mathfrak{G}: [\mu, \nu] \to \mathbb{R}_I$ be interval-valued function and $\mathfrak{G} \in \mathfrak{T}\mathcal{R}_{[\mu,\nu]}$. Then interval Riemann-Liouville-type integrals of $\mathfrak{G}$ are defined as

$$\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{G}(y) = \frac{1}{\Gamma(\alpha)}\int_\mu^y (y - \mathrm{t})^{\alpha-1}\mathfrak{G}(\mathrm{t})\, dt \quad (y > \mu), \tag{12}$$

$$\mathcal{J}_{\nu^-}^{\alpha}\ \mathfrak{G}(y)=\frac{1}{\Gamma(\alpha)}\int_y^{\nu}(\mathrm{t}-y)^{\alpha-1}\mathfrak{G}(\mathrm{t})\,dt\quad (y<\nu), \tag{13}$$

where $\alpha>0$ and Γ is the gamma function.

Theorem 2. [20] Let $\mathfrak{G}:[\varsigma,\rho]\to\mathbb{R}_I{}^+$ be a LR-convex I-V.F such that $\mathfrak{G}(y)=[\mathfrak{G}_*(y),\mathfrak{G}^*(y)]$ for all $y\in[\varsigma,\rho]$. If $\mathfrak{G}\in L([\varsigma,\rho],\mathbb{R}_I^+)$, then

$$\mathfrak{G}\left(\frac{\varsigma+\rho}{2}\right)\leq_p\frac{\Gamma(\alpha+1)}{2(\rho-\varsigma)^{\alpha}}\left[\mathcal{J}_{\varsigma^+}^{\alpha}\ \mathfrak{G}(\rho)+\mathcal{J}_{\rho^-}^{\alpha}\ \mathfrak{G}(\varsigma)\right]\leq_p\frac{\mathfrak{G}(\varsigma)+\mathfrak{G}(\rho)}{2}. \tag{14}$$

Theorem 3. [20] Let $\mathfrak{G},\mathfrak{S}:[\varsigma,\rho]\to\mathbb{R}_I^+$ be two LR-convex I-V.Fs such that $\mathfrak{G}(x)=[\mathfrak{G}_*(x),\mathfrak{G}^*(x)]$ and $\mathfrak{S}(x)=[\mathfrak{S}_*(x),\mathfrak{S}^*(x)]$ for all $x\in[\varsigma,\rho]$. If $\mathfrak{G}\times\mathfrak{S}\in L([\varsigma,\rho],\mathbb{R}_I^+)$ is fuzzy Riemann integrable, then

$$\frac{\Gamma(\alpha+1)}{2(\rho-\varsigma)^{\alpha}}\left[\mathcal{J}_{\varsigma^+}^{\alpha}\ \mathfrak{G}(\rho)\times\mathfrak{S}(\rho)+\mathcal{J}_{\rho^-}^{\alpha}\mathfrak{G}(\varsigma)\times\mathfrak{S}(\varsigma)\right]$$

$$\leq_p\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\mathcal{M}(\varsigma,\rho)+\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\mathcal{N}(\varsigma,\rho), \tag{15}$$

and

$$\mathfrak{G}\left(\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\frac{\varsigma+\rho}{2}\right)$$

$$\leq_p\frac{\Gamma(\alpha+1)}{4(\rho-\varsigma)^{\alpha}}\left[\mathcal{J}_{\varsigma^+}^{\alpha}\ \mathfrak{G}(\rho)\times\mathfrak{S}(\rho)+\mathcal{J}_{\rho^-}^{\alpha}\ \mathfrak{G}(\varsigma)\times\mathfrak{S}(\varsigma)\right]$$

$$+\frac{1}{2}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\mathcal{M}(\varsigma,\rho)+\frac{1}{2}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\mathcal{N}(\varsigma,\rho), \tag{16}$$

where $\mathcal{M}(\varsigma,\rho)=\mathfrak{G}(\varsigma)\times\mathfrak{S}(\varsigma)+\ \mathfrak{G}(\rho)\times\mathfrak{S}(\rho)$, $\mathcal{N}(\varsigma,\rho)=\mathfrak{G}(\varsigma)\times\mathfrak{S}(\rho)+\mathfrak{G}(\rho)\times\mathfrak{S}(\varsigma)$, and $\mathcal{M}(\varsigma,\rho)=[\mathcal{M}_*(\varsigma,\rho),\mathcal{M}^*(\varsigma,\rho)]$ and $\mathcal{N}(\varsigma,\rho)=[\mathcal{N}_*(\varsigma,\rho),\mathcal{N}^*(\varsigma,\rho)]$.

Note that, the Theorem 1 is also true for interval double integrals. The collection of all double integrable *I-V-F* is denoted $\mathfrak{T}\mathfrak{D}_{\Delta}$, respectively.

Theorem 4. [35] Let $\Delta=[\varsigma,\rho]\times[\mu,\nu]$. If $\mathfrak{G}:\Delta\to\mathbb{R}_I$ is interval-valued doubl integrable (ID-integrable) on Δ. Then, we have

$$(ID)\int_{\varsigma}^{\rho}\int_{\mu}^{\nu}\mathfrak{G}(x,y)dydx=(IR)\int_{\varsigma}^{\rho}(IR)\int_{\mu}^{\nu}\mathfrak{G}(x,y)dydx.$$

Definition 2. [36] Let $\mathfrak{G}:\Delta\to\mathbb{R}_I^+$ and $\mathfrak{G}\in\mathfrak{T}\mathfrak{D}_{\Delta}$. The interval Riemann-Liouville-type integrals $\mathcal{J}_{\mu^+,\varsigma^+}^{\alpha,\beta},\mathcal{J}_{\mu^+,\rho^-}^{\alpha,\beta},\ \mathcal{J}_{\nu^-,\varsigma^+}^{\alpha,\beta},\mathcal{J}_{\nu^-,\rho^-}^{\alpha,\beta}$ of $\mathfrak{G}$ order $\alpha,\beta>0$ are defined by

$$\mathcal{J}_{\mu^+,\varsigma^+}^{\alpha,\beta}\ \mathfrak{G}(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\mu}^{x}\int_{\varsigma}^{y}(x-\mathrm{t})^{\alpha-1}(y-\mathrm{s})^{\beta-1}\mathfrak{G}(\mathrm{t},\mathrm{s})\,dsdt\ \ (x>\mu,y>\varsigma), \tag{17}$$

$$\mathcal{J}_{\mu^+,\rho^-}^{\alpha,\beta}\ \mathfrak{G}(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\mu}^{x}\int_{y}^{\rho}(x-\mathrm{t})^{\alpha-1}(\mathrm{s}-y)^{\beta-1}\mathfrak{G}(\mathrm{t},\mathrm{s})\,dsdt\ \ (x>\mu,y<\rho), \tag{18}$$

$$\mathcal{J}^{\alpha,\beta}_{\nu^-,\varsigma^+}\ \mathfrak{G}(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^{\nu}\int_{\varsigma}^{y}(\mathrm{t}-x)^{\alpha-1}(y-\mathrm{s})^{\beta-1}\mathfrak{G}(\mathrm{t},\mathrm{s})\,dsdt\ \ (x<\nu,y>\varsigma), \quad (19)$$

$$\mathcal{J}^{\alpha,\beta}_{\nu^-,\rho^-}\ \mathfrak{G}(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^{\nu}\int_{y}^{\rho}(\mathrm{t}-x)^{\alpha-1}(\mathrm{s}-y)^{\beta-1}\mathfrak{G}(\mathrm{t},\mathrm{s})\,dsdt\ \ (x<\nu,y<\rho). \quad (20)$$

Definition 3. [38] The I-V.F $\mathfrak{G}:\Delta\to\mathbb{R}_I^+$ is said to be coordinated *LR*-convex I-V.F on Δ if

$$\mathfrak{G}(\tau\mu+(1-\tau)\nu,s\varsigma+(1-s)\rho)$$

$$\leq_p \tau s\mathfrak{G}(\mu,\varsigma)+\tau(1-s)\mathfrak{G}(\mu,\rho)+(1-\tau)s\mathfrak{G}(\nu,\varsigma)+(1-\tau)(1-s)\mathfrak{G}(\nu,\rho), \quad (21)$$

for all $(\mu,\nu),(\varsigma,\rho)\in\Delta$, and $\tau,s\in[0,1]$. If inequality (21) is reversed, then $\mathfrak{G}$ is called coordinate *LR*-concave I-V.F on Δ.

Lemma 1. [38] Let $\mathfrak{G}:\Delta\to\mathbb{R}_I^+$ be an coordinated I-V.F on Δ. Then, $\mathfrak{G}$ is coordinated *LR*-convex I-V.F on Δ, if and only if there exist two coordinated *LR*-convex I-V.Fs $\mathfrak{G}_x:[\varsigma,\rho]\to\mathbb{R}_I^+$, $\mathfrak{G}_x(w)=\mathfrak{G}(x,w)$ and $\mathfrak{G}_y:[\mu,\nu]\to\mathbb{R}_I^+$, $\mathfrak{G}_y(z)=\mathfrak{G}(z,y)$.

Theorem 5. [38] Let $\mathfrak{G}:\Delta\to\mathbb{R}_I^+$ be a I-V.F on Δ such that

$$\mathfrak{G}(x,\varpi)=[\mathfrak{G}_*(x,\varpi),\mathfrak{G}^*(x,\varpi)], \quad (22)$$

for all $(x,\varpi)\in\Delta$. Then, $\mathfrak{G}$ is coordinated *LR*-convex I-V.F on Δ, if and only if, $\mathfrak{G}_*(x,\varpi)$ and $\mathfrak{G}^*(x,\varpi)$ are coordinated convex functions.

Example 1. We consider the I-V.Fs $\mathfrak{G}:[0,1]\times[0,1]\to\mathbb{R}_I^+$ defined by,

$$\mathfrak{G}(x)(\sigma)=\begin{cases}\dfrac{\sigma}{2(6+e^x)(6+e^{\varpi})}, & \sigma\in[0,2(6+e^x)(6+e^{\varpi})]\\ \dfrac{4(6+e^x)(6+e^{\varpi})-\sigma}{2(6+e^x)(6+e^{\varpi})}, & \sigma\in(2(6+e^x)(6+e^{\varpi}),4(6+e^x)(6+e^{\varpi})]\\ 0, & \text{otherwise,}\end{cases}$$

Then, for each $\theta\in[0,1]$, we have $\mathfrak{G}(x)=[2\theta(6+e^x)(6+e^{\varpi}),(4+2\theta)(6+e^x)(6+e^{\varpi})\,]$. Since end point functions $\mathfrak{G}_*\big((x,\varpi),\theta\big)$, $\mathfrak{G}^*\big((x,\varpi),\theta\big)$ are coordinate concave functions for each $\theta\in[0,1]$. Hence $\mathfrak{S}(x,\varpi)$ is coordinate *LR*-concave I-V.F.

From Lemma 1, we can easily note that each *LR*-convex I-V.F is coordinated *LR*-convex I-V.F. But the converse is not true.

Remark 2. If one takes $\mathfrak{G}_*(x,\varpi)=\mathfrak{G}^*(x,\varpi)$, then $\mathfrak{G}$ is known as coordinated function if $\mathfrak{G}$ satisfies the coming inequality

$$\mathfrak{G}(\tau\mu+(1-\tau)\nu,s\varsigma+(1-s)\rho)$$

$$\leq \tau s\mathfrak{G}(\mu,\varsigma)+\tau(1-s)\mathfrak{G}(\mu,\rho)+(1-\tau)s\mathfrak{G}(\nu,\varsigma)+(1-\tau)(1-s)\mathfrak{G}(\nu,\rho),$$

is valid which is defined by Dragomir [5]

Let one takes $\mathfrak{G}_*(x,\varpi)\neq\mathfrak{G}^*(x,\varpi)$, where $\mathfrak{G}_*(x,\varpi)$ is affine function and $\mathfrak{G}^*(x,\varpi)$ is a concave function. If coming inequality,

$$\mathfrak{G}(\tau\mu+(1-\tau)\nu,s\varsigma+(1-s)\rho)$$

$$\supseteq \tau s\mathfrak{G}(\mu,\varsigma)+\tau(1-s)\mathfrak{G}(\mu,\rho)+(1-\tau)s\mathfrak{G}(\nu,\varsigma)+(1-\tau)(1-s)\mathfrak{G}(\nu,\rho),$$

is valid, then $\mathfrak{G}$ is named as coordinated IVF which is defined by Zhao et al. [37, Definition 2 and Example 2]

3. Main results

In this section, we shall continue with the following fractional HH-inequality for coordinated LR-convex I-V.Fs, and we also give fractional HH-Fejér inequality for coordinated LR-convex I-V.F through fuzzy order relation.

Theorem 6. Let $\mathfrak{G}: \Delta \to \mathbb{R}_I^+$ be a coordinate LR-convex I-V.F on Δ such that $\mathfrak{G}(x, y) = [\mathfrak{G}_*(x, y), \mathfrak{G}^*(x, y)]$ for all $(x, y) \in \Delta$. If $\mathfrak{G} \in \mathfrak{TO}_\Delta$, then following inequalities holds:

$$\mathfrak{G}\left(\frac{\mu+\nu}{2}, \frac{\varsigma+\rho}{2}\right) \leq_p \frac{\Gamma(\alpha+1)}{4(\nu-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^\alpha \mathfrak{G}\left(\nu, \frac{\varsigma+\rho}{2}\right) + \mathcal{J}_{\nu^-}^\alpha \mathfrak{G}\left(\mu, \frac{\varsigma+\rho}{2}\right)\right]$$
$$+\frac{\Gamma(\beta+1)}{4(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\varsigma^+}^\beta \mathfrak{G}\left(\frac{\mu+\nu}{2}, \rho\right) + \mathcal{J}_{\rho^-}^\beta \mathfrak{G}\left(\frac{\mu+\nu}{2}, \varsigma\right)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^\alpha(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\mu^+,\varsigma^+}^{\alpha,\beta} \mathfrak{G}(\nu, \rho) + \mathcal{J}_{\mu^+,\rho^-}^{\alpha,\beta} \mathfrak{G}(\nu, \varsigma) + \mathcal{J}_{\nu^-,\varsigma^+}^{\alpha,\beta} \mathfrak{G}(\mu, \rho) + \mathcal{J}_{\nu^-,\rho^-}^{\alpha,\beta} \mathfrak{G}(\mu, \varsigma)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)}{8(\nu-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^\alpha \mathfrak{G}(\nu, \varsigma) + \mathcal{J}_{\mu^+}^\alpha \mathfrak{G}(\nu, \rho) + \mathcal{J}_{\nu^-}^\alpha \mathfrak{G}(\mu, \varsigma) + \mathcal{J}_{\nu^-}^\alpha \mathfrak{G}(\mu, \rho)\right]$$
$$+\frac{\Gamma(\beta+1)}{4(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\varsigma^+}^\beta \mathfrak{G}(\mu, \rho) + \mathcal{J}_{\rho^-}^\beta \mathfrak{G}(\nu, \varsigma) + \mathcal{J}_{\varsigma^+}^\beta \mathfrak{G}(\mu, \rho) + \mathcal{J}_{\rho^-}^\beta \mathfrak{G}(\nu, \varsigma)\right]$$
$$\leq_p \frac{\mathfrak{G}(\mu,\varsigma)+\mathfrak{G}(\nu,\varsigma)+\mathfrak{G}(\mu,\rho)+\mathfrak{G}(\nu,\rho)}{4}. \tag{23}$$

If $\mathfrak{G}(x)$ coordinated LR-concave I-V.F, then

$$\mathfrak{G}\left(\frac{\mu+\nu}{2}, \frac{\varsigma+\rho}{2}\right) \geq_p \frac{\Gamma(\alpha+1)}{4(\nu-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^\alpha \mathfrak{G}\left(\nu, \frac{\varsigma+\rho}{2}\right) + \mathcal{J}_{\nu^-}^\alpha \mathfrak{G}\left(\mu, \frac{\varsigma+\rho}{2}\right)\right]$$
$$+\frac{\Gamma(\beta+1)}{4(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\varsigma^+}^\beta \mathfrak{G}\left(\frac{\mu+\nu}{2}, \rho\right) + \mathcal{J}_{\rho^-}^\beta \mathfrak{G}\left(\frac{\mu+\nu}{2}, \varsigma\right)\right]$$
$$\geq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^\alpha(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\mu^+,\varsigma^+}^{\alpha,\beta} \mathfrak{G}(\nu, \rho) + \mathcal{J}_{\mu^+,\rho^-}^{\alpha,\beta} \mathfrak{G}(\nu, \varsigma) + \mathcal{J}_{\nu^-,\varsigma^+}^{\alpha,\beta} \mathfrak{G}(\mu, \rho) + \mathcal{J}_{\nu^-,\rho^-}^{\alpha,\beta} \mathfrak{G}(\mu, \varsigma)\right]$$
$$\geq_p \frac{\Gamma(\alpha+1)}{8(\nu-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^\alpha \mathfrak{G}(\nu, \varsigma) + \mathcal{J}_{\mu^+}^\alpha \mathfrak{G}(\nu, \rho) + \mathcal{J}_{\nu^-}^\alpha \mathfrak{G}(\mu, \varsigma) + \mathcal{J}_{\nu^-}^\alpha \mathfrak{G}(\mu, \rho)\right]$$
$$+\frac{\Gamma(\beta+1)}{4(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\varsigma^+}^\beta \mathfrak{G}(\mu, \rho) + \mathcal{J}_{\rho^-}^\beta \mathfrak{G}(\nu, \varsigma) + \mathcal{J}_{\varsigma^+}^\beta \mathfrak{G}(\mu, \rho) + \mathcal{J}_{\rho^-}^\beta \mathfrak{G}(\nu, \varsigma)\right]$$
$$\geq_p \frac{\mathfrak{G}(\mu,\varsigma)+\mathfrak{G}(\nu,\varsigma)+\mathfrak{G}(\mu,\rho)+\mathfrak{G}(\nu,\rho)}{4}. \tag{24}$$

Proof. Let $\mathfrak{G}: [\mu, \nu] \to \mathbb{R}_I^+$ be a coordinated LR-convex I-V.F. Then, by hypothesis, we have

$$4\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right) \leq_p \mathfrak{G}(\tau\mu+(1-\tau)\nu,\tau\varsigma+(1-\tau)\rho)+\mathfrak{G}\big((1-\tau)\mu+\tau\nu,(1-\tau)\varsigma+\tau\rho\big).$$

By using Theorem 5, we have

$$4\mathfrak{G}_*\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$
$$\leq \mathfrak{G}_*(\tau\mu+(1-\tau)\nu,\tau\varsigma+(1-\tau)\rho)+\mathfrak{G}_*\big((1-\tau)\mu+\tau\nu,(1-\tau)\varsigma+\tau\rho\big),$$
$$4\mathfrak{G}^*\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$
$$\leq \mathfrak{G}^*(\tau\mu+(1-\tau)\nu,\tau\varsigma+(1-\tau)\rho)+\mathfrak{G}^*\big((1-\tau)\mu+\tau\nu,(1-\tau)\varsigma+\tau\rho\big).$$

By using Lemma 1, we have

$$\begin{aligned} 2\mathfrak{G}_*\left(x,\tfrac{\varsigma+\rho}{2}\right) &\leq \mathfrak{G}_*(x,\tau\varsigma+(1-\tau)\rho)+\mathfrak{G}_*(x,(1-\tau)\varsigma+\tau\rho),\\ 2\mathfrak{G}^*\left(x,\tfrac{\varsigma+\rho}{2}\right) &\leq \mathfrak{G}^*(x,\tau\varsigma+(1-\tau)\rho)+\mathfrak{G}^*(x,(1-\tau)\varsigma+\tau\rho), \end{aligned} \tag{25}$$

and

$$\begin{aligned} 2\mathfrak{G}_*\left(\tfrac{\mu+\nu}{2},y\right) &\leq \mathfrak{G}_*(\tau\mu+(1-\tau)\nu,y)+\mathfrak{G}_*\big((1-\tau)\mu+t\nu,y\big),\\ 2\mathfrak{G}^*\left(\tfrac{\mu+\nu}{2},y\right) &\leq \mathfrak{G}^*(\tau\mu+(1-\tau)\nu,y)+\mathfrak{G}^*\big((1-\tau)\mu+t\nu,y\big). \end{aligned} \tag{26}$$

From (25) and (26), we have

$$2\left[\mathfrak{G}_*\left(x,\frac{\varsigma+\rho}{2}\right),\mathfrak{G}^*\left(x,\frac{\varsigma+\rho}{2}\right)\right]$$
$$\leq_p [\mathfrak{G}_*(x,\tau\varsigma+(1-\tau)\rho),\mathfrak{G}^*(x,\tau\varsigma+(1-\tau)\rho)]$$
$$+[\mathfrak{G}_*(x,(1-\tau)\varsigma+\tau\rho),\mathfrak{G}^*(x,(1-\tau)\varsigma+\tau\rho)],$$

and

$$2\left[\mathfrak{G}_*\left(\frac{\mu+\nu}{2},y\right),\mathfrak{G}^*\left(\frac{\mu+\nu}{2},y\right)\right]$$
$$\leq_p [\mathfrak{G}_*(\tau\mu+(1-\tau)\nu,y),\mathfrak{G}^*(\tau\mu+(1-\tau)\nu,y)]$$
$$+[\mathfrak{G}_*(\tau\mu+(1-\tau)\nu,y),\mathfrak{G}^*(\tau\mu+(1-\tau)\nu,y)],$$

It follows that

$$\mathfrak{G}\left(x,\tfrac{\varsigma+\rho}{2}\right) \leq_p \mathfrak{G}(x,\tau\varsigma+(1-\tau)\rho)+\mathfrak{G}(x,(1-\tau)\varsigma+\tau\rho), \tag{27}$$

and

$$\mathfrak{G}\left(\tfrac{\mu+\nu}{2},y\right) \leq_p \mathfrak{G}(\tau\mu+(1-\tau)\nu,y)+\mathfrak{G}(\tau\mu+(1-\tau)\nu,y). \tag{28}$$

Since $\mathfrak{G}(x,.)$ and $\mathfrak{G}(.,y)$, both are coordinated *LR*-convex-IVFs, then from inequality (14), inequalities (27) and (28) we have

$$\mathfrak{G}_x\left(\frac{\varsigma+\rho}{2}\right) \leq_p \frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\varsigma^+}^{\beta}\ \mathfrak{G}_x(\rho) + \mathcal{J}_{\rho^-}^{\beta}\ \mathfrak{G}_x(\varsigma)\right] \leq_p \frac{\mathfrak{G}_x(\varsigma)+\mathfrak{G}_x(\rho)}{2}. \tag{29}$$

and

$$\mathfrak{G}_y\left(\frac{\mu+\nu}{2}\right) \leq_p \frac{\Gamma(\alpha+1)}{2(\nu-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\ \mathfrak{G}_y(\nu) + \mathcal{J}_{\nu^-}^{\alpha}\ \mathfrak{G}_y(\mu)\right] \leq_p \frac{\mathfrak{G}_y(\mu)+\mathfrak{G}_y(\nu)}{2} \tag{30}$$

Since $\mathfrak{G}_x(w) = \mathfrak{G}(x, w)$, the inequality (29) can be written as

$$\mathfrak{G}\left(x, \frac{\varsigma+\rho}{2}\right) \leq_p \frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\varsigma^+}^{\alpha}\ \mathfrak{G}(x,\rho) + \mathcal{J}_{\rho^-}^{\alpha}\ \mathfrak{G}(x,\varsigma)\right] \leq_p \frac{\mathfrak{G}(x,\varsigma)+\mathfrak{G}(x,\rho)}{2}. \tag{31}$$

That is

$$\mathfrak{G}\left(x, \frac{\varsigma+\rho}{2}\right) \leq_p \frac{\beta}{2(\rho-\varsigma)^\beta}\left[\int_\varsigma^\rho (\rho-s)^{\beta-1}\mathfrak{G}(x,s)ds + \int_\varsigma^\rho (s-\varsigma)^{\beta-1}\mathfrak{G}(x,s)ds\right] \leq_p \frac{\mathfrak{G}(x,\varsigma)+\mathfrak{G}(x,\rho)}{2}.$$

Multiplying double inequality (31) by $\frac{\alpha(\nu-x)^{\alpha-1}}{2(\nu-\mu)^\alpha}$and integrating with respect to x over $[\mu,\nu]$, we have

$$\frac{\alpha}{2(\nu-\mu)^\alpha}\int_\mu^\nu \mathfrak{G}\left(x, \frac{\varsigma+\rho}{2}\right)(\nu-x)^{\alpha-1}dx$$

$$\leq_p \int_\mu^\nu \int_\varsigma^\rho (\nu-x)^{\alpha-1}(\rho-s)^{\beta-1}\mathfrak{G}(x,s)\,dsdx + \int_\mu^\nu \int_\varsigma^\rho (\nu-x)^{\alpha-1}(s-\varsigma)^{\beta-1}\mathfrak{G}(x,s)\,dsdx$$

$$\leq_p \frac{\alpha}{4(\nu-\mu)^\alpha}\left[\int_\mu^\nu (\nu-x)^{\alpha-1}\mathfrak{G}(x,\varsigma)dx + \int_\mu^\nu (\nu-x)^{\alpha-1}\mathfrak{G}(x,\rho)dx\right]. \tag{32}$$

Again multiplying double inequality (31) by $\frac{\alpha(x-\mu)^{\alpha-1}}{2(\nu-\mu)^\alpha}$ and integrating with respect to x over $[\mu,\nu]$, we have

$$\frac{\alpha}{2(\nu-\mu)^\alpha}\int_\mu^\nu \mathfrak{G}\left(x, \frac{\varsigma+\rho}{2}\right)(\nu-x)^{\alpha-1}dx$$

$$\leq_p \frac{\alpha\beta}{4(\nu-\mu)^\alpha(\rho-\varsigma)^\beta}\int_\mu^\nu \int_\varsigma^\rho (x-\mu)^{\alpha-1}(\rho-s)^{\beta-1}\mathfrak{G}(x,s)\,dsdx$$

$$+\frac{\alpha\beta}{4(\nu-\mu)^\alpha(\rho-\varsigma)^\beta}\int_\mu^\nu \int_\varsigma^\rho (x-\mu)^{\alpha-1}(s-\varsigma)^{\beta-1}\mathfrak{G}(x,s)\,dsdx$$

$$\leq_p \frac{\alpha}{4(\nu-\mu)^\alpha}\left[\int_\mu^\nu (x-\mu)^{\alpha-1}\mathfrak{G}(x,\varsigma)dx + \int_\mu^\nu (x-\mu)^{\alpha-1}\mathfrak{G}(x,d)dx\right]. \tag{33}$$

From (32), we have

$$\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}\left(\nu,\frac{\varsigma+\rho}{2}\right)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}}\left[\mathcal{J}_{\mu^+,\varsigma^+}^{\alpha,\beta}\,\mathfrak{G}(\nu,\rho)+\mathcal{J}_{\nu^-,\varsigma^+}^{\alpha,\beta}\,\mathfrak{G}(\nu,\varsigma)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)}{4(\nu-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\varsigma)+\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\rho)\right]. \tag{34}$$

From (33), we have

$$\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left[\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}\left(\mu,\frac{\varsigma+\rho}{2}\right)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}}\left[\mathcal{J}_{\nu^-,\varsigma^+}^{\alpha,\beta}\,\mathfrak{G}(\mu,\rho)+\mathcal{J}_{\nu^-,\rho^-}^{\alpha,\beta}\,\mathfrak{G}(\mu,\varsigma)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)}{4(\nu-\mu)^{\alpha}}\left[\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\varsigma)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\rho)\right]. \tag{35}$$

Similarly, since $\mathfrak{G}_y(z)=\mathfrak{G}(z,y)$ then, from (34) and (35), (30) we have

$$\frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left[\mathcal{J}_{\varsigma^+}^{\beta}\,\mathfrak{G}\left(\frac{\mu+\nu}{2},\rho\right)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}}\left[\mathcal{J}_{\mu^+,\varsigma^+}^{\alpha,\beta}\,\mathfrak{G}(\nu,\rho)+\mathcal{J}_{\nu^-,\varsigma^+}^{\alpha,\beta}\,\mathfrak{G}(\mu,\rho)\right]$$
$$\leq_p \frac{\Gamma(\beta+1)}{4(\rho-\varsigma)^{\beta}}\left[\mathcal{J}_{\varsigma^+}^{\beta}\,\mathfrak{G}(\mu,\rho)+\mathcal{J}_{\varsigma^+}^{\beta}\,\mathfrak{G}(\nu,\rho)\right], \tag{36}$$

and

$$\frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\alpha}}\left[\mathcal{J}_{\rho^-}^{\beta}\,\mathfrak{G}\left(\frac{\mu+\nu}{2},\varsigma\right)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}}\left[\mathcal{J}_{\mu^+,\rho^-}^{\alpha,\beta}\,\mathfrak{G}(\nu,\varsigma)+\mathcal{J}_{\nu^-,\rho^-}^{\alpha,\beta}\,\mathfrak{G}(\mu,\varsigma)\right]$$
$$\leq_p \frac{\Gamma(\beta+1)}{4(\rho-\varsigma)^{\beta}}\left[\mathcal{J}_{\rho^-}^{\beta}\,\mathfrak{G}(\mu,\varsigma)+\mathcal{J}_{\rho^-}^{\beta}\,\mathfrak{G}(\nu,\varsigma)\right]. \tag{37}$$

After adding the inequalities (46), (35), (36) and (37), we will obtain as resultant second, third and fourth inequalities of (23).
Now, from left part of inequality (14), we have

$$\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)\leq_p \frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left[\mathcal{J}_{\varsigma^+}^{\beta}\,\mathfrak{G}\left(\frac{\mu+\nu}{2},\rho\right)+\mathcal{J}_{\rho^-}^{\beta}\,\mathfrak{G}\left(\frac{\mu+\nu}{2},\varsigma\right)\right], \tag{38}$$

and

$$\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right) \leq_p \frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}\left(\nu,\frac{\varsigma+\rho}{2}\right)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}\left(\mu,\frac{\varsigma+\rho}{2}\right)\right]. \tag{39}$$

Summing the inequalities (38) and (39), we obtain the following inequality:

$$\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$

$$\leq_p \frac{\Gamma(\alpha+1)}{4(\nu-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}\left(\nu,\frac{\varsigma+\rho}{2}\right)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}\left(\mu,\frac{\varsigma+\rho}{2}\right)\right]+\frac{\Gamma(\beta+1)}{4(\rho-\varsigma)^{\beta}}\left[\mathcal{J}_{\varsigma^+}^{\beta}\,\mathfrak{G}\left(\frac{\mu+\nu}{2},\rho\right)+\mathcal{J}_{\rho^-}^{\beta}\,\mathfrak{G}\left(\frac{\mu+\nu}{2},\varsigma\right)\right], \tag{40}$$

this is the first inequality of (23).
Now, from right part of inequality (14), we have

$$\frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left[\mathcal{J}_{\varsigma^+}^{\beta}\mathfrak{G}(\mu,\rho)+\mathcal{J}_{\rho^-}^{\beta}\,\mathfrak{G}(\mu,\varsigma)\right] \leq_p \frac{\mathfrak{G}(\mu,\varsigma)+\mathfrak{G}(\mu,\rho)}{2}, \tag{41}$$

$$\frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left[\mathcal{J}_{\varsigma^+}^{\beta}\mathfrak{G}(\nu,\rho)+\mathcal{J}_{\rho^-}^{\beta}\,\mathfrak{G}(\nu,\varsigma)\right] \leq_p \frac{\mathfrak{G}(\nu,\varsigma)+\mathfrak{G}(\nu,\rho),}{2} \tag{42}$$

$$\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\mathfrak{G}(\nu,\varsigma)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\varsigma)\right] \leq_p \frac{\mathfrak{G}(\mu,\varsigma)+\mathfrak{G}(\nu,\varsigma)}{2}, \tag{43}$$

$$\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\mathfrak{G}(\nu,\rho)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\rho)\right] \leq_p \frac{\mathfrak{G}(\mu,\rho)+\mathfrak{G}(\nu,\rho)}{2}. \tag{44}$$

Summing inequalities (41), (42), (43) and (44), and then taking multiplication of the resultant with $\frac{1}{4}$, we have

$$\frac{\Gamma(\alpha+1)}{8(\nu-\mu)^{\alpha}}\left[\mathcal{J}_{\mu^+}^{\alpha}\mathfrak{G}(\nu,\varsigma)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\varsigma)+\mathcal{J}_{\mu^+}^{\alpha}\mathfrak{G}(\nu,\rho)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\rho)\right]$$

$$+\frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left[\mathcal{J}_{\varsigma^+}^{\beta}\mathfrak{G}(\mu,\rho)+\mathcal{J}_{\rho^-}^{\beta}\,\mathfrak{G}(\mu,\varsigma)+\mathcal{J}_{\varsigma^+}^{\beta}\mathfrak{G}(\nu,\rho)+\mathcal{J}_{\rho^-}^{\beta}\,\mathfrak{G}(\nu,\varsigma)\right]$$

$$\leq_p \frac{\mathfrak{G}(\mu,\varsigma)+\mathfrak{G}(\mu,\rho)+\mathfrak{G}(\nu,\varsigma)+\mathfrak{G}(\nu,\rho)}{4}. \tag{45}$$

This is last inequality of (23) and the result has been proven.
Remark 3. If one to take $\alpha=1$ and $\beta=1$, then from (23), we achieve the coming inequality, see [38]:

$$\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$

$$\leq_p \frac{1}{2}\left[\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\mathfrak{G}\left(x,\frac{\varsigma+\rho}{2}\right)dx+\frac{1}{\rho-\varsigma}\int_{\varsigma}^{\rho}\mathfrak{G}\left(\frac{\mu+\nu}{2},y\right)dy\right] \leq_p \frac{1}{(\nu-\mu)(\rho-\varsigma)}\int_{\mu}^{\nu}\int_{\varsigma}^{\rho}\mathfrak{G}(x,y)dydx$$

$$\leq_p \frac{1}{4(\nu-\mu)}\left[\int_{\mu}^{\nu}\mathfrak{G}(x,\varsigma)\,dx+\int_{\mu}^{\nu}\mathfrak{G}(x,\rho)dx\right]+\frac{1}{4(\rho-\varsigma)}\left[\int_{\varsigma}^{\rho}\mathfrak{G}(\mu,y)dy+\int_{\varsigma}^{\rho}\mathfrak{G}(\nu,y)dy\right]$$

$$\leq_p \frac{\mathfrak{G}(\mu,\varsigma)+\mathfrak{G}(\nu,\varsigma)+\mathfrak{G}(\mu,\rho)+\mathfrak{G}(\nu,\rho)}{4}. \tag{46}$$

Let one takes $\mathfrak{G}_*(x,y)$ is an affine function and $\mathfrak{G}^*(x,y)$ is concave function. If $\mathfrak{G}_*(x,y) \neq \mathfrak{G}^*(x,y)$, then from Remark 2 and (24), we acquire the coming inequality, see [31]:

$$\mathfrak{G}\left(\frac{\mu+\nu}{2}, \frac{\varsigma+\rho}{2}\right) \supseteq \frac{\Gamma(\alpha+1)}{4(\nu-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\, \mathfrak{G}\left(\nu, \frac{\varsigma+\rho}{2}\right) + \mathcal{J}_{\nu^-}^{\alpha}\mathfrak{G}\left(\mu, \frac{\varsigma+\rho}{2}\right)\right] + \frac{\Gamma(\beta+1)}{4(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\varsigma^+}^{\beta}\, \mathfrak{G}\left(\frac{\mu+\nu}{2}, \rho\right) + \mathcal{J}_{\rho^-}^{\beta}\, \mathfrak{G}\left(\frac{\mu+\nu}{2}, \varsigma\right)\right]$$

$$\supseteq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^\alpha(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\mu^+,\varsigma^+}^{\alpha,\beta}\, \mathfrak{G}(\nu,\rho) + \mathcal{J}_{\mu^+,\rho^-}^{\alpha,\beta}\, \mathfrak{G}(\nu,\varsigma) + \mathcal{J}_{\nu^-,\varsigma^+}^{\alpha,\beta}\, \mathfrak{G}(\mu,\rho) + \mathcal{J}_{\nu^-,\rho^-}^{\alpha,\beta}\, \mathfrak{G}(\mu,\varsigma)\right]$$

$$\supseteq \frac{\Gamma(\alpha+1)}{8(\nu-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\, \mathfrak{G}(\nu,\varsigma)\mathfrak{G}\mathcal{J}_{\mu^+}^{\alpha}\, \mathfrak{G}(\nu,\rho) + \mathcal{J}_{\nu^-}^{\alpha}\, \mathfrak{G}(\mu,\varsigma) + \mathcal{J}_{\nu^-}^{\alpha}\, \mathfrak{G}(\mu,\rho)\right]$$

$$+\frac{\Gamma(\beta+1)}{4(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\varsigma^+}^{\beta}\, \mathfrak{G}(\mu,\rho)\widetilde{+}\mathcal{J}_{\rho^-}^{\beta}\, \mathfrak{G}(\nu,\varsigma) + \mathcal{J}_{\varsigma^+}^{\beta}\, \mathfrak{G}(\mu,\rho) + \mathcal{J}_{\rho^-}^{\beta}\, \mathfrak{G}(\nu,\varsigma)\right]$$

$$\supseteq \frac{\mathfrak{G}(\mu,\varsigma)+\mathfrak{G}(\nu,\varsigma)+\mathfrak{G}(\mu,\rho)+\mathfrak{G}(\nu,\rho)}{4}. \tag{47}$$

Let one takes $\alpha = 1$ and $\beta = 1$, $\mathfrak{G}_*(x,y)$ is an affine function and $\mathfrak{G}^*(x,y)$ is concave function. If $\mathfrak{G}_*(x,y) \neq \mathfrak{G}^*(x,y)$, then Remark 2 and from (24), we acquire the coming inequality, see [37]:

$$\mathfrak{G}\left(\frac{\mu+\nu}{2}, \frac{\varsigma+\rho}{2}\right)$$

$$\supseteq \frac{1}{2}\left[\frac{1}{\nu-\mu}\int_\mu^\nu \mathfrak{G}\left(x, \frac{\varsigma+\rho}{2}\right)dx + \frac{1}{\rho-\varsigma}\int_\varsigma^\rho \mathfrak{G}\left(\frac{\mu+\nu}{2}, y\right)dy\right] \supseteq \frac{1}{(\nu-\mu)(\rho-\varsigma)}\int_\mu^\nu\int_\varsigma^\rho \mathfrak{G}(x,y)dydx$$

$$\supseteq \frac{1}{4(\nu-\mu)}\left[\int_\mu^\nu \mathfrak{G}(x,\varsigma)\,dx + \int_\mu^\nu \mathfrak{G}(x,\rho)dx\right] + \frac{1}{4(\rho-\varsigma)}\left[\int_\varsigma^\rho \mathfrak{G}(\mu,y)dy + \int_\varsigma^\rho \mathfrak{G}(\nu,y)dy\right]$$

$$\supseteq \frac{\mathfrak{G}(\mu,\varsigma)+\mathfrak{G}(\nu,\varsigma)+\mathfrak{G}(\mu,\rho)+\mathfrak{G}(\nu,\rho)}{4}. \tag{48}$$

Example 2. We consider the I-V-Fs $\mathfrak{G}: [0,1]\times[0,1] \to \mathbb{R}_I^+$ defined by,

$$\mathfrak{G}(x) = [2,6](6+e^x)(6+e^y).$$

Since end point functions $\mathfrak{G}_*(x,y)$, $\mathfrak{G}^*(x,y)$ are convex functions on coordinate, then $\mathfrak{G}(x,y)$ is convex I-V-F on coordinate. Then for $\alpha = 1$ and $\beta = 1$, we have

$$\mathfrak{G}\left(\frac{\mu+\nu}{2}, \frac{\varsigma+\rho}{2}\right) = \left[2\left(5+e^{\frac{1}{2}}\right)^2, 6\left(6+e^{\frac{1}{2}}\right)^2\right],$$

$$\frac{\Gamma(\alpha+1)}{4(\nu-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\, \mathfrak{G}\left(\nu, \frac{\varsigma+\rho}{2}\right) + \mathcal{J}_{\nu^-}^{\alpha}\mathfrak{G}\left(\mu, \frac{\varsigma+\rho}{2}\right)\right] + \frac{\Gamma(\beta+1)}{4(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\varsigma^+}^{\beta}\, \mathfrak{G}\left(\frac{\mu+\nu}{2}, \rho\right) + \mathcal{J}_{\rho^-}^{\beta}\, \mathfrak{G}\left(\frac{\mu+\nu}{2}, \varsigma\right)\right]$$

$$= \left[4\left(6+e^{\frac{1}{2}}\right)(5+e), 12\left(6+e^{\frac{1}{2}}\right)(5+e)\right],$$

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\mu^+,\varsigma^+}\,\mathfrak{G}(\nu,\rho)+\mathcal{J}^{\alpha,\beta}_{\mu^+,\rho^-}\,\mathfrak{G}(\nu,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\nu^-,\varsigma^+}\,\mathfrak{G}(\mu,\rho)+\mathcal{J}^{\alpha,\beta}_{\nu^-,\rho^-}\,\mathfrak{G}(\mu,\varsigma)\right]$$

$$= [2(5+e)^2, 6(5+e)^2],$$

$$\frac{\Gamma(\alpha+1)}{8(\nu-\mu)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\mu^+}\,\mathfrak{G}(\nu,\varsigma)\mathfrak{G}\mathcal{J}^{\alpha}_{\mu^+}\,\mathfrak{G}(\nu,\rho)+\mathcal{J}^{\alpha}_{\nu^-}\,\mathfrak{G}(\mu,\varsigma)+\mathcal{J}^{\alpha}_{\nu^-}\,\mathfrak{G}(\mu,\rho)\right]$$

$$+\frac{\Gamma(\beta+1)}{4(\rho-\varsigma)^{\beta}}\left[\mathcal{J}^{\beta}_{\varsigma^+}\,\mathfrak{G}(\mu,\rho)\tilde{+}\mathcal{J}^{\beta}_{\rho^-}\,\mathfrak{G}(\nu,\varsigma)+\mathcal{J}^{\beta}_{\varsigma^+}\,\mathfrak{G}(\mu,\rho)+\mathcal{J}^{\beta}_{\rho^-}\,\mathfrak{G}(\nu,\varsigma)\right]$$

$$= [(5+e)(13+e), 3(5+e)(13+e)]$$

$$\frac{\mathfrak{G}(\mu,\varsigma)+\mathfrak{G}(\nu,\varsigma)+\mathfrak{G}(\mu,\rho)+\mathfrak{G}(\nu,\rho)}{4} = \left[\frac{(6+e)(20+e)+49}{2}, \frac{6\big((6+e)(20+e)+49\big)}{2}\right].$$

That is

$$\left[2\left(5+e^{\frac{1}{2}}\right)^2, 6\left(6+e^{\frac{1}{2}}\right)^2\right] \leq_p \left[4\left(6+e^{\frac{1}{2}}\right)(5+e), 12\left(6+e^{\frac{1}{2}}\right)(5+e)\right]$$

$$\leq_p [2(5+e)^2, 6(5+e)^2]$$

$$\leq_p [(5+e)(13+e), 3(5+e)(13+e)]$$

$$\leq_p \left[\frac{(6+e)(20+e)+49}{2}, 3\big((6+e)(20+e)+49\big)\right].$$

Hence, Theorem 3.1 has been verified

Next both results obtain Hermite-Hadamard type inequalities for the product of two coordinate LR-convex I-V.Fs

Theorem 7. Let $\mathfrak{G}, \mathfrak{S}: \Delta \to \mathbb{R}_I^+$ be a coordinate LR-convex I-V.Fs on Δ such that $\mathfrak{G}(x,y) = [\mathfrak{G}_*(x,y), \mathfrak{G}^*(x,y)]$ and $\mathfrak{S}(x,y) = [\mathfrak{S}_*(x,y), \mathfrak{S}^*(x,y)]$ for all $(x,y) \in \Delta$. If $\mathfrak{G} \times \mathfrak{S} \in \mathfrak{TD}_{\Delta}$, then following inequalities holds:

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\mu^+,\varsigma^+}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}^{\alpha,\beta}_{\mu^+,\rho^-}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\nu^-,\varsigma^+}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)\right.$$
$$\left.+\mathcal{J}^{\alpha,\beta}_{\nu^-,\rho^-}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\right]$$

$$\leq_p \left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)K(\mu,\nu,\varsigma,\rho)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)L(\mu,\nu,\varsigma,\rho)$$

$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{M}(\mu,\nu,\varsigma,\rho)+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{(\alpha+1)(\alpha+2)}\mathcal{N}(\mu,\nu,\varsigma,\rho). \quad (49)$$

If $\mathfrak{G}$ and $\mathfrak{S}$ both are coordinate LR-concave I-V.Fs on Δ, then above inequality can be written as

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\mu^{+},\varsigma^{+}}\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}^{\alpha,\beta}_{\mu^{+},\rho^{-}}\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\nu^{-},\varsigma^{+}}\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)\right.$$
$$\left.+\mathcal{J}^{\alpha,\beta}_{\nu^{-},\rho^{-}}\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\right]$$

$$\geq_{p}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)K(\mu,\nu,\varsigma,\rho)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)L(\mu,\nu,\varsigma,\rho)$$

$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{M}(\mu,\nu,\varsigma,\rho)+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{(\alpha+1)(\alpha+2)}\mathcal{N}(\mu,\nu,\varsigma,\rho). \quad (50)$$

Where

$$K(\mu,\nu,\varsigma,\rho)=\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)+\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho),$$
$$L(\mu,\nu,\varsigma,\rho)=\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)\,\tilde{\mp}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)+\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\nu,\rho),$$
$$\mathcal{M}(\mu,\nu,\varsigma,\rho)=\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\rho)+\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\rho)+\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\varsigma)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\varsigma),$$
$$\mathcal{N}(\mu,\nu,\varsigma,\rho)=\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\nu,\rho)+\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\mu,\rho)+\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\nu,\varsigma)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\mu,\varsigma).$$

and $K(\mu,\nu,\varsigma,\rho)$, $\tilde{L}(\mu,\nu,\varsigma,\rho)$, $\mathcal{M}(\mu,\nu,\varsigma,\rho)$ and $\mathcal{N}(\mu,\nu,\varsigma,\rho)$ are defined as follows:

$$K(\mu,\nu,\varsigma,\rho)=[K_{*}(\mu,\nu,\varsigma,\rho),K^{*}(\mu,\nu,\varsigma,\rho)],$$
$$L(\mu,\nu,\varsigma,\rho)=[L_{*}(\mu,\nu,\varsigma,\rho),L^{*}(\mu,\nu,\varsigma,\rho)],$$
$$\mathcal{M}(\mu,\nu,\varsigma,\rho)=[\mathcal{M}_{*}(\mu,\nu,\varsigma,\rho),\mathcal{M}^{*}(\mu,\nu,\varsigma,\rho)],$$
$$\mathcal{N}(\mu,\nu,\varsigma,\rho)=[\mathcal{N}_{*}(\mu,\nu,\varsigma,\rho),\mathcal{N}^{*}(\mu,\nu,\varsigma,\rho)].$$

Proof. Let $\mathfrak{G}$ and $\mathfrak{S}$ both are coordinated *LR*-convex I-V.Fs on $[\mu,\nu]\times[\varsigma,\rho]$. Then

$$\mathfrak{G}(\tau\mu+(1-\tau)\nu,s\varsigma+(1-s)\rho)$$
$$\leq_{p}\tau s\mathfrak{G}(\mu,\varsigma)+\tau(1-s)\mathfrak{G}(\mu,\rho)+(1-\tau)s\mathfrak{G}(\nu,\varsigma)+(1-\tau)(1-s)\mathfrak{G}(\nu,\rho),$$

and

$$\mathfrak{S}(\tau\mu+(1-\tau)\nu,s\varsigma+(1-s)\rho)$$
$$\leq_{p}\tau s\mathfrak{S}(\mu,\varsigma)+\tau(1-s)\mathfrak{S}(\mu,\rho)+(1-\tau)s\mathfrak{S}(\nu,\varsigma)+(1-\tau)(1-s)\mathfrak{S}(\nu,\rho).$$

Since $\mathfrak{G}$ and $\mathfrak{S}$ both are coordinated *LR*-convex I-V.Fs, then by Lemma 1, there exist

$$\mathfrak{G}_{x}:[\varsigma,\rho]\to\mathbb{R}_{I}^{+},\ \mathfrak{G}_{x}(y)=\mathfrak{G}(x,y),\qquad \mathfrak{S}_{x}:[\varsigma,\rho]\to\mathbb{R}_{I}^{+},\ \mathfrak{S}_{x}(y)=\mathfrak{S}(x,y),$$

Since $\mathfrak{G}_{x}$, and $\mathfrak{S}_{x}$ are I-V.Fs, then by inequality (15), we have

$$\frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left[\mathcal{J}^{\beta}_{\varsigma^{+}}\mathfrak{G}_{x}(\rho)\times\mathfrak{S}_{x}(\rho)+\mathcal{J}^{\beta}_{\rho^{-}}\mathfrak{G}_{x}(\varsigma)\times\mathfrak{S}_{x}(\varsigma)\right]$$
$$\leq_{p}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{G}_{x}(\varsigma)\times\mathfrak{S}_{x}(\varsigma)+\mathfrak{G}_{x}(\rho)\times\mathfrak{S}_{x}(\rho)\right)$$

$$+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{G}_x(\varsigma)\times\mathfrak{S}_x(\rho)+\ \mathfrak{G}_x(\rho)\times\mathfrak{S}_x(\varsigma)\right).$$

That is

$$\frac{\beta}{2(\rho-\varsigma)^\beta}\left[\int_\varsigma^\rho(\rho-y)^{\beta-1}\mathfrak{G}(x,y)\times\mathfrak{S}(x,y)\,\rho y+\int_\varsigma^\rho(y-\varsigma)^{\beta-1}\mathfrak{G}(x,y)\times\mathfrak{S}(x,y)\,\rho y\right]$$

$$\leq_p\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{G}(x,\varsigma)\times\mathfrak{S}(x,\varsigma)+\mathfrak{G}(x,\rho)\times\mathfrak{S}(x,\rho)\right)$$

$$+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{G}(x,\varsigma)\times\mathfrak{S}(x,\rho)+\mathfrak{G}(x,\rho)\times\mathfrak{S}(x,\varsigma)\right). \quad (51)$$

Multiplying double inequality (51) by $\frac{\alpha(\nu-x)^{\alpha-1}}{2(\nu-\mu)^\alpha}$ and integrating with respect to x over $[\mu,\nu]$, we get

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^\alpha(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\mu^+,\varsigma^+}^{\alpha,\beta}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}_{\mu^+,\rho^-}^{\alpha,\beta}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)\right]$$

$$\leq_p\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^\alpha}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)\right)$$

$$+\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^\alpha}\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\rho)+\ \mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\varsigma)\right). \quad (52)$$

Again, multiplying double inequality (51) by $\frac{\alpha(x-\mu)^{\alpha-1}}{2(\nu-\mu)^\alpha}$ and integrating with respect to x over $[\mu,\nu]$, we gain

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^\alpha(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\nu^-,\varsigma^+}^{\alpha,\beta}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}_{\nu^-,\rho^-}^{\alpha,\beta}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\right]$$

$$\leq_p\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^\alpha}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)\right)$$

$$+\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^\alpha}\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\rho)+\ \mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\varsigma)\right). \quad (53)$$

Summing (52) and (53), we have

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^\alpha(\rho-\varsigma)^\beta}\begin{bmatrix}\mathcal{J}_{\mu^+,\varsigma^+}^{\alpha,\beta}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}_{\mu^+,\rho^-}^{\alpha,\beta}\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)\\+\mathcal{J}_{\nu^-,\varsigma^+}^{\alpha,\beta}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}_{\nu^-,\rho^-}^{\alpha,\beta}\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\end{bmatrix}$$

$$\leq_p\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^\alpha}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\right)$$

$$+\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^\alpha}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)\right)$$

$$+\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\rho)\right)$$

$$+\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\varsigma)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\varsigma)\right). \tag{54}$$

Now, again with the help of integral inequality (15) for first two integrals on the right-hand side of (54), we have the following relation

$$\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left(\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\right)$$

$$\leq_p\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)+\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)\right)$$

$$+\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\right). \tag{55}$$

$$\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left(\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)\right)$$

$$\leq_p\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)\right)$$

$$+\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\mu,\rho)\right). \tag{56}$$

$$\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left(\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\rho)\right)$$

$$\leq_p\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\rho)+\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\rho)\right)$$

$$+\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\nu,\rho)+\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\mu,\rho)\right). \tag{57}$$

And

$$\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left(\mathcal{J}_{\mu^+}^{\alpha}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\varsigma)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\varsigma)\right)$$

$$\leq_p\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\varsigma)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\varsigma)\right)$$

$$+\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\nu,\varsigma)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\mu,\varsigma)\right). \tag{58}$$

From (55)–(58), inequality (54) we have

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}}\left[\begin{array}{c}\mathcal{J}_{\mu^+,\varsigma^+}^{\alpha,\beta}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}_{\mu^+,\rho^-}^{\alpha,\beta}\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)\\+\mathcal{J}_{\nu^-,\varsigma^+}^{\alpha,\beta}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}_{\nu^-,\rho^-}^{\alpha,\beta}\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\end{array}\right]$$

$$\leq_p \left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)K(\mu,\nu,\varsigma,\rho) + \frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)L(\mu,\nu,\varsigma,\rho)$$

$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{M}(\mu,\nu,\varsigma,\rho)+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{(\alpha+1)(\alpha+2)}\mathcal{N}(\mu,\nu,\varsigma,\rho).$$

Hence, the result has been proven.

Remark 4. If one to take $\alpha = 1$ and $\beta = 1$, then from (49), we achieve the coming inequality, see [38]:

$$\frac{1}{(\nu-\mu)(\rho-\varsigma)}\int_{\mu}^{\nu}\int_{\varsigma}^{\rho}\mathfrak{G}(x,y)\times\mathfrak{S}(x,y)dydx$$

$$\leq_p \frac{1}{9}K(\mu,\nu,\varsigma,\rho)+\frac{1}{18}[L(\mu,\nu,\varsigma,\rho)+\mathcal{M}(\mu,\nu,\varsigma,\rho)]+\frac{1}{36}\mathcal{N}(\mu,\nu,\varsigma,\rho). \tag{59}$$

Let one takes $\mathfrak{G}_*(x,y)$ is an affine function and $\mathfrak{G}^*(x,y)$ is concave function. If $\mathfrak{G}_*(x,y) \neq \mathfrak{G}^*(x,y)$, then by Remark 2 and (50), we acquire the coming inequality, see [36]:

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}}\begin{bmatrix}\mathcal{J}^{\alpha,\beta}_{\mu^+,\varsigma^+}\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}^{\alpha,\beta}_{\mu^+,\rho^-}\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)\\+\mathcal{J}^{\alpha,\beta}_{\nu^-,\varsigma^+}\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}^{\alpha,\beta}_{\nu^-,\rho^-}\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\end{bmatrix}$$

$$\supseteq \left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)K(\mu,\nu,\varsigma,\rho)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)L(\mu,\nu,\varsigma,\rho)$$

$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{M}(\mu,\nu,\varsigma,\rho)+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{(\alpha+1)(\alpha+2)}\mathcal{N}(\mu,\nu,\varsigma,\rho). \tag{60}$$

Let one takes $\mathfrak{G}_*(x,y)$ is an affine function and $\mathfrak{G}^*(x,y)$ is concave function. If $\mathfrak{G}_*(x,y) \neq \mathfrak{G}^*(x,y)$, then by Remark 2 and (50), we acquire the coming inequality, see [37]:

$$\frac{1}{(\nu-\mu)(\rho-\varsigma)}\int_{\mu}^{\nu}\int_{\varsigma}^{\rho}\mathfrak{G}(x,y)\times\mathfrak{S}(x,y)dydx$$

$$\supseteq \frac{1}{9}K(\mu,\nu,\varsigma,\rho)+\frac{1}{18}[L(\mu,\nu,\varsigma,\rho)+\mathcal{M}(\mu,\nu,\varsigma,\rho)]+\frac{1}{36}\mathcal{N}(\mu,\nu,\varsigma,\rho). \tag{61}$$

If $\mathfrak{G}_*(x,y) = \mathfrak{G}^*(x,y)$ and $\mathfrak{S}_*(x,y) = \mathfrak{S}^*(x,y)$, then from (49), we acquire the coming inequality, see [39]:

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}}\begin{bmatrix}\mathcal{J}^{\alpha,\beta}_{\mu^+,\varsigma^+}\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}^{\alpha,\beta}_{\mu^+,\rho^-}\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)\\+\mathcal{J}^{\alpha,\beta}_{\nu^-,\varsigma^+}\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}^{\alpha,\beta}_{\nu^-,\rho^-}\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\end{bmatrix}$$

$$\leq \left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)K(\mu,\nu,\varsigma,\rho)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)L(\mu,\nu,\varsigma,\rho)$$

$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{M}(\mu,\nu,\varsigma,\rho)+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{(\alpha+1)(\alpha+2)}\mathcal{N}(\mu,\nu,\varsigma,\rho). \quad (62)$$

Theorem 8. Let $\mathfrak{G},\mathfrak{S}:\Delta\to\mathbb{R}_I^+$ be a coordinate *LR*-convex I-V.F on Δ such that $\mathfrak{G}(x,y)=[\mathfrak{G}_*(x,y),\mathfrak{G}^*(x,y)]$ and $\mathfrak{S}(x,y)=[\mathfrak{S}_*(x,y),\mathfrak{S}^*(x,y)]$ for all $(x,y)\in\Delta$. If $\mathfrak{G}\times\mathfrak{S}\in\mathfrak{TO}_\Delta$, then following inequalities holds:

$$4\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$

$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^\alpha(\rho-\varsigma)^\beta}\begin{bmatrix}\mathcal{J}^{\alpha,\beta}_{\mu^+,\varsigma^+}\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}^{\alpha,\beta}_{\mu^+,\rho^-}\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)\\+\mathcal{J}^{\alpha,\beta}_{\nu^-,\varsigma^+}\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}^{\alpha,\beta}_{\nu^-,\rho^-}\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\end{bmatrix}$$

$$+\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]K(\mu,\nu,\varsigma,\rho)$$

$$+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]L(\mu,\nu,\varsigma,\rho)$$

$$+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{M}(\mu,\nu,\varsigma,\rho)$$

$$+\left[\frac{1}{4}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{N}(\mu,\nu,\varsigma,\rho). \quad (63)$$

If $\mathfrak{G}$ and $\mathfrak{S}$ both are coordinate *LR*-concave I-V.Fs on Δ, then above inequality can be written as

$$4\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$

$$\geq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^\alpha(\rho-\varsigma)^\beta}\begin{bmatrix}\mathcal{J}^{\alpha,\beta}_{\mu^+,\varsigma^+}\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}^{\alpha,\beta}_{\mu^+,\rho^-}\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)\\+\mathcal{J}^{\alpha,\beta}_{\nu^-,\varsigma^+}\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}^{\alpha,\beta}_{\nu^-,\rho^-}\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\end{bmatrix}$$

$$+\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]K(\mu,\nu,\varsigma,\rho)$$

$$+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]L(\mu,\nu,\varsigma,\rho)$$

$$+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{M}(\mu,\nu,\varsigma,\rho)$$

$$+\left[\frac{1}{4}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{N}(\mu,\nu,\varsigma,\rho). \quad (64)$$

Where $K(\mu,\nu,\varsigma,\rho)$, $L(\mu,\nu,\varsigma,\rho)$, $\mathcal{M}(\mu,\nu,\varsigma,\rho)$ and $\mathcal{N}(\mu,\nu,\varsigma,\rho)$ are given in Theorem 7.
Proof. Since $\mathfrak{G},\mathfrak{S}:\Delta\to\mathbb{R}_I^+$ be two *LR*-convex I-V.Fs, then from inequality (16), we have

$$2\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$

$$\leq_p \frac{\alpha}{2(\nu-\mu)^\alpha}\begin{bmatrix}\int_\mu^\nu(\nu-x)^{\alpha-1}\mathfrak{G}\left(x,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(x,\frac{\varsigma+\rho}{2}\right)dx\\+\int_\mu^\nu(x-\mu)^{\alpha-1}\mathfrak{G}\left(x,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(x,\frac{\varsigma+\rho}{2}\right)dx\end{bmatrix}$$

$$+\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{G}\left(\mu,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\mu,\frac{\varsigma+\rho}{2}\right)+\mathfrak{G}\left(\nu,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\nu,\frac{\varsigma+\rho}{2}\right)\right)$$
$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{G}\left(\mu,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\nu,\frac{\varsigma+\rho}{2}\right)+\mathfrak{G}\left(\nu,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\mu,\frac{\varsigma+\rho}{2}\right)\right), \tag{65}$$

and

$$2\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$
$$\leq_p \frac{\beta}{2(\rho-\varsigma)^\beta}\left[\begin{array}{l}\int_\varsigma^\rho(\rho-y)^{\beta-1}\mathfrak{G}\left(\frac{\mu+\nu}{2},y\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},y\right)dy\\+\int_\varsigma^\rho(y-\varsigma)^{\beta-1}\mathfrak{G}\left(\frac{\mu+\nu}{2},y\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},y\right)dy\end{array}\right]$$
$$+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{G}\left(\frac{\mu+\nu}{2},\varsigma\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\varsigma\right)+\mathfrak{G}\left(\frac{\mu+\nu}{2},\rho\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\rho\right)\right)$$
$$+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{G}\left(\frac{\mu+\nu}{2},\varsigma\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\rho\right)+\mathfrak{G}\left(\frac{\mu+\nu}{2},\rho\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\varsigma\right)\right), \tag{66}$$

Adding (73) and (74), and then taking the multiplication of the resultant one by 2, we obtain

$$8\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$
$$\leq_p \frac{\alpha}{2(\nu-\mu)^\alpha}\left[\begin{array}{l}\int_\mu^\nu 2(\nu-x)^{\alpha-1}\mathfrak{G}\left(x,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(x,\frac{\varsigma+\rho}{2}\right)dx\\+\int_\mu^\nu 2(x-\mu)^{\alpha-1}\mathfrak{G}\left(x,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(x,\frac{\varsigma+\rho}{2}\right)dx\end{array}\right]$$
$$+\frac{\beta}{2(\rho-\varsigma)^\beta}\left[\begin{array}{l}\int_\varsigma^\rho 2(\rho-y)^{\beta-1}\mathfrak{G}\left(\frac{\mu+\nu}{2},y\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},y\right)dy\\+\int_\varsigma^\rho 2(y-\varsigma)^{\beta-1}\mathfrak{G}\left(\frac{\mu+\nu}{2},y\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},y\right)dy\end{array}\right]$$
$$+\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(2\mathfrak{G}\left(\mu,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\mu,\frac{\varsigma+\rho}{2}\right)+2\mathfrak{G}\left(\nu,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\nu,\frac{\varsigma+\rho}{2}\right)\right)$$
$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(2\mathfrak{G}\left(\mu,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\nu,\frac{\varsigma+\rho}{2}\right)+2\mathfrak{G}\left(\nu,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\mu,\frac{\varsigma+\rho}{2}\right)\right)$$
$$+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(2\mathfrak{G}\left(\frac{\mu+\nu}{2},\varsigma\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\varsigma\right)+2\mathfrak{G}\left(\frac{\mu+\nu}{2},\rho\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\rho\right)\right)$$
$$+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(2\mathfrak{G}\left(\frac{\mu+\nu}{2},\varsigma\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\rho\right)+2\mathfrak{G}\left(\frac{\mu+\nu}{2},\rho\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\varsigma\right)\right). \tag{67}$$

Again, with the help of integral inequality (16) and Lemma 1 for each integral on the right-hand side of (67), we have

$$\frac{\alpha}{2(\nu-\mu)^\alpha}\int_\mu^\nu 2(\nu-x)^{\alpha-1}\mathfrak{G}\left(x,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(x,\frac{\varsigma+\rho}{2}\right)dx$$
$$\leq_p \frac{\alpha\beta}{4(\nu-\mu)^\alpha(\rho-\varsigma)^\beta}\left[\begin{array}{l}\int_\mu^\nu\int_\varsigma^\rho(\nu-x)^{\alpha-1}(\rho-y)^{\beta-1}\mathfrak{G}(x,y)\,dydx\\+\int_\mu^\nu\int_\varsigma^\rho(\nu-x)^{\alpha-1}(y-\varsigma)^{\beta-1}\mathfrak{G}(x,y)\,dydx\end{array}\right]$$
$$+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{2(\nu-\mu)^\alpha}\int_\mu^\nu(\nu-x)^{\alpha-1}\big(\mathfrak{G}(x,\varsigma)\times\mathfrak{S}(x,\varsigma)+\mathfrak{G}(x,\rho)\times\mathfrak{S}(x,\rho)\big)dx$$
$$+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\frac{\alpha}{2(\nu-\mu)^\alpha}\int_\mu^\nu(\nu-x)^{\alpha-1}\big(\mathfrak{G}(x,\varsigma)\times\mathfrak{S}(x,\rho)+\mathfrak{G}(x,\rho)\times\mathfrak{S}(x,\varsigma)\big)\,dx,$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}} \left[\mathcal{J}^{\alpha,\beta}_{\mu^+,\varsigma^+} \mathfrak{G}(\nu,\rho) \times \mathfrak{S}(\nu,\rho) + \mathcal{J}^{\alpha,\beta}_{\mu^+,\rho^-} \mathfrak{G}(\nu,\varsigma) \times \mathfrak{S}(\nu,\varsigma) \right]$$
$$+ \frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}} \left(\frac{\beta}{(\beta+1)(\beta+2)} \right) \left(\mathcal{J}^{\alpha}_{\mu^+} \mathfrak{G}(\nu,\varsigma) \times \mathfrak{S}(\nu,\varsigma) + \mathcal{J}^{\alpha}_{\mu^+} \mathfrak{G}(\nu,\rho) \times \mathfrak{S}(\nu,\rho) \right)$$
$$+ \frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}} \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) \left(\mathcal{J}^{\alpha}_{\mu^+} \mathfrak{G}(\nu,\varsigma) \times \mathfrak{S}(\nu,\rho) + \mathcal{J}^{\alpha}_{\mu^+} \mathfrak{G}(\nu,\rho) \times \mathfrak{S}(\nu,\varsigma) \right). \tag{68}$$

$$\frac{\alpha}{2(\nu-\mu)^{\alpha}} \int_{\mu}^{\nu} 2(x-\mu)^{\alpha-1} \mathfrak{G}\left(x, \frac{\varsigma+\rho}{2}\right) \times \mathfrak{S}\left(x, \frac{\varsigma+\rho}{2}\right) dx$$
$$\leq_p \frac{\alpha\beta}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}} \begin{bmatrix} \int_{\mu}^{\nu} \int_{\varsigma}^{\rho} (x-\mu)^{\alpha-1}(\rho-y)^{\beta-1} \mathfrak{G}(x,y)\, dy dx \\ + \int_{\mu}^{\nu} \int_{\varsigma}^{\rho} (x-\mu)^{\alpha-1}(y-\varsigma)^{\beta-1} \mathfrak{G}(x,y)\, dy dx \end{bmatrix}$$
$$+ \frac{\beta}{(\beta+1)(\beta+2)} \frac{\alpha}{2(\nu-\mu)^{\alpha}} \int_{\mu}^{\nu} (x-\mu)^{\alpha-1} \left(\mathfrak{G}(x,\varsigma) \times \mathfrak{S}(x,\varsigma) + \mathfrak{G}(x,\rho) \times \mathfrak{S}(x,\rho) \right) dx$$
$$+ \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) \frac{\alpha}{2(\nu-\mu)^{\alpha}} \int_{\mu}^{\nu} (x-\mu)^{\alpha-1} \left(\mathfrak{G}(x,\varsigma) \times \mathfrak{S}(x,\rho) + \mathfrak{G}(x,\rho) \times \mathfrak{S}(x,\varsigma) \right) dx,$$
$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}} \left[\mathcal{J}^{\alpha,\beta}_{\nu^-,\varsigma^+} \mathfrak{G}(\mu,\rho) \times \mathfrak{S}(\mu,\rho) + \mathcal{J}^{\alpha,\beta}_{\nu^-,\rho^-} \mathfrak{G}(\mu,\varsigma) \times \mathfrak{S}(\mu,\varsigma) \right]$$
$$+ \frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}} \left(\frac{\beta}{(\beta+1)(\beta+2)} \right) \left(\mathcal{J}^{\alpha}_{\nu^-} \mathfrak{G}(\mu,\varsigma) \times \mathfrak{S}(\mu,\varsigma) + \mathcal{J}^{\alpha}_{\nu^-} \mathfrak{G}(\mu,\rho) \times \mathfrak{S}(\mu,\rho) \right)$$
$$+ \frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}} \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) \left(\mathcal{J}^{\alpha}_{\nu^-} \mathfrak{G}(\mu,\varsigma) \times \mathfrak{S}(\mu,\rho) + \mathcal{J}^{\alpha}_{\nu^-} \mathfrak{G}(\mu,\rho) \times \mathfrak{S}(\mu,\varsigma) \right). \tag{69}$$

$$\frac{\beta}{2(\rho-\varsigma)^{\beta}} \left[\int_{\varsigma}^{\rho} 2(\rho-y)^{\beta-1} \mathfrak{G}\left(\frac{\mu+\nu}{2}, y\right) \times \mathfrak{S}\left(\frac{\mu+\nu}{2}, y\right) dy \right]$$
$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}} \left[\mathcal{J}^{\alpha,\beta}_{\mu^+,\varsigma^+} \mathfrak{G}(\nu,\rho) \times \mathfrak{S}(\nu,\rho) + \mathcal{J}^{\alpha,\beta}_{\nu^-,\varsigma^+} \mathfrak{G}(\mu,\rho) \times \mathfrak{S}(\mu,\rho) \right]$$
$$+ \frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \left(\mathcal{J}^{\beta}_{\varsigma^+} \mathfrak{G}(\mu,\rho) \times \mathfrak{S}(\mu,\rho) + \mathcal{J}^{\beta}_{\varsigma^+} \mathfrak{G}(\nu,\rho) \times \mathfrak{S}(\nu,\rho) \right)$$
$$+ \frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \left(\mathcal{J}^{\beta}_{\varsigma^+} \mathfrak{G}(\mu,\rho) \times \mathfrak{S}(\nu,\rho) + \mathcal{J}^{\beta}_{\varsigma^+} \mathfrak{G}(\nu,\rho) \times \mathfrak{S}(\nu,\rho) \right). \tag{70}$$

$$\frac{\beta}{2(\rho-\varsigma)^{\beta}} \left[\int_{\varsigma}^{\rho} 2(y-\varsigma)^{\beta-1} \mathfrak{G}\left(\frac{\mu+\nu}{2}, y\right) \times \mathfrak{S}\left(\frac{\mu+\nu}{2}, y\right) dy \right]$$
$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}} \left[\mathcal{J}^{\alpha,\beta}_{\mu^+,\rho^-} \mathfrak{G}(\nu,\varsigma) \times \mathfrak{S}(\nu,\varsigma) + \mathcal{J}^{\alpha,\beta}_{\nu^-,\rho^-} \mathfrak{G}(\nu,\varsigma) \times \mathfrak{S}(\nu,\varsigma) \right]$$
$$+ \frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \left(\mathcal{J}^{\beta}_{\rho^-} \mathfrak{G}(\mu,\varsigma) \times \mathfrak{S}(\mu,\varsigma) + \mathcal{J}^{\beta}_{\rho^-} \mathfrak{G}(\nu,\varsigma) \times \mathfrak{S}(\nu,\varsigma) \right)$$
$$+ \frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \left(\mathcal{J}^{\beta}_{\rho^-} \mathfrak{G}(\mu,\varsigma) \times \mathfrak{S}(\nu,\varsigma) + \mathcal{J}^{\beta}_{\rho^-} \mathfrak{G}(\nu,\varsigma) \times \mathfrak{S}(\nu,\varsigma) \right). \tag{71}$$

And

$$2\mathfrak{G}\left(\frac{\mu+\nu}{2}, \varsigma\right) \times \mathfrak{S}\left(\frac{\mu+\nu}{2}, \varsigma\right)$$

$$\leq_p \frac{\Gamma(\alpha+1)}{2(\nu-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\ \mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathcal{J}_{\nu^-}^{\alpha}\ \mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\right]$$
$$+\frac{\alpha}{(\alpha+1)(\alpha+2)}\big(\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)+\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)\big)$$
$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\big(\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\big), \tag{72}$$

$$2\mathfrak{G}\left(\frac{\mu+\nu}{2},\rho\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\rho\right)$$
$$\leq_p \frac{\Gamma(\alpha+1)}{2(\nu-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\ \mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}_{\nu^-}^{\alpha}\ \mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)\right]$$
$$+\frac{\alpha}{(\alpha+1)(\alpha+2)}\big(\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)\big)$$
$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\big(\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\mu,\rho)\big), \tag{73}$$

$$2\mathfrak{G}\left(\frac{\mu+\nu}{2},\varsigma\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\rho\right)$$
$$\leq_p \frac{\Gamma(\alpha+1)}{2(\nu-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\ \mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}_{\nu^-}^{\alpha}\ \mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\rho)\right]$$
$$+\frac{\alpha}{(\alpha+1)(\alpha+2)}\big(\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\rho)+\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\rho)\big)$$
$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\big(\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\nu,\rho)+\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\mu,\rho)\big), \tag{74}$$

$$2\mathfrak{G}\left(\frac{\mu+\nu}{2},\rho\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\varsigma\right)$$
$$\leq_p \frac{\Gamma(\alpha+1)}{2(\nu-\mu)^\alpha}\left[\mathcal{J}_{\mu^+}^{\alpha}\ \mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\varsigma)+\mathcal{J}_{\nu^-}^{\alpha}\ \mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\varsigma)\right]$$
$$+\frac{\alpha}{(\alpha+1)(\alpha+2)}\big(\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\varsigma)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\varsigma)\big)$$
$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\big(\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\nu,\varsigma)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\mu,\varsigma)\big), \tag{75}$$

$$2\mathfrak{G}\left(\mu,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\mu,\frac{\varsigma+\rho}{2}\right)$$
$$\leq_p \frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\varsigma^+}^{\beta}\ \mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}_{\rho^-}^{\beta}\ \mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\varsigma)\right]$$
$$+\frac{\beta}{(\beta+1)(\beta+2)}\big(\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)+\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)\big)$$
$$+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\big(\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\rho)+\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\varsigma)\big), \tag{76}$$

$$2\mathfrak{G}\left(\nu,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}_{\varphi}\left(\nu,\frac{\varsigma+\rho}{2}\right)$$
$$\leq_p \frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^\beta}\left[\mathcal{J}_{\varsigma^+}^{\beta}\ \mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}_{\rho^-}^{\beta}\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\varsigma)\right]$$

$$+\frac{\beta}{(\beta+1)(\beta+2)}\big(\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)\big)$$
$$+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\big(\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\rho)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\varsigma)\big), \quad (77)$$

$$2\mathfrak{G}\left(\mu,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\nu,\frac{\varsigma+\rho}{2}\right)$$
$$\leq_p \frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left[\mathcal{J}^{\beta}_{\varsigma^+}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}^{\beta}_{\rho^-}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\nu,\varsigma)\right]$$
$$+\frac{\beta}{(\beta+1)(\beta+2)}\big(\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\nu,\rho)\big)$$
$$+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\big(\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\nu,\rho)+\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\nu,\varsigma)\big), \quad (78)$$

and

$$2\mathfrak{G}\left(\nu,\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\mu,\frac{\varsigma+\rho}{2}\right)$$
$$\leq_p \frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left[\mathcal{J}^{\beta}_{\varsigma^+}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}^{\beta}_{\rho^-}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\mu,\varsigma)\right]$$
$$+\frac{\beta}{(\beta+1)(\beta+2)}\big(\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\mu,\rho)\big)$$
$$+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\big(\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\mu,\rho)+\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\mu,\varsigma)\big), \quad (79)$$

From inequalities (68) to (79), inequality (67) we have

$$8\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$
$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}}\begin{bmatrix}\mathcal{J}^{\alpha,\beta}_{\mu^+,\varsigma^+}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}^{\alpha,\beta}_{\mu^+,\rho^-}\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)\\+\mathcal{J}^{\alpha,\beta}_{\nu^-,\varsigma^+}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}^{\alpha,\beta}_{\nu^-,\rho^-}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\end{bmatrix}$$
$$+\left(\frac{2\alpha}{(\alpha+1)(\alpha+2)}\right)\begin{bmatrix}\frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left(\mathcal{J}^{\beta}_{\varsigma^+}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}^{\beta}_{\varsigma^+}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)\right)\\+\frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left(\mathcal{J}^{\beta}_{\rho^-}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)+\mathcal{J}^{\beta}_{\rho^-}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)\right)\end{bmatrix}$$
$$+2\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\begin{bmatrix}\frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left(\mathcal{J}^{\beta}_{\varsigma^+}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}^{\beta}_{\varsigma^+}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\mu,\rho)\right)\\+\frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left(\mathcal{J}^{\beta}_{\rho^-}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathcal{J}^{\beta}_{\rho^-}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\right)\end{bmatrix}$$
$$+2\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\begin{bmatrix}\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left(\mathcal{J}^{\alpha}_{\mu^+}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathcal{J}^{\alpha}_{\mu^+}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)\right)\\+\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left(\mathcal{J}^{\alpha}_{\nu^-}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)+\mathcal{J}^{\alpha}_{\nu^-}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)\right)\end{bmatrix}$$

$$+2\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left[\begin{array}{l}\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left(\mathcal{J}_{\mu^{+}}^{\alpha}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}_{\mu^{+}}^{\alpha}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\varsigma)\right)\\+\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left(\mathcal{J}_{\nu^{-}}^{\alpha}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}_{\nu^{-}}^{\alpha}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\varsigma)\right)\end{array}\right]$$

$$+\frac{2\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}K(\mu,\nu,\varsigma,\rho)++\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{2\beta}{(\beta+1)(\beta+2)}L(\mu,\nu,\varsigma,\rho)$$

$$+\frac{2\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{M}(\mu,\nu,\varsigma,\rho)$$

$$+2\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{N}(\mu,\nu,\varsigma,\rho). \tag{80}$$

Again, with the help of integral inequality (15) and Lemma 1, for each integral on the right-hand side of (80), we have

$$\frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left(\mathcal{J}_{\varsigma^{+}}^{\beta}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}_{\varsigma^{+}}^{\beta}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)\right)$$
$$+\frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left(\mathcal{J}_{\rho^{-}}^{\beta}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)+\mathcal{J}_{\rho^{-}}^{\beta}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)\right)$$
$$\leq_{p}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)K(\mu,\nu,\varsigma,\rho)+\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{M}(\mu,\nu,\varsigma,\rho). \tag{81}$$

$$\frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left(\mathcal{J}_{\varsigma^{+}}^{\beta}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}_{\varsigma^{+}}^{\beta}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\mu,\rho)\right)$$
$$+\frac{\Gamma(\beta+1)}{2(\rho-\varsigma)^{\beta}}\left(\mathcal{J}_{\rho^{-}}^{\beta}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathcal{J}_{\rho^{-}}^{\beta}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\right)$$
$$\leq_{p}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)L(\mu,\nu,\varsigma,\rho)+\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{N}(\mu,\nu,\varsigma,\rho). \tag{82}$$

$$\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left(\mathcal{J}_{\mu^{+}}^{\alpha}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)+\mathcal{J}_{\mu^{+}}^{\alpha}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)\right)$$
$$+\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left(\mathcal{J}_{\nu^{-}}^{\alpha}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)+\mathcal{J}_{\nu^{-}}^{\alpha}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)\right)$$
$$\leq_{p}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)K(\mu,\nu,\varsigma,\rho)+\frac{\alpha}{(\alpha+1)(\alpha+2)}L(\mu,\nu,\varsigma,\rho). \tag{83}$$

$$\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left(\mathcal{J}_{\nu^{-}}^{\alpha}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}_{\nu^{-}}^{\alpha}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\varsigma)\right)$$
$$+\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left(\mathcal{J}_{\nu^{-}}^{\alpha}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}_{\nu^{-}}^{\alpha}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\varsigma)\right)$$
$$\leq_{p}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\mathcal{M}(\mu,\nu,\varsigma,\rho)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\mathcal{N}(\mu,\nu,\varsigma,\rho). \tag{84}$$

From (77) to (84), (80) we have

$$4\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$

$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}} \begin{bmatrix} \mathcal{J}^{\alpha,\beta}_{\mu^+,\varsigma^+}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}^{\alpha,\beta}_{\mu^+,\rho^-}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma) \\ +\mathcal{J}^{\alpha,\beta}_{\nu^-,\varsigma^+}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}^{\alpha,\beta}_{\nu^-,\rho^-}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma) \end{bmatrix}$$
$$+\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]K(\mu,\nu,\varsigma,\rho)$$
$$+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]L(\mu,\nu,\varsigma,\rho)$$
$$+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{M}(\mu,\nu,\varsigma,\rho)$$
$$+\left[\frac{1}{4}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{N}(\mu,\nu,\varsigma,\rho). \tag{85}$$

This concludes the proof of Theorem 8 result has been proven.

Remark 5. If we take $\alpha=1$ and $\beta=1$, then from (63), we achieve the coming inequality, see [38]:

$$4\,\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$
$$\leq_p \frac{1}{(\nu-\mu)(\rho-\varsigma)}\int_{\mu}^{\nu}\int_{\varsigma}^{\rho}\mathfrak{G}(x,y)\times\mathfrak{S}(x,y)dydx+\frac{5}{36}K(\mu,\nu,\varsigma,\rho)$$
$$+\frac{7}{36}[L(\mu,\nu,\varsigma,\rho)+\mathcal{M}(\mu,\nu,\varsigma,\rho)]+\frac{2}{9}\mathcal{N}(\mu,\nu,\varsigma,\rho). \tag{86}$$

Let one takes $\mathfrak{G}_*(x,y)$ is an affine function and $\mathfrak{G}^*(x,y)$ is convex function. If $\mathfrak{G}_*(x,y)\neq\mathfrak{G}^*(x,y)$, then from Remark 2 and (64), we acquire the coming inequality, see [37]:

$$4\,\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$
$$\supseteq \frac{1}{(\nu-\mu)(\rho-\varsigma)}\int_{\mu}^{\nu}\int_{\varsigma}^{\rho}\mathfrak{G}(x,y)\times\mathfrak{S}(x,y)dydx+\frac{5}{36}K(\mu,\nu,\varsigma,\rho)$$
$$+\frac{7}{36}[L(\mu,\nu,\varsigma,\rho)+\mathcal{M}(\mu,\nu,\varsigma,\rho)]+\frac{2}{9}\mathcal{N}(\mu,\nu,\varsigma,\rho). \tag{87}$$

Let one takes $\mathfrak{G}_*(x,y)$ is an affine function and $\mathfrak{G}^*(x,y)$ is convex function. If $\mathfrak{G}_*(x,y)\neq\mathfrak{G}^*(x,y)$, then from Remark 2 and (64) we acquire the coming inequality, see [36]:

$$4\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$
$$\supseteq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}} \begin{bmatrix} \mathcal{J}^{\alpha,\beta}_{\mu^+,\varsigma^+}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}^{\alpha,\beta}_{\mu^+,\rho^-}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma) \\ +\mathcal{J}^{\alpha,\beta}_{\nu^-,\varsigma^+}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}^{\alpha,\beta}_{\nu^-,\rho^-}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma) \end{bmatrix}$$
$$+\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]K(\mu,\nu,\varsigma,\rho)$$
$$+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]L(\mu,\nu,\varsigma,\rho)$$
$$+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{M}(\mu,\nu,\varsigma,\rho)$$

$$+\left[\frac{1}{4}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{N}(\mu,\nu,\varsigma,\rho). \tag{88}$$

If we take $\mathfrak{G}_*(x,y)=\mathfrak{G}^*(x,y)$ and $\mathfrak{S}_*(x,y)=\mathfrak{S}^*(x,y)$, then from (63), we acquire the coming inequality, see [39]:

$$4\mathfrak{G}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)\times\mathfrak{S}\left(\frac{\mu+\nu}{2},\frac{\varsigma+\rho}{2}\right)$$

$$\leq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\nu-\mu)^{\alpha}(\rho-\varsigma)^{\beta}}\begin{bmatrix}\mathcal{J}^{\alpha,\beta}_{\mu^+,\varsigma^+}\,\mathfrak{G}(\nu,\rho)\times\mathfrak{S}(\nu,\rho)+\mathcal{J}^{\alpha,\beta}_{\mu^+,\rho^-}\,\mathfrak{G}(\nu,\varsigma)\times\mathfrak{S}(\nu,\varsigma)\\+\mathcal{J}^{\alpha,\beta}_{\nu^-,\varsigma^+}\,\mathfrak{G}(\mu,\rho)\times\mathfrak{S}(\mu,\rho)+\mathcal{J}^{\alpha,\beta}_{\nu^-,\rho^-}\,\mathfrak{G}(\mu,\varsigma)\times\mathfrak{S}(\mu,\varsigma)\end{bmatrix}$$

$$+\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]K(\mu,\nu,\varsigma,\rho)$$

$$+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]L(\mu,\nu,\varsigma,\rho)$$

$$+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{M}(\mu,\nu,\varsigma,\rho)$$

$$+\left[\frac{1}{4}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{N}(\mu,\nu,\varsigma,\rho). \tag{89}$$

4. Conclusions

In this study, with the help of coordinated *LR*-convexity for interval-valued functions, several novel Hermite-Hadamard type inequalities are presented. It is also demonstrated that the conclusions reached in this study represent a possible extension of previously published equivalent results. Similar inequalities may be discovered in the future using various forms of convexities. This is a novel and intriguing topic, and future study will be able to find equivalent inequalities for various types of convexity and coordinated m-convexity by using different fractional integral operators.

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Conflict of interest

The authors declare that they have no competing interests.

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