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Research article

Rough topological structure based on reflexivity with some applications

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Abstract: Recently, topological structures have emerged as one of the most popular rough sets (RS) research topics. It can be stated that it is a fundamental and significant subject in the theory of RS. This study introduces a debate about the structure of rough topological space based on the reflexive relation. To create the rough topological space, we use the representation of RS. We also look at the relationships between approximation operators, closure operators, and interior operators. Also, the relationship between topological space in the universe that is not limited or restricted to be ended, and RS induced by reflexive relations is investigated. Furthermore, we define the relationships between the set of all topologies that satisfy the requirement of compactness C_2 and the set of all reflexive relations. Finally, we present a medical application that addresses the issue of dengue fever. The proposed structures are used to determine the impact factors for identifying dengue fever.

Keywords: rough sets; reflexive relation; topological structure; interior operator; closure operator; generalized approximation space (GAS); minimal neighborhood; medical applications
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1. Introduction

Pawlak [34, 35] introduced the notion of the theory of RS. The equivalence relation is the establishment of its object identification. Where, the upper and lower approximation operations are the heartier center concepts of RS, which the operations are caused by an equivalent relation on a field. They may additionally stay seen as like closure and interior operators of the topology caused by an equivalence relation on a field. The theory of RS based on an equivalence relation has been extended to general binary relations [10, 13–15, 19, 20, 31, 39, 44, 47, 51], tolerance relations [1], dominance relations [29, 44], similarity relations [6, 8], topological structures [9, 16, 18, 23, 32, 36, 47], soft rough sets [11, 21, 22, 24] and coverings [27, 33, 43, 49, 50].

There exist near connections between RS and topology. Topological forms concerning RS were examined by many authors [6,8] and coverings [12,30,33,36]. Pawlak RS was extended to generalized rough sets by Lin [31] using neighborhood systems and topology to base a model for granular computing. Furthermore, the links between rough sets and digital topology were studied by Abo Tabl [5]. The "hit-or-miss" topology on RS was defined and the mathematical morphology within the general paradigm of soft computing was approximated by Polkowski [37,38]. Kondo introduced some properties of topology and rough sets for a kind of relation [28]. Qin et al. [42] and Zhang et al. [47] presented a further examination of the pair of relation-based operators of approximation studied in [28]. Pomykala studied some properties of topology for two pairs of covering RS approximation operators [39]. Furthermore, the connections between topology over multiset and rough multiset theory were also investigated (see [2–4, 7, 25, 26]).

The main contributions and innovations of the article are to introduce an integrate about the structure of rough topological space based on the reflexive relation and RS. First, we use the representation of RS to produce the rough topological space and thus the relationships among approximation operators, closure operators, and interior operators are investigated. Additionally, we explain the relationships between the set of all topologies that satisfy the requirement of compactness C_2 and the set of all reflexive relations. Therefore, the present paper is organized as follows:

In Section 2, we explore a review of some essential ideas of RS and topological space. Moreover, we use the representation of RS to construct the rough topological space. Also, we investigate the relationships among approximation operators, closure operators, and interior operators in Section 3. Furthermore, in Section 4, the relationship between topological space on the universe which is not limited restricted to be ended and RS induced by reflexive relations is investigated. Moreover, we explain the relationships between the set of all topologies which satisfy the requirement of compactness C_2 and the set of all reflexive relations. At last, in Section 5 we present a medical application of dengue fever for illustrating the suggested techniques.

2. Preliminaries

Some essential concepts of Pawlak RS and topological space are introduced in this section.

The class τ of subset of U is called topology on U if the conditions below are satisfied :

(1) $\phi, U \in \tau$.

(2) $Q_1 \cap Q_2 \in \tau \ \forall \ Q_1, Q_2 \in \tau$.

(3) $\cup_i(Q_i) \in \tau \ \forall \ Q_i \in \tau$, $i \in I$, (I is an index set).

The pair (U, τ) is called a topological space, every element belonging to τ is called open, and their complement is called closed [45].

Moreover, in this space

$$\kappa(\mathbf{Q}) = \cap\{C \subseteq U | \mathbf{Q} \subseteq C, \text{C is closed}\}$$

called τ-closure of $\mathbf{Q}$,

$$\mu(\mathbf{Q}) = \cup\{O \subseteq U | O \subseteq \mathbf{Q}, \text{O is open}\}$$

called τ-interior of $\mathbf{Q}$.

Definition 2.1. [45] In the topological space (U, τ) the closure (resp. interior) operator $\kappa : U \to \tau^c$ (resp.

$\mu : U \to \tau$) satisfies the Kuratowski axioms if the following conditions hold for every $Q_1, Q_2 \in U$:
(i) $\kappa(\phi) = \phi$ (resp. $\mu(U) = U$),
(ii) $\kappa(Q_1 \cup Q_2) = \kappa(Q_1) \cup \kappa(Q_2)$ (resp. $\mu(Q_1 \cap Q_2) = \mu(Q_1) \cap \mu(Q_2)$),
(iii) $Q_1 \subseteq \kappa(Q_1)$ (resp. $\mu(Q_1) \subseteq Q_1$),
(iv) $\kappa(\kappa(Q_1)) = \kappa(Q_1)$ (resp. $\mu(Q_1) = \mu(\mu(Q_1))$).
Definition 2.2. [35] Assume that R is an equivalence relation on a non-empty set U. We can use the equivalence class $[a]_R$ of $a \in U$ to define the lower and upper approximations of a subset Q of U as follows:

$$\underline{R}(Q) = \{a \in U : [a]_R \subseteq Q\}$$
$$\overline{R}(Q) = \{a \in U : [a]_R \cap Q \neq \phi\}$$

Also, the boundary region of the set Q is $BND(Q) = \overline{R}(Q) - \underline{R}(Q)$.

3. Novel generalization of rough sets on reflexive relations

Suppose that U is a universal set and R is a binary reflexive relation on U, we call (U, R) as the (GAS). Also, $R_s(a) = \{b \in U : (a, b) \in R\}$ is called the right set of a and $R_p(a) = \{b \in U : (b, a) \in R\}$ is called the left set of a for all $a \in U$.
Definition 3.1. [8] For a universal set U and a reflexive relation R on U, the intersection of all right set containing a is called the minimal right neighborhood of a and denoted by $\langle a\rangle R$, i.e.,

$$\langle a\rangle R = \bigcap_{a\in R_s(b)} (R_s(b))$$

Also, the intersection of all left set containing a is called the minimal left neighborhood of a and denoted by $R\langle a\rangle$, i.e.,

$$R\langle a\rangle = \bigcap_{a\in R_p(b)} (R_p(b))$$

Definition 3.2. [6] For a universal set U and a reflexive relation R on U, we define two neighborhoods of a subset Q as follows:
The first is the minimal right neighborhood of Q

$$\langle Q\rangle R = \bigcup_{a\in Q} \langle a\rangle R$$

and the second is the minimal left neighborhood of Q

$$R\langle Q\rangle = \bigcup_{a\in Q} R\langle a\rangle$$

for any $Q \subseteq U$.
Definition 3.3. [8] For a universal set U and a reflexive relation R on U, the lower and upper approximations of Q was defined as follows:

$$\underline{R}(Q) = \{a \in U : \langle a\rangle R \subseteq Q\}$$

$$\overline{R}(Q) = \{a \in U : \langle a \rangle R \cap Q \neq \phi\}$$

For any $Q \subseteq U$. The accuracy of the approximations is given by:

$$\mho(Q) = \frac{|\underline{R}(Q)|}{|\overline{R}(Q)|}$$

Theorem 3.4. [8] For a (GAS) (U, R), the conditions below are equivalent:
(1) the operator of lower approximation $\underline{\underline{R}} : P(U) \to P(U)$ is the operator of interior;
(2) the operator of upper approximation $\overline{\overline{R}} : P(U) \to P(U)$ is the operator of closure.

We can select a representative element from every $\langle a \rangle R$ for all $a \in U$ and it is not repeat, where R is a reflexive relation on U.

Note that: S_0 is the set of representative element of the minimal neighborhood of each element in the universal set U.

Example 3.5. Assume that $U = \{h_1, h_2, h_3, h_4\}$ is a universal set and R is a reflexive relation on U such that $R = \{(h_1, h_1), (h_2, h_2), (h_3, h_3), (h_4, h_4), (h_1, h_2), (h_2, h_4), (h_3, h_4)\}$, then $R_s(h_1) = \{h_1, h_2\}$, $R_s(h_2) = \{h_2, h_4\}$, $R_s(h_3) = \{h_3, h_4\}$, $R_s(h_4) = \{h_4\}$, and $\langle h_1 \rangle R = \{h_1, h_2\}$, $\langle h_2 \rangle R = \{h_2\}$, $\langle h_3 \rangle R = \{h_3, h_4\}$, $\langle h_4 \rangle R = \{h_4\}$. Then $S_0 = \{h_1, h_2, h_3, h_4\}$.

Definition 3.6. For a (GAS) (U, R) and $Q \subseteq U$.
(1) The set Q is called right-composed set if $Q = \langle Q \rangle R$.
(2) The set Q is called left-composed set if $Q = R\langle Q \rangle$.
(3) $\tau_R = \{Q \subseteq U : \langle Q \rangle R = Q\}$ is the family of all right-composed sets in U.
(4) $\tau_L = \{Q \subseteq U : R\langle Q \rangle = Q\}$ is the family of all left-composed sets in U.

Proposition 3.7. For a (GAS) (U, R), the class τ_R is a topology on U.

Proof. Firstly, since R is reflexive, then $\langle \phi \rangle R = \phi$ and $\langle U \rangle R = U$, hence $\phi, U \in \tau_R$.
Secondly, if $Q_1, Q_2 \in \tau_R$, then $\langle Q_1 \rangle R = Q_1$, $\langle Q_2 \rangle R = Q_2$. From Proposition 3.1 in [6] we have $\langle Q_1 \cap Q_2 \rangle R \subseteq \langle Q_1 \rangle R \cap \langle Q_2 \rangle R$, i.e., $\langle Q_1 \cap Q_2 \rangle R \subseteq Q_1 \cap Q_2$. Also, let $a \in \langle Q_1 \rangle R \cap \langle Q_2 \rangle R$, then $a \in \langle Q_1 \rangle R = Q_1$ and $a \in \langle Q_2 \rangle R = Q_2$, hence, $a \in Q_1 \cap Q_2$, i.e., $Q_1 \cap Q_2 \subseteq \langle Q_1 \cap Q_2 \rangle R$. That is $Q_1 \cap Q_2 = \langle Q_1 \cap Q_2 \rangle R$, thus $Q_1 \cap Q_2 \in \tau_R$. Thirdly, assume that $Q_i \in \tau_R$ for all $i \in I$, then $\langle Q_i \rangle R = Q_i$. We have $\cup_i Q_i = \cup_i \langle Q_i \rangle = \langle \cup_i Q_i \rangle$, i.e., $\cup_i Q_i \in \tau_R$. Thus, τ_R is a topology on U.

Theorem 3.8. For a (GAS) (U, R), the topology τ_R is the complement of the topology τ_L.

We define the class of minimal right neighborhood of all subsets of U as follows $\tau_* = \{\langle A \rangle R : A \subseteq U\}$.

Lemma 3.9. For a (GAS) (U, R). $\{\langle Q \rangle R : Q \subseteq U\} = \{Q \subseteq U : \langle Q \rangle R = Q\}$.

Proof. Suppose that $P \in \{\langle Q \rangle R : Q \subseteq U\}$, then $\exists\ Q \subseteq U$ such that $P = \langle Q \rangle R = \cup_{a \in Q}(\langle a \rangle R)$. Since R is reflexive, then $Q \subseteq P$. Also, $P \subseteq Q$, if $P \nsubseteq Q$, then $\exists\ b \in P$ and $b \notin Q$, hence there is an element $a \in Q$ such that $b \in \langle a \rangle R$, thus not necessary $P = Q$. But, there exist a set $Q \cup \{b\}$, such that $P = \langle Q \cup \{b\} \rangle R = Q \cup \{b\}$, and so $P \subseteq \{Q \subseteq U : \langle Q \rangle R = Q\}$.
Conversely, if $P \subseteq \{Q \subseteq U : \langle Q \rangle R = Q\}$, then $\langle P \rangle R = P$, hence $P \in \{\langle Q \rangle R : Q \subseteq U\}$.
Thus, $\{\langle Q \rangle R : Q \subseteq U\} = \{Q \subseteq U : \langle Q \rangle R = Q\}$.

We can use Lemma 3.9 to prove the following theorem.

Theorem 3.10. For a (GAS) (U, R), the class τ_* is a topology on U and $\tau_* = \tau_R$.

We introduce the following definition from [46], For any $Q \in P(U)$,

$$\begin{aligned} Q_g &= \underline{R}(Q) \cup (BND(Q) \cap S_0) \\ &= \underline{R}(Q) \cup ((\overline{R}(Q) - \underline{R}(Q)) \cap S_0) \\ &= \overline{R}(Q) \cap (\underline{R}(Q) \cup S_0). \end{aligned}$$

Also, we define $S = \{a \in U : |\langle a \rangle R| = 1\}$.

Theorem 3.11. For a (GAS) (U, R), the right-composed set P and the left-composed set Q. The pair (P, Q) is (RS) if and only if $P \subseteq Q$ and $(Q - P) \cap S = \phi$.

Proof. Assume that (P, Q) is a generalized (RS), Then there exists $D \subseteq U$ such that $\underline{R}(D) = P$, $\overline{R}(D) = Q$, hence $P = \underline{R}(D) \subseteq D \subseteq \overline{R}(D) = Q$.

If $s \in S$, then $\langle s \rangle R = \{s\}$.

If $s \in Q = \overline{R}(D) = \{s : \langle s \rangle R \cap D \neq \phi\}$, then $\{s\} = \langle s \rangle R \subseteq D$, hence $s \in \underline{R}(D) = P$. That is, $(Q - P) \cap S = \phi$.

Conversely, let $P \subseteq Q$, $(Q - P) \cap S = \phi$ and $D = P \cup ((Q - P) \cap S_0)$. Firstly, we want to prove that $\underline{R}(D) = P$. Since P is right-composed set, thus $\underline{R}(D) = \underline{R}(P \cup ((Q - P) \cap S_0)) \supseteq \underline{R}(P) = P$.

Now, we want to show that $\underline{R}(D) \subseteq P$. If $s \in \underline{R}(D)$, then $\langle s \rangle R \subseteq D$.

Now, there are two cases.

Case 1. If $|\langle s \rangle R| = 1$, then $s \in S$. Since $(Q - P) \cap S = \phi$, hence $s \notin Q - P$, so, $s \notin (Q - P) \cap S_0$, thus, $s \in P$.

Case 2. If $|\langle s \rangle R| > 1$, since $(Q - P) \cap S_0$ contains only one element of $\langle s \rangle R$. If s is not representative element, hence, $s \in P$. If s is representative element, provided $s \in (Q - P) \cap S_0$, hence, at least exist $d \in P$ and $d \in \langle s \rangle R$, i.e., $\langle d \rangle R \subseteq \langle s \rangle R$, thus, $| \langle d \rangle R | > | \langle a \rangle R |$, which it is a contradiction to that s is a representative element, that is $s \in P$.

Hence, by Case 1 and Case 2, it follows $\underline{R}(D) \subseteq P$, and so $\underline{R}(D) = P$.

Secondly, we want to show that $\overline{R}(D) = Q$. Since B is a left- composed set, so, $\overline{R}(P \cup ((Q - P) \cap S_0)) = \overline{R}(Q \cap (P \cup S_0)) \subseteq \overline{R}(Q) = Q$.

Now, we want to show that $Q \subseteq \overline{R}(D)$. Let $s \in Q$, then there are two cases.

Case 1. If $s \in P$, by $s \in D$, we have $s \in \overline{R}(D)$.

Case 2. If $s \notin P$ we have $s \in (Q - P)$, since $(Q - P) \cap S = \phi$, so $s \notin S$, i.e., there is $d \in P$ such that $d \in \langle s \rangle R$, hence $\langle d \rangle R \subseteq \langle s \rangle R$ and $\langle d \rangle R \subseteq D$, thus $\langle s \rangle R \cap D \neq \phi$, we have $s \in \overline{R}(D)$.

According by Case 1 and Case 2, it follows $Q \subseteq \overline{R}(D)$, and so $Q = \overline{R}(D)$.

Thus (P, Q) is a generalized (RS).

Definition 3.12. For a (GAS) (U, R), someone can define the binary relation " $\approx$ " on $P(U)$ as follows: $Q \approx P$ if and only if $\underline{R}(Q) = \underline{R}(P)$, $\overline{R}(Q) = \overline{R}(P)$. Note that $Q \approx P$ is an equivalence relation on $P(U)$. Also, $[Q]_{\approx} = \{P \in P(U); Q \approx P\}$ is an equivalence class of Q. Moreover, the set of all equivalence classes denote by $P(U)/ \approx = \{[Q]_{\approx}; Q \in P(U)\}$.

Theorem 3.13. For a (GAS) (U, R), we have $Q_g \in [Q]_{\approx}$.

Proof. Let $Q \in P(U)$, then we get two definable sets $\underline{R}(Q)$ and $\overline{R}(Q)$, so $\underline{R}(Q) \subseteq \overline{R}(Q)$ and $(\overline{R}(Q) - \underline{R}(Q)) \cap S = \phi$, this proof is similar to the proof of Theorem 3.11. Hence $\underline{R}(Q_g) = \underline{R}(Q)$, $\overline{R}(Q_g) = \overline{R}(Q)$. That is $Q_g \in [Q]_{\approx}$.

Theorem 3.14. For a (GAS) (U, R) and $M = \{Q_g : Q \in P(U)\}$, we have

(1)For any $Q_g, P_g \in M$, then $Q_g \cup P_g \in M$,

(2)For any $Q_g, P_g \in M$, then $Q_g \cap P_g \in M$.
Proof.
(1) Firstly, since $Q \cup P \in P(U)$, then $(Q \cup P)_g \in M$, hence
$\overline{R}(Q \cup P)_g = \overline{R}(Q \cup P) = \overline{R}(Q) \cup \overline{R}(P)$, also
$\overline{R}(Q_g \cup P_g) = \overline{R}(Q_g) \cup \overline{R}(P_g) = \overline{R}(Q) \cup \overline{R}(P) = \overline{R}(Q \cup P)$.
Secondly, $\underline{R}(Q \cup P)_g = \underline{R}(Q \cup P)$, also
$\underline{R}(Q_g \cup P_g) \supseteq \underline{R}(Q_g) \cup \underline{R}(P_g) = \underline{R}(Q) \cup \underline{R}(P) \subseteq \underline{R}(Q \cup P)$.
Hence, $\overline{R}(Q \cup P)_g = \overline{R}(Q \cup P)$ and $\underline{R}(Q_g \cup P_g)\underline{R}(Q \cup P)$.
That is $Q_g \cup P_g \in M$.
(2) The proof is similar to (1).
Theorem 3.15. For a (GAS) (U, R). If Q is an element of τ_L, then $Q_g = Q$.
Proof. Since, $Q \in \tau_L$, then $Q = R\langle Q\rangle$ and so $\overline{R}(Q) = Q$, and so, $Q_g = \underline{R}(Q) \cup ((\overline{R}(Q) - \underline{R}(Q)) \cap S_0) = \underline{R}(Q) \cup ((Q - \underline{R}(Q)) \cap S_0) \subseteq Q$.
If $Q \nsubseteq Q_g$, then there exists $a \in Q_g$ and $a \notin Q$, hence $a \in (\underline{R}(Q) \cup ((Q - \underline{R}(Q)) \cap S_0))$, that is $a \in \underline{R}(Q) \subseteq Q$ or $a \in (Q - \underline{R}(Q)) \subseteq Q$, which it is a contradiction, then $Q \subseteq Q_g$ thus, $Q_g = Q$.
Theorem 3.16. For a (GAS) (U, R), the class M is a topological space on U.
Proof. Firstly, from Theorem 3.14, we have M is closed under intersection and union, also $U, \phi \in M$. Secondly, since U is finite set, then M is a topology on U.
Theorem 3.17. For a (GAS) (U, R), the topology τ_L is less than the topology M on U.
Proof. The proof immediately from Theorem 3.15.
Theorem 3.18. For any $Q \subseteq U$ we have, $\kappa_R(Q) = Q \cup (\overline{R}(Q) - S_0)$, and $\mu_R(Q) = \underline{R}(Q) \cup (Q \cap S_0)$. Where $\kappa_R(\mu_R)$ is a closure (interior) operator of the topological space M.
Proof. Firstly, since $\mu_R(Q) = \cup\{P_g \in M : P_g \subseteq Q\}$, then
$\mu_R(Q) = \cup\{\underline{R}(P) \cup ((\overline{R}(P) - \underline{R}(P)) \cap S_0) : (\underline{R}(P) \cup ((\overline{R}(P) - \underline{R}(P)) \cap S_0)) \subseteq Q\}$, then $\mu_R(Q) = \underline{R}(Q) \cup (Q \cap S_0)$.
Secondly,

$$\begin{aligned}
\kappa_R(Q) &= (\mu_R(Q^c)^c \\
&= \{\underline{R}(Q^c) \cup (Q^c \cap S_0)\}^c \\
&= (\underline{R}(Q^c))^c \cup (Q^c \cap S_0)^c \\
&= \overline{R}(Q) \cap (Q \cup (S_0)^c) \\
&= \{\overline{R}(Q) \cap Q\} \cup \{\overline{R}(Q) \cap ((S_0)^c)\} \\
&= Q \cup (\overline{R}(Q) - S_0).
\end{aligned}$$

By Theorem 3.18, any element contains in the topology M and its complement have the form $\underline{R}(Q) \cup (Q \cap S_0)$ and $Q \cup (\overline{R}(Q) - S_0)$ respectively; also $\underline{R}(Q)$ is the union of elements in τ_R, thus we get the following corollary.
Corollary 3.19. The family $\{\{a\}_g : a \in U\}$ is a base of the topology M.
Example 3.20. Assume that $U = \{h_1, h_2, h_3, h_4, h_5\}$ is a universal set and R is a reflexive relation on U such that
$R = \{(h_1, h_1), (h_2, h_2), (h_3, h_3), (h_4, h_4), (h_5, h_5), (h_1, h_4), (h_1, h_5), (h_2, h_1), (h_2, h_3), (h_2, h_4),$
$(h_2, h_5), (h_3, h_5), (h_4, h_1)\}$, then $R_s(h_1) = \{h_1, h_4, h_5\}$, $R_s(h_2) = U$, $R_s(h_3) = \{h_3, h_5\}$, $R_s(h_4) = \{h_1, h_4\}$,

$R_S(h_5) = \{h_5\}$ and $\langle h_1\rangle R = \{h_1, h_4\}$, $\langle h_2\rangle R = U$, $\langle h_3\rangle R = \{h_3, h_5\}$, $\langle h_4\rangle R = \{h_1, h_4\}$, $\langle h_5\rangle R = \{h_5\}$. Then $S_0 = \{h_1, h_2, h_3, h_5\}$. and $\tau_R = \tau_* = \{\phi, U, \{h_5\}, \{h_3, h_5\}, \{h_1, h_4\}, \{h_1, h_4, h_5\}, \{h_1, h_3, h_4, h_5\}\}$
If $Q = \{h_1, h_2, h_4\}$, then $\underline{R}(Q) = \{h_1, h_4\}$, $\overline{R}(Q) = \{h_1, h_2, h_4\}$ and
$Q_g = \underline{R}(Q) \cup ((\overline{R}(Q) - \underline{R}(Q)) \cap S_0) = \{h_1, h_4\} \cup (\{h_2\} \cap \{h_1, h_2, h_3, h_5\}) = \{h_1, h_2, h_4\} = Q$.
If $Q = \{h_1, h_4\}$, then $\underline{R}(Q) = \{h_1, h_4\}$, $\overline{R}(Q) = \{h_1, h_2, h_4\}$ and
$Q_g = \{h_1, h_4\} \cup (\{h_2\} \cap \{h_1, h_2, h_3, h_5\}) = \{h_1, h_2, h_4\}$. Also $\underline{R}(Q_g) = \{h_1, h_4\} = \underline{R}(Q)$, $\overline{R}(Q_g) = \{h_1, h_2, h_4\} = \overline{R}(Q)$, that is $Q_g \in [Q]_\approx$.
If $Q = \{h_3, h_5\}$, then $\underline{R}(Q) = \{h_3, h_5\}$, $\overline{R}(Q) = \{h_2, h_3, h_5\}$ and
$Q_g = \{h_3, h_5\} \cup (\{h_2\} \cap \{h_1, h_2, h_3, h_5\}) = \{h_2, h_3, h_5\}$. Also $\underline{R}(Q_g) = \{h_3, h_5\} = \underline{R}(Q)$, $\overline{R}(Q_g) = \{h_2, h_3, h_5\} = \overline{R}(Q)$, that is $Q_g \in [Q]_\approx$.
If $Q = \{h_2, h_5\}$, then $\underline{R}(Q) = \{h_5\}$, $\overline{R}(Q) = \{h_2, h_3, h_5\}$ and
$Q_g = \{h_5\} \cup (\{h_2, h_3\} \cap \{h_1, h_2, h_3, h_5\}) = \{h_2, h_3, h_5\}$. Also $\underline{R}(Q_g) = \{h_3, h_5\} \neq \underline{R}(Q)$, $\overline{R}(Q_g) = \{h_2, h_3, h_5\} = \overline{R}(Q)$, that is $Q_g \notin [Q]_\approx$.
The base of M is $\{\{a\}_g : a \in U\} = \{\{h_2\}, \{h_1, h_2\}, \{h_2, h_3\}, \{h_2, h_4\}, \{h_2, h_3, h_5\}\}$ also, the topology M is $M = \{\varphi, U, \{h_2\}, \{h_1, h_2\}, \{h_2, h_3\}, \{h_2, h_4\}, \{h_2, h_3, h_5\}, \{h_1, h_2, h_3\},$
$\{h_1, h_2, h_4\}, \{h_2, h_3, h_4\}, \{h_1, h_2, h_3, h_4\}, \{h_1, h_2, h_3, h_5\}, \{h_2, h_3, h_4, h_5\}\}$.

Theorem 3.21. M is a topological space on the universe which is not necessary to be finite.
Proof. This theorem can be proven in the same way as previously explained.

4. Topological structures of generalized rough sets by reflexive relations

We will study in this section, the relationship between topologies on the universe which is not restricted to be finite and the generalized RS induced by reflexive relations. Moreover, the relationships between the set of all topologies which satisfy the requirement C_2 of compactness and the set of all reflexive relations are studied.

For this study we define the famous class $\tau(R) = \{Q \subseteq U : \underline{R}(Q) = Q\}$.

Theorem 4.1. For a (GAS) (U, R), the class $\tau(R)$ is a topology on U.

Lemma 4.2. For a (GAS) (U, R), $\{Q \subseteq U : \langle Q\rangle R = Q\} = \{Q \subseteq U : \underline{R}(Q) = Q\}$.
Proof. Suppose that $P \in \{Q \subseteq U : \langle Q\rangle R = Q\}$, then $\langle P\rangle R = P$, hence $\langle p\rangle R \subseteq P$ for all $p \in P$, so $\underline{R}(P) = P\}$, that is $P \in \{Q \subseteq U : \underline{R}(Q) = Q\}$. (1)
Conversely, let $P \in \{Q \subseteq U : \underline{R}(Q) = Q\}$, then $\underline{R}(Q) = Q\}$. Since R is reflexive, hence $\langle p\rangle R \subseteq P$ for all $p \in P$, so $\langle P\rangle R = P$, that is $P \in \{Q \subseteq U : \langle Q\rangle R = Q\}$. (2)
From (1) and (2), the proof is complete.

From Lemma 4.2 we can proof the next theorem.

Theorem 4.3. For a (GAS) (U, R), $\tau_R = \tau(R)$.

Theorem 4.4. If a topological space (U, τ) satisfies the condition:
(C_1) [8]: For all $P \subseteq U$ and $Q_i \in \tau$; $i \in I$, if $(\cap Qi) \cap P = \phi$, then there are a finite subset $\{Q_i : i \leq n\}$ of $\{Q_i : i \in I\}$ such that $Q_1 \cap Q_2 \cap ... \cap Q_n \cap P = \phi$, then there is a reflexive relation $R(\tau)$ on U such that $\underline{R(\tau)}(Q) = \mu(P)$, $\overline{R(\tau)}(P) = \kappa(P)$, for all $P \subseteq U$.

In the following example, note that the topological space $(U, \tau(R))$ does not satisfy (C_1) in general, for any reflexive relation R.

Example 4.5. In fact the identity relation $R = \{(a, a) : a \in U\}$ in an infinite set U is equivalence and $\langle a\rangle R = \{a\}$. Hence, $\tau(R)$ is a discrete topology on U. Also, note that

$$\bigcap_{a\in U}(U-\{a\})\cap U=\phi$$

and for each finite set Q of U we have,

$$\bigcap_{a\in Q}(U-\{a\})\cap U\neq\phi$$

We define another class $\tau^*=\{\underline{R}(A):A\subseteq U\}$

Lemma 4.6. For a (GAS) (U,R), $\{\underline{R}(Q):Q\subseteq U\}=\{Q\subseteq U:\underline{R}(Q)=Q\}$.

Proof. Assume that $P\in\{\underline{R}(Q):Q\subseteq U\}$, then $\exists\ Q\subseteq U$ such that $\underline{R}(Q)=P$, hence $\underline{R}(\underline{R}(Q))=\underline{R}(P)$. Since R is reflexive, then $\underline{R}(\underline{R}(Q))=\underline{R}(Q)$, that is $\underline{R}(P)=P$, hence $P\in\{Q\subseteq U:\underline{R}(Q)=Q\}$.
Conversely, let $P\in\{Q\subseteq U:\underline{R}(Q)=Q\}$, then $\underline{R}(P)=P$, thus $P\in\{\underline{R}(Q):Q\subseteq U\}$.

From Lemma 4.6 we can proof the next theorem.

Theorem 4.7. For a (GAS) (U,R), the class τ^* is a topology on U and $\tau^*=\tau(R)$.

From Theorems 3.10, 4.3 and 4.7, we can proof the next theorem.

Theorem 4.8. For a (GAS) (U,R), $\tau_*=\tau_R=\tau^*=\tau(R)$.

We introduce another condition (C_2), which used to study the relationship between generalized (RS) induced by reflexive relation and topologies which satisfy (C_2).

Lemma 4.9. If (U,τ) satisfies (C_1), then it is satisfies the condition:
(C_2) [19]: For all $a\in U$ and $Q\subseteq U$, if $a\in\kappa(Q)$, then $\exists\ b\in Q$ such that $a\in\kappa(\{b\})$.

Proof. $\forall\ a\in U$ and $P\subseteq U$, we assume that $a\in\kappa(P)$. Then, $a\notin\mu(P^c)$ by $\kappa(P)=(\mu(P^c))^c$. We take

$$\Theta=\{Q:a\in\mu(Q)\}\cup\{P\}$$

Then we can conclude $\cap\Theta\neq\phi$. Otherwise, we suppose $\cap\Theta=\phi$. We get $\cap\{\mu(Q):a\in\mu(Q)\}\cap P=\phi$ by $\mu(Q)\subseteq Q$. We have from C_1 that there are a finite subset $\{int(Q_i:i\leq n\}$ of $\{\mu(Q):a\in\mu(Q)\}$ such that $\mu(Q_1)\cap\mu(Q_2)\cap...\mu(Q_n)\cap P=\phi$, and hence $\mu(Q_1)\cap\mu(Q_2)\cap...\mu(Q_n)\subseteq P^c$. Since $\mu(Q_1\cap\mu(Q_2\cap...\mu(Q_n)$ is open, we have $\mu(Q_1)\cap\mu(Q_2)\cap...\mu(Q_n)\subseteq\mu(P^c)$. Thus $a\in\mu(P^c)$. Which it is a contradiction. That is $\cap\Theta\neq\phi$.

From the definition of Θ, we get $b\in P$ such that for any $Q\subseteq U$, $a\in\mu(Q)$ this means that $b\in Q$. That is $a\notin\mu(U-\{b\})=(\kappa(\{b\}))^c$ by $b\notin U-\{b\}$, thus $a\in\kappa(\{b\})$.

Example 4.10. Let τ be the topology on the set of natural numbers $N=\{0,1,2,...,n,...\}$ defined by

$$\tau=\{N,\phi,Q_n=\{n+1,n+2,...\}:n\in N\}.$$

(1) In fact, τ satisfies C_2. Assume that $Q\subseteq N$ and $a\in cl(Q)$. There are two cases, the first, when Q is finite, we get $b\in Q$ such that $d\leq b$ for any $d\in Q$. That is $a\in\kappa(Q)=\{0,1,...,b\}=\kappa(\{b\})$. The second, when X is infinite, we have $b\in Q$ such that $a\leq b$, thus $a\in\{0,1,...,b\}=\kappa(\{b\})$.
(2) Someone can prove that

$$(\bigcap_{a\in N}(Q_i))\cap N=\phi$$

also for any finite set Q of N we get,

$$(\bigcap_{a\in Q}(A_i))\cap N\neq\phi$$

This means that, τ does not satisfy C_1.

Theorem 4.11. Assume that (U, τ^*) is a topological space, then $\overline{R}$ and $\underline{R}$ are a closure operator and an interior operator of τ^*, respectively.

Proof. Let κ and μ be a closure operator and an interior operator of τ^* respectively. Since $\underline{R}(P)$ is open and $\underline{R}(P) \subseteq P$, we get $\underline{R}(P) \subseteq \mu(P)$. Also, for all $Q \subseteq P$ with $\underline{R}(Q) = Q$, we get $Q = \underline{R}(Q) \subseteq \underline{R}(P)$, that is $\mu(P) = \cup\{Q : \underline{R}(Q) = Q, Q \subseteq P\} \subseteq \underline{R}(P)$. Also, we can prove that $\overline{R}$ is a closure operator of τ^*.

Theorem 4.12. For a (GAS) (U, R), $\tau(R)$ satisfies C_2.

Proof. Suppose that $q \in \kappa(Q)$. Then $q \in \overline{R}(Q)$, thus $\exists\ p \in Q$ such that $p \in \langle q \rangle R$, hence, $q \in \overline{R}(\{p\}) = \kappa(\{p\})$.

Assume that (U, τ) is a topological space, μ and κ its interior and closure operators respectively. Someone can define the relation R_τ on U in the form $(q, p) \in R_\tau$ if and only if $q \in \kappa(\{p\})$, $\forall\ q, p \in U$. And so, the relation R_τ is reflexive.

Theorem 4.13. $(q, p) \in R_\tau$ if and only if, for any $Q \subseteq U$, $q \in \mu(Q)$ implies $p \in Q$.

Proof. Assume that $(q, p) \in R_\tau$. For each $Q \subseteq U$, if $q \in int(Q)$, then $q \notin (\mu(Q))^c$. Since $q \in \kappa(\{p\})$, hence $p \in Q^c$ and $p \in Q$.

Conversely, assume that for all $Q \subseteq U$, $q \in \mu(Q)$, then $p \in Q$. Since $p \notin U - \{p\}$, that is $q \notin \mu(U - \{p\})$ and thus $q \in (\mu(U - \{p\}))^c = \kappa(U - \{p\})^c = \kappa(\{p\})$.

Theorem 4.14. If (U, τ) satisfies the condition C_2, then $\overline{R_\tau}(Q) = \kappa(Q)$ and $\underline{R_\tau}(Q) = \mu(Q)\ \forall\ Q \subseteq U$.

Proof. Suppose that $Q \subseteq U$ and $q \in U$, if $q \in \overline{R_\tau}(Q)$, then $\exists\ p \in Q$ such that $(q, p) \in R_\tau$, this means that, $q \in \kappa(\{p\}) \subseteq \kappa(Q)$ and $\overline{R_\tau}(Q) \subseteq \kappa(Q)$.

Conversely, assume that $q \in \kappa(Q)$. By C_2, there is $p \in Q$ such that $q \in \kappa(\{p\})$, thus, $(q, p) \in R_\tau$, so $q \in \overline{R_\tau}(Q)$. Consequently, $\kappa(Q) \subseteq \overline{R_\tau}(Q)$. By the duality, $\underline{R_\tau}(Q) = \mu(Q)$ holds.

Theorem 4.15. (1) If the topological space (U, τ) satisfy C_2, then $\tau(R_\tau)) = \tau$.

(2) $R_{\tau(R)} = R$, if the relation R on U is reflexive.

Proof. (1) Assume that μ is an interior operator of τ. Then $R(\tau)$ is reflexive. By Theorem 4.11, if $Q \in \tau(R_\tau)$, hence $\underline{R_\tau}(Q) = Q$, and then $\mu(Q) = Q$. Thus, $Q \in \tau$. Conversely, assume that $Q \in \tau$. From Theorem 4.14, $\underline{R_\tau}(Q) = \mu(Q) = Q$, that is $Q \in \tau(R_\tau)$.

(2) Assume that κ is a closure operator of the topology $\tau(R)$. For all $q, p \in U$, if $(q, p) \in R_{\tau(R)}$, then according to Theorem 4.11, $q \in \kappa(\{p\}) = \overline{R}(\{p\})$, and $(q, p) \in R$ by the definition. Conversely, assume that $(q, p) \in R$. This means that $q \in \overline{R}(\{p\}) = \kappa(\{p\})$, then $(q, p) \in R_{\tau(R)}$.

Assume that Ω is the set of all topologies on U which satisfies C_2 and Θ is the set of all reflexive relations on U.

Corollary 4.16. There is a one-to-one correspondence between Θ and Ω.

Proof. One can define a function $f : \Theta \to \Omega$ by $f(R) = \tau(R)$ and by Theorem 4.15, can prove that it is a one-to-one correspondence. Also, a function g from Ω to Θ defined by $g(\tau) = R_\tau$ is a one-to-one correspondence.

5. Medical applications

Recently, several medical applications of rough sets and its applications (for instance, [9, 10, 13, 15, 17, 19–23]. In this section, we are considering the problem of dengue fever. This disease is transmitted to humans via virus-carrying Dengue mosquitoes [17, 40]. Symptoms of Dengue fever begin three to four days after infection. Recovery usually takes between two and seven days [40]. It is common in more than 120 countries around the world, mainly Asia and South America [52]. It causes about 60 million symptomatic infections worldwide and 13,600 deaths worldwide. Consequently, we deal with this problem and have tried to analyze it through a minimal structure space, the reduction of condition attributes, and the accuracy of decision attributes. The data discuss the problem of dengue fever. Columns of the following Table 1 are the attributes (symptoms of Dengue fever), such that the set of attributes is $\{J, F, S, H\}$ where J interpreted as (muscle and joint pains), F interpreted as (fever), S interpreted as (characteristic skin rash) and H interpreted as (headache) [17, 40]. Attribute D is the decision of problem and the rows of attributes $P = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8\}$ are the patients.

Note that: The present illustrative example shows the importance of the proposed approaches in the reduction of attributes where Pawlak's rough sets cannot be applied in the information system of Table 1 since the used relation is not an equivalence relation.

Table 1. Dengue fever information system.

P	J	F	S	H	**Dengue fever**
m_1	✓	✓	✓	×	✓
m_2	✓	×	×	×	×
m_3	✓	×	×	×	✓
m_4	×	×	×	✓	×
m_5	×	✓	✓	×	×
m_6	✓	✓	×	✓	✓
m_7	✓	✓	×	×	×
m_8	✓	✓	×	✓	✓

From Table 1, we obtain the symptoms of every patient are:
$v(m_1) = \{J, F, S\}$, $v(m_2) = \{J\}$, $v(m_3) = \{J\}$, $v(m_4) = \{H\}$, $v(m_5) = \{F, S\}$, $v(m_6) = \{J, F, H\}$, $v(m_7) = \{J, F\}$, and $v(m_8) = \{J, F, H\}$.

Now, we construct the right neighborhoods via the following relation, that is related to the nature of the studied problem:

$$m_i R m_j \Longleftrightarrow v(m_i) \subseteq v(m_j)$$

Note that: The relation in each issue is defined according to the expert's requirements. Thus, the relation for all attributes is:
$R = \{(m_1, m_1), (m_2, m_2), (m_2, m_1), (m_2, m_3), (m_2, m_6), (m_2, m_7), (m_2, m_8), (m_3, m_3), (m_3, m_1), (m_3, m_2), (m_3, m_6), (m_3, m_7), (m_3, m_8), (m_4, m_4), (m_4, m_6), (m_4, m_8), (m_5, m_5), (m_5, m_1), (m_6, m_6), (m_6, m_8), (m_7, m_7), (m_7, m_1), (m_7, m_6), (m_7, m_8), (m_8, m_8), (m_8, m_6)\}$.

Therefore, from Table 1, the minimal right neighborhood of all elements in P are:

$\langle m_1 \rangle R = \{m_1\}$, $\langle m_2 \rangle R = \langle m_3 \rangle R = \{m_1, m_2, m_3, m_6, m_7, m_8\}$, $\langle m_4 \rangle R = \{m_4, m_6, m_8\}$, $\langle m_5 \rangle R = \{m_1, m_5\}$, $\langle m_6 \rangle R = \{m_6, m_8\}$, $\langle m_7 \rangle R = \{m_1, m_6, m_7, m_8\}$, and $\langle m_8 \rangle R = \{m_6, m_8\}$.

From Table 1, we have two cases are:

Case 1. (Patients infected with dengue fever) $U_1 = \{m_1, m_3, m_6, m_8\}$.

Therefore, using Definition 3.3, we calculate the accuracy of U_1, through lower and upper approximations respectively as

$\underline{R}(U_1) = \{m_1, m_6, m_8\}$ and $\overline{R}(U_1) = P$. Thus, the accuracy measure is $\mho(U_1) = 3/8$.

Now, if we remove the attribute J, then the symptoms of every patient are:

$v(m_1) = \{F, S\}$, $v(m_2) = \phi$, $v(m_3) = \phi$, $v(m_4) = \{H\}$, $v(m_5) = \{F, S\}$, $v(m_6) = \{F, H\}$, $v(m_7) = \{F\}$, and $v(m_8) = \{F, H\}$.

Therefore, the minimal right neighborhood of all elements in P are:

$\langle m_1\rangle R = \langle m_5\rangle R = \{m_1, m_5\}$, $\langle m_2\rangle R = \langle m_3\rangle R = P$, $\langle m_4\rangle R = \{m_4, m_6, m_8\}$, $\langle m_6\rangle R = \langle m_8\rangle R = \{m_6, m_8\}$, and $\langle m_7\rangle R = \{m_1, m_5, m_6, m_7, m_8\}$.

Accordingly, lower and upper approximations of U_1 respectively are

$\underline{R}(U_1) = \{m_6, m_8\}$ and $\overline{R}(U_1) = P$. Thus, the accuracy measure is $\mho(U_1) = 1/4$ which differs than the accuracy of the original information system in Table1. Hence, the attribute J is not dispensable.

Again, if we remove the attribute F, then the symptoms of every patient are:

$v(m_1) = \{J, S\}$, $v(m_2) = \{J\}$, $v(m_3) = \{J\}$, $v(m_4) = \{H\}$, $v(m_5) = \{S\}$, $v(m_6) = \{J, H\}$, $v(m_7) = \{J\}$, and $v(m_8) = \{J, H\}$.

Therefore, the minimal right neighborhood of all elements in P are:

$\langle m_1\rangle R = \{m_1\}$, $\langle m_2\rangle R = \langle m_3\rangle R = \langle m_7\rangle R = \{m_1, m_2, m_3, m_6, m_7, m_8\}$, $\langle m_4\rangle R = \{m_4, m_6, m_8\}$, $\langle m_5\rangle R = \{m_1, m_5\}$, and $\langle m_6\rangle R = \langle m_8\rangle R = \{m_6, m_8\}$.

Accordingly, lower and upper approximations of U_1 respectively are

$\underline{R}(U_1) = \{m_1, m_6, m_8\}$ and $\overline{R}(U_1) = P$. Thus, the accuracy measure is $\mho(U_1) = 3/8$ which is the same as the accuracy of the original information system in Table1. Hence, the attribute F is dispensable.

Another step, if we remove the attribute S, then the symptoms of every patient are:

$v(m_1) = \{J, F\}$, $v(m_2) = \{J\}$, $v(m_3) = \{J\}$, $v(m_4) = \{H\}$, $v(m_5) = \{F\}$, $v(m_6) = \{J, F, H\}$, $v(m_7) = \{J, F\}$, and $v(m_8) = \{J, F, H\}$.

Therefore, the minimal right neighborhood of all elements in P are:

$\langle m_1\rangle R = \langle m_7\rangle R = \{m_1, m_6, m_7, m_8\}$, $\langle m_2\rangle R = \langle m_3\rangle R = \{m_1, m_2, m_3, m_6, m_7, m_8\}$, $\langle m_4\rangle R = \{m_4, m_6, m_8\}$, $\langle m_5\rangle R = \{m_1, m_5, m_7, m_8\}$, and $\langle m_6\rangle R = \langle m_8\rangle R = \{m_6, m_8\}$.

Accordingly, lower and upper approximations of U_1 respectively are

$\underline{R}(U_1) = \{m_6, m_8\}$ and $\overline{R}(U_1) = P$. Thus, the accuracy measure is $\mho(U_1) = 1/4$ which differs than the accuracy of the original information system in Table1. Hence, the attribute S is not dispensable.

Finally, if we remove the attribute H, then the symptoms of every patient are:

$v(m_1) = \{J, F, S\}$, $v(m_2) = \{J\}$, $v(m_3) = \{J\}$, $v(m_4) = \phi$, $v(m_5) = \{F, S\}$, $v(m_6) = \{J, F\}$, $v(m_7) = \{J, F\}$, and $v(m_8) = \{J, F\}$.

Therefore, the minimal right neighborhood of all elements in P are:

$\langle m_1\rangle R = \{m_1\}$, $\langle m_2\rangle R = \langle m_3\rangle R = \langle m_4\rangle R = P$, $\{m_4, m_6, m_8\}$, $\langle m_5\rangle R = \{m_1, m_5\}$, and $\langle m_6\rangle R = \langle m_7\rangle R = \langle m_8\rangle R = \{m_1, m_6, m_7, m_8\}$.

Accordingly, lower and upper approximations of U_1 respectively are

$\underline{R}(U_1) = \{m_1\}$ and $\overline{R}(U_1) = P$. Thus, the accuracy measure is $\mho(U_1) = 1/8$ which differs than the accuracy of the original information system in Table1. Hence, the attribute H is not dispensable.

Case 2. (Patients are not infected with dengue fever) $U_2 = \{m_2, m_4, m_5, m_7\}$.

By following the same steps like Case 1, we obtain that the attributes J, S, and H are not dispensable.

Concluding remark: From the above discussion, we notice that the attributes $\{J, S, H\}$ cannot be removed, and then they represent the core attributes of the original information system. Therefore, $\{J, S, H\}$ represent the basic factors for identifying the dengue fever.

6. Conclusions

The debate of structure for rough topological space based on reflexive relation has been introduced in this research. We used the representation of RS to construct the rough topological space. Moreover, we have investigated the relationships among approximation operators, closure operators, and interior operators. Besides, the relationships between topological spaces in the universe which are not limited restricted to being ended, and RS induced by reflexive relations were investigated. Additionally, we have established the relationships between the set of all topologies which satisfy the requirement of compactness C_2 and the set of all reflexive relations. Finally, a medical application for our proposals was established. In future work, we will investigate the topological structure of the other models.

List of symbols and abbreviations

RS	rough sets	R	Binary relation
GAS	Generalized approximation space	$\langle a\rangle R$	Minimal right neighborhood of a
U	Universal set	$R\langle a\rangle$	Minimal left neighborhood of a
τ	Topology	$\underline{R}(Q)$	Lower approximation of **Q**
τ^c	Class of all closed sets	$\overline{R}(Q)$	Upper approximation of **Q**
$\kappa(\mathbf{Q})$	τ-closure of **Q**	$BND(Q)$	The boundary region of **Q**
$\mu(\mathbf{Q})$	τ-interior of **Q**	$\mho(Q)$	The accuracy of the approximations

Conflict of interest

All authors declare that there is no conflict of interest regarding the publication of this manuscript.

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