Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 8206–8223.
DOI: 10.3934/math.2022457
Received: 10 December 2021
Revised: 11 February 2022
Accepted: 14 February 2022
Published: 25 February 2022

Research article

Asymptotic stability for quaternion-valued BAM neural networks via a contradictory method and two Lyapunov functionals

Ailing Li[1], **Mengting Lv**[2] **and Yifang Yan**[1,*]

[1] College of Science, Hebei North University, Zhangjiakou, 075000, China
[2] School of Mathematics, Hunan University, Changsha, 410082, China

* **Correspondence:** Email: yanyifang1990@163.com.

Abstract: We explore the existence and asymptotic stability of equilibrium point for a class of quaternion-valued BAM neural networks with time-varying delays. Firstly, by employing Homeomorphism theorem and a contradictory method with novel analysis skills, a criterion ensuring the existence of equilibrium point of the considered quaternion-valued BAM neural networks is acquired. Secondly, by constructing two Lyapunov functionals, a criterion assuring the global asymptotic stability of equilibrium point for above discussed quaternion-valued BAM is presented. Applying a contradictory method to study the equilibrium point and applying two Lyapunov functionals to study stability of equilibrium point are completely new methods.

Keywords: quaternion-valued BAM neural networks; contradictory method; asymptotic stability; Lyapunov functionals
Mathematics Subject Classification: 34K24

1. Introduction

Quaternion was found by the Irish mathematician W. R. Hamilton in 1843, which didn't give rise to much attention for a long time and not to mention the practical applications. The skew field of quaternion is expressed with

$$\mathbf{Q} = \{z = z_0 + iz_1 + jz_2 + k^* z_3\},$$

in which z_0, z_1, z_2, z_3 are real numbers. Quaternion-valued neural networks can be regarded as a generic extension of complex-valued neural networks or real-valued neural networks and they have much more complicated structure than complex-valued neural networks on their quaternion-valued states, quaternion-valued connection weights and quaternion-valued activation functions. Because of some practical applications, as a class of hypercomplex system, recently, the dynamical behaviors of quaternion-valued neural networks have attracted great interest from wide researchers and some good results have been reported in some international journal of mathematics and engineering [1–23]. In [1], the existence and global exponential stability of anti-periodic solution for a quaternion-valued high-order Hopfield neural networks were discussed. By separating the discussed quaternion-valued neural networks into four real-valued systems and applying a novel continuation theorem of coincidence degree theory, some novel sufficient conditions to guarantee the existence and global exponential stability of anti-periodic solutions were acquired in [1]. In [2], the existence and global exponential stability of anti-periodic solutions for a class of inertial quaternion-valued high-order Hopfield neural networks were studied. Without decomposing the discussed neural networks into real-valued systems, by applying wirtinger inequality and a continuation theorem of coincidence degree theory, new sufficient conditions ensuring the existence and exponential stability of anti-periodic solutions for discussed networks were obtained in [2]. In [3], the authors discussed the multistability of a class of quaternion-valued neural networks with time delays by using inequality craftsmanships. In [4,5], the authors discussed respectively the stability and robust stability for respectively discussed quaternion-valued neural networks. In [6], the authors discussed a class of quaternion-valued Cohen-Grossberg neural networks. By applying Homeomorphism theorem and constructing Lyapunov functional, by decomposing and direct approaches, several new sufficient conditions were acquired to assure the existence and global asymptotic stability and global exponential stability for system (1) in [6]. In [11], the global μ-stability of a class of quaternion-valued neural networks with mixed time-varying delays was explored. In [13], the existence, uniqueness and stability of the equilibrium point of quaternion-valued neural networks with both discrete and distributed delays were explored. On the basis of homeomorphic mapping theorem and linear matrix inequality method, several sufficient conditions ascertaining the equilibrium point is globally asymptotically stable were gained. In [14], the multistability of equilibrium point of a class of quaternion-valued neural networks with nonmonotonic piecewise nonlinear activation functions were discussed. In [15], under the fixed-time stability and some analytical skills, criterion of fixed-tme synchronization for a class of quaternion-valued neural networks was gained. In [16], a class of quaternion-valued fuzzy cellular neural networks with time-varying delays on time scales was put forward. On the basis of inequality analysis techniques on time scales, the exponential stability of anti-periodic solutions for the networks was attained.

Up to now, in the study of the existence and uniqueness of equilibrium point for almost real-valued neural networks [24–27], almost complex-valued neural networks [28,29] and almost quaternion-valued neural networks [6,7,9,13], the results of the existence of the equilibrium point of the considered neural networks have been obtained mainly by applying the Homeomorphism theorem. In the study of global asymptotic/exponential stability of equilibrium point for almost the considered neural networks [15,24–26,28,29], by firstly constructing a Lyapunov functional $V(t)$, then by proving $V'(t) < 0$, the global asymptotic/exponential stability of equilibrium point for discussed neural networks was proved.

So far, the existence and global asymptotic/expoential stability of equilibrium point for quaternion-valued neural networks have been investigated mainly by decomposing method (namely separating quaternion-valued neural networks into four real-valued systems), see [1,3,5–11]. The shortage of decomposing method is that the computation of the proof is too complicated, as a result, the obtained results are also so complicated that they cannot be easily verified. However, the results on the global asymptotic/exponential stability obtained by applying the direct approach method are less [2,4].

Up to present, the stability results of quaternion-valued BAM neural networks have been also discussed [20,23,30–38]. In [20], the existence and global asymptotic stability of periodic solutions for a class of discrete quaternion-valued BAM neural networks were investigated. By applying continuation theorem of coincidence degree theory and constructing Lyapunov discrete sequences, the criteria to ensure the existence and global asymptotic stability of periodic solutions for the considered neural networks were established. In [23], the global stability analysis for the fractional-order BAM quaternion-valued neural networks was discussed. By using the principle of homeomorphism, Lyapunov fractional-order method and linear matrix inequality approach, the result for the existence, uniqueness and global asymptotic stability of the equilibrium point were obtained. In [34], the global exponential stability in Lagrange sense of BAM quaternion-valued inertial neural networks was considered by non-reduced order and un-decomposed approach. Several criteria for Lagrange stability were acquired in the form of linear matrix inequalities. In [35], the global stability for BAM quaternion-valued inertial neural networks with time delay was investigated. Based on nonlinear measure approach and some inequality techniques, a new sufficient condition was obtained to ensure the existence and uniqueness of the equilibrium point. Meanwhile, some new Lyapunov functionals were constructed to directly propose the asymptotic stability for the discussed system and some stability criteria in linear matrix inequality form were derived by means of Barbalat Lemma and inequality techniques. Up to now, the stability results for quaternion-valued non-BAM neural networks and quaternion-valued neural networks have been obtained mainly by using LMI approach, matrix measure and non-reduced order approach. This make us to look for new approach to study the stability of quaternion-valued BAM neural networks. This constitutes the motivation of this paper.

Based on above discussions, in this paper, we are concerned with the following quaternion-valued BAM neural networks with time-varying delays for $p = 1, 2, \cdots, n, q = 1, 2, \cdots, m$:

$$
\begin{aligned}
u_p'(t) &= -\alpha_p u_p(t) + \sum_{q=1}^{m} a_{pq} F_q(v_q(t)) + \sum_{q=1}^{m} c_{pq} F_q(v_q(t-\tau(t))) + I_p, \\
v_q'(t) &= -\gamma_q v_q(t) + \sum_{p=1}^{n} b_{qp} G_p(u_p(t)) + \sum_{p=1}^{n} d_{qp} G_p(u_p(t-\sigma(t))) + J_q,
\end{aligned} \tag{1.1}
$$

where, $u_p(t), v_q(t) \in Q$ are the neuron states, $\alpha_p > 0$ and $\gamma_q > 0$ are constants which denote the rate with which the ith neurons and the jth neurons will reset its potential to the resetting state in isolation when disconnected from the networks and external inputs, the connection weights $a_{pq}, c_{pq}, b_{qp}, d_{qp} \in Q$ are the strength of the neuron interconnections, $I_p, J_q \in Q$ are the external inputs, $F_q(u_q(t)), F_q(u_q(t)), G_p(v_p(t)), G_p(v_p(t)) : Q \to Q$ are the activation functions, $\tau(t) > 0$ and $\sigma(t) > 0$ are time delays.

The initial values of system (1.1) are expressed as:

$$
u_p(s) = \phi_p(s), v_q(s) = \psi_q(s), s \in [0, \tau], \tag{1.2}
$$

where, $\tau = \max\{\max_{t\in[0,\infty)}\{\tau(t)\}, \max_{t\in[0,\infty)}\{\sigma(t)\}\}$.

In the paper, our objective is to gain novel sufficient conditions ensuring the existence and asymptotic stability of equilibrium point for system (1.1) by applying direct method of quaternion-valued neural networks and by applying the Homeomorphism theorem but with a contradictory method of proving the existence of equilibrium point and constructing two Lyapunov functionals to study stability of equilibrium point. In the study of the existence of equilibrium point for system (1.1), we also construct a mapping $H(u-\hat{u}, v-\hat{v})$ as in the existing papers. But we use the contradictory approach to obtain the results. In the study of the asymptotic stability of equilibrium point, by constructing two functionals $V_1(t) = K_1(t) + \int_{t-\tau(t)}^{t} V_1(s)ds, V_2(t) = K_2(t) + \int_{t-\sigma(t)}^{t} V_2(s)ds$, then by proving respectively $V_1'(t) < 0, V_2'(t) < 0$, the proof of global asymptotic stability of equilibrium point is accomplished. The novelty is that in the proof of the existence-uniqueness of equilibrium point, a contradictory approach is introduced, in the proof of the global asymptotic stability, two Lyapunov functionals are introduced. Consequently, the contribution of the paper is embodied the following two points: (1) Novel study methods of proving the existence and stability of equilibrium point of neural networks are introduced in our paper: using a contradictory method with new analysis techniques proving the existence of equilibrium point and constructing two Lyapunov functionals to discuss the stability of equilibrium point; (2) New concise sufficient conditions to assure the existence and global asymptotic stability of equilibrium point for system (1.1) are gained by applying direct method of quaternion-valued neural networks.

2. Preliminary

For each $u \in \mathbf{Q}$, the conjugate of u is $u^* = u^R - iu^I - ku^K - ju^J$ and the norm of u is defined as

$$\|u\|_Q = \sqrt{uu^*} = \sqrt{(u^R)^2 + (u^I)^2 + (u^J)^2 + (u^K)^2}.$$

For $h = (h_1, h_2, \cdots, h_n) \in Q^n$, the norm of h is defined as

$$\|h\|_{Q^n} = \sqrt{\sum_{p=1}^{n} h_p^* h_p}, h_p(p = 1, 2, \cdots, n) \in Q.$$

Lemma 2.1. For $c, d \in H, c^*d + d^*c \leq c^*c + d^*d$.

Lemma 2.2. (Chen et al [5]) Assume that $H(x) : Q^n \to Q^n$ is a continuous map and satisfies the following conditions:

(a) $H(x)$ is injective on Q^n;

(b) $\lim_{\|x\|_{Q^n} \to \infty} \|H(x)\|_{Q^n} = \infty$,

Then, $H(x)$ is a homeomorphism of Q^n onto itself.

We make the following assumptions:

(D_1) There exist positive constants L_q and L_p such that

$$\|F_q(u) - F_q(v)\|_Q \leq L_q \|u - v\|_Q,$$

$$\|G_p(u) - G_p(v)\|_Q \leq L_p \|u - v\|_Q;$$

$$p = 1, 2, \cdots, n; q = 1, 2, \cdots, m; \quad u, v \in Q;$$

(D_2)

$$2\alpha > 1 + 2mA + 2mA_1, \alpha \neq 1,$$

(D_3)

$$2\gamma > 1 + 2nB + 2nB_1, \gamma \neq 1,$$

where

$$A = \max_{1\leq q\leq m}\{\sum_{p=1}^{n} \|a_{pq}\|_Q^2 L_q^2\}, A_1 = \max_{1\leq q\leq m}\{\sum_{p=1}^{n} \|c_{pq}\|_Q^2 L_q^2\},$$

$$B = \max_{1\leq p\leq n}\{\sum_{q=1}^{m} \|b_{qp}\|_Q^2 L_p^2\}, B_1 = \max_{1\leq p\leq n}\{\sum_{q=1}^{m} \|d_{qp}\|_Q^2 L_p^2\},$$

$$\alpha = \min_{1\leq p\leq n}\{\alpha_p\}, \gamma = \min_{1\leq q\leq m}\{\gamma_q\}.$$

Claim1. (D_2) implies
(D_4)

$$\alpha^2 > 2m(A + A_1).$$

(D_3) implies
(D_5)

$$\gamma > 2n(B + B_1).$$

Proof. Since $\alpha \neq 1$, then $(\alpha - 1)^2 > 0$. Thus $\alpha^2 > 2\alpha - 1 > 2m(A + A_1)$. So (D_4) holds. Similarly, (D_5) can be proved.

We introduce the following notations:

$$U_1 = \sum_{p=1}^{n} \|u_p - \hat{u}_p\|_Q^2, U_2 = \sum_{q=1}^{m} \|v_q - \hat{v}_q\|_Q^2,$$

$$F_1(t) = \sum_{p=1}^{n} \|u_p(t) - u_p^*\|_Q^2, F_2(t) = \sum_{q=1}^{m} \|v_q(t) - v_q^*\|_Q^2,$$

where $\hat{u}_p, \hat{v}_q$ are defined in Theorem 3.1, u_p^*, v_q^* are defined in Theorem 4.1.

3. Existence of equilibrium point

In this section, we prove that under some conditions, system (1.1) has a unique equilibrium point.
Theorem 3.1. Assume that $(D_1) - (D_3)$ hold. Then system (1.1) with (1.2) has a unique equilibrium point.
Proof. We employ Lemma 2.2 to prove that system (1.1) has a unique equilibrium point. Firstly, we prove that (a) in Lemma 2.2 is satisfied. By system (1.1), it is clear that an equilibrium point $(\breve{u}, \breve{v}) = (\breve{u}_1, \breve{u}_2, \cdots, \breve{u}_n, \breve{v}_1, \breve{v}_2, \cdots, \breve{v}_m)$ of system (1.1) satisfies :

$$\begin{aligned}
&\alpha_p u_p - \sum_{q=1}^{m} a_{pq} F_q(v_q) - \sum_{q=1}^{m} c_{pq} F_q(v_q) - I_p = 0,\\
&\gamma_q v_q - \sum_{p=1}^{n} b_{qp} G_p(u_q) - \sum_{p=1}^{n} d_{qp} G_p(u_p) - J_q = 0.
\end{aligned} \tag{3.1}$$

By (3.1), define a mapping as follows:

$$H(u,v)=\begin{pmatrix} \alpha_1u_1-\sum\limits_{q=1}^{m}a_{1q}F_q(v_q)-\sum\limits_{q=1}^{m}c_{1q}F_q(v_q)-I_1\\ \alpha_2u_2-\sum\limits_{q=1}^{m}a_{2q}F_q(v_q)-\sum\limits_{q=1}^{m}c_{2q}F_q(v_q)-I_2\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \alpha_nu_n-\sum\limits_{q=1}^{m}a_{nq}F_q(v_q)-\sum\limits_{q=1}^{m}c_{nq}F_q(v_q)-I_n\\ \gamma_1v_1-\sum\limits_{p=1}^{n}b_{1p}G_p(u_p)-\sum\limits_{p=1}^{n}d_{1p}G_p(u_p)-J_1\\ \gamma_2v_2-\sum\limits_{p=1}^{n}b_{2p}G_p(u_p)-\sum\limits_{p=1}^{n}d_{2p}G_p(u_p)-J_2\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \gamma_mv_m-\sum\limits_{p=1}^{n}b_{mp}G_p(u_p)-\sum\limits_{p=1}^{m}d_{mp}G_p(u_p)-J_m \end{pmatrix}, \tag{3.2}$$

where

$$u=(u_1,u_2,\cdots,u_n), v=(v_1,v_2,\cdots,v_m).$$

Set

$$(u,v)\neq(\hat{u},\hat{v}),\hat{u}=(\hat{u}_1,\hat{u}_2,\cdots,\hat{u}_n),\hat{v}=(\hat{v}_1,\hat{v}_2,\cdots,\hat{v}_m).$$

In order to prove that $H(u,v)$ is an injective, we only need prove that $H(u,v)\neq H(\hat{u},\hat{v})$. If $H(u,v)=H(\hat{u},\hat{v})$, by (3.2), one has for $p=1,2,\cdots,n, q=1,2,\cdots,m$,

$$\sum_{p=1}^{n}\Big\{(u_p-\hat{u}_p)^*\Big(\alpha_p(u_p-\hat{u}_p)-\sum_{q=1}^{m}(a_{pq}+c_{pq})[F_q(v_q)-F_q(\hat{v}_q)]\Big)+\Big(\alpha_p(u_p-\hat{u}_p)-\sum_{q=1}^{m}(a_{pq}+c_{pq})\times [F_q(v_q)-F_q(\hat{v}_q)]\Big)^*(u_p-\hat{u}_p)\Big\}=0 \tag{3.3}$$

and

$$\sum_{q=1}^{m}\Big\{(v_q-\hat{v}_q)^*\Big(\gamma_q(v_q-\hat{v}_q)-\sum_{p=1}^{n}(b_{qp}+d_{qp})[G_p(u_p)-G_p(\hat{v}_p)]\Big)+\Big(\gamma_q(v_q-\hat{v}_q)-\sum_{p=1}^{n}(b_{qp}+d_{qp})\times [G_p(u_q)-G_p(\hat{u}_p)]\Big)^*(v_q-\hat{v}_q)\Big\}=0. \tag{3.4}$$

By (D_1) and Lemma 2.1, we have from (3.3) and (3.4),

$$\begin{aligned}&\sum_{p=1}^{n}2\alpha_p(u_p-\hat{u}_p)^*(u_p-\hat{u}_p)\\ \leq\ &\sum_{p=1}^{n}\Big\{\Big(\sum_{q=1}^{m}(a_{pq}+c_{pq})[F_q(v_q)-F_q(\hat{v}_q)]\Big)^*(u_p-\hat{u}_p)+(u_p-\hat{u}_p)^*\Big(\sum_{q=1}^{m}(a_{pq}+c_{pq})[F_q(v_q)-F_q(\hat{v}_q)]\Big)\Big\}\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{p=1}^{n}\Big\{\Big(\sum_{q=1}^{m}(a_{pq}+c_{pq})[F_q(v_q)-F_q(\hat{v}_q)]\Big)^*\Big(\sum_{q=1}^{m}(a_{pq}+c_{pq})[F_q(v_q)-F_q(\hat{v}_q)]\Big)+(u_p-\hat{u}_p)^*(u_p-\hat{u}_p)\Big\}\\
&\le \sum_{p=1}^{m}\Big\{\Big\|\sum_{q=1}^{m}a_{pq}[F_q(v_q)-F_q(\hat{v}_q)]+\sum_{q=1}^{m}c_{pq}[F_q(v_q)-F_q(\hat{v}_q)]\Big\|_Q^2+\|u_p-\hat{u}_p\|_Q^2\Big\}\\
&\le \sum_{p=1}^{n}\Big\{\|u_p-\hat{u}_p\|_Q^2+\Big[\sum_{q=1}^{m}\|a_{pq}\|_Q L_q\|v_q-\hat{v}_q\|_Q+\sum_{q=1}^{m}\|c_{pq}\|_Q L_q\|v_q-\hat{v}_q\|_Q\Big]^2\Big\}\\
&\le \sum_{p=1}^{n}\Big\{\|u_p-\hat{u}_p\|_Q^2+m\sum_{q=1}^{m}L_q^2\Big(\|a_{pq}\|_Q\|v_q-\hat{v}_q\|_Q+\|c_{pq}\|_Q\|v_q-\hat{v}_q\|_Q\Big)^2\Big\}\\
&\le \sum_{p=1}^{n}\Big\{\|u_p-\hat{u}_p\|_Q^2+2m\sum_{q=1}^{m}L_q^2\Big[\|a_{pq}\|_Q^2\|v_q-\hat{v}_q\|_Q^2+\|c_{pq}\|_Q^2\|v_q-\hat{v}_q\|_Q^2\Big]\Big\} \qquad (3.5)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{q=1}^{m}2\gamma_q(v_q-\hat{v}_q)^*(v_q-\hat{v}_q)\\
\le\ &\sum_{q=1}^{m}\Big\{\Big(\sum_{p=1}^{n}(b_{qp}+d_{qp})[G_p(u_p)-G_p(\hat{u}_p)]\Big)^*(v_q-\hat{v}_q)+(v_q-\hat{v}_q)^*\Big(\sum_{p=1}^{n}(b_{qp}+d_{qp})[G_p(u_p)-G_p(\hat{u}_p)]\Big)\Big\}\\
\le\ &\sum_{q=1}^{m}\Big\{\Big(\sum_{p=1}^{n}(b_{qp}+d_{qp})[G_p(u_p)-G_p(\hat{u}_p)]\Big)^*\Big(\sum_{p=1}^{n}(b_{qp}+d_{qp})[G_p(u_p)-G_p(\hat{u}_p)]\Big)+(v_q-\hat{v}_q)^*(v_q-\hat{v}_q)\Big\}\\
\le\ &\sum_{q=1}^{m}\Big\{\|v_q-\hat{v}_q\|_Q^2+2n\sum_{p=1}^{n}L_p^2\Big[\|b_{qp}\|_Q^2\|u_p-\hat{u}_p\|_Q^2+\|d_{qp}\|_Q^2\|u_p-\hat{u}_p\|_Q^2\Big]\Big\}. \qquad (3.6)
\end{aligned}
$$

By (3.5) and (3.6), one has

$$2\alpha U_1 \le 2mAU_2 + 2mA_1U_2 + U_1 \qquad (3.7)$$

and

$$2\gamma U_2 \le 2nBU_1 + 2nB_1U_1 + U_2. \qquad (3.8)$$

We explore two possible cases: (1) $U_2 \le U_1$; (2) $U_2 > U_1$.
(1) When $U_2 \le U_1$, (3.7) implies

$$(2\alpha - 1 - 2mA - 2mA_1)U_1 \le 0. \qquad (3.9)$$

Since $U_1 > 0, 2\alpha - 1 - 2mA - 2mA_1 > 0$, then (3.9) leads a contradiction.
(2) When $U_2 > U_1$, (3.8) implies

$$(2\gamma - 1 - 2nB - 2nB_1)U_2 \le 0. \qquad (3.10)$$

Since $U_2 > 0, 2\gamma - 1 - 2nB - 2nB_1 > 0$, then (3.10) leads a contradiction.

From the discussions of (1) and (2), it follows that $H(u,v)$ is an injective. Consequently, (a) in Lemma 2.2 is fulfilled. Next we prove that (b) in Lemma 2.2 is fulfilled. Namely, we will prove that when $\|(u,v)\|_{Q^n} \to \infty, \|H(u,v)\|_{Q^n} \to \infty$. If $\lim\limits_{\|(u,v)\|_{Q^n}\to\infty} \|H(u,v)\|_{Q^n} \neq \infty$, then there exist two positive constants r and A^* with

$$r > \sqrt{2}\max\Big\{\frac{D\sqrt{m} + \sqrt{D^2m + [\alpha^2 - 2m(A+A_1)\sum_{p=1}^{n} C_p^2}}{\alpha^2 - 2m(A+A_1)}, \frac{D_1\sqrt{n} + \sqrt{D_1^2 n + [\gamma^2 - 2n(B+B_1)\sum_{q=1}^{m}(C_q^*)^2]}}{\gamma^2 - 2n(B+B_1)}\Big\},$$

such that

$$\|H(u,v)\|_{Q^n} \le A^*, \|(u,v)\|_{Q^n} > r, \tag{3.11}$$

in which C_p, C_q^*, D, D_1 are defined respectively in (3.14) and (3.15). By (3.2), one has

$$\sum_{p=1}^{n}[\alpha_p u_p - \sum_{q=1}^{m}(a_{pq} + c_{pq}) - I_p]^*[\alpha_p u_p - \sum_{q=1}^{m}(a_{pq} + c_{pq}) - I_p] < (A^*)^2$$

and

$$\sum_{q=1}^{m}[\gamma_q v_q - \sum_{p=1}^{n}(b_{qp} + d_{qp}) - J_q]^*[\gamma_q v_q - \sum_{p=1}^{n}(b_{pq} + d_{qp}) - J_q] < (A^*)^2.$$

Then by the definition of $\|.\|_Q$, one has

$$\|\alpha_p u_p - \sum_{q=1}^{m}(a_{pq} + c_{pq})F_q(v_q)\|_Q \le A^* + \|I_p\|_Q \tag{3.12}$$

and

$$\|\gamma_q v_q - \sum_{p=1}^{n}(b_{qp} + d_{qp})G_p(u_p)\|_Q \le A^* + \|J_q\|_Q. \tag{3.13}$$

It follows from (3.12) that

$$\begin{aligned}\alpha_p\|u_p\|_Q &\le \Big\|\sum_{q=1}^{m}\Big[(a_{pq} + c_{pq})[F_q(v_q) - F_q(0)] + (a_{pq} + c_{pq})F_q(0)\Big\|_Q + A^* + \|I_p\|_Q \\ &\le \sum_{q=1}^{m}\Big[\|a_{pq}\|_Q L_q\|v_q\|_Q + \|c_{pq}\|_Q L_q\|v_q\|_Q + \|a_{pq} + c_{pq}|_Q\|\|F_q(0)\|_Q\Big] + A^* + \|I_p\|_Q,\end{aligned}$$

which implies

$$\alpha^2\sum_{p=1}^{n}\|u_p\|_Q^2$$

$$
\begin{aligned}
&\le \sum_{p=1}^{n}\alpha_p^2\|u_p\|_Q^2\\
&\le \sum_{p=1}^{n}\Big[\sum_{q=1}^{m}(\|a_{pq}\|_Q+\|c_{pq}\|_Q)L_q\|v_q\|_Q+C_p\Big]^2\\
&= \sum_{p=1}^{n}\Big\{\Big(\sum_{q=1}^{m}(\|a_{pq}\|_Q+\|c_{pq}\|_Q)L_q\|v_q\|_Q\Big)^2+C_p^2+2C_p\sum_{q=1}^{m}(\|a_{pq}\|_Q+\|c_{pq}\|_Q)L_q\|v_q\|_Q\Big\}\\
&\le \sum_{p=1}^{n}\Big\{m\sum_{q=1}^{m}\Big((\|a_{pq}\|_Q+\|c_{pq}\|_Q)L_q\|v_q\|_Q\Big)^2+C_p^2+2\sum_{p=1}^{n}C_p\sum_{q=1}^{m}[\|a_{pq}\|_Q+\|c_{pq}\|_Q]L_q\|v_q\|_Q\Big\}\\
&\le 2m\sum_{p=1}^{n}\sum_{q=1}^{m}\Big(\|a_{pq}\|_Q^2L_q^2\|v_q\|_Q^2+\|c_{pq}\|_Q^2L_q^2\|v_q\|_Q^2\Big)+\sum_{p=1}^{n}C_p^2+2\sum_{p=1}^{n}C_p\sum_{q=1}^{m}[\|a_{pq}\|_Q+\|c_{pq}\|_Q]L_q\|v_q\|_Q\\
&\le 2m(A+A_1)\sum_{q=1}^{m}\|v_q\|_Q^2+\sum_{p=1}^{n}C_p^2+2D\sum_{q=1}^{m}\|v_q\|_Q\\
&\le 2m(A+A_1)\sum_{q=1}^{m}\|v_q\|_Q^2+\sum_{p=1}^{n}C_p^2+2D\sqrt{m}\sqrt{\sum_{q=1}^{m}\|v_q\|_Q^2},
\end{aligned}
\tag{3.14}
$$

where $C_p = A^* + \|I_p\|_Q + \sum_{q=1}^{m}\|a_{pq}+c_{pq}\|_Q\|F_q(0)\|_Q, D = \max_{1\le q\le m}\{\sum_{p=1}^{n} C_p[\|a_{pq}\|_Q+\|c_{pq}\|_Q]L_q.$

Similarly, from (3.13), we can get

$$
\gamma^2\sum_{q=1}^{m}\|v_q\|_Q^2 \le 2n(B+B_1)\sum_{p=1}^{n}\|u_p\|_Q^2+\sum_{q=1}^{m}(C_q^*)^2+2D_1\sqrt{n}\sqrt{\sum_{p=1}^{n}\|u_p\|_Q^2}, \tag{3.15}
$$

where $C_q^* = A^* + \|J_p\|_Q + \sum_{p=1}^{n}[\|b_{qp}\|_Q+\|d_{qp}\|_Q]\|G_p(0)\|_Q, D_1 = \max_{1\le p\le n}\{\sum_{q=1}^{m} C_q^*[\|b_{qp}\|_Q+\|d_{qp}\|_Q]L_p.$

Denoting $M_1^2 = \sum_{p=1}^{n}\|u_p\|_Q^2, M_2^2 = \sum_{q=1}^{m}\|v_q\|_Q^2$, on the basis of (3.14) and (3.15), we have

$$
\alpha^2 M_1^2 \le 2m(A+A_1)M_2^2+\sum_{p=1}^{n}C_p^2+2D\sqrt{m}M_2 \tag{3.16}
$$

and

$$
\gamma M_2^2 \le 2n(B+B_1)M_1^2+\sum_{q=1}^{m}(C_q^*)^2+2D_1\sqrt{n}M_1. \tag{3.17}
$$

Next we consider two possible cases: (1) $M_2 \le M_1$; (2) $M_2 > M_1$.
(1) When $M_2 \le M_1$, (3.16) implies

$$
[\alpha^2-2m(A+A_1)]M_1^2-\sum_{p=1}^{n}C_p^2-2D\sqrt{m}M_1 \le 0.
$$

As a result,

$$\begin{aligned} M_2 &\leq M_1 \\ &\leq \frac{D\sqrt{m} + \sqrt{D^2 m + [\alpha^2 - 2m(A + A_1)] \sum_{p=1}^{n} c_p^2}}{\alpha^2 - 2m(A + A_1)} \\ &= d_1. \end{aligned}$$

Then

$$\begin{aligned} \|(u, v)\|_{Q^n} &= \sqrt{\sum_{p=1}^{n} u_p^* u_p + \sum_{q=1}^{m} v_q^* v_q} \\ &\leq \sqrt{d_1^2 + d_1^2} \\ &= \sqrt{2} d_1. \end{aligned}$$

This contradicts the choice of r in (3.11).

(2) When $M_2 > M_1$, (3.17) implies

$$(\gamma - 2nB)M_2^2 - 2mC_1^2 - 4C_1 D_1 \sqrt{n} M_2 \leq 0.$$

As a result,

$$\begin{aligned} M_1 &\leq M_2 \\ &\leq \frac{D_1\sqrt{n} + \sqrt{D_1^2 n + [2\gamma - 2n(B + B_1) \sum_{q=1}^{m} (C_q^*)^2}}{\gamma^2 - 2n(B + B_1)} \\ &= d_2. \end{aligned}$$

Then

$$\begin{aligned} \|(u, v)\|_{Q^n} &= \sqrt{\sum_{p=1}^{n} u_p^* u_p + \sum_{q=1}^{m} v_q^* v_q} \\ &\leq \sqrt{d_2^2 + d_2^2} \\ &= \sqrt{2} d_2. \end{aligned}$$

This contradicts the choice of r in (3.11).

By the discussions of (1) and (2), we have

$$\lim_{\|(u,v)\|_{Q^n} \to \infty} \|H(u, v)\|_{Q^n} = \infty.$$

Thus (b) in Lemma 2.2 is fulfilled. By Lemma 2.2, system (1.1) has a unique equilibrium point.

4. Asymptotic stability of equilibrium point

Theorem 4.1. Assume that (D_1) holds. Further assume that $\tau'(t) \leq \tau^+ < 1, \sigma'(t) \leq \sigma^+ < 1$. Then system (1.1) has a unique equilibrium point which is globally asymptotically stable if the following conditions hold:

(D_6)

$$2\alpha > 1 + 2mA + \frac{2mA_1}{1-\tau^+}, \alpha \neq 1,$$

(D_7)

$$2\gamma > 1 + 2nB + \frac{2nB_1}{1-\sigma^+}, \gamma \neq 1.$$

Proof. Since $0 < 1 - \tau^+ < 1$, then $\frac{2mA_1}{1-\tau^+} > 2mA_1$. Thus by (3.4), we obtain $2\alpha > 1 + 2m(A + A_1)$. Consequently (D_2) holds. Similarly, by (3.5), we can prove (D_3) holds. By Theorem 3.1, system (1.1) has a unique equilibrium point, say, $(u^+, v^+) = (u_1^+, u_2^+, \cdots, u_n^+, v_1^+, v_2^+, \cdots, v_m^+)$. Let $(u(t), v(t)) = (u_1(t), u_2(t), \cdots, u_n(t), v_1(t), v_2(t), \cdots, v_m(t))$ be a arbitrary solution of system (1.1).

Construct two Lyapunov functions as follows:

$$K_1(t) = \sum_{p=1}^{n} [u_p(t) - u_p^+]^* [u_p(t) - u_p^+],$$

$$K_2(t) = \sum_{q=1}^{m} [v_q(t) - v_q^+]^* [v_q(t) - v_q^+].$$

By system (1.1), we have

$$\begin{aligned}
K_1'(t) &= \sum_{p=1}^{n} \Big(([u_p(t) - u_p^+]^*)' [u_p(t) - u_p^+] + [u_p(t) - u_p^+]^* [u_p(t) - u_p^+]' \Big) \\
&= \sum_{p=1}^{n} \Big\{ \Big[-\alpha_p [u_p(t) - u_p^+]^* + \sum_{q=1}^{m} a_{pq}^* [F_q(v_q(t)) - F_q(v_q^+)]^* + \sum_{q=1}^{m} c_{pq}^* [F_q(v_q(t-\tau(t))) - \\
&\quad F_q(v_q^+)]^* \Big] [u_p(t) - u_p^+] + [u_p(t) - u_p^+]^* \Big[-\alpha_p [u_p(t) - u_p^+] + \sum_{q=1}^{m} a_{pq} [F_q(v_q(t)) - F_q(v_q^+)] + \\
&\quad \sum_{q=1}^{n} c_{pq} [F_q(v_q(t-\tau(t))) - F_q(v_q^+)] \Big] \Big\}. \qquad (4.1)
\end{aligned}$$

By employing Lemma 2.1 and (D_1), it follows that

$$\begin{aligned}
&[u_p(t) - u_p^+]^* \Big(\sum_{q=1}^{m} a_{pq} [F_q(v_q(t)) - F_q(v_q^+)] + \sum_{q=1}^{n} c_{pq} [F_q(v_q(t-\tau(t))) - F_q(v_q^+)] \Big) + \\
&\sum_{q=1}^{m} a_{pq}^* [F_q(v_q(t)) - F_q(v_q^+)]^* + \sum_{q=1}^{m} c_{pq}^* [F_q(v_q(t-\tau(t))) - F_q(v_q^+)]^* \Big] [u_p(t) - u_p^+] \\
\leq\ & [u_p(t) - u_p^+]^* [u_p(t) - u_p^+] + \Big(\sum_{q=1}^{m} a_{pq} [F_q(v_q(t)) - F_q(v_q^+)] + \sum_{q=1}^{m} c_{pq} [F_q(v_q(t-\tau(t))) -
\end{aligned}$$

$$
\begin{aligned}
& F_q(v_q^+)]\Big)^*\Big(\sum_{q=1}^m a_{pq}[F_q(v_q(t)) - F_q(v_q^+)] + \sum_{q=1}^m c_{pq}[F_q(v_q(t-\tau(t))) - F_q(v_q^+)]\Big) \\
= \;& \|u_p(t) - u_p^+\|_Q^2 + \Big\|\sum_{q=1}^m a_{pq}[F_q(v_q(t)) - F_q(v_q^+)] + \sum_{q=1}^m c_{pq}[F_q(v_q(t-\tau(t))) - F_q(v_q^+)]\Big\|_Q^2 \\
\le \;& \|u_p(t) - u_p^+\|_Q^2 + \Big[\sum_{q=1}^m \|a_{pq}\|_Q L_q \|v_q(t) - v_q^+\|_Q + \sum_{q=1}^m \|c_{pq}\|_Q L_q \|v_q(t-\tau(t)) - v_q^+\|_Q\Big]^2 \\
\le \;& \|u_p - u_p^+\|_Q^2 + m\sum_{q=1}^m L_q^2\Big(\|a_{pq}\|_Q\|v_q(t) - v_q^+\|_Q + \|c_{pq}\|_Q\|v_q(t-\tau(t)) - v_q^+\|_Q\Big)^2 \\
\le \;& \|u_p - u_p^+\|_Q^2 + 2m\sum_{q=1}^m L_q^2\Big[\|a_{pq}\|_Q^2\|v_q(t) - v_q^+\|_Q^2 + \|c_{pq}\|_Q^2\|v_q(t-\tau(t)) - v_q^+\|_Q^2\Big].
\end{aligned} \tag{4.2}
$$

Substituting (4.2) into (4.1) gives

$$
\begin{aligned}
& K_1'(t) \\
\le \;& \sum_{p=1}^n \Big\{(1-2\alpha_p)\|u_p(t) - u_p^+\|_Q + 2m\sum_{q=1}^m L_q^2\Big[\|a_{pq}\|_Q^2|v_q(t) - v_q^+\|_Q + \|c_{pq}\|_Q^2\|v_q(t-\tau(t)) - v_q^+\|_Q^2\Big]\Big\} \\
\le \;& (1-2\alpha)K_1(t) + 2mAK_2(t) + 2mA_1K_2(t-\tau(t)).
\end{aligned} \tag{4.3}
$$

Similarly, we can get

$$
K_2'(t) \le (1-2\gamma)K_2(t) + 2nBK_1(t) + 2nB_1K_1(t-\sigma(t)). \tag{4.4}
$$

Because $(u_1(t), u_2(t), \cdots, u_n(t), v_1(t), v_2(t), \cdots, v_m(t))$ is a solution of system (3.3) with initial values, according to the uniqueness and existence theorem of solutions of differential equations, the solution of system (3.3) which satisfies the initial conditions exists and is unique. Consequently, $K_1(t) = \sum_{p=1}^n [v_p(t) - v_p^+]^*[v_p(t) - v_p^+]$ and $K_2(t) = \sum_{q=1}^m [v_q(t) - v_q^+][v_q(t) - v_q(t)]$ converge to one point.

Discuss two possible cases: (1) $K_2(t) \le K_1(t)$; (2) $K_2(t) > K_1(t)$.

(1) When $K_2(t) \le K_1(t)$, from (4.3), we have

$$
K_1'(t) \le (1-2\alpha)K_1(t) + 2mAK_1(t) + 2mA_1K_1(t-\tau(t)). \tag{4.5}
$$

Define $V_1(t) = K_1(t) + \frac{2mA_1}{1-\tau^+}\int_{t-\tau(t)}^t K_1(s)ds$, then in view of (4.5), it follows that

$$
\begin{aligned}
V_1'(t) \;=\;& K_1'(t) + 2mA_1\Big[\frac{K_1(t)}{1-\tau^+} - \frac{1-\tau'(t)}{1-\tau^+}K_1(t-\tau(t)\Big] \\
\le \;& \Big(1 - 2\alpha + 2mA + \frac{2mA_1}{1-\tau^+}\Big)K_1(t) \\
\le \;& 0.
\end{aligned} \tag{4.6}
$$

(2) When $K_2(t) > K_1(t)$, from (4.4), we have

$$
K_2'(t) \le (1-2\gamma)K_2(t) + 2nBK_2(t) + 2nB_1K_2(t-\sigma(t)). \tag{4.7}
$$

Define $V_2(t) = K_2(t) + \frac{2nB_1}{1-\sigma^+}\int_{t-\sigma(t)}^{t} K_2(s)ds$, then in view of (4.7), it follows that

$$\begin{aligned} V_2'(t) &= K_2'(t) + 2nB_1\Big[\frac{K_2(t)}{1-\sigma^+} - \frac{1-\sigma'(t)}{1-\sigma^+}K_2(t-\sigma(t)\Big] \\ &\le \Big(1 - 2\gamma + 2nB + \frac{2nB_1}{1-\sigma^+}\Big)K_2(t) \\ &\le 0. \end{aligned} \tag{4.8}$$

From the discussions of (1) and (2), namely from (4.6) and (4.8), it follows that system has a unique equilibrium point which is globally asymptotically stable.

Claim2. So far, in many papers which studied the existence and uniqueness of equilibrium point of neural networks, the results have been gained mainly employing Homeomorphism theorem and applying $[H(x) - H(\overline{x})]^T M[H(x) - H(\overline{x})] > 0(< 0)$(M is a positive or negative constant, M can be a positive or negative definite matrix). In our paper, we employ Homeomorphism theorem, but without applying $[H(x) - H(\overline{x})]^T M[H(x) - H(\overline{x})] > 0(< 0)$, we apply a contradictory method to discuss the equilibrium point. Namely, the method of proving the existence of equilibrium point for quaternion-valued neural networks is different from those in the existing papers [6–8,24–26,29].

Claim3. In many papers [4–15,19,21–30] which investigated the global exponential/asymptotic stability of equilibrium point for neural networks, a Lyapunov functional is constructed to obtain the sufficient conditions of global asymptotic/exponential stability for neural networks, while in our paper, two Lyapunov functionals respectively are constructed to reach the sufficient condition on global asymptotic stability of equilibrium point for neural networks.

Claim4. Even if when the quaternion-valued BAM neural networks discussed by us degenerate into real-valued neural networks [4–14,24–26] and complex-valued neural networks [28,29], the results and study method in our paper are also completely new.

Claim5. The study approach in our paper can be applied to dealing with the global asymptotic/exponential stability for the real-valued BAM neural networks, complex-valved neural networks and Octonion-valued neural networks.

Claim6. The inequality techniques used in our are different from those used in [30–33].

5. An example

Example 5.1. We discuss the following quaternion BAM neural networks with time-varying delays for $n = 2, m = 2, p = q = 1, 2,$

$$\begin{aligned} u_p'(t) &= -\alpha_p u_p(t) + \sum_{q=1}^{m} a_{pq}F_q(v_q(t)) + \sum_{q=1}^{m} c_{pq}F_q(v_q(t-\tau(t)) + I_p, \\ v_q'(t) &= -\gamma_q v_q(t) + \sum_{p=1}^{n} b_{qp}G_p(u_p(t))] + \sum_{p=1}^{n} d_{qp}G_p(u_p(t-\sigma(t)) + J_q, \end{aligned} \tag{5.1}$$

where $a_{11} = a_{21} = 0.5 + i - 0.6j + 0.2k$, $a_{12} = a_{22} = 0.3 - 0.1i + j - 0.4k$, $b_{11} = b_{21} = 0.7 + 0.6i + 0.9j - 0.3k$, $b_{12} = b_{22} = 0.3 + 0.4i - 0.6j + 0.8k$, $c_{11} = c_{21} = 0.1 - 0.2i - 0.3j + 0.4k$, $c_{12} = c_{22} = 0.2-0.5i+0.7j-01k$, $d_{11} = d_{21} = -0.4-0.5i+0.6j-0.3k$, $d_{12} = d_{22} = 0.3+0.4i-0.6j+0.8k$, $I_p = 1 + 2i + 3j + 4k, J_p = 0.4 + 06i + 0.8j + k, F_1(u) = F_2(u) = \frac{1}{5}|u^R| + i\frac{1}{6}|u^I| + j\frac{1}{7}|u^J| + k\frac{3}{8}|u^K|, G_1(u) =$

$G_2(u) = -\frac{1}{3}|u^R| + i\frac{3}{4}|u^I| + j\frac{1}{5}|u^J| + k\frac{2}{7}|u^K|, \tau(t) = 1 + 0.5 \sin t, \sigma(t) = 1 + 0.3 \cos t$. Consequently, in Theorem 4.1, $\tau^+ = 0.5, \sigma^+ = 0.5, L_q = \frac{3}{8}, L_p = \frac{3}{4}, \alpha_1 = 3, \alpha_2 = 5, \gamma_1 = 10, \gamma_2 = 9$. It is easy to know $A = 0.4640, A_1 = 0.2222, B = 1.9688, B_1 = 0.9675, \alpha = 3, \gamma = 9$. It is easy to verify that $(D_1) - (D_7)$ are all satisfied with these parameters.

Through using the Matlab software, the curves of variables are abstracted in the following figures.

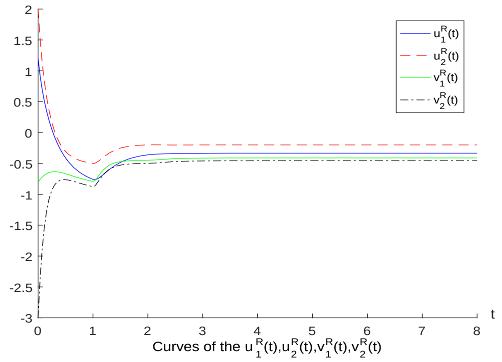

Figure 1. Curves of U_p^R, V_q^R.

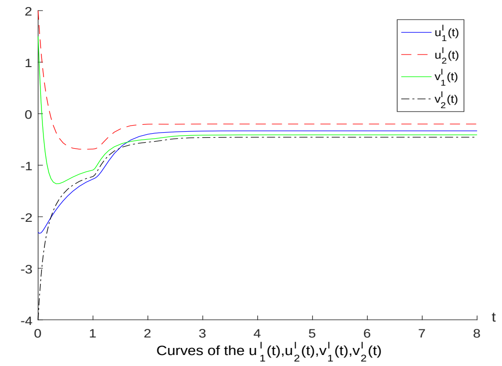

Figure 2. Curves of U_p^I, V_q^I.

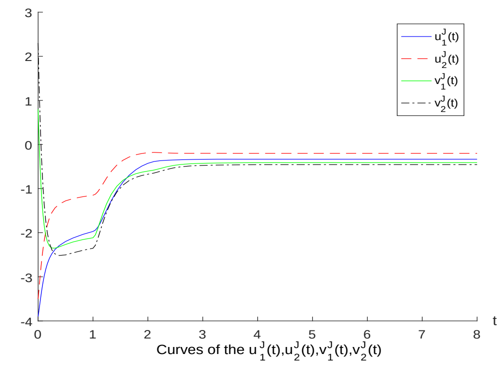

Figure 3. Curves of U_p^J, V_q^J.

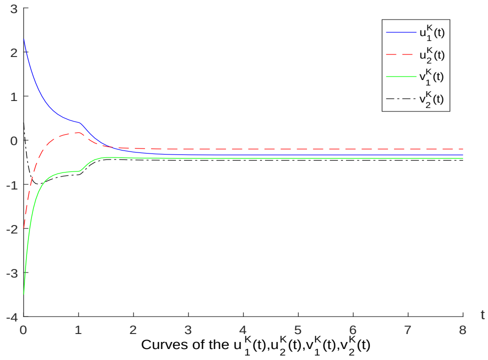

Figure 4. Curves of U_p^K, V_q^K.

Figures 1–4 show that the neuron states'curves of quaternion-valued neural networks:

$$u_p(t) = u_p(t)^R - u_p(t)^I i - u_p(t)^j j - u_p(t)^K k \in Q,$$
$$v_q(t) = v_q(t)^R - v_q(t)^I i - v_q(t)^j j - v_q(t)^K k \in Q.$$

It can be seen from the Figures 1–4 that each component solution tends to be globally asymptotically stable after a fixed point with the change of time T ,which point that the curves of the components behind become horizontal, meaning the components' derivatives is 0, so this fixed point is the only equilibrium point. It is a clear argument that the assumptions are all met with our conditions. As a result, under the all assumptions in Theorems 3.1 and 4.1, the system has a unique equilibrium solution which can be globally asymptotically stable.

6. Conclusion

Since the quaternion-valued BAM neural networks have the behaviors of the real-valued BAM neural networks and complex-valued BAM neural networks, then the study of the global asymptotic stability for a class of quaternion-valued BAM neural networks is of definite meaning in theory. In this paper, we explore the existence and stability of the equilibrium point for a class of quaternion-valued BAM neural networks with time-varying delays. By non-decomposing the system into eight real-valued systems and by applying a contradictory method and constructing two Lyapunov functionals, novel sufficient conditions to ensure the existence and stability of equilibrium point of above networks are gained. Contradictory method and constructing two Lyapunov functionals are completely novel study approach of global asymptotic stability. In the near future, we will study the finite-time synchronization of quaternion-valued neural networks.

Acknowledgments

The work was supported by the Science and Technology Research Project of the Education Department of Hebei Province (No. QN2021009).

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

1. Y. K. Li, J. L. Qin, B. Li, Anti-periodic solutions for quaternion-valued high-order Hopfield neural networks with time-varying delays, *Neural Process. Lett.*, **49** (2019), 1217–1237. https://doi.org/10.1007/s11063-018-9867-8
2. N. N. Huo, B. Li, Y. K. Li, Existence and exponential stability of anti-periodic solutions for inertial quaternion-valued high-order Hopfield neural networks with state-dependent delays, *IEEE Access*, **7** (2019), 60010–60019. https://doi.org/10.1109/ACCESS.2019.2915935
3. Q. K. Song, X. F. Chen, Multistability analysis of quaternion-valued neural networks with time delays, *IEEE T. Neur. Net. Lear.*, **29** (2018), 5430–5440. https://doi.org/10.1109/TNNLS.2018.2801297
4. X. F. Chen, Q. K. Song, Z. S. Li, Z. J. Zhao, Y. R. Liu, Stability analysis of continuous-time and discrete-time quaternion-valued neural networks with linear threshold neurons, *IEEE T. Neur. Net. Lear.*, **29** (2018), 2769–2781. https://doi.org/10.1109/TNNLS.2017.2704286
5. X. F. Chen, Z. S. Li, Q. K. Song, J. Hu, Y. S. Tan, Robust stability analysis of quaternion-valued neural networks with time delays and parameter uncertainties, *Neural Networks*, **91** (2017), 55–65. https://doi.org/10.1016/j.neunet.2017.04.006
6. R. X. Li, X. B. Gao, J. D. Cao, K. Zhang, Stability analysis of quaternion-valued Cohen-Grossberg-Grossberg neural networks, *Math. Method. Appl. Sci.*, **42** (2019), 3721–3738. https://doi.org/10.1002/mma.5607
7. X. J. Yang, C. D. Li, Q. K. Song, J. Y. Chen, J. J. Huang, Global mittag-leffler stability and synchronization analysis of fractional-order quaternion-valued neural networks with linear threshold neurons, *Neural Networks*, **105** (2018), 88–103. https://doi.org/10.1016/j.neunet.2018.04.015
8. Y. K. Li, J. L. Qin, B. Li, Periodic solutions for quaternion-valued fuzzy cellular neural networks with time-varying delays, *Adv. Differ. Equ.*, **2019** (2019), 63. https://doi.org/10.1186/s13662-019-2008-5
9. J. W. Zhu, J. T. Sun, Stability of quaternion-valued neural networks with mixed delay, *Neural Process Lett.*, **49** (2019), 819–833. https://doi.org/10.1007/s11063-018-9849-x
10. Y. K. Li, J. L. Qin, Existence and global exponential stability of periodic solutions for quaternion-valued cellular neural networks with time-varying delays, *Neurocomputing*, **292** (2018), 91–103. https://doi.org/10.1016/j.neucom.2018.02.077
11. X. X. You, Q. K. Song, J. Liang, Y. R. Liu, F. E. Alsaadi, Global μ-stability of quaternion-valued neural networks with mixed time-varying delays, *Neurocomputing*, **290** (2018), 12–25. https://doi.org/10.1016/j.neucom.2018.02.030
12. X. W. Liu, Z. G. Li, Global μ-stability of quaternion-valued neural networks with unbounded and asynchronous time-varying delays, *IEEE Access*, **7** (2019), 9128–9141. https://doi.org/10.1109/ACCESS.2019.2891721
13. Z. W. Tu, Y. X. Zhao, N. Ding, Y. M. Teng, W. Zhang, Stability analysis of quaternion-valued neural networks with both discrete and distributed delays, *Appl. Math. Comput.*, **343** (2019), 342–353. https://doi.org/10.1016/j.amc.2018.09.049

14. M. C. Tan, Y. F. Liu, D. S. Xu, Multistability analysis of delayeed quaternion-valued neural networks with nonmonotonic piecewise nonlinear activation functions, *Appl. Math. Comput.*, **341** (2019), 229–255. https://doi.org/10.1016/j.amc.2018.08.033
15. R. Y. Wei, J. D. Cao, Fixed-time synchronization of quaternion-valued memristive neural networks with time delays, *Neural Networks*, **113** (2019), 1–10. https://doi.org/10.1016/j.neunet.2019.01.014
16. S. P. Shen, B. Li, Y. K. Li, Anti-periodic dynamics of quaternion-valued fuzzy cellular neural networks with time-varying delays on time scales, *Discrete Dyn. Nat. Soc.*, **2018** (2018), 5290786. https://doi.org/10.1155/2018/5290786
17. C. A. Popa, E. Kaslik, Multistability and muitiperiodicity in impulsive hybird quaternion-valued neural networks with mixed delays, *Neural Networks*, **99** (2018), 1–18. https://doi.org/10.1016/j.neunet.2017.12.006
18. R. Y. Wei, J. D. Cao, Synchronization control of quaternion-valued menristive neural networks with and without event-triggered scheme, *Cogn. Neyrodyn.*, **13** (2019), 489–502. https://doi.org/10.1007/s11571-019-09545-w
19. H. Q. Shen, Q. K. Song, J. Liang, Z. J. Zhao, Y. R. Liu, F. E. Alsaadi, Glibal exponential stability in lagrange sense for quaternion-valued neural networks with leakage delay and mixed time-varying delays, *Int. J. Syst. Sci.*, **50** (2019), 858–870. https://doi.org/10.1080/00207721.2019.1586001
20. D. H. Li, Z. Q. Zhang, X. L. Zhang, Periodic solutions of discrete-time Quaternion-valued BAM neural networks, *Chaos Soliton. Fract.*, **138** (2020), 110144. https://doi.org/10.1016/j.chaos.2020.110144
21. Q. K. Song, L. Y. Long, Z. J. Zhao, Y. R. Liu, F. E. Alsaadi, Stability criteria of quaternion-valued neutral-type delayed neural networks, *Neurocomputing*, **412** (2020), 287–294. https://doi.org/10.1016/j.neucom.2020.06.086
22. H. M. Wang, J. Tan, S. P. Wen, Exponential stability analysis of mixed delayed quaternion-valued neural networks via decomposed approach, *IEEE Access*, **8** (2020), 91501–91509. https://doi.org/10.1109/ACCESS.2020.2994554
23. U. Humphries, G. Rajchakit, P. Kaewmesri, P. Chanthorn, R. Sriraman, R. Samidurai, et al., Global stability analysis of fractional-order quaternion-valued bidirectional associative memory neural networks, *Mathematics*, **8** (2020), 801.https://doi.org/10.3390/math8050801
24. Z. Q. Zhang, W. B. Liu, D. M. Zhou, Global asymptotic stability to a generalized Cohen-Grossberg BAM neural networks of neutral type delays, *Neural Networks*, **25** (2012), 94–105. https://doi.org/10.1016/j.neunet.2011.07.006
25. Z. Q. Zhang, J. D. Cao, D. M. Zhou, Novel LMI-based conditioon on global asymptotic stability for a class of Cohen-Grossberg BAM networks with extended activation functions, *IEEE T. Neur. Net. Lear.*, **25** (2014), 1161–1172. https://doi.org/10.1109/TNNLS.2013.2289855
26. W. L. Peng, Q. X. Wu, Z. Q. Zhang, LMI-based global exponential stability of equilibrium point for neutral delayed BAM neural networks with delays in leakage terms via new inequality technique, *Neurocomputing*, **199** (2016), 103–113. https://doi.org/10.1016/j.neucom.2016.03.030
27. H. L. Li, X. B. Gao, R. X. Li, Exponential stability and sampled-data synchronization of

delayed complex-valued memristive neural networks, *Neural Process. Lett.*, **51** (2020), 193–209. https://doi.org/10.1007/s11063-019-10082-0

28. Z. Q. Zhang, S. H. Yu, Global asymptotic stability for a class of complex-valued Cohen-Grossberg neural networks with time delays, *Neurocomputing*, **171** (2016), 1158–1166. https://doi.org/10.1016/j.neucom.2015.07.051
29. Z. Q. Zhang, D. L. Hao, D. M. Zhou, Global asymptotic stability by complex-valued inequalities for complex-valued neural networks with delays on periodic time scales, *Neurocomputing*, **219** (2017), 494–501. https://doi.org/10.1016/j.neucom.2016.09.055
30. C. J. Xu, M. X. Liao, P. L. Li, Z. X. Liu, S. Yuan, New results on pseudo almost periodic solutions of quaternion-valued fuzzy cellular neural networks with delays, *Fuzzy Set. Syst.*, **411** (2021), 25–47. https://doi.org/10.1016/j.fss.2020.03.016
31. C. J. Xu, Z. X. Liu, M. X. Liao, P. L. Li, Q. M. Xiao, S. Yuan, Fractional-order bidirectional associate memory (BAM) neural networks with multiple delays: The case of Hopf bifurcation, *Math. Comput. Simulat.*, **182** (2021), 471–494. https://doi.org/10.1016/j.matcom.2020.11.023
32. C. J. Xu, Z. X. Liu, L. Y. Yao, C. Aouit, Further exploration on bifurcation of fractional-order sixneuron bidirectional associative memory neural networks with multi-delays, *Appl. Math. Comput.*, **410** (2021), 126458. https://doi.org/10.1016/j.amc.2021.126458
33. C. J. Xu, M. X. Liao, P. L. Li, Y. Guo, Q. M. Xiao, S. Yuan, Influence of multiple time delays on bifurcation of fractional-order neural networks, *Appl. Math. Comput.*, **361** (2019), 565–582. https://doi.org/10.1016/j.amc.2019.05.057
34. R. Zhao, B. X. Wang, J. G. Jian, Lagrange stability of BAM quaternion-valued inertial neural networks via auxiliary function-based integral inequalities, *Neural Process. Lett.*, 2022. https://doi.org/10.1007/s11063-021-10685-6
35. J. Liu, J. G. Jian, B. X. Wang, Stability analysis for quaternion-valued BAM inertial neural networks with time delay via nonlinear measure approach, *Math. Comput. Simulat.*, **174** (2020), 134–152. https://doi.org/10.1016/j.matcom.2020.03.002
36. C. J. Xu, M. X. Liao, P. L. Li, Y. Guo, Z. X. Liu, Bifurcation properties for fractional order delayed BAM neural networks, *Cogn. Comput.*, **13** (2021), 322–356. https://doi.org/10.1007/s12559-020-09782-w
37. C. J. Xu, W. Zhang, C. Aouit, Z. X. Liu, M. X. Liao, P. L. Li, Further investigation on bifurcation and their control of fractional-order bidirectional associative memory neural networks involving four neurons and multiple delays, *Math. Method. Appl. Sci.*, 2021. https://doi.org/10.1002/mma.7581
38. C. J. Xu, M. X. Liao, P. L. Li, S. Yuan, Impact of leakage delay on bifurcation in fractional-order complex-valued neural networks, *Chaos Soliton. Fract.*, **142** (2021), 110535. https://doi.org/10.1016/j.chaos.2020.110535

AIMS Press

© 2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)