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Research article

Existence fixed-point theorems in the partial b-metric spaces and an application to the boundary value problem

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Abstract: In this paper, we prove some results on the Hausdorff partial b-metrics. We prove some new Lemmas regarding convergence of the sequences in the Hausdorff partial b-metric spaces. The obtained results generalize and improve many existing fixed-point results. The examples are given for the explanation of theory. The existence of the solution to the boundary value problem is proved via fixed-point approach.

Keywords: fixed point; multivalued almost contractions; partial b-metric space; application
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1. Introduction

The almost contractions form a class of generalized contractions that includes several contractive type mappings like usual contractions, Kannan mappings, Zamfirescu mappings, etc. Since any usual contraction is continuous, while a Kannan mapping is not generally continuous but is continuous at the fixed point. The almost (multivalued) contractions are not continuous. However, the almost contraction is continuous at its fixed point(s) (see [6, 18] for details). This work has been extended to generalized multivalued almost contractions in b-metric spaces [12].The b-metric space was formally introduced by Czerwick [7]. Every metric is a b-metric, but converse is not true. The fixed point theorems in the b- metric spaces have been established by many authors (see [1, 2, 4, 13, 22] and references therein).

On the other hand, Matthews [14] introduced the notion of the partial metric space as a part of study of denotational semantics of data flow network. Every metric is a partial metric, but converse is not true. Matthews also initiated the fixed point theory in the partial metric space. He proved Banach contraction principle in this space to be applied in program verification. We can find so many fixed-point theorems in the partial metric spaces by many fixed-point theorists (see [17] and references there in). Shukla [23] extended the concept of partial metric to partial b-metric and investigated fixed points of Banach contraction and Kannan contraction in the partial b-metric spaces. Mustafa [16] modified the triangle property of partial b-metric and established convergence criterion and some working rules in partial b- metric spaces. Diana Dolicanin-Dekic [8], obtained the fixed-point theorems for Ciric type contractions in the partial b-metric spaces.

Nadler [20] extended the contraction rule to the multivalued mappings and find fixed points of such mappings. Rhoades [11], Feng and Liu [10], Altun et al. [3] and Miculescu et al. [15] added more fixed point theorems for multivalued mappings. Recently, Ameer et al. [4] presented some fixed-point results in the Hausdorff partial b-metric spaces. However, many supporting results are yet to prove. In this paper, we prove all the supporting results in the Hausdorff partial b-metric spaces and hence prove some fixed-point theorems which state some conditions for the existence of fixed points of the multivalued almost contractions. The examples and an application are presented to support this theory.

2. Preliminaries

Definition 2.1. [22] Let $X \neq \phi$ and $s \geq 1$. A mapping $p_b : X \times X \to \mathbb{R}^+$ is referred as a partial b-metric if for all $x, y, \check{t} \in X$, p_b satisfies the following conditions:

(i) $p_b(x, x) = p_b(x, y) = p_b(y, y) \Leftrightarrow x = y$.
(ii) $p_b(x, y) \leq s[p_b(x, \check{t}) + p_b(\check{t}, y)] - p_b(\check{t}, \check{t})$.
(iii) $p_b(x, y) = p_b(y, x)$.
(iv) $p_b(x, x) \leq p_b(x, y)$.

The pair (X, p_b) is called a partial b-metric space.

Each partial b-metric p_b on X induces a T_0 topology $\tau(p_b)$ on X which has as a base the family of open balls $\{B_{p_b}(x_0, \varepsilon) : x_0 \in X, \varepsilon > 0\}$, where $B_{p_b}(x_0, \varepsilon) = \{y \in X : p_b(x_0, y) < p_b(x_0, x_0) + \varepsilon\}$ for some $x_0 \in X$ and $\varepsilon > 0$. Also $\overline{B(x_0, r)} = \{y \in X : p_b(x_0, y) \leq p_b(x_0, x_0) + \varepsilon\}$ is a closed ball in (X, p_b).

It is clear that $p_b(x, y) = 0$ implies $x = y$ by (P_1) and (P_2). But if $x = y$, then $p_b(x, y)$ may not be 0. A basic example of a partial b-metric space is the pair (R_0^+, p_b), where $p_b(x, y) = (\max\{x, y\})^2$ for all $x, y \in R_0^+$.

Remark 2.2. Every partial b-metric space is a generalization of the partial metric space and the b-metric space. However, converse is not true in general.

Example 2.3. Let $X = [0, \infty)$ and $k > 1$. Define $p_b : X \times X \to [0, \infty)$ by

$$p_b(x, y) = \{(x \vee y)^k + |x - y|^k\}$$

for $x, y \in X$ is a partial b-metric on X with $s = 2^k$.
Also note that $p_b(x, x) = x^k \neq 0$. Thus p_b is not a b-metric on X.
Now let x, y and $\check{t} \in X$ such that $x > \check{t} > y$.
Then

$$(x - y)^k \quad > \quad (x - \check{t})^k + (\check{t} - y)^k.$$

$$\begin{aligned} p_b(x,y) &= x^k + (x-y)^k \\ p_b(x,\check{t}) + p_b(\check{t},y) - p_b(\check{t},\check{t}) &= x^k + (x-\check{t})^k + (\check{t}-y)^k. \end{aligned}$$

$$p_b(x,y) > p_b(x,\check{t}) + p_b(\check{t},y) - p_b(\check{t},\check{t}).$$

Hence, p_b is not a partial metric on X.

Definition 2.4. [19] Let $(X,\ p_b)$ be a partial b-metric space and $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ is said to be convergent to x if $\lim\limits_{n\to\infty} p_b(x_n,x) = p_b(x,x)$.
(ii) $\{x_n\}$ is said to be Cauchy sequence if $\lim\limits_{p,q\to\infty} p_b(x_n,x_m)$ exists and is finite.

Let $(X,\ p_b)$ be a partial b-metric space. The function $b : X\times X \to [0,\infty)$ defined by

$$b(x,\acute{u}) = 2p_b(x,\acute{u}) - p_b(x,x) - p_b(\acute{u},\acute{u}), \forall x,\acute{u} \in X, \tag{2.1}$$

satisfies all axioms of the b-metric. The pair (X,b) is a b-metric space. It is called an associated b-metric space.

Another associated b-metric is defined as follows:

Let $(X,\ p_b)$ be a partial b-metric space and the mapping $b : X\times X \to [0,\infty)$ be defined by

$$b(x,y) = \begin{cases} p_b(x,y), & x \neq y \\ 0, & x = y \end{cases}$$

for $x,y \in X$. Then b is a b-metric associated with p_b.

The following theorem states the convergence in both b-metric space and partial b-metric space.

Theorem 2.5. *Let (X,p_b) be a partial b-metric space. Define $b : X\times X \to [0,\infty)$ by*

$$b(x,y) = \begin{cases} p_b(x,y), & x \neq y \\ 0, & x = y \end{cases}$$

for $x,y \in X$. If $\lim\limits_{n\to\infty} x_n = x$ in (X,b), then $\lim\limits_{n\to\infty} x_n = x$ in (X,p_b).

Proof. Let $x_n = x$ for some n, then $\lim\limits_{n\to\infty} p_b(x_n,x) = p_b(x,x)$. This proves that $\lim\limits_{n\to\infty} x_n = x$ in (X,p_b). So we may assume that $x_n \neq x$ for all $n \in N$. Then $b(x_n,x) = p_b(x_n,x)$ for all $n \in N$. Since, $\lim\limits_{n\to\infty} x_n = x$ in (X,b), we have $\lim\limits_{n\to\infty} b(x_n,x) = 0$. Therefore, $\lim\limits_{n\to\infty} b(x_n,x) = \lim\limits_{n\to\infty} p_b(x_n,x) = 0$. Note that $0 \leq p_b(x,x) \leq p_b(x_n,x)$ for $n \in N$, then $0 \leq p_b(x,x) \leq p_b(x_n,x) = 0$. This proves $\lim\limits_{n\to\infty} p_b(x_n,x) = 0 = p_b(x,x)$, that is, $\lim\limits_{n\to\infty} x_n = x$ in (X,p_b). □

Converse of the above theorem is not true, in general. The following example explains this fact.

Example 2.6. Let $X = [0,1]$ and $p_b(x,y) = |x-y|^5 + c$ for $x,y \in X$ and $c \geq 1$. Then (X,p_b) is a partial b-metric space with $s = 16$. Note that for a sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ in X,

$$\lim_{n\to\infty} p_b\left(\frac{1}{n},0\right) = \lim_{n\to\infty}\left[\left|\frac{1}{n}-0\right|^5 + c\right] = c = p_b(0,0).$$

This prove that $\lim_{n\to\infty}\frac{1}{n} = 0$ in the partial b-metric space (X, p_b). However,

$$\lim_{n\to\infty} b\left(\frac{1}{n},0\right) = \lim_{n\to\infty}\left[\left|\frac{1}{n}-0\right|^5 + c\right] = c \neq 0.$$

This proves that the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ does not converge in the associated b-metric space (X, b).

The following Lemma relates the properties of the sequences in (X, p_b) and (X, b).

Lemma 2.7. [16]

(1) A sequence $\{x_n\}$ is a Cauchy sequence in (X, p_b) if and only if it is a Cauchy sequence in b-metric space (X, b).
(2) (X, p_b) is complete if and only if (X, b) is complete.
(3) A sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to a point $x \in X$ in (X, b) if and only if

$$\lim_{n\to\infty} p_b(x, x_n) = p_b(x, x) = \lim_{n,m\to\infty} p_b(x_n, x_m).$$

The following lemmas state the conditions for a sequence to be Cauchy-sequence in the partial b-metric spaces.

Lemma 2.8. *Let (X, p_b, s) be a partial b-metric space and the function $g : \{1, 2, 3, \cdots\} \to \{0, 1, 2, 3, \cdots\}$ be defined by $g(n) = -[-\log_2 n]$ for all $n \in \{1, 2, 3, \cdots\}$. Then for $(x_0, x_1, \cdots, x_n) \in X^{n+1}$, the following inequality holds.*

$$p_b(x_0,x_n) < s^{g(n)}\sum_{i=0}^{n-1} p_b(x_i, x_{i+1}).$$

Proof. We observed that $2^{g(n)-1} < n \leq 2^{g(n)}$ for all $n \in \{1, 2, 3, \cdots\}$. Let

$$p(n) := p_b(x_0,x_n) < s^{g(n)}\sum_{i=0}^{n-1} p_b(x_i, x_{i+1}).$$

For $p(1)$ we have $p_b(x_0,x_n) \leq s^{g(1)}\sum_{i=0}^{n-1} p_b(x_i\ , x_{i+1})$. Suppose that $p(n)$ holds for $n \leq 2^K$ for some $K \in \{0, 1, 2, 3, \cdots\}$. For all those n lying in $2^K < n \leq 2^{K+1}$, we have $g(n) = K + 1$, $g\left(2^K\right) = K$ and $g\left(n - 2^K\right) \leq K$. By triangle property of the partial b-metric, we have

$$\begin{aligned}
p_b(x_0,x_n) &\leq s p_b(x_0,x_{2^K}) + s p_b(x_{2^K}, x_n) - p_b(x_{2^K}, x_{2^K}) \\
&< s p_b(x_0,x_{2^K}) + s p_b(x_{2^K},x_n) \\
&< s s^{g(2^K)}\sum_{i=0}^{2^K-1} p_b(x_i,x_{i+1}) + s s^{g(n-2^K)}\sum_{i=2^K}^{n-1} p_b(x_i,x_{i+1})
\end{aligned}$$

$$
\begin{aligned}
&\leq s^{K+1}\sum_{i=0}^{n-1}p_b(x_i\,,x_{i+1})\\
&= s^{g(n)}\sum_{i=0}^{n-1}p_b(x_i\,,x_{i+1}).
\end{aligned}
$$

Thus (by induction), $p(n)$ is true for all n. □

The Lemma 2.8 is useful to obtain the following lemma.

Lemma 2.9. *Let (X, p_b, s) be a partial b-metric space. If there exists $\lambda \in [0, 1)$ and the sequence $\{x_n\} \subset X$ meets the following condition:*

$$p_b(x_{n+1}, x_{n+2}) < \lambda p_b(x_n, x_{n+1}),\ \textit{for all}\ n \in \{1, 2, 3, \cdots\}. \tag{2.2}$$

Then $\{x_n\}$ is a Cauchy sequence.

Proof. If $\lambda = 0$, then, (2.2) holds. Let $0 < \lambda < 1$, and choose a natural number $\acute{q}$ such that $s\lambda^{2^{\acute{q}}} < 1$. By Lemma 2.8, we have

$$
\begin{aligned}
p_b(x_n,x_m) &< s^{g(m-n)}\sum_{i=n}^{m-1}p_b(x_i, x_{i+1}),\ \text{for } n < m \leq n + 2^{\acute{q}}\\
&\leq s^{\acute{q}}\sum_{i=n}^{m-1}\lambda^{i-1}p_b(x_1, x_2)\\
&\leq s^{\acute{q}}\sum_{i=n}^{\infty}\lambda^{i-1}p_b(x_1, x_2)\\
&= s^{\acute{q}}\frac{\lambda^{n-1}}{1-\lambda}p_b(x_1, x_2).
\end{aligned}
$$

For $m > n + 2^{\acute{q}}$, the inequality (2.2) also holds. □

Definition 2.10. [20] Let $CB(X)$ be the class of all non-empty, closed and bounded subsets of the metric space (X, d). For $A, E \in CB(X)$, we define

$$H(A, E) = \max\left\{\sup_{\gamma\in A} d(\gamma, E), \sup_{\alpha\in E} d(\alpha, A)\right\},$$

where, $d(x, A) = \inf\{d(x, a) : a \in A\}$ is the distance of a point x to the set A. It is known that H is a metric on $CB(X)$, called the Hausdorff metric induced by the metric d.

Definition 2.11. [20] Let (X, d) be the metric space. An element $x \in X$ is labeled as a fixed point of a multivalued mapping $\psi : X \to 2^X$, if $x \in \psi(x)$.

A multivalued mapping $\psi : X \to CB(X)$ is called contraction if there exists $\lambda \in [0, 1)$ such that

$$H(\psi(x), \psi(y)) \leq \lambda d(x, y)$$

for each $x, y \in X$. It is known as multivalued contraction.

3. The properties of the Hausdorff partial b-metric spaces

The concept of Hausdorff metric or Hausdorff distance was first introduced by Hausdorff in his book Grundzuge der Mengenlehre [21]. The second name of Hausdorff distance is Pompeiu–Hausdorff distance. The Hausdorff distance has many applications in the computer field. The use of Hausdorff distance is to find a given template in an arbitrary target image in computer vision. The most important application of the Hausdorff metric in computer graphics is to measure the difference between two different representations of the same 3D object specifically when generating the level of detail for efficient display of complex 3D models.

In this section, we state and prove the supporting properties of the Hausdorff partial b-metric. These properties are comparable to Proposition 2.2 and proposition 2.3 presented in [5].

Let (X, p_b) denote the partial b-metric space and $CB_{p_b}(X)$ denote the family of all non-empty bounded and closed subsets of X with respect to partial b-metric. Note that the closedness is taken from (X, τ_{p_b}) and the boundedness is given as follows: A is a bounded subset in (X, p_b) if there exists $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_{p_b}(x_0, M)$, that is, $p_b(x_0, a) < p_b(x_0, x_0) + M$. The following distance functions are required in the proofs.

(1) Let the mapping $f_{p_b} : X \times CB_{p_b}(X) \to [0, \infty)$ be defined by

$$f_{p_b}(x, A) = \inf\{p_b(x, \acute{u}), \acute{u} \in A\}.$$

(2) Let the mapping $g_{p_b} : CB_{p_b}(X) \times CB_{p_b}(X) \to [0, \infty)$ be defined by

$$g_{p_b}(A, D) = \sup\{f_{p_b}(\acute{u}, D) : \acute{u} \in A\} \text{ and } g_{p_b}(D, A) = \sup\{f_{p_b}(u, A) : u \in D\}.$$

(3) Let the mapping $\Upsilon_{p_b} : CB_{p_b}(X) \times CB_{p_b}(X) \to [0, \infty)$ be defined by

$$\Upsilon_{p_b}(A, D) = \max\{g_{p_b}(A, D), g_{p_b}(D, A)\}, \text{ for all } A, D \in CB_{p_b}(X).$$

Let A be any non-empty set in (X, p_b), then, $\acute{u} \in \overline{A}$ if and only if $f_{p_b}(\acute{u}, A) = p_b(\acute{u}, \acute{u})$ for all $\acute{u} \in A$. The set $\overline{A}$ denotes the closure of A with respect to partial b-metric space (X, p_b). Moreover, A is closed in (X, p_b) if and only if $A = \overline{A}$. The following Lemmas have been stated without proofs in [9]. We give their proofs in detail.

Lemma 3.1. *Let (X, p_b) be a partial b-metric space. For all $A, D, \hat{H} \in CB_{p_b}(X)$ the following equations hold:*

(i) $g_{p_b}(A, A) = \sup\{p_b(\acute{u}, \acute{u}) : \acute{u} \in A\}$.
(ii) $g_{p_b}(A, A) \leq g_{p_b}(A, D)$.
(iii) $g_{p_b}(A, D) = 0$ *implies that* $A \subseteq D$.
(iv) $g_{p_b}(A, D) \leq s[g_{p_b}(A, \hat{H}) + g_{p_b}(\hat{H}, D)] - \inf_{\sigma \in \hat{H}} p_b(\sigma, \sigma)$.

Proof. (i). If $A \in CB_{p_b}(X)$, then for all $\acute{u} \in A$, we have $f_{p_b}(x, A) = p_b(x, \acute{u})$. Therefore

$$g_{p_b}(A, A) = \sup\{p_b(\acute{u}, A) : \acute{u} \in A\} = \sup\{p_b(\acute{u}, \acute{u}) : \acute{u} \in A\}.$$

(ii). Let $\acute{u} \in A$. Since, $p_b(\acute{u}, \acute{u}) \leq p_b(\acute{u}, b)$ for all $b \in D$, therefore we have

$$p_b(\acute{u}, \acute{u}) \leq f_{p_b}(\acute{u}, D) \leq g_{p_b}(A, D)$$

by (i), we have

$$g_{p_b}(A,A) = \sup\{p_b(ú,ú) : ú \in A\} \leq g_{p_b}(A,D).$$

Hence, $g_{p_b}(A,A) \leq g_{p_b}(A,D)$.
(iii). Suppose that $ú \in A$ and $g_{p_b}(A,D) = 0$, then it implies $f_{p_b}(ú,D) = 0$ for each $ú \in A$. By (i) and (ii) it follows that $p_b(ú,ú) \leq g_{p_b}(A,D) = 0$. That is $p_b(ú,ú) = 0$ for all $ú \in A$, and hence $p_b(ú,ú) = f_{p_b}(ú,D)$ for all $ú \in A$. Since, D is closed, we have $ú \in \overline{D} = D$ and $A \subseteq D$.
(iv). Assume that $ú \in A, \varkappa \in D$ and $\sigma \in \hat{H}$. By triangle property, we have

$$p_b(ú,\varkappa) \leq s[p_b(ú,\sigma) + p_b(\sigma,\varkappa)] - p_b(\sigma,\sigma).$$

Then

$$\begin{aligned}
\inf_{\varkappa\in D} p_b(ú,\varkappa) &\leq s[\inf_{\varkappa\in D} p_b(ú,\sigma) + \inf_{\varkappa\in D} p_b(\sigma,\varkappa)] - \inf_{\varkappa\in D} p_b(\sigma,\sigma).\\
f_{p_b}(ú,D) &\leq s\left[p_b(ú,\sigma) + f_{p_b}(\sigma,D)\right] - p_b(\sigma,\sigma).\\
f_{p_b}(ú,D) + p_b(\sigma,\sigma) &\leq s[p_b(ú,\sigma) + f_{p_b}(\sigma,D)].
\end{aligned}$$

Now

$$\begin{aligned}
\sup_{\sigma\in\hat{H}} f_{p_b}(ú,D) + \sup_{\sigma\in\hat{H}} p_b(\sigma,\sigma) &\leq s[\sup_{\sigma\in\hat{H}} p_b(ú,\sigma) + \sup_{\sigma\in\hat{H}} f_{p_b}(\sigma,D)].\\
f_{p_b}(ú,D) + p_b(\sigma,\sigma) &\leq s[p_b(ú,\sigma) + g_{p_b}(\hat{H},D)].
\end{aligned}$$

Taking sup with respect to $ú$, we have

$$g_{p_b}(A,D) \leq s[g_{p_b}(A,\hat{H}) + g_{p_b}(\hat{H},D)] - \inf_{\sigma\in\hat{H}} p_b(\sigma,\sigma).$$

□

Lemma 3.2. *Let (X,p_b) be a partial b-metric space. Then, for all $A,D,\hat{H} \in CB_{p_b}(X)$, we have*

(i) $\Upsilon_{p_b}(A,A) \leq \Upsilon_{p_b}(A,D)$.
(ii) $\Upsilon_{p_b}(A,D) = \Upsilon_{p_b}(D,A)$.
(iii) $\Upsilon_{p_b}(A,D) \leq s[\Upsilon_{p_b}(A,\hat{H}) + \Upsilon_{p_b}(\hat{H},D)] - \inf_{\sigma\in\hat{H}} p_b(\sigma,\sigma)$.

Proof. (i). By definition $\Upsilon_{p_b}(A,A) = \max\{g_{p_b}(A,A), g_{p_b}(A,A)\}$, so, that $\Upsilon_{p_b}(A,A) = g_{p_b}(A,A)$ and $\Upsilon_{p_b}(A,D) = \max\{g_{p_b}(A,D), g_{p_b}(D,A)\}$. By Lemma 3.1, $g_{p_b}(A,A) \leq g_{p_b}(A,D)$ so that $\Upsilon_{p_b}(A,A) \leq \Upsilon_{p_b}(A,D)$.
(ii). Obvious.

$$\begin{aligned}
\text{(iii). } \Upsilon_{p_b}(A,D) &= \max\{g_{p_b}(A,D), g_{p_b}(D,A)\} \leq \max\{s[g_{p_b}(A,\hat{H}) + g_{p_b}(\hat{H},D)]\\
&\quad -\inf_{\sigma\in\hat{H}} p_b(\sigma,\sigma), s[g_{p_b}(\hat{H},A) + g_{p_b}(D,\hat{H})] - \inf_{\sigma\in\hat{H}} p_b(\sigma,\sigma)\}\\
&= \max\{sg_{p_b}(A,\hat{H}) + sg_{p_b}(\hat{H},D), sg_{p_b}(\hat{H},A) + sg_{p_b}(D,\hat{H})\} - \inf_{\sigma\in\hat{H}} p_b(\sigma,\sigma)\\
&= \max\{sg_{p_b}(A,\hat{H}) + sg_{p_b}(\hat{H},A), sg_{p_b}(D,\hat{H}) + sg_{p_b}(\hat{H},D)\} - \inf_{\sigma\in\hat{H}} p_b(\sigma,\sigma)
\end{aligned}$$

$$= \quad s[\max\{g_{p_b}(A,\hat{H}), g_{p_b}(\hat{H},A)\} + \max\{g_{p_b}(D,\hat{H}), g_{p_b}(\hat{H},D)\}] - \inf_{\sigma\in\hat{H}} p_b(\sigma,\sigma)$$
$$= \quad s[\Upsilon_{p_b}(A,\hat{H}) + \Upsilon_{p_b}(\hat{H},D)] - \inf_{\sigma\in\hat{H}} p_b(\sigma,\sigma).$$

□

Corollary 3.3. *Let (X, p_b) be a partial b metric space. For all $A, D, \in CB_{p_b}(X)$, the following holds*

$$\Upsilon_{p_b}(A,D) = 0 \iff A = D.$$

Lemma 3.4. *Let (X, p_b) be a partial b-metric space and $A, D, \in CB_{p_b}(X)$. For all $x \in A$, there exists $y = y(x) \in D$ and $h > 1$ such that*

$$p_b(x,y) \le h\Upsilon_{p_b}(A,D).$$

Proof. **Case 1.** If $A = D$, then we have

$$\Upsilon_{p_b}(A,D) = \Upsilon_{p_b}(A,A) = g_{p_b}(A,D) = \sup_{x\in D} p_b(x,x).$$

Let $x \in A$, since $h > 1$, we have

$$p_b(x,x) \le \sup_{x\in D} p_b(x,x) = \Upsilon_{p_b}(A,D) \le h\Upsilon_{p_b}(A,D).$$

Case 2. If $A \neq D$, and suppose that there exists $x \in A$ such that $p_b(x,y) > \Upsilon_{p_b}(A,D)$ for all $y \in D$. This implies that

$$\inf\{p_b(x,y) : y \in D\} \ge h\Upsilon_{p_b}(A,D).$$

Thus,

$$f_{p_b}(x,D) \ge h\Upsilon_{p_b}(A,D).$$

Consider,

$$\Upsilon_{p_b}(A,D) \ge g_{p_b}(A,D) = \sup_{x\in A} p_b(x,D) \ge f_{p_b}(x,D) \ge h\Upsilon_{p_b}(A,D).$$

This implies that $h \le 1$, a contradiction. Hence,

$$p_b(x,y) \le h\Upsilon_{p_b}(A,D).$$

□

The following is the main result on the multivalued almost contractions.

Theorem 3.5. *Let (X, p_b) be a complete partial b-metric space. Suppose that $\Psi : X \to CB_{p_b}(X)$ is an multivalued almost contraction, that is, for all $x, y \in X$, there exist $\mu, \upsilon \in (0,1)$ satisfying $2(\mu+\upsilon) < 1$ and $s < \frac{\upsilon}{\mu} + 2$ such that*

$$\Upsilon_{p_b}(\Psi x, \Psi y) \le \mu p_b(x,y) + \upsilon f_{p_b}(y, \Psi x). \tag{3.1}$$

Then Ψ admits a fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in \Psi(x_0)$. We construct an iterative sequence $\{x_n\}$ such that $x_{n+1} \in \Psi(x_n)$ for all $n \in \mathbb{N}$. By using Lemma 3.4 and taking $h = \frac{1}{2(\mu+\upsilon)}$, for $x_1 \in \Psi(x_0)$ there exists $x_2 \in \Psi(x_1)$ such that

$$\begin{aligned}
p_b(x_1, x_2) &\leq \frac{1}{2(\mu+\upsilon)} \Upsilon_{p_b}(\Psi(x_0), \Psi(x_1)) \\
&\leq \frac{1}{2(\mu+\upsilon)} \left\{\mu p_b(x_0, x_1) + \upsilon f_{p_b}(x_1, \Psi(x_0))\right\} \\
&\leq \frac{1}{2(\mu+\upsilon)} \{\mu p_b(x_0, x_1) + \upsilon p_b(x_1, x_1)\} \\
p_b(x_1, x_2) &\leq \frac{1}{2(\mu+\upsilon)} \{\mu p_b(x_0, x_1) + \upsilon p_b(x_1, x_2)\} \\
&\leq \frac{1}{2(\mu+\upsilon)} \{\mu p_b(x_0, x_1) + \upsilon p_b(x_1, x_2)\} \\
&= \frac{\mu}{2(\mu+\upsilon)} p_b(x_0, x_1) + \frac{\upsilon}{2(\mu+\upsilon)} p_b(x_1, x_2).
\end{aligned}$$

Then

$$\begin{aligned}
\left[1 - \frac{\upsilon}{2(\mu+\upsilon)}\right] p_b(x_1, x_2) &\leq \frac{\mu}{2(\mu+\upsilon)} p_b(x_0, x_1) \\
\left[\frac{2\mu + 2\upsilon - \upsilon}{2(\mu+\upsilon)}\right] p_b(x_1, x_2) &\leq \frac{\mu}{2(\mu+\upsilon)} p_b(x_0, x_1) \\
\left[\frac{2\mu + \upsilon}{2(\mu+\upsilon)}\right] p_b(x_1, x_2) &\leq \frac{\mu}{2(\mu+\upsilon)} p_b(x_0, x_1)
\end{aligned}$$

$$p_b(x_1, x_2) \leq \left(\frac{\mu}{2\mu+\upsilon}\right) p_b(x_0, x_1). \tag{3.2}$$

By Lemma 3.4, for $x_2 \in \Psi(x_1)$ there exists $x_3 \in \Psi(x_2)$ such that

$$\begin{aligned}
p_b(x_2, x_3) &\leq \frac{1}{2(\mu+\upsilon)} \Upsilon_{p_b}(\Psi(x_1), \Psi(x_2)) \\
&\leq \frac{1}{2(\mu+\upsilon)} \left\{\mu p_b(x_1, x_2) + \upsilon f_{p_b}(x_2, \Psi(x_1))\right\} \\
&\leq \frac{1}{2(\mu+\upsilon)} \{\mu p_b(x_1, x_2) + \upsilon p_b(x_2, x_2)\} \\
&\leq \frac{1}{2(\mu+\upsilon)} \{\mu p_b(x_1, x_2) + \upsilon p_b(x_2, x_3)\} \\
&= \frac{\mu}{2(\mu+\upsilon)} p_b(x_1, x_2) + \frac{\upsilon}{2(\mu+\upsilon)} p_b(x_2, x_3).
\end{aligned}$$

Thus,

$$\begin{aligned}
\left[1 - \frac{\upsilon}{2(\mu+\upsilon)}\right] p_b(x_2, x_3) &\leq \frac{\mu}{2(\mu+\upsilon)} p_b(x_1, x_2) \\
\left[\frac{2\mu + 2\upsilon - \upsilon}{2(\mu+\upsilon)}\right] p_b(x_2, x_3) &\leq \frac{\mu}{2(\mu+\upsilon)} p_b(x_1, x_2)
\end{aligned}$$

$$\left[\frac{2\mu+\upsilon}{2(\mu+\upsilon)}\right]p_b(x_2,x_3) \quad \le \quad \frac{\mu}{2(\mu+\upsilon)}p_b(x_1,x_2)$$

$$p_b(x_2,x_3) \le \left(\frac{\mu}{2\mu+\upsilon}\right)p_b(x_1,x_2). \tag{3.3}$$

By (3.2) and (3.3), we have

$$p_b(x_2,x_3) \le \frac{\mu}{2\mu+\upsilon}\left(\frac{\mu}{2\mu+\upsilon}\right)p_b(x_0,x_1).$$

Thus

$$p_b(x_2,x_3) \le \left(\frac{\mu}{2\mu+\upsilon}\right)^2 p_b(x_0,x_1).$$

In general, for $x_n \in \Psi(x_{n-1})$ there exists $x_{n+1} \in \Psi(x_n)$ such that

$$p_b(x_n,x_{n+1}) \le \left(\frac{\mu}{2\mu+\upsilon}\right)^n p_b(x_0,x_1). \tag{3.4}$$

To show that $\{x_n\}$ is a Cauchy sequence, we proceed by using triangle property.

$$\begin{aligned}
p_b(x_n,x_m) &\le s\left[p_b(x_n,x_{n+1})+p_b(x_{n+1},x_m)\right]-p_b(x_{n+1},x_{n+1})\\
&\le s\left[p_b(x_n,x_{n+1})+p_b(x_{n+1},x_m)\right]\\
&\le sp_b(x_n,x_{n+1})+sp_b(x_{n+1},x_m)\\
&\le sp_b(x_n,x_{n+1})+s\left[s\left\{p_b(x_{n+1},x_{n+2})+p_b(x_{n+2},x_m)\right\}\right]-p_b(x_{n+2},x_{n+2})\\
&\le sp_b(x_n,x_{n+1})+s\left[s\left\{p_b(x_{n+1},x_{n+2})+p_b(x_{n+2},x_m)\right\}\right]\\
&\le sp_b(x_n,x_{n+1})+s^2p_b(x_{n+1},x_{n+2})+s^2p_b(x_{n+2},x_m)\\
&\le sp_b(x_n,x_{n+1})+s^2p_b(x_{n+1},x_{n+2})+s^3p_b(x_{n+2},x_{n+3})+\cdots+s^{m-n}p_b(x_{m-1},x_m)\\
&\le s\left(\frac{\mu}{2\mu+\upsilon}\right)^n p_b(x_0,x_1)+s^2\left(\frac{\mu}{2\mu+\upsilon}\right)^{n+1} p_b(x_0,x_1)\\
&\quad +s^3\left(\frac{\mu}{2\mu+\upsilon}\right)^{n+2} p_b(x_0,x_1)+\cdots+s^{m-n}\left(\frac{\mu}{2\mu+\upsilon}\right)^{m-1} p_b(x_0,x_1)\\
&\le s\left(\frac{\mu}{2\mu+\upsilon}\right)^n\left[1+\left(\frac{s\mu}{2\mu+\upsilon}\right)+\left(\frac{s\mu}{2\mu+\upsilon}\right)^2+\cdots\right]p_b(x_0,x_1)\\
&= \frac{s\left(\frac{\mu}{2\mu+\upsilon}\right)^n}{1-\left(\frac{s\mu}{2\mu+\upsilon}\right)}p_b(x_0,x_1)\to 0 \quad \text{as } n\to\infty \qquad \text{since} \quad 0\le\frac{s\mu}{2\mu+\upsilon}<1.
\end{aligned}$$

That is

$$\lim_{n,m\to\infty}p_b(x_m,x_n)=0.$$

Thus, $\{x_n\}$ is a Cauchy sequence in (X,p_b). Since $p_b(x_n,x_n)\le p_b(x_m,x_n)$ for all $n\neq m$, this implies that

$$\lim_{n\to\infty}p_b(x_n,x_n)=0.$$

By (2.1), $\lim_{n,m\to\infty} b(x_m, x_n) = 2\lim_{n,m\to\infty} p_b(x_m, x_n) = 0$. Thus, $\{x_n\}$ is a Cauchy sequence in (X, b). The completeness of (X, p_b) implies that of (X, b), so, there exists $x^* \in X$, such that

$$\lim_{n\to\infty} b(x_n, x^*) = 0.$$

By Lemma 2.7, we have

$$\lim_{n,m\to\infty} p_b(x_m, x_n) = \lim_{n\to\infty} p_b(x_n, x^*) = p_b(x^*, x^*). \tag{3.5}$$

By (3.1) and (3.5), we get

$$\Upsilon_{p_b}(\Psi(x_n), \Psi(x^*)) \leq \mu p_b(x_n, x^*) + \upsilon f_{p_b}(x^*, \Psi(x_n)) \leq \mu p_b(x_n, x^*) + \upsilon p_b(x^*, x_{n+1}).$$

This implies that $\lim_{n\to\infty} \Upsilon_{p_b}(\Psi(x_n), \Psi(x^*)) = 0$. Now consider,

$$\begin{aligned} f_{p_b}(x_{n+1}, \Psi(x^*)) &\leq g_{p_b}(\Psi(x_n), \Psi(x^*)) \leq \Upsilon_{p_b}(\Psi x_n, \Psi x^*). \\ \lim_{n\to\infty} p_b(x_{n+1}, \Psi(x^*)) &\leq \lim_{n\to\infty} g_{p_b}(\Psi(x_n), \Psi(x^*)) \leq \lim_{n\to\infty} \Upsilon_{p_b}(\Psi(x_n), \Psi(x^*)). \end{aligned}$$

This implies that $\lim_{n\to\infty} p_b(x_{n+1}, \Psi(x^*)) = 0$. Also

$$\begin{aligned} f_{p_b}(x^*, \Psi(x^*)) &\leq s\left[p_b(x^*, x_{n+1}) + f_{p_b}(x_{n+1}, \Psi(x^*))\right] - p_b(x_{n+1}, x_{n+1}) \\ f_{p_b}(x^*, \Psi(x^*)) &\leq s\left[p_b(x^*, x_{n+1}) + f_{p_b}(x_{n+1}, \Psi(x^*))\right] \\ \lim_{n\to\infty} f_{p_b}(x^*, \Psi(x^*)) &\leq s\left[\lim_{n\to\infty} p_b(x^*, x_{n+1}) + \lim_{n\to\infty} f_{p_b}(x_{n+1}, \Psi(x^*))\right]. \end{aligned}$$

This implies that $f_{p_b}(x^*, \Psi(x^*)) = 0 = p_b(x^*, x^*)$. Hence,

$$x^* \in \overline{\Psi(x^*)} = \Psi(x^*).$$

□

Corollary 3.6. *Let (X, P_b) be a complete partial b-metric space and $\Psi : X \to CB_{P_b}(X)$ satisfies the following condition:*

$$H(\Psi(x), \Psi(y)) \leq \mu p_b(x, y), \textit{ for all } x, y \in X \textit{ and } 0 \leq \mu < 1.$$

Then Ψ has a fixed point.

Example 3.7. Let $X = \{0, 1, 4\}$. Define the function $p_b : X \times X \to [0, \infty)$ by

$$p_b(x, y) = (\max\{x, y\})^2 + |x - y|^2,$$

for all $x, y \in X$.

Note that $p_b(1,1) = (\max\{1,1\})^2 + |1-1|^2 = 1 \neq 0$. This implies that p_b is not b-metric on X. In the following, we show that $\{0\}$ and $\{0,1\}$ are bounded and closed sets in (X, p_b). Consider,

$$\begin{aligned} x \in \overline{\{0\}} &\Leftrightarrow p_b(x,\{0\}) = p_b(x,x) \\ &\Leftrightarrow \left(\max\{x,0\})^2 + |x-0|^2\right) = \left[(\max\{x,x\})^2 + |x-x|^2\right] \\ &\Leftrightarrow x^2 + x^2 = x^2 + 0^2 \\ &\Leftrightarrow 2x^2 = x^2 \\ &\Leftrightarrow x \in \{0\}. \end{aligned}$$

Therefore the set $\{0\}$ in respect of the partial b-metric is closed.
Similarly,

$$\begin{aligned} x \in \overline{\{0,1\}} &\Leftrightarrow p_b(x,\{0,1\}) = p_b(x,x) \\ &\Leftrightarrow \inf\{(\max\{x,0\})^2 + |x-0|^2, (\max\{x,1\})^2 + |x-1|^2\} = (\max\{x,x\})^2 + |x-x|^2) \\ &\Leftrightarrow \inf\{x^2 + x^2, (\max\{x,1\})^2 + |x-1|^2\} = x^2 \\ &\Leftrightarrow \inf\{2x^2, (\max\{x,1\})^2 + |x-1|^2\} = x^2 \\ &\Leftrightarrow x \in \{0,1\}. \end{aligned}$$

Therefore $\{0,1\}$ in respect of the partial b-metric is closed.
Now define

$$\Psi : X \to CB_{p_b}(X) \text{ by } \Psi(0) = \Psi(1) = \{0\} \text{ and } \Psi(4) = \{0,1\}.$$

We show that for all $x, y \in X$ the contractive condition (3.1) is satisfied for $\mu = \frac{1}{5}$ and $\upsilon = \frac{1}{5}$.
Case 1. If $x, y \in \{0,1\}$, then,

$$\Upsilon_{p_b}(\Psi(0),\Psi(0)) = \Upsilon_{p_b}(\Psi(1),\Psi(1)) = \Upsilon_{p_b}(\Psi(0),\Psi(1)).$$

$$\Upsilon_{p_b}(\Psi(0),\Psi(0)) = \Upsilon_{p_b}(0,0) = 0. \tag{3.6}$$

For $x = y = 0$, we have

$$\begin{aligned} \mu p_b(x,y) + \upsilon f_{p_b}(y,\Psi(x)) &= \frac{1}{5}p_b(0,0) + \frac{1}{5}\inf_{0\in\{0\}} f_{p_b}(0,\Psi(0)) \\ &= \frac{1}{5}p_b(0,0) + \frac{1}{5}\inf_{0\in\{0\}} p_b(0,\{0\}) = 0. \end{aligned}$$

$$\mu p_b(x,y) + \upsilon f_{p_b}(y,\Psi(x)) = 0. \tag{3.7}$$

By (3.6) and (3.7), $\Upsilon_{p_b}(\Psi(x),\Psi(y)) \leq \mu p_b(x,y) + \upsilon f_{p_b}(y,\Psi(x))$ holds for $x, y \in \{0,1\}$.
Case 2. If $x \in \{0,1\}$ and $y = 4$

$$\begin{aligned} \Upsilon_{p_b}(\Psi(0),\Psi(4)) &= \Upsilon_{p_b}(\Psi(1),\Psi(4)) \\ &= \Upsilon_{p_b}(\{0\},\{0,1\}) = \max\{g_{p_b}(\{0\},\{0,1\}), g_{p_b}(\{0,1\},\{0\}). \end{aligned}$$

Note that

$$g_{p_b}(\{0\},\{0,1\}) = \sup\{0\} = 0 \text{ and } g_{p_b}(\{0,1\},\{0\}) = \sup\{2,0\} = 2.$$

Thus,

$$\Upsilon_{p_b}(\{0\},\{0,1\}) = \max\{0,2\} = 2. \tag{3.8}$$

For $x = 0$ and $y = 4$, we have

$$\begin{aligned}\mu p_b(x,y) + \upsilon f_{p_b}(y,\Psi(x)) &= \frac{1}{5}p_b(0,4) + \frac{1}{5}f_{p_b}(4,\Psi(0))\\ &= \frac{1}{5}(32) + \frac{1}{5}f_{p_b}(4,\{0\})\\ &= \frac{32}{5} + \frac{1}{5}\inf_{0\in\{0\}} p_b(4,0)\\ &= \frac{32}{5} + \frac{32}{5} = \frac{65}{5}.\end{aligned}$$

For $x = 1$ and $y = 4$, we have

$$\begin{aligned}\mu p_b(x,y) + \upsilon f_{p_b}(y,\Psi(x)) &= \frac{1}{5}p_b(1,4) + \frac{1}{5}f_{p_b}(4,\Psi(1))\\ &= \frac{1}{5}(25) + \frac{1}{5}f_{p_b}(4,\{0\})\\ &= \frac{25}{5} + \frac{1}{5}\inf_{0\in\{0\}} p_b(4,0)\\ &= \frac{25}{5} + \frac{32}{5} = \frac{57}{5}.\end{aligned}$$

So, the contractive condition (3.1) holds in this case, that is, we have

$$\begin{aligned}\Upsilon_{p_b}(\{0\},\{0,1\}) &= 2 \leq \frac{1}{5}p_b(0,4) + \frac{1}{5}f_{p_b}(4,\Psi(0)) = \frac{65}{5}\\ \Upsilon_{p_b}(\{0\},\{0,1\}) &= 2 \leq \frac{1}{5}p_b(1,4) + \frac{1}{5}f_{p_b}(4,\Psi(1)) = \frac{57}{5}.\end{aligned}$$

Case 3. For $x = y = 4$

$$\begin{aligned}\Upsilon_{p_b}(\Psi(x),\Psi(y)) &= \Upsilon_{p_b}(\Psi(4),\Psi(4))\\ g_{p_b}(\{0,1\},\{0,1\} &= \sup\{0,1\} = 1.\end{aligned}$$

And

$$\begin{aligned}\mu p_b(x,y) + \upsilon f_{p_b}(y,\Psi(x)) &= \frac{1}{5}p_b(4,4) + \frac{1}{5}f_{p_b}(4,\Psi(4))\\ &= \frac{16}{5} + \inf_{0\in\{0,1\}} p_b(4,\{0,1\}) = \frac{16}{5} + \frac{25}{5} = \frac{41}{5}.\end{aligned}$$

So, in this case (3.1) also holds true.

$$\Upsilon_{p_b}(\{0,1\},\{0,1\}) = 1 \leq \frac{1}{5}p_b(4,4) + \frac{1}{5}f_{p_b}(4,\Psi(4)) = \frac{41}{5}.$$

Hence, this example verifies Theorem 3.5 and $x = 0$ is a fixed-point of Ψ. Since, the mapping p_b is not a partial metric and a b-metric, so, Theorem 3.5 is only useful in the partial b-metric space.

For the single valued almost contraction, we have the following theorem.

Theorem 3.8. *Let (X, p_b) be a complete partial b-metric space. Suppose that $T : X \to X$ is an almost contraction, that is, for all $x, y \in X$, there exist $\mu, \upsilon \in (0, 1)$ satisfying $2(\mu + \upsilon) < 1$ and $s < \frac{\upsilon}{\mu} + 2$ such that*

$$p_b(Tx, Ty) \leq \mu p_b(x, y) + \upsilon p_b(y, Tx). \tag{3.9}$$

Then T admits a fixed point.

Proof. In Theorem 3.5, define $\Psi(x) = \{T(x)\}$ for all $x \in X$. Then,

$$\Upsilon_{p_b}(\Psi x, \Psi y) = p_b(Tx, Ty).$$

Hence, proof follows the proof of Theorem 3.5. □

4. Multivalued almost contraction II

In this section, we define another multivalued almost contraction defined on a partial b-metric space (X, p_b). We will investigate the conditions under which such contractions admit at least one fixed point point.

Definition 4.1. Let (X, p_b) be a partial b-metric space with $s \geq 1$. The mapping $T \colon X \to CB_{p_b}(X)$ is said to be an multivalued almost contraction II, if there exist $\mu, \upsilon \in (0, 1)$ satisfying $2(\mu + \upsilon) < 1$ such that

$$\Upsilon_{p_b}(T(x), T(y)) \leq \mu M(x, y) + \upsilon N(x, y), \text{ for all } x, y \in X, \tag{4.1}$$

where

$$M(x, y) = \max\left\{p_b(x, y), \frac{f_{p_b}(x, Tx) f_{p_b}(y, Ty)}{1 + p_b(x, y)}, \frac{f_{p_b}(x, Tx) f_{p_b}(y, Ty)}{1 + \Upsilon_{p_b}(Tx, Ty)}\right\},$$
$$N(x, y) = \min\left\{f_{p_b}(x, Tx), f_{p_b}(x, Ty), f_{p_b}(y, Tx), f_{p_b}(y, Ty)\right\}.$$

Theorem 4.2. *Let (X, p_b) be a complete partial b-metric space with $s > 1$. Suppose that $T \colon X \to CB_{p_b}(X)$ is an multivalued almost contraction II. If $s < \frac{\upsilon}{\mu} + 2$, then T has a fixed point.*

Proof. Let there exists $x_0 \in X$ such that $x_1 \in T(x_0)$. We construct an iterative sequence x_n of points in X such a way that, $x_n \in T(x_{n-1})$ where $n = 1, 2, \ldots$. We observe that if $x_n = x_{n+1}$, then x_n is a fixed point of T. Thus, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. By Lemma 3.4 and taking $h = \frac{1}{2(\mu+\upsilon)}$, for $x_{n+1} \in T(x_n)$ there exists $x_{n+2} \in T(x_{n+1})$ such that

$$p_b(x_{n+1}, x_{n+2}) \leq \frac{1}{2(\mu + \upsilon)} \Upsilon_{p_b}(T(x_n, T(x_{n+1})) \leq \frac{1}{2(\mu + \upsilon)} [\mu M(x_n, x_{n+1}) + \upsilon N(x_n, x_{n+1})]. \tag{4.2}$$

Note that

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\left\{\begin{array}{l} p_b(x_n, x_{n+1}), \dfrac{f_{p_b}(x_n, T(x_n)) f_{p_b}(x_{n+1}, T(x_{n+1}))}{1 + p_b(x_n, x_{n+1})}, \\ \dfrac{f_{p_b}(x_n, T(x_n)) f_{p_b}(x_{n+1}, T(x_{n+1}))}{1 + \Upsilon_{p_b}(T(x_n), T(x_{n+1}))} \end{array}\right\} \\ &\leq \max\{p_b(x_n, x_{n+1}), p_b(x_{n+1}, x_{n+2})\}, \end{aligned}$$

and

$$\begin{aligned} N(x_n, x_{n+1}) &= \min \left\{ \begin{array}{l} f_{p_b}(x_n, T(x_n)), f_{p_b}(x_n, T(x_{n+1})), f_{p_b}(x_{n+1}, T(x_n)), \\ f_{p_b}(x_{n+1}, T(x_{n+1})) \end{array} \right\} \\ &\leq \min\{p_b(x_n, x_{n+1}), p_b(x_n, x_{n+2}), p_b(x_{n+1}, x_{n+1}), p_b(x_{n+1}, x_{n+2})\} \\ &= p_b(x_{n+1}, x_{n+1}). \end{aligned}$$

Observe that if $M(x_n, x_{n+1}) = p_b(x_{n+1}, x_{n+2})$, then inequality (4.2) does not hold, therefore, substituting $M(x_n, x_{n+1}) = p_b(x_n, x_{n+1})$ and $N(x_n, x_{n+1}) = p_b(x_{n+1}, x_{n+1})$ in (4.2), we obtain

$$\begin{aligned} p_b(x_{n+1}, x_{n+2}) &\leq \frac{1}{2(\mu+\upsilon)}[\mu p_b(x_n, x_{n+1}) + \upsilon p_b(x_{n+1}, x_{n+1})] \\ &\leq \frac{1}{2(\mu+\upsilon)}[\mu p_b(x_n, x_{n+1}) + \upsilon p_b(x_{n+1}, x_{n+2})]. \end{aligned}$$

Thus, we have

$$p_b(x_{n+1}, x_{n+2}) \leq \frac{\mu}{2\mu+\upsilon} p_b(x_n, x_{n+1}), \text{ for all } n = 0, 1, 2, \cdots.$$

This implies that

$$p_b(x_{n+1}, x_{n+2}) \leq \left(\frac{\mu}{2\mu+\upsilon}\right)^{n+1} p_b(x_0, x_1).$$

The remaining part of the proof is omitted. It follows directly from the proof of Theorem 3.5. □

Example 4.3. Let $A = \{1, 2, 3, \cdots, 40\}$ and $X = A \cup \{\infty\}$. Let $p_b : X \times X \to \mathbb{R}$ be given by the rule

$$p_b(x, y) = \begin{cases} \frac{\mu}{760}, & \text{if } x = y, \\ |\frac{1}{x} - \frac{1}{y}|, & \text{if one of } x, y \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } x, y \text{ is odd and the other is odd (and } x \neq y) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then, (X, p_b) is a partial b-metric space with $s = 5/2$. Let the mapping $T\colon X \to CB_{p_b}(X)$ be defined by

$$T(x) = \begin{cases} \{3, 6\} & \text{if } x \in \mathbb{N} - \{3, 6\} \\ \{\infty\} & \text{if } x \in \{\infty, 3, 6\} \end{cases}$$

The mapping T satisfies (4.1). Indeed,

Case 1. If $x, y \neq 6$ are even numbers. Then $\Upsilon_{p_b}(T(x), T(y)) = \Upsilon_{p_b}(\{3, 6\}, \{3, 6\}) = \frac{\mu}{760}$, and for $x = 6$, $y = \infty$, $\Upsilon_{p_b}(T(x), T(y)) = \frac{\mu}{760}$, and if $x = 6$, y is any other even number, then $\Upsilon_{p_b}(T(x), T(y)) = \Upsilon_{p_b}(\{\infty\}, \{3, 6\}) = \frac{1}{6}$. In all these sub-cases, there exist $\mu, \upsilon \in (0, 1)$ satisfying $2(\mu+\upsilon) < 1$ such that the RHS of (4.1) is greater than $\Upsilon_{p_b}(T(x), T(y))$.

Case 2. If $x, y \neq 3$ are odd numbers (and $x \neq y$). Then $\Upsilon_{p_b}(T(x), T(y)) = \Upsilon_{p_b}(\{3, 6\}, \{3, 6\}) = \frac{\mu}{760}$, and for $x = 3$, $y = \infty$, $\Upsilon_{p_b}(T(x), T(y)) = \frac{\mu}{760}$, and if $x = 3$, y is any other odd number, then $\Upsilon_{p_b}(T(x), T(y)) = \Upsilon_{p_b}(\{\infty\}, \{3, 6\}) = \frac{1}{6}$. In all these sub-cases, there exist $\mu, \upsilon \in (0, 1)$ satisfying $2(\mu+\upsilon) < 1$ such that the RHS of (4.1) is greater than $\Upsilon_{p_b}(T(x), T(y))$.

Case 3. If x, y are natural numbers of different parity. Then $\Upsilon_{p_b}(T(x), T(y)) = \frac{\mu}{760}$, $M(x, y) = 2$ and $N(x, y) \geq \frac{\mu}{760}$. We can find $\mu, \upsilon \in (0, 1)$ satisfying $2(\mu + \upsilon) < 1$ such that the RHS of (4.1) is greater than $\Upsilon_{p_b}(T(x), T(y))$.

Similarly, for all other cases, we have same conclusion. The point $x = \infty$ is a fixed point of T. Since, the mapping p_b is not a partial metric and a b-metric, so, Theorem 4.2 is only useful in the partial b-metric space.

5. Application to the BVP

This section contains an existence theorem for the solution to the following boundary value problem (BVP):

$$-\frac{d^2\acute{y}}{dt^2} = g(t, \acute{y}(t)); \forall t \in [0, 1] = J,\ \acute{y}(0) = \acute{y}(1) = 0. \tag{5.1}$$

The associated Green's function $\mathcal{G} : J \times J \to J$ to (5.1) can be defined as follows:

$$\mathcal{G}(t,\ l) = \left\{ \begin{array}{l} t(1-l) \text{ if } 0 \leq t \leq l \leq 1 \\ l(1-t) \text{ if } 0 \leq l \leq t \leq 1 \end{array} \right\}$$

Let $C(J)$ represents the set of continuous functions defined on J. Let the mapping $b : C(J) \times C(J) \to [0,\ \infty)$ be defined by

$$b(f, h) = \left\| (f-h)^2 \right\| = \sup |f(t) - h(t)|^2, \forall f, h \in C(J), \text{ and } t \in J.$$

The pair $(C(J), b)$ is a complete b-metric space with $s = 2$. The associated integral operator $\mathcal{S} : C(J) \to C(J)$ to (5.1) is defined by:

$$\mathcal{S}(f)(t) \ = \ \int_0^1 \mathcal{G}(t, b) g(b, f(b)) db.$$

It is remarked that the fixed point of the operator $\mathcal{S}$ is a solution to (5.1). The following theorem states the condition under which the BVP has a solution.

Theorem 5.1. *Let the function $g : J \times C(J) \to \mathbb{R}$ is continuous and satisfies the following condition:*

$$|g(t, f) - g(t, h)|^2 \leq 64 \left(\mu |f(t) - h(t)|^2 + \upsilon |h(t) - \mathcal{S}(f)(t)|^2 \right),$$

for all $t \in J$, $f, h \in C(J)$ and $\mu, \upsilon \in (0, 1)$ satisfying $2(\mu + \upsilon) < 1$ and $s < \frac{\upsilon}{\mu} + 2$.Then the BVP (5.1) has a solution.

Proof. This proof will be done by the application of Theorem 3.8. Since, the function g is continuous, so, the operator $\mathcal{S} : C(J) \to C(J)$ defined above is continuous. To show that the mapping $\mathcal{S}$ form an almost contraction, we proceed as follow:

$$\begin{aligned} |\mathcal{S}(f)(t) - \mathcal{S}(h)(t)|^2 &= \left| \int_0^1 \mathcal{G}(t, b)(g(b, f) - g(b, h)) db \right|^2 \\ &\leq \left(\int_0^1 \mathcal{G}(t, b) \sqrt{64 \left(\mu |f(t) - h(t)|^2 + \upsilon |h(t) - \mathcal{S}(f)(t)|^2 \right)} dt \right)^2. \end{aligned}$$

Since, $\left(\sup \int_0^1 \mathcal{G}(t,b)\,db\right)^2 = \frac{1}{64}$, for all $t \in J$, thus, taking supremum on both sides of above inequality, we have

$$p_b(\mathcal{S}(f), \mathcal{S}(h)) \leq \mu d(f,h) + \upsilon d(h, \mathcal{S}(f)) \; \forall \, f,h \in C(J).$$

Now for any partial b-metric p_b on $C(J)$, we can have a b-metric b on $C(J)$ by

$$b(f,h) = \begin{cases} p_b(f,h) \text{ if } f \neq h \\ 0 \text{ if } f = h \end{cases}$$

The last inequality can be written as:

$$p_b(\mathcal{S}(f), \mathcal{S}(h)) \leq \mu p_b(f,h) + \upsilon p_b(h, \mathcal{S}(f)) \; \forall \, f,h \in C(J).$$

Hence, by Theorem 3.8, the BVP (5.1) has a solution in $C(J)$. □

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Conflict of interest

The authors declare that they have no competing interests.

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