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Research article

Periodic solution of a stage-structured predator-prey model with Crowley-Martin type functional response

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Abstract: In this paper, the existence of positive periodic solution of stage-structured predator-prey model with Crowley-Martin type functional response is investigated. The prey population fall into two categories: mature and immature prey. The predator population is dependent only on mature prey and is influenced by Crowley-Martin type functional response. Based on the Mawhin's coincidence degree theory and nontrivial estimation techniques for a priori bounds of unknown solutions to the operator equation $Fz = \mu Nz$, we prove the existence of positive periodic solution. Finally, the effectiveness of our result is verified by an example and numerical simulation.

Keywords: periodic solution; stag-structure; Crowley-Martin type functional response
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1. Introduction

The predator-prey relationship has become one of the most important relationships in ecology in recent decades due to the prevalence and importance of species predation. The predator-prey model [1, 2] generally takes the form of

$$\frac{dz_i(t)}{dt} = z_i(t)\left[a_i(t) + \sum_{j=1}^{n} b_{ij}z_j(t)\right], \quad i = 1, 2, \cdots, n.$$

Many scholars have made contributions to it (see e.g. [3–18]). In population models, stage-structure is one of the important factors to explain the dynamics of predator-prey model. Recently, many studies (see for example [19–24]) have considered the predator-prey system with the stage-structure of predator, prey or both.

On the other hand, in the study of predator-prey system, functional response also plays an important role, which can represent the quantity of prey killed by a predator per unit time and describe the amount of biological transfer between different nutritional levels. Holling type-II functional response [16,25,26] takes into account the average feeding rate of a predator to its prey. It can be expressed as: $f(x,y) = \frac{a_1 x}{1+a_2 x}$, where a_1 and a_2 are positive constants, which denote the capture rate and the influence of the processing time, respectively. In this functional response, competition between predators for food only occurs when prey is depleted. Another functional response function is the Beddinton-DeAnglis type [22,27,28], which is similar to the Holling type-II. However, it takes into account the interference between predators. Therefore, Beddinton-DeAnglis type functional response function describes that individuals from two or more predator groups not only take prey, but also meet and compete with other predators. It has the type: $f(x,y) = \frac{a_1 x}{1+a_2 x+a_3 y}$, where a_3 is a positive constant, describing the degree of disturbance between predators. Moreover, it is assumed that its influence on the predation rate can be ignored in the case of high prey density. In this paper, we consider the Crowley-Martin type functional response [29–32]:

$$f(x,y) = \frac{a_1 x}{1 + a_2 x + a_3 y + a_2 a_3 x y} = \frac{a_1 x}{(1 + a_2 x)(1 + a_3 y)}.$$

It also takes the interference between predators into account, but the biggest difference between it and Beddinton-DeAnglis type functional response is: the influence of predator disturbance on the predation rate is always an important factor, which cannot be ignored. Hence, Crowley-Martin type functional response is more consistent with the phenomenon in ecology and has more research value.

Maiti et al. [32] studied the global dynamics of an *autonomous* stage-structured predator-prey model with Crowley-Martin type functional response. However, they did not cosider the periodic behavior of this model with periodic parameters. The assumption of periodicity of the parameters is a way of incorporating the periodicity of the environment. The periodic oscillation of the parameters seems reasonable in view of seasonal factors, e.g. mating habits, availability of food, weather conditions, harvesting and hunting, etc. Cai et al. [30] presented the existence of positive periodic solutions of an Eco-Epidemic model with Crowley- Martin type functional response. Inspired by the above works, we study a stage-structured predator-prey system with Crowley-Martin type functional response:

$$\begin{cases} \frac{dx_1(t)}{dt} = s(t)x_2(t) - r(t)x_1(t) - d(t)x_1(t), \\ \frac{dx_2(t)}{dt} = r(t)x_1(t) - \alpha(t)x_2^2(t) - \frac{\beta(t)x_2(t)y(t)}{(1+ax_2(t))(1+by(t))} - d_1(t)x_2(t), \\ \frac{dy(t)}{dt} = \frac{\beta_1(t)x_2(t)y(t)}{(1+ax_2(t))(1+by(t))} - d_2(t)y(t) - \gamma(t)y^2(t), \end{cases} \tag{1.1}$$

where $x_1(t)$ and $x_2(t)$ are the population density of immature and mature prey at time t, $y(t)$ is the population density of predator at time t, and all the following parameters involved are continuous positive periodic functions: (I) for immature prey: (1) the ratio function $s(t)$ represents the ratio of birth rate to available mature prey; (2) the ratio of the conversion of immature prey to mature prey to existing immature prey is denoted by $r(t)$, and the ratio of the death rate of immature prey to existing immature prey is represented by $d(t)$. (II) for mature prey: (1) $\alpha(t)$ is an internally specific interference function; (2) the ratio of the death rate to existing mature prey is denoted by $d_1(t)$; (3) the interaction between predator and mature prey is a Crowley-Martin type functional response with rate $\beta(t)$. (III) for predator: (1) $\beta_1(t)$ denotes the intake of predators, and $0 < \beta_1(t) < \beta(t)$; (2) $d_2(t)$ denotes the death rate of the predator; (3) $\gamma(t)$ denotes the internal specific disturbance function for the predator.

In terms of the number of creatures, the initial conditions for model (1.1) are given by

$$(x_1(t), x_2(t), y(t)) \in C_+ = C(0, \mathbb{R}_+^3), x_1(0) > 0, x_2(0) > 0, y(0) > 0.$$

The purpose in the present paper is to find some suitable conditions of the existence of positive periodic solution for system (1.1). The method is based on Mawhin's coincidence degree theory.

2. Existence of positive periodic solution

In this section, we establish the existence of positive periodic solution for the system (1.1). For this purpose, we first assume that the parameters in the system (1.1) are all ω-period. To obtain a positive periodic solution for system (1.1), we summarize the following lemmas.

Lemma 2.1. *[35–37] Let $U \in Z$ be an open bounded set on Banach space Z. Assume that F is a Fredholm operator of index zero and N is F-compact on $\bar{U}$. If the following conditions hold:*
(1) for any fixed $\mu \in (0,1)$, $z \in \partial U \cap DomF$, $Fz \neq \mu Nz$;
(2) for any fixed $z \in \partial U \cap \ker F$, $QNz \neq 0$ and the Brouwer's degree: $\deg[JQN, U \cap \ker F, 0] \neq 0$.
Then $Fz = Nz$ has at least one solution on $\bar{U} \cap DomF$.

For the sake of simplicity, we use the notations:
$\bar{\vartheta} = \frac{1}{\omega}\int_0^\omega \vartheta(t)dt, \qquad \vartheta^M = \max\limits_{t\in[0,\omega]} \vartheta(t), \qquad \vartheta^L = \min\limits_{t\in[0,\omega]} \vartheta(t),$
$l_\pm = \frac{-(\gamma^L+bd_2^L)\pm\sqrt{\Delta_1}}{2b\gamma^L}, \qquad \Delta_1 = (\gamma^L - bd_2^L)^2 + 4a^{-1}b\gamma^L\beta_1^M,$
$h_\pm = \frac{-(\gamma^M+bd_2^M)\pm\sqrt{\Delta_2}}{2b\gamma^M}, \qquad \Delta_2 = (\gamma^M - bd_2^M)^2 + \frac{4b\gamma^M\beta_1^L}{1+a},$
$q_+ = \frac{r^M s^M - \bar{d}_1\omega(r^L+d^L)}{\alpha\omega(r^L+d^L)}, \qquad m_+ = \ln\frac{\bar{s}p_+}{\bar{r}+\bar{d}} - 2\bar{d}_1\omega,$
$\Delta_3 = [\bar{\beta} + \bar{d}_1(1+a)(b + l_+^{-1}e^{-2\bar{d}_2\omega}]^2 + 4\bar{\alpha}\bar{r}(1+a)^2(b + l_+^{-1}e^{-2\bar{d}_2\omega})^2,$
$p_\pm = \{-[\bar{\beta} + \bar{d}_1(1+a)(b + l_+^{-1}e^{-2\bar{d}_2\omega}] \pm \sqrt{\Delta_3}\}\{2\bar{\alpha}(1+a)(b + l_+^{-1}e{-}2\bar{d}_2\omega)\}^{-1}.$
Furthermore, we assume that:

$(H_1)\ \beta_1^M > ad_2^L, \qquad (H_2)\ \beta_1^L > (1+a)d_2^M, \qquad (H_3)\ \bar{s}p_+ > (\bar{r}+\bar{d})e^{2(\bar{d}+\bar{d}_1)\omega}.$

From a biological viewpoint, the assumptions (H_1) and (H_2) imply that the intake of a mature predator is greater than its death rate, while the assumption (H_3) implies that the birth rate of the prey is influenced by the stage structure and the death rate of mature and immature prey.

Now, we present a theorem on the existence.

Theorem 2.1. *If (H_1), (H_2), (H_3) hold, then system (1.1) has at least a positive periodic solution.*

Proof: Firstly, replacing the variables by

$$z_1(t) = \ln x_1(t), \quad z_2(t) = \ln x_2(t), \quad z_3(t) = \ln y(t).$$

Then system (1.1) changes into

$$\begin{cases} \frac{dz_1(t)}{dt} = s(t)e^{z_2(t)-z_1(t)} - r(t) - d(t), \\ \frac{dz_2(t)}{dt} = r(t)e^{z_1(t)-z_2(t)} - \alpha(t)e^{z_2(t)} - d_1(t) - \frac{\beta(t)e^{z_3(t)}}{(1+ae^{z_2(t)})(1+be^{z_3(t)})}, \\ \frac{dz_3(t)}{dt} = \frac{\beta_1(t)e^{z_2(t)}}{(1+ae^{z_2(t)})(1+be^{z_3(t)})} - \gamma(t)e^{z_3(t)} - d_2(t). \end{cases} \tag{2.1}$$

Define

$$Z = W = \{z = (z_1, z_2, z_3) \in C(\mathbb{R}, \mathbb{R}^3) | z(t+\omega) = z(t)\}.$$

Z, W are both Banach space with the norm $\|\cdot\|$ as follows

$$\|z\| = \max_{t\in(0,\omega)} \sum_{i=1}^{3} |z_i|, z = (z_1, z_2, z_3) \in Z \text{ or } W.$$

For any $z = (z_1, z_2, z_3) \in Z$, the periodicity of system (2.1) implies:

$$s(t)e^{z_2(t)-z_1(t)} - r(t) - d(t) := \Gamma_1(z, t),$$

$$r(t)e^{z_1(t)-z_2(t)} - \alpha(t)e^{z_2(t)} - d_1(t) - \frac{\beta(t)e^{z_3(t)}}{(1 + ae^{z_2(t)})(1 + be^{z_3(t)})} := \Gamma_2(z, t)$$

and

$$\frac{\beta_1(t)e^{z_2(t)}}{(1 + ae^{z_2(t)})(1 + be^{z_3(t)})} - \gamma(t)e^{z_3(t)} - d_2(t) := \Gamma_3(z, t)$$

are ω-period functions. In fact,

$$\begin{aligned}\Gamma_1(z(t + \omega), t + \omega) =& s(t + \omega)e^{z_2(t+\omega)-z_1(t+\omega)} - r(t + \omega) - d(t + \omega)\\ =& s(t)e^{z_2(t)-z_1(t)} - r(t) - d(t)\\ =& \Gamma_1(z, t).\end{aligned}$$

Obviously, $\Gamma_2(z, t), \Gamma_3(z, t)$ are also both periodic functions by a similar way. Set

$$F : DomF \bigcap V, \qquad F(z_1, z_2, z_3) = \left(\frac{dz_1}{dt}, \frac{dz_2}{dt}, \frac{dz_3}{dt}\right),$$

where $DomF = \{(z_1, z_2, z_3) \in C(\mathbb{R}, \mathbb{R}^3)\}$ and $N : Z \to Z$ is defined by

$$N\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \Gamma_1(z, t) \\ \Gamma_2(z, t) \\ \Gamma_3(z, t) \end{pmatrix}.$$

Define

$$P\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = Q\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega}\int_0^\omega z_1(t)dt \\ \frac{1}{\omega}\int_0^\omega z_2(t)dt \\ \frac{1}{\omega}\int_0^\omega z_3(t)dt \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in Z = W.$$

From the above definition, we have

$$\ker F = \{z \in Z | z = C_0, C_0 \in \mathbb{R}^3\} \quad \text{and} \quad \Im F = \left\{w \in W | \int_0^\omega w(t)dt \equiv 0\right\},$$

and $\text{codim}\Im F = \dim\ker F = 3 < \infty$. Hence, F is a Fredholm map of index zero. Moreover, it is clear that P and Q are continuous projection operators with

$$\ker F = \text{Im}P, \quad \text{and} \quad \text{Im}F = \ker Q = \text{Im}(I - Q).$$

Therefore, the inverse $K_p : \Im F \to DomF \cap \ker P$ exists and is given by

$$K_p\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \int_0^t z_1(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t z_1(s)dsdt \\ \int_0^t z_2(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t z_2(s)dsdt \\ \int_0^t z_3(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t z_3(s)dsdt \end{pmatrix}.$$

Thus, we have

$$QN\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega}\int_0^\omega \Gamma_1(z,t)dt \\ \frac{1}{\omega}\int_0^\omega \Gamma_2(z,t)dt \\ \frac{1}{\omega}\int_0^\omega \Gamma_3(z,t)dt \end{pmatrix}$$

and

$$K_p(I-Q)Nz = \int_0^t Nz(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t Nz(s)dsdt - (\frac{t}{\omega} - \frac{1}{2})\int_0^\omega Nz(s)ds.$$

Clearly, QN and $K_p(I-Q)N$ are continuous. Due to Z is a Banach space, using the Arzala-Ascoli theorem, we have that N is F-compact on $\bar{U}$ for any open bounded set $U \subset Z$.

Next, in order to apply the continuation theorem, we need construct an appropriate open bounded subset U. Therefore, the operator equation is defined by $Fz = \mu Nz,\ \ \mu \in (0,1)$, that is,

$$\begin{cases} \dot{z}_1(t) = \mu[s(t)e^{z_2(t)-z_1(t)} - r(t) - d(t)], \\ \dot{z}_2(t) = \mu\left[r(t)e^{z_1(t)-z_2(t)} - \alpha(t)e^{z_2(t)} - d_1(t) - \frac{\beta(t)e^{z_3(t)}}{(1+ae^{z_2(t)})(1+be^{z_3(t)})}\right], \\ \dot{z}_3(t) = \mu\left[\frac{\beta_1(t)e^{z_2(t)}}{(1+ae^{z_2(t)})(1+be^{z_3(t)})} - \gamma(t)e^{z_3(t)} - d_2(t)\right]. \end{cases} \tag{2.2}$$

We assume that $z = (z_1, z_2, z_3)^T \in Z$ is a solution of system (2.1) for any fixed $\mu \in (0,1)$. Now, integrating system (2.1) from 0 to ω leads to

$$\begin{cases} \bar{d}\omega = \int_0^\omega [s(t)e^{z_2(t)-z_1(t)} - r(t)]dt, \\ \bar{d}_1\omega = \int_0^\omega \left[r(t)e^{z_1(t)-z_2(t)} - \alpha(t)e^{z_2(t)} - \frac{\beta(t)e^{z_3(t)}}{(1+ae^{z_2(t)})(1+be^{z_3(t)})}\right]dt, \\ \bar{d}_2\omega = \int_0^\omega \left[\frac{\beta_1(t)e^{z_2(t)}}{(1+ae^{z_2(t)})(1+be^{z_3(t)})} - \gamma(t)e^{z_3(t)}\right]dt. \end{cases} \tag{2.3}$$

From (2.2) and (2.3), we can deduce that

$$\begin{cases} \int_0^\omega |\dot{z}_1(t)|dt \le \mu\left[\int_0^\omega |s(t)e^{z_2(t)-z_1(t)} - r(t)|dt + \int_0^\omega |d(t)|dt\right] < 2\bar{d}\omega, \\ \int_0^\omega |\dot{z}_2(t)|dt \le \mu\left[\int_0^\omega |r(t)e^{z_1(t)-z_2(t)} - \alpha(t)e^{z_2(t)} - \frac{\beta(t)e^{z_3(t)}}{(1+ae^{z_2(t)})(1+be^{z_3(t)})}|dt + \int_0^\omega |d_1(t)|dt\right] < 2\bar{d}_1\omega, \\ \int_0^\omega |\dot{z}_3(t)|dt \le \mu\left[\int_0^\omega |\frac{\beta_1(t)e^{z_2(t)}}{(1+ae^{z_2(t)})(1+be^{z_3(t)})} - \gamma(t)e^{z_3(t)}|dt + \int_0^\omega |d_2(t)|dt\right] < 2\bar{d}_2\omega. \end{cases} \tag{2.4}$$

Since $(z_1, z_2, z_3) \in Z$, there exist $\eta_i, \xi_i \in [0, \omega]$ such that

$$z_i(\eta_i) = \max_{t\in[0,\omega]} z_i(t), \quad z_i(\xi_i) = \min_{t\in[0,\omega]} z_i(t), \quad i = 1,2,3.$$

Integrating the third equation of (2.1) from 0 to ω, we obtain

$$\int_0^\omega [d_2(t) + \gamma(t)e^{z_3(t)}]dt = \int_0^\omega \frac{\beta_1(t)e^{z_2(t)}}{(1+ae^{z_2(t)})(1+be^{z_3(t)})}dt. \tag{2.5}$$

From the equation (2.5), we immediately have

$$
\begin{aligned}
d_2^L + \gamma^L e^{z_3(\xi_3)} \le \int_0^\omega [d_2(t) + \gamma(t)e^{z_3(t)}]dt &= \int_0^\omega \frac{\beta_1(t)e^{z_2(t)}}{(1+ae^{z_2(t)})(1+be^{z_3(t)})}dt \\
&\le \int_0^\omega \frac{\beta_1(t)}{a(1+be^{z_3(t)})} \cdot \frac{ae^{z_2(t)}}{(1+ae^{z_2(t)})}dt \\
&\le \frac{\beta_1^M}{a} \cdot \frac{1}{(1+be^{z_3(t)})}
\end{aligned}
$$

which implies

$$
b\gamma^L e^{2z_3(\xi_3)} + (\gamma^L + bd_2^L)e^{z_3(\xi_3)} + (d_2^L - \frac{\beta_1^M}{a}) \le 0.
$$

Since $\triangle_1 = (\gamma^L - bd_2^L)^2 + 4a^{-1}b\gamma^L\beta_1^M > 0$, we have

$$
l_\pm := \frac{-(\gamma^L + bd_2^L) \pm \sqrt{\triangle_1}}{2b\gamma^L}.
$$

In view of (H_1), $l_- = \frac{-(\gamma^L+bd_2^L)-\sqrt{\triangle_1}}{2b\gamma^L} < 0$, it does not exist. Hence,

$$
z_3(\xi_3) < \ln l_+. \tag{2.6}
$$

It follows from (2.4), (2.6) that

$$
z_3(t) \le z_3(\xi_3) + \int_0^\omega |\dot{z}_3(t)|dt < \ln l_+ + 2\bar{d}_2\omega,
$$

thus,

$$
z_3(\eta_3) < \ln l_+ + 2\bar{d}_2\omega = M_1.
$$

In view of equation (2.5) again, we have that

$$
\begin{aligned}
d_2^M + \gamma^M e^{z_3(\eta_3)} \ge \int_0^\omega [d_2(t) + \gamma(t)e^{z_3(t)}]dt &= \int_0^\omega \frac{\beta_1(t)e^{z_2(t)}}{(1+ae^{z_2(t)})(1+be^{z_3(t)})}dt \\
&\ge \frac{\beta_1^L}{(1+be^{z_3(\eta_3)})} \cdot \frac{1}{(1+ae^{z_2(\xi_3)})} \\
&\ge \frac{\beta_1^L}{(1+be^{z_3(\eta_3)})(1+a)},
\end{aligned}
$$

which implies

$$
b\gamma^M e^{2z_3(\eta_3)} + (\gamma^M + bd_2^M)e^{z_3(\eta_3)} + (d_2^M - \frac{\beta_1^L}{1+a}) \ge 0.
$$

Since $\triangle_2 = (\gamma^M - bd_2^M)^2 + \frac{4b\gamma^M\beta_1^L}{1+a} > 0$, we have

$$
h_\pm := \frac{-(\gamma^M + bd_2^M) \pm \sqrt{\triangle_2}}{2b\gamma^M},
$$

and in view of (H_2), $h_- < 0$ does not exist. Consequently,

$$z_3(\eta_3) > \ln h_+. \tag{2.7}$$

It follows from (2.4), (2.7) that

$$z_3(t) \geq z_3(\eta_3) - \int_0^\omega |\dot{z}_3(t)|dt > \ln h_+ - 2\bar{d}_2\omega,$$

in particular,

$$z_3(\xi_3) > \ln h_+ - 2\bar{d}_2\omega = M_2.$$

Thus, we take

$$\max_{t\in[0,\omega]} |z_3(t)| < \max\{|M_1|, |M_2|\} = C_3.$$

It follows the second equation of (2.3) that

$$\bar{d}_1\omega \geq \frac{\bar{r}\omega}{e^{z_2(\eta_2)}} - \bar{\alpha}\omega e^{z_2(\eta_2)} - \frac{\bar{\beta}\omega}{(1+a)} \cdot \frac{1}{b + 1/e^{z_3(\eta_3)}},$$

that is,

$$\bar{\alpha}(1+a)(b + l_+^{-1}e^{2z_2(\eta_2)} + [\bar{\beta} + \bar{d}_1(1+a)(b + l_+^{-1}e^{-2\bar{d}_2\omega}]e^{z_2(\eta_2)} - \bar{r}(1+a)(b + l_+^{-1}e^{-2\bar{d}_2\omega}) \geq 0.$$

Due to

$$\triangle_3 = [\bar{\beta} + \bar{d}_1(1+a)(b + l_+^{-1}e^{-2\bar{d}_2\omega}]^2 + 4\bar{\alpha}\bar{r}(1+a)^2(b + l_+^{-1}e^{-2\bar{d}_2\omega})^2 > 0,$$

we obtain

$$p_\pm := \frac{-[\bar{\beta} + \bar{d}_1(1+a)(b + l_+^{-1}e^{-2\bar{d}_2\omega}] \pm \sqrt{\triangle_3}}{2\bar{\alpha}(1+a)(b + l_+^{-1}e{-}2\bar{d}_2\omega)},$$

Notice that $p_- < 0$ does not exist, we have

$$z_2(\eta_2) > \ln p_+. \tag{2.8}$$

It follows from (2.4), (2.8) that

$$z_2(t) \geq z_2(\eta_2) - \int_0^\omega |\dot{z}_2(t)|dt > \ln p_+ - 2\bar{d}_1\omega,$$

in particular,

$$z_2(\xi_2) > \ln p_+ - 2\bar{d}_1\omega = M_3. \tag{2.9}$$

Multiplying the first equation of (2.1) by $e^{z_1(t)-z_2(t)}$ and integrating over $[0,\omega]$, we deduce

$$\int_0^\omega [r(t) + d(t)]e^{z_1(t)-z_2(t)}dt = \int_0^\omega s(t)dt.$$

It implies that

$$\int_0^\omega e^{z_1(t)-z_2(t)}dt \leq \frac{s^M}{r^L + d^L}.$$

In view of the second equation of (2.3), we find that

$$\bar{d}_1 \leq \frac{r^M}{\omega}\int_0^{\omega} e^{z_1(t)-z_2(t)}dt - \bar{\alpha}e^{z_2(\xi_2)} \leq \frac{r^M \cdot s^M}{\omega(r^L+d^L)} - \bar{\alpha}e^{z_2(\xi_2)}. \tag{2.10}$$

It can be rewritten as

$$z_2(\xi_2) \leq \ln\frac{r^M s^M - \bar{d}_1\omega(r^L+d^L)}{\alpha\omega(r^L+d^L)} = \ln q_+,$$

which implies,

$$z_2(\eta_2) < \ln q_+ + 2\bar{d}_1\omega = M_4.$$

Hence,

$$\max_{t\in[0,\omega]}|z_2(t)| < \max\{|M_3|,|M_4|\} = C_2.$$

Combining the first equation of (2.3) with (2.9) leads to

$$\bar{d} \geq \frac{\bar{s}e^{z_2(\xi_2)}}{e^{z_1(\eta_1)}} - \bar{r},$$

which is

$$z_1(\eta_1) > \ln\frac{\bar{s}p_+}{\bar{r}+\bar{d}} - 2\bar{d}_1\omega = m_+. \tag{2.11}$$

It follows from (2.4), (2.11) that

$$z_1(\xi_1) > m_+ - 2\bar{d}\omega = M_5.$$

Finally, from the first equation of (2.3), we deduce

$$\bar{r}+\bar{d} \leq \bar{s}\cdot\frac{e^{z_2(\eta_2)}}{e^{z_1(\xi_1)}} < \bar{s}q_+e^{2\bar{d}_1\omega}\frac{1}{e^{z_1(\xi_1)}},$$

which can be rewritten as

$$z_1(\xi_1) < \ln\frac{\bar{s}q_+e^{2\bar{d}_1\omega}}{\bar{r}+\bar{d}}. \tag{2.12}$$

From (2.4), (2.12), we have

$$z_1(t) \leq z_1(\xi_1) + 2\bar{d}\omega < \ln\frac{\bar{s}q_+e^{2\bar{d}_1\omega}}{\bar{r}+\bar{d}} + 2\bar{d}\omega,$$

which is

$$z_1(\eta_1) < \ln\frac{\bar{s}q_+e^{2\bar{d}_1\omega}}{\bar{r}+\bar{d}} + 2\bar{d}\omega = M_6.$$

Therefore, we take

$$\max_{t\in[0,\omega]}|z_1(t)| < \max\{|M_5|,|M_6|\} = C_1.$$

Now, we consider QNz with $z = (z_1, z_2, z_3)^T \in \mathbb{R}^3$. Note that

$$QN(z_1,z_2,z_3)^T = \Big[-(\bar{r}+\bar{d}) + \bar{s}e^{z_2(t)-z_1(t)}, \bar{r}e^{z_1(t)-z_2(t)} - \bar{\alpha}e^{z_2(t)} - \bar{d}_1$$
$$-\frac{\bar{\beta}e^{z_3(t)}}{(1+ae^{z_2(t)})(1+be^{z_3(t)})}, \frac{\bar{\beta}_1e^{z_2(t)}}{(1+ae^{z_2(t)})(1+be^{z_3(t)})} - \bar{\gamma}e^{z_3(t)} - \bar{d}_2\Big].$$

In view of $(H_i)_{i=1}^3$, we know that the equation $QN(z_1,z_2,z_3)^T = 0$ has a solution $\tilde{z} = (\tilde{z}_1,\tilde{z}_2,\tilde{z}_3)$ with

$$M_5 < \tilde{z}_1 < M_6, \quad M_3 < \tilde{z}_2 < M_4, \quad M_2 < \tilde{z}_3 < M_1.$$

where $M_i > 0$, $i = 1...6$ are positive constants. Take $B = \max\{C_1 + C_0, C_2 + C_0, C_3 + C_0\}$, where $C_0 > 0$ is sufficiently large constant such that $\|(\tilde{z}_1,\tilde{z}_2,\tilde{z}_3)\| < C_0$. We set

$$U = \{z(t) = (z_1(t), z_2(t), z_3(t))^T \in Z : \|z\| < B\}.$$

Then the open bounded set U of Z satisfies the condition (1) of Lemma 2.1. If $(z_1,z_2,z_3) \in \partial U \cap \ker F = \partial U \cap \mathbb{R}^3$, then (z_1,z_2,z_3) is a constant vector on $\mathbb{R}^3$, and it satisfies $\|(z_1,z_2,z_3)\| = |z_1| + |z_2| + |z_3| = B$. Hence, we obtain that $QN(z_1,z_2,z_3)^T \neq (0,0,0)^T$. In order to calculate Brouwer's degree, we consider a homotopic mapping as follows:

$$H_\lambda((z_1,z_2,z_3)^T) = \lambda QN((z_1,z_2,z_3)^T) + (1-\lambda)G((z_1,z_2,z_3)^T), \quad \lambda \in [0,1],$$

where

$$G((z_1,z_2,z_3)^T) = \begin{pmatrix} \bar{s}e^{z_2(t)-z_1(t)} - (\bar{r} + \bar{d}) \\ \bar{r}e^{z_1(t)-z_2(t)} - \bar{\alpha}e^{z_2(t)} - \bar{d}_1 \\ \frac{\bar{\beta}_1 e^{z_2(t)}}{(1+ae^{z_2(t)})(1+be^{z_3(t)})} - \bar{\gamma}e^{z_3(t)} - \bar{d}_2 \end{pmatrix}.$$

Then for any $\lambda \in [0,1]$, $0 \notin H_\lambda(\partial U \cap \ker F)$. Moreover, we see that equation $G((z_1,z_2,z_3)^T) = 0$ has a unique solution on $\mathbb{R}^3$. Indeed, through the first and second components of G, we can get the unique expression of z_2, i.e., $z_2(t) = \ln\left[\alpha^{-1}\left(\frac{\bar{s}}{\bar{r}+\bar{d}} + \bar{d}_1\right)\right]$. Further, the expression of z_1 is uniquely determined, i.e. $z_1(t) = z_2(t) - \ln\left(\frac{\bar{r}+\bar{d}}{\bar{s}}\right)$. Now through the third component of G, we have

$$b\bar{\gamma}(e^{z_3(t)})^2 + (\gamma + d_2 b)e^{z_3(t)} + d_2 - \frac{\bar{\beta}_1 e^{z_2(t)}}{(1 + ae^{z_2(t)})} = 0.$$

In view of (H_1) and (H_2), the product of two solutions of the above equation must be less than zero, thus there is only one positive solution $z_3(t)$. Since $\Im Q = \ker F$, we deduce that $J = I$. Thus

$$\deg(JQN, U \cap \ker F, 0) = \deg(QN, U \cap \ker F, 0) = \deg(G, U \cap \ker F, 0) \neq 0,$$

where $\deg(\cdot,\cdot,\cdot)$ is Brouwer's degree. Hence, the requirement (2) of Lemma 2.1 also holds. From Lemma 2.1 and the periodicity of the system, we know that system (1.1) has at least one positive periodic solution on $DomF \cap \bar{U}$. This completes the proof of Theorem 2.1.

3. Example and numerical simulations

As an example, we consider the following nonautonomous predator-prey model with Crowley-Martin type functional response:

$$\begin{cases} \frac{dx_1(t)}{dt} = (10 + \sin t)x_2(t) - (7 + \sin t)x_1(t) - 0.3x_1(t), \\ \frac{dx_2(t)}{dt} = (7 + \sin t)x_1(t) - 0.1x_2^2(t) - \frac{(2+\sin t)x_2(t)y(t)}{(1+0.5x_2(t))(1+0.5y(t))} - 0,2x_2(t), \\ \frac{dy(t)}{dt} = \frac{(1.5+\sin t)x_2(t)y(t)}{(1+0.5x_2(t))(1+0.5y(t))} - 0.1y(t) - 0.05y^2(t). \end{cases} \tag{3.1}$$

where $x_1(t)$ and $x_2(t)$ are the population density of immature and mature prey at time t, $y(t)$ is the population density of predator at time t. $s(t) = 10 + \sin t$ denotes the ratio of birth rate to available mature prey, $r(t) = 7 + \sin t$ represents the ratio of the conversion of immature prey to mature prey to existing immature prey, $d(t) = 0.3$ and $d_1(t) = 0.2$ are the ratio of the death rate of immature prey and mature prey, respectively. $\alpha(t) = 0.1$ is an internally specific interference coefficient. The term $\frac{(2+\sin t)x_2(t)y(t)}{(1+0.5x_2(t))(1+0.5y(t))}$ stands for the Crowley-Martin type functional response with rate $\beta(t) = 2 + \sin t$. Moreover, $\beta_1(t) = 1.5 + \sin t$ denotes the intake of predator; $d_2(t) = 0.1$ is the death rate of the predator and $\gamma(t) = 0.05$ is the internal specific disturbance coefficient for the predator. Simple computation shows (H_1)-(H_3) in Theorem 2.1 are satisfied, we conclude that system (3.1) has at least one positive periodic solution. Now we take the initial values $x_1(0) = 0.5, x_2(0) = 0.5, y(0) = 5$, then the following figure is obtained through Maple software.

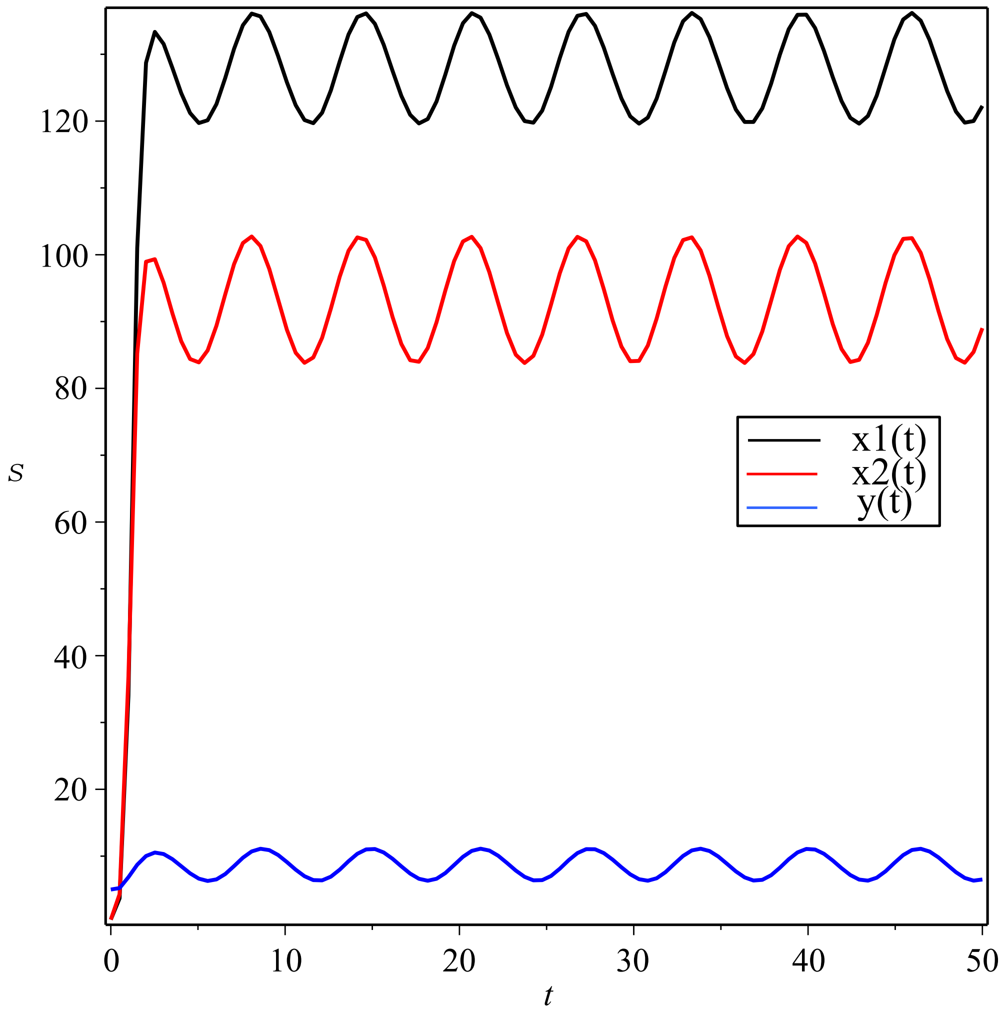

Figure 1. The periodic solution

As can be seen from the above figure, system (3.1) has a periodic solution with a period 2π. Namely, immature prey, mature prey and predator all have a periodic solution with period 2π in the same periodic environment. On the other hand, it also shows that our results are feasible based on the method of Mawhins coincidence degree theory and some nontrivial estimation techniques.

4. Conclusion

In the present paper, a stage-structured predator-prey model with Crowley-Martin type functional response is considered. It is assumed that the prey population can divide into two parts: mature and immature prey. The predator population is only dependent on mature prey and is influenced by Crowley-Martin type functional response. Based on the method of Mawhin's coincidence degree theory and novel estimation techniques for a priori bounds of unknown solutions to the operator equation $Fz = \mu Nz$, we obtain some interesting and novel sufficient conditions for the existence of positive periodic solution of the ecological model. Another interesting topic is the existence of analytic periodic solution. Kosov and Semenov [33, 34] studied the existence of (analytic) exact periodic solutions of some nonlinear differential equations, and showed that these periodic solutions are analytic functions under some sufficiently conditions. However, it is impossible to get the analytic periodic solutions in this model due to its great complexity. The analytic periodic solutions can be obtained for some particular systems, it is difficult for the very complicated system. Moreover, our method is based on Mawhin's coincidence degree theory. Thus, we can prove the existence of the periodic solutions by this method.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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