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Research article

Finite fractal dimension of pullback attractors for a nonclassical diffusion equation

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Abstract: In this paper, we investigate the finite fractal dimension of pullback attractors for a nonclassical diffusion equation in $H_0^1(\Omega)$. First, we prove the existence of pullback attractors for a nonclassical diffusion equation with arbitrary polynomial growth condition by applying the operator decomposition method. Then, by the fractal dimension theorem of pullback attractors given by [6], we prove the finite fractal dimension of pullback attractors for a nonclassical diffusion equation in $H_0^1(\Omega)$.

Keywords: nonclassical diffusion equations; finite fractal dimension; pullback attractors
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1. Introduction

The general nonclassical diffusion equations describe some physical phenomena, for example, non-Newtonian flows, soil mechanics and heat conduction theory. In this paper, we investigate the existence of pullback attractors and obtain the finite fractal dimension of pullback attractors in $H_0^1(\Omega)$ for the following nonclassical diffusion equations:

$$\begin{cases} u_t - \beta\Delta u_t - \Delta u + g(u) = f(t), \ x \in \Omega, \ t > \tau, \\ u|_{\partial\Omega} = 0, \ t > \tau, \\ u|_{t=\tau} = u_\tau(x), \ x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, the nonlinearity $g(u)$ and the given external force term $f(t)$ satisfy some assumptions later. It is called the nonclassical diffusion equation when $\beta > 0$ and it is named the classical reaction-diffusion equation in the case $\beta = 0$.

Before exhibiting the main results in this paper, let us briefly review some known results to the problem (1.1). In recent decades, the existence and asymptotic behavior of the solutions to problem (1.1) have been investigated by many authors under the different assumption conditions. There have been some results [1–3] when the nonlinearity $g(u)$ satisfies the arbitrary polynomial growth condition. Anh and Toan [1] investigated the existence of pullback attractors for a non-autonomous nonclassical diffusion equation in a non-cylindrical domain with the homogeneous Dirichlet boundary condition and got the upper semi-continuity of pullback attractors. Xie, Li and Zhu [2] studied the upper semi-continuity and regularity of global attractors for the autonomous nonclassical diffusion equations by applying the operator decomposition method. Yuan et al. [3] studied the existence of exponential attractors for problem (1.1). There have also been some results [4] for the fractal dimension of the attractor. Chen, Chen and Tang [4] considered the fractal dimension of the global attractor of nonclassical diffusion equation with fading memory and critical nonlinearity in a periodic boundary value domain by using the fractal dimension theorem given in [5]. To be more precise, our motivation in this paper is to use the operator decomposition method to prove the existence of pullback attractors in $H_0^1(\Omega)$ for the non-autonomous nonclassical diffusion equations with arbitrary polynomial growth condition, then we prove the finite fractal dimension of pullback attractors in $H_0^1(\Omega)$, which is still an open problem before the present paper solved it.

According to the problem (1.1), we can observe that the equation contains the term $-\Delta u_t$, it is different from the original reaction diffusion equation. For instance, the solution of the usual reaction diffusion equation has some smoothing effect, in other words, if the initial data only belong to a weaker topology space, then the solution of the usual reaction diffusion equation will belong to a stronger topology space with higher regularity. However, for problem (1.1), both the initial data and the solution belong to the same space, but the solution has no higher regularity because of appearance of $-\Delta u_t$. The main purpose of this paper is to prove the finite fractal dimension of pullback attractors. We first prove the existence of pullback attractors for a nonclassical diffusion equation with arbitrary polynomial growth condition by applying the operator decomposition method. Then, by the fractal dimension theorem of pullback attractors given by [6], we prove the finite fractal dimension of pullback attractors for a nonclassical diffusion equation in $H_0^1(\Omega)$.

Besides, under the conditions of nonlinearity g satisfies the following growth and dissipation conditions,

$$\begin{cases} g(u)u \geq -\lambda u^2 - C, \\ g'(u) \geq -\ell, \ g(0) = 0, \\ |g'(u)| \leq C(1 + |u|^\gamma), \\ \liminf\limits_{|u|\to\infty} \frac{ug(u)-\kappa G(u)}{u^2} \geq 0, \ \text{for some } \kappa > 0, \\ \liminf\limits_{|u|\to\infty} \frac{G(u)}{u^2} \geq 0, \end{cases}$$

there have also been some results [7–17] for problem (1.1) when the perturbation parameter β is a fixed constant. Recently, Anh and Bao [7] established the existence of pullback $\mathcal{D}$ attractors and the upper semi-continuity of pullback $\mathcal{D}$ attractors in $H_0^1(\Omega)$ for the non-autonomous nonclassical diffusion equation by using the asymptotic a priori estimate method. Wu and Zhang [15] discussed the long-time behaviour of the nonclassical diffusion equation with critical nonlinearity in $H_0^1(\Omega)$. Zhang and Ma [16] investigated the existence of exponential attractors of the nonclassical diffusion equation with critical nonlinearity and lower regular forcing term and got the finite fractal dimension of the global attractors. Wang, Zhu and Li [17] proved the regularity of pullback attractors for a three dimensional non-autonomous nonclassical diffusion equation with critical nonlinearity. In addition to the above results, the asymptotic behavior of solutions for a nonclassical diffusion equation with delay and memory has been extensively investigated by a lot of authors in [18–33] and references therein.

Moreover, there have also some known results in [34–37] for a nonclassical diffusion equation when $\beta(t)$ depends on time. Besides, there have also some results in [38–40] for the fractional nonclassical diffusion equations.

When the parameter $\beta = 0$, the problem (1.1) reduces to the one of the usual reaction diffusion equations. To our best knowledge, many results have been obtained about the usual reaction diffusion equations under the different assumption conditions. The existence of attractors for reaction diffusion equations has been investigated in many literatures, see [41–48] and the references therein for the recent progress.

Finally, the rest of the paper is arranged as follows. In Section 2, we give some definitions and lemmas used frequently in this paper. In Section 3, we prove the existence of pullback attractors for problem (1.1) in $H_0^1(\Omega)$ by the operator decomposition method. In Section 4, we prove the finite fractal dimension of pullback attractors in $H_0^1(\Omega)$.

Hereafter, let capital letter C be a general positive constant, which may vary from line to line to each step.

2. Preliminaries

In this section, we first introduce some notations on the function spaces and norms which will be used later to study the existence and finite fractal dimension of pullback attractors. Without loss of generality, we assume $\beta = 1$.

As in [10], let

$$A = -\Delta, \text{ with domain } D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

and consider the family of Hilbert spaces $D(A^{s/2})$, $s \in \mathbb{R}$, with the standard inner products and norms, respectively,

$$(\cdot,\cdot)_{D(A^{s/2})} = (A^{s/2}\cdot, A^{s/2}\cdot) \text{ and } \|\cdot\|_{D(A^{s/2})} = |A^{s/2}\cdot|_2.$$

Let $H = L^2(\Omega)$, $V_1 = H_0^1(\Omega)$, $V_2 = D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Denote by $(\cdot,\cdot)$ and $|\cdot|_2$ the inner product and norm of H, respectively. We denote by $\|\cdot\|_1$ and $\|\cdot\|_2$ the norms of V_1 and V_2, respectively.

To investigate problem (1.1), we need the following assumption conditions on g (see [2]),

(i) The nonlinearity $g \in C^1(\mathbb{R})$ fulfills $g(0) = 0$ and satisfies the following condition

$$\gamma_1|s|^p - \beta_1 \le g(s)s \le \gamma_2|s|^p + \beta_2, \ p \ge 2, \tag{2.1}$$

and the dissipative condition

$$g'(s) \ge -\ell, \tag{2.2}$$

where γ_i, β_i $(i = 1, 2)$ and ℓ are positive constants. Assume $G(u) = \int_0^u g(s)ds$, then there exist positive constants $\tilde{\gamma}_i$ and $\tilde{\beta}_i$ $(i = 1, 2)$ such that

$$\tilde{\gamma}_1|s|^p - \tilde{\beta}_1 \le G(s) \le \tilde{\gamma}_2|s|^p + \tilde{\beta}_2, \ p \ge 2. \tag{2.3}$$

(ii) We assume that the external $f \in L^2_{loc}(\mathbb{R}, H)$ satisfies

$$\int_{-\infty}^{t} e^{\alpha_1 s}|f|_2^2 ds < +\infty, \text{ for any } t \in \mathbb{R},$$

where $\alpha_1 = \min\{1, \lambda_1, 2(\mu - \ell)\}$, $\mu \geq 2\ell$ is a positive constant and λ_1 is the first eigenvalue of the operator $A = -\Delta$.

Definition 2.1. *[49] A two parameter family of mappings $U(t,\tau) : X \to X,\ t \geq \tau,\ \tau \in \mathbb{R}$, is called to be a process if*
1) $U(\tau,\tau)x = x,\ \forall \tau \in \mathbb{R},\ x \in X$,
2) $U(t,s)U(s,\tau)x = U(t,\tau)x,\ t \geq s \geq \tau,\ \tau \in \mathbb{R},\ x \in X$.

Definition 2.2. *[49] A family of bounded sets $\hat{B} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$ is called pullback $\mathcal{D}$-absorbing for the process $\{U(t,\tau)\}$ if for any $t \in \mathbb{R}$ and for any $\hat{D} \in \mathcal{D}$, there exists $\tau_0(t,\hat{D}) \leq t$ such that*

$$U(t,\tau)D(\tau) \subset B(t), \text{ for all } \tau \leq \tau_0(t,\hat{D}).$$

Definition 2.3. *[14] Let $\mathcal{D} = \{D(t)\}_{t\in\mathbb{R}}$ be a family of sets in X. A process $\{U(\cdot,\cdot)\}$ is said to be pullback $\mathcal{D}$-asymptotically compact in X if for any $t \in \mathbb{R}$ any sequences $\tau_n \to \infty$ and $x_n \in D(t-\tau_n)$, the sequence $\{U(t, t-\tau_n)x_n\}$ is relatively compact in X.*

Lemma 2.4. *[6] Let $U(t,\tau)$ be a process in a separable Hilbert space H, B be a uniformly pullback absorbing set in H, $\hat{A} = \{A(t) : t \in \mathbb{R}\}$ be a pullback attractor for $U(t,\tau)$, for all $u_1,\ u_2 \in B$, if there exists a finite dimensional projection P in the space H such that*

$$\|P(U(t,\tau)u_1 - U(t,\tau)u_2)\|_X \leq L(T_0)\|u_1 - u_2\|_X \tag{2.4}$$

with some existed constants $T_0,\ L(T_0) > 0$ being independent on the choice of t and for all $u_1,\ u_2 \in B$

$$\|(I-P)(U(t,\tau)u_1 - U(t,\tau)u_2)\|_X \leq \delta\|u_1 - u_2\|_X \tag{2.5}$$

with $\delta < 1$. Then the family of pullback attractors $\hat{A} = \{A(t) : t \in \mathbb{R}\}$ possesses a finite fractal dimension, specifically

$$dim_F(A(t)) \leq dimP \cdot log\Big(1 + \frac{8L}{1-\delta}\Big)\Big[log\Big(\frac{2}{1+\delta}\Big)\Big]^{-1},\ \forall t \in \mathbb{R}.$$

3. Pullback attractors in V_1

In this section, we shall show the existence of pullback attractors for Eq (1.1).

Theorem 3.1. *Assume $\mathcal{D} = \{D(t)\}_{t\in\mathbb{R}}$ is pullback absorbing and $U(\cdot,\cdot)$ is pullback $\mathcal{D}$-asymptotically compact in V_1. Then, there exists a pullback attractor $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$, and*

$$A(t) = \bigcap_{s\geq 0}\overline{\bigcup_{\tau\geq s} U(t, t-\tau)D(t-\tau)},\ \forall t \in \mathbb{R}.$$

Indeed, we prove Theorems 3.1 by the following a series of lemmas.

3.1. Existence and uniqueness of solutions

We know that the existence and uniqueness of the solutions to the nonclassical diffusion Eq (1.1) in $H_0^1(\Omega)$ was proved by *Galerkin* approximation methods [2]. Here we only state the result.

Lemma 3.2. *[2] For any $\tau \in \mathbb{R}$, $T > \tau$, for each $u_\tau \in H_0^1(\Omega)$, Eq (1.1) has a unique solution $u = u(t) = u(t; u_\tau)$. Moreover, we have also the following result: for any $u_i(\tau) \in H_0^1(\Omega)$, $u_i(t)$ $(i = 1, 2)$ denote the corresponding solutions of Eq (1.1), then for all $0 \le t \le T$,*

$$|u_1(t) - u_2(t)|_2^2 + \|u_1(t) - u_2(t)\|_1^2 \le Ce^{\kappa_1(T-\tau)}(\|u_1(\tau) - u_2(\tau)\|_1^2), \tag{3.1}$$

where κ_1 is a positive constant.

Thanks to Lemma 3.2, we can define a continuous process $\{U(t,\tau)\}_{t\ge\tau}$ in V_1 by

$$U(t,\tau)u_\tau = u(t),\ t \ge \tau, \tag{3.2}$$

where $u(t)$ is the unique solution of the Eq (1.1) with $f(t) = f \in L^2_{loc}(\mathbb{R}; H)$ and $u(\tau) = u_\tau \in V_1$.

3.2. Pullback absorbing balls in V_1

In this subsection, we shall establish the existence of pullback absorbing sets for n-dimensional nonclassical diffusion equation with Dirichlet boundary condition. Throughout this subsection, we always assume that the initial data belong to a bounded set of corresponding suitable space.

Next, we show the existence of pullback absorbing balls $\{\mathcal{B}(t)\}$ in V_1 for the nonclassical diffusion Eq (1.1).

Lemma 3.3. *Let g satisfy the assumption conditions (2.1), (2.2) and $f \in L^2_{loc}(\mathbb{R}; H)$ satisfy*

$$\int_{-\infty}^{t} e^{\alpha_1 s}|f(s)|_2^2 ds < +\infty, \text{ for any } t \in \mathbb{R}, \tag{3.3}$$

then the process $\{U(t,\tau)\}_{t\ge\tau}$ possesses a family of pullback absorbing balls $\{\mathcal{B}(t)\}$ in V_1 with the center zero and $R(t) = C\Big(1 + e^{-\alpha_1 t}\int_{-\infty}^{t} e^{\alpha_1 s}|f(s)|_2^2 ds\Big)$ with $\alpha_1 = \min\{1, \lambda_1, 2(\mu - \ell)\}$, for all $\tau \le \tau_B \le t$,

$$B = \Big\{u \in V_1 : \|u\|_{V_1}^2 \le R(t)\Big\}. \tag{3.4}$$

Proof. Multiplying (1.1) by u and integrating the resulting equation over Ω, we attain

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \frac{1}{2}\frac{d}{dt}\|u\|_1^2 + \|u\|_1^2 + (g(u), u) = (f(t), u). \tag{3.5}$$

Using the assumption condition (2.1), we have

$$(g(u), u) \ge \gamma_1|u|_{L^p}^p - \beta_1|\Omega|. \tag{3.6}$$

Applying the Young's inequality to the right-hand side term in (3.5), we deduce

$$(f(t), u) \le \frac{1}{2\lambda_1}|f|_2^2 + \frac{\lambda_1}{2}|u|_2^2. \tag{3.7}$$

Inserting (3.6) and (3.7) into (3.5), we obtain

$$\frac{d}{dt}|u|_2^2+\frac{d}{dt}\|u\|_1^2+\|u\|_1^2+2\gamma_1|u|_{L^p}^p\le\frac{1}{\lambda_1}|f|_2^2+C. \tag{3.8}$$

Noting that $\|u\|_1^2\ge\lambda_1|u|^2$ and taking $\alpha_1=\min\{1,\lambda_1,2(\mu-\ell)\}$, it follows from (3.8) that

$$\frac{d}{dt}(|u|_2^2+\|u\|_1^2)+\alpha_1(|u|_2^2+\|u\|_1^2)+2\gamma_1|u|_{L^p}^p\le C\Big(1+|f|_2^2\Big). \tag{3.9}$$

Applying the Gronwall's inequality [50–52] to (3.9), we deduce

$$\begin{aligned}|u(t)|_2^2+\|u(t)\|_1^2 &\le e^{-\alpha_1(t-\tau)}(|u(\tau)|_2^2+\|u(\tau)\|_1^2)+C\Big(1+e^{-\alpha_1 t}\int_\tau^t e^{\alpha_1 s}|f(s)|_2^2ds\Big)\\ &\le e^{-\alpha_1(t-\tau)}(|u(\tau)|_2^2+\|u(\tau)\|_1^2)+C\Big(1+e^{-\alpha_1 t}\int_{-\infty}^t e^{\alpha_1 s}|f(s)|_2^2ds\Big)\\ &\le R(t),\ \tau\le\tau_B\le t.\end{aligned} \tag{3.10}$$

The proof is thus complete. □

By Lemma 3.3, it is easy to check that the $\mathcal{B}=\{B(t)\}_{t\in\mathbb{R}}$, where

$$B(t)=\{u\in V_1;\|u\|_{V_1}^2\le R(t)\}, \tag{3.11}$$

is (V_1,V_1)-pullback absorbing for the process U defined by (3.2). Moreover,

$$e^{\alpha_1\tau}(R(t))\to 0,\ \text{as } \tau\to-\infty. \tag{3.12}$$

3.3. Asymptotically compact in V_1

In this subsection, we shall establish the existence of the pullback attractors in V_1 for the Eq (1.1). The main difficulty is to attain a higher regularity estimates to ensure the asymptotic compactness of process. To overcome this difficulty, we shall use the decomposition method from [2]. We now decompose the solution $U(t,\tau)u_\tau=u(t)$ into

$$U(t,\tau)u_\tau=U_1(t)u_\tau+K(t)u_\tau, \tag{3.13}$$

where $U_1(t)u_\tau=v(t)$ and $K(t)u_\tau=w(t)$ are solutions to the following equations respectively:

$$\begin{cases} v_t-\Delta v_t-\Delta v+g(u)-g(w)+\mu v=0,\ x\in\Omega,\ t>\tau,\\ v|_{\partial\Omega}=0,\ t>\tau,\\ v|_{t=\tau}=u_\tau(x),\ x\in\Omega,\end{cases} \tag{3.14}$$

and

$$\begin{cases} w_t-\Delta w_t-\Delta w+g(w)+\mu w=f(t)+\mu u,\ x\in\Omega,\ t>\tau,\\ w|_{\partial\Omega}=0,\ t>\tau,\\ w|_{t=\tau}=0,\ x\in\Omega.\end{cases} \tag{3.15}$$

First, we prove the higher regularity of the solution w.

Lemma 3.4. *Assume $f \in L^2_{loc}(\mathbb{R}; H)$ satisfies*

$$\int_{-\infty}^{t} e^{\alpha_1 s}|f(s)|_2^2 ds < +\infty, \text{ for any } t \in \mathbb{R}, \tag{3.16}$$

then

$$\|w(t)\|_1^2 + \|w(t)\|_2^2 \le C\Big(1 + e^{-\alpha_1 t}\int_{-\infty}^{t} e^{\alpha_1 s}|f(s)|_2^2 ds\Big),\ t \in \mathbb{R}. \tag{3.17}$$

Proof. Multiplying (3.15) by $-\Delta w$ and integrating over Ω, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|w\|_1^2 + \|w\|_2^2) + \|w\|_2^2 + \int_\Omega g(w)(-\Delta w)dx + \mu\|w\|_1^2\\ &= \int_\Omega f(-\Delta w)dx + \mu\int_\Omega u(-\Delta w)dx.\end{aligned} \tag{3.18}$$

Integrating by parts over Ω in (3.18) and applying the Young's inequality to (3.18), we derive

$$\frac{d}{dt}(\|w\|_1^2 + \|w\|_2^2) + \|w\|_2^2 + 2(\mu - \ell)\|w\|_1^2 \le |f|_2^2 + 2\mu^2|u|_2^2,$$

which implies

$$\frac{d}{dt}(\|w\|_1^2 + \|w\|_2^2) + \alpha_1(\|w\|_1^2 + \|w\|_2^2) \le |f|_2^2 + 2\mu^2|u|_2^2. \tag{3.19}$$

Applying Lemma 3.3 and the Gronwall's inequality [50–52] to (3.19), we find

$$\begin{aligned}&\|w(s)\|_1^2 + \|w(s)\|_2^2 \le e^{-\alpha_1 t}\int_\tau^t e^{\alpha_1 s}(|f(s)|_2^2 + 2\mu^2|u(s)|_2^2)ds\\ &\le e^{-\alpha_1 t}\int_\tau^t e^{\alpha_1 s}\Big(|f(s)|_2^2 + 2\mu^2 R(s)\Big)ds\\ &\le e^{-\alpha_1 t}\int_{-\infty}^t e^{\alpha_1 s}|f(s)|_2^2 ds + \frac{2\mu^2}{\alpha_1}R(t)\\ &\le C\Big(1 + e^{-\alpha_1 t}\int_{-\infty}^t e^{\alpha_1 s}|f(s)|_2^2 ds\Big)\\ &< +\infty,\ \text{for any fixed } t \in \mathbb{R},\end{aligned} \tag{3.20}$$

where we have used the fact $w(0) = 0$. The proof is now complete. □

Finally, we prove the dissipation of the solutions v.

Lemma 3.5. *For any $v(\tau) \in B$, there exists a positive constant $\alpha_2 = \min\{2(\mu - \ell), 2\}$ such that*

$$|v(t)|_2^2 + \|v(t)\|_1^2 \le (|v(\tau)|_2^2 + \|v(\tau)\|_1^2)e^{-\alpha_2(t-\tau)}. \tag{3.21}$$

Proof. Multiplying (3.14) by $v(t)$ and integrating the resulting equation over Ω, we attain

$$\frac{1}{2}\frac{d}{dt}(|v(t)|_2^2 + \|v(t)\|_1^2) + \int_\Omega (g(u) - g(w))v dx + \|v(t)\|_1^2 + \mu|v|_2^2 \le 0. \tag{3.22}$$

Using the assumption (2.2), we have

$$\int_\Omega (g(u) - g(w))v dx \geq -\ell |v|_2^2. \tag{3.23}$$

Inserting (3.23) into (3.22), we obtain

$$\frac{1}{2}\frac{d}{dt}(|v(t)|_2^2 + \|v(t)\|_1^2) + (\mu - \ell)|v|_2^2 + \|v\|_1^2 \leq 0.$$

Letting $\alpha_2 = \min\{2(\mu - \ell), 2\}$, we get

$$\frac{d}{dt}(|v(t)|_2^2 + \|v(t)\|_1^2) + \alpha_2(|v|_2^2 + \|v\|_1^2) \leq 0. \tag{3.24}$$

Applying the Gronwall's inequality ([50–52]) to (3.24), we conclude

$$|v(t)|_2^2 + \|v(t)\|_1^2 \leq (|v(\tau)|_2^2 + \|v(\tau)\|_1^2)e^{-\alpha_2(t-\tau)}, \tag{3.25}$$

which completes the proof. □

Proof of Theorem 3.1. Combining Lemmas 3.2–3.5, we can get the desired result.
The proof is thus complete. □

4. Finite fractal dimension of pullback attractors

In this section, we shall use the finite fractal dimension theorem given in [6, 53] to obtain the fractal dimension of pullback attractors for a nonclassical reaction diffusion Eq (1.1) is finite.

Theorem 4.1. *The fractal dimension of the pullback attractors $\mathcal{A}$ for the process $U(\cdot,\cdot)$ generated by* (1.1) *is finite.*

Proof. Let $u(t)$ and $v(t)$ be two weak solutions to the non-autonomous nonclassical reaction diffusion Eq (1.1) belonging to the pullback attractors $\mathcal{A}$. Let us define $w(t) = u(t) - v(t)$, then $w(t)$ solves the following equation

$$w_t - \Delta w_t - \Delta w + g(u) - g(v) = 0, \ x \in \Omega, \ t > \tau. \tag{4.1}$$

Multiplying (4.1) by $w(t)$ and integrating it over Ω, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|w(t)|_2^2 + \|w(t)\|_1^2) + \|w(t)\|_1^2 \\ &= -\int_\Omega (g(u) - g(v))w dx \\ &= -\int_\Omega g'(\eta)w^2 dx \\ &\leq \ell |w|_2^2, \end{aligned} \tag{4.2}$$

with $\eta = \theta_1 u + (1 - \theta_1)v$, where θ_1 is a constant and $\theta_1 \in [0, 1]$.
Then

$$\frac{d}{dt}(|w(t)|_2^2 + \|w(t)\|_1^2) \leq 2\ell(|w|_2^2 + \|w\|_1^2) \leq \mu_1(|w|_2^2 + \|w\|_1^2), \tag{4.3}$$

with $\mu_1 = 2\ell$.
Therefore, applying the Gronwall inequality to (4.3), we conclude

$$|w(t)|_2^2 + \|w(t)\|_1^2 \leq (|w(\tau)|_2^2 + \|w(\tau)\|_1^2)e^{\mu_1(t-\tau)},\ t > \tau, \tag{4.4}$$

and

$$\|w(t)\|_1^2 \leq (1 + \lambda_1^{-1})\|w(\tau)\|_1^2 e^{\mu_1(t-\tau)},\ t > \tau. \tag{4.5}$$

Let $\{\omega_N\}$ be an orthonormal basis of H_N which consists of eigenvectors of $A = -\Delta$. λ_k denote the corresponding eigenvalues, $k = 1, 2, ...$ and $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \to +\infty$ as $j \to +\infty$. Then $\{\omega_N\}$ is also an orthonormal basis of V_1. We write $H_N = span\{\omega_1, \cdot\cdot\cdot, \omega_N\}$ and $P_N : V_1 \to H_N$ is the orthogonal projection.
Let $w = P_N w + Q_N w$, where $P_N w \in H_N$, then

$$\|P_N w(t)\|_1^2 \leq (1 + \lambda_1^{-1})\|w(\tau)\|_1^2 e^{\mu_1(t-\tau)},\ t > \tau. \tag{4.6}$$

Multiplying (4.1) by $Q_N w := (I - P_N)w$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|Q_N w(t)|_2^2 + \|Q_N w(t)\|_1^2) + \|Q_N w(t)\|_1^2 \\ &= -\int_\Omega (g(u) - g(v))Q_N w dx \\ &\leq \int_\Omega |(g(u) - g(v))||Q_N w| dx. \end{aligned} \tag{4.7}$$

Next, for dealing with (4.7), we divide into the following two cases that $p \geq 2, n \leq 2$ and $2 \leq p \leq \frac{2n-2}{n-2}$, $n \geq 3$.

Case 1: $p \geq 2, n \leq 2$. We know that both v and u lie in $\mathcal{A}$, they are bounded in L^∞ ($u, v \in V_1 \hookrightarrow L^\infty$), and

$$|(g(u) - g(v))|_2 \leq C|v(t) - u(t)|_2 = C|w(t)|_2. \tag{4.8}$$

Thus,

$$\frac{1}{2}\frac{d}{dt}(|Q_N w(t)|_2^2 + \|Q_N w(t)\|_1^2) + \|Q_N w(t)\|_1^2 \leq C|w|_2|Q_N w(t)|_2. \tag{4.9}$$

Utilizing the inequality

$$|Q_N \omega|_2 \leq \lambda_{N+1}^{-1/2}\|Q_N \omega\|_1,\ \omega \in (H_N)^\perp, \tag{4.10}$$

and combining (4.9) with (4.10), and using the Young inequality, we attain

$$\begin{aligned} &\frac{d}{dt}(|Q_N w(t)|_2^2 + \|Q_N w(t)\|_1^2) + \lambda_1 |Q_N w(t)|_2^2 + \|Q_N w(t)\|_1^2 \\ &\leq 2\lambda_{N+1}^{-1/2}|w|_2\|Q_N w\|_1 \\ &\leq \lambda_{N+1}^{-1/2}|w|_2^2 + \lambda_{N+1}^{-1/2}\|Q_N w\|_1^2. \end{aligned} \tag{4.11}$$

Thus,

$$\frac{d}{dt}(|Q_N w(t)|_2^2 + \|Q_N w(t)\|_1^2) + \lambda_1 |Q_N w(t)|_2^2 + (1 - \lambda_{N+1}^{-1/2})\|Q_N w(t)\|_1^2 \leq \lambda_{N+1}^{-1/2}|w|_2^2. \tag{4.12}$$

Let us choose N large enough, so that $1-\lambda_{N+1}^{-1/2}>0$ and let

$$\alpha_3=\min\{\lambda_1, 1-\lambda_{N+1}^{-1/2}\}.$$

Inserting (4.5) into (4.12) implies

$$\begin{aligned}&\frac{d}{dt}(|Q_Nw(t)|_2^2+\|Q_Nw(t)\|_1^2)+\alpha_3(|Q_Nw(t)|_2^2+\|Q_Nw(t)\|_1^2)\\&\leq \lambda_{N+1}^{-1/2}\lambda_1^{-1}(1+\lambda_1^{-1})\|w(\tau)\|_1^2e^{\mu_1(t-\tau)},\ t>\tau.\end{aligned}\tag{4.13}$$

Integrating the above inequality, we derive

$$\begin{aligned}&e^{\alpha_3 t}(|Q_Nw(t)|_2^2+\|Q_Nw(t)\|_1^2)\\&\leq e^{\alpha_3\tau}(|Q_Nw(\tau)|_2^2+\|Q_Nw(\tau)\|_1^2)\\&\quad+\lambda_{N+1}^{-1/2}\lambda_1^{-1}(1+\lambda_1^{-1})\|w(\tau)\|_1^2(\alpha_3+\mu_1)^{-1}e^{-\mu_1\tau}(e^{(\mu_1+\alpha_3)t}-e^{(\mu_1+\alpha_3)\tau})\\&\leq e^{\alpha_3\tau}(1+\lambda_{N+1}^{-1})\|Q_Nw(\tau)\|_1^2\\&\quad+\lambda_{N+1}^{-1/2}\lambda_1^{-1}(1+\lambda_1^{-1})\|w(\tau)\|_1^2(\alpha_3+\mu_1)^{-1}e^{-\mu_1\tau}(e^{(\mu_1+\alpha_3)t}-e^{(\mu_1+\alpha_3)\tau}),\ t>\tau,\end{aligned}$$

i.e.,

$$\begin{aligned}\|Q_Nw(t)\|_1^2&\leq e^{-\alpha_3(t-\tau)}(1+\lambda_{N+1}^{-1})\|w(\tau)\|_1^2\\&\quad+C\lambda_{N+1}^{-1/2}\|w(\tau)\|_1^2e^{\mu_1(t-\tau)},\ t>\tau.\end{aligned}\tag{4.14}$$

Let us choose again N large enough and $t-\tau=T_0=\frac{ln2(1+\lambda_{N+1}^{-1})}{\alpha_3}$ such that

$$\|Q_Nw(t)\|_1^2 \leq \Big(\frac{1}{2}+C\lambda_{N+1}^{-1/2}(1+\lambda_{N+1}^{-1})^{\frac{\mu_1}{\alpha_3}}\Big)\|w(\tau)\|_1^2.\tag{4.15}$$

Since $\lambda_{N+1}\to+\infty$, $\frac{1}{2}+C\lambda_{N+1}^{-1/2}(1+\lambda_{N+1}^{-1})^{\frac{\mu_1}{\alpha_3}}<1$ when N is sufficiently large. Clearly,

$$\|P_Nw(t)\|_1^2\leq L\|w(\tau)\|_1^2 \text{ and } \|Q_Nw(t)\|_1^2\leq\delta_1\|w(\tau)\|_1^2,\tag{4.16}$$

where $L=(1+\lambda_1^{-1})e^{\mu_1 T_0}$ and $\delta_1=\frac{1}{2}+C\lambda_{N+1}^{-1/2}(1+\lambda_{N+1}^{-1})^{\frac{\mu_1}{\alpha_3}}$.
Therefore, we conclude the estimate of the fractal dimension of the pullback attractor $\mathcal{A}$ by Lemma 2.4 for $p\geq 2$, $n\leq 2$,

$$dim_F(A(t))\leq dimP\cdot log\Big(1+\frac{8L}{1-\delta_1}\Big)\Big[log\Big(\frac{2}{1+\delta_1}\Big)\Big]^{-1}.$$

Case 2: $2\leq p\leq\frac{2n-2}{n-2}$, $n\geq 3$. Using the Sobolev embedding theorem ($V_1\hookrightarrow L^{\frac{2n}{n-2}}$), we infer

$$\begin{aligned}\int_\Omega|g(u)-g(v)|_2^2dx &= \int_\Omega|g'(u+\theta(v-u))|^2|u-v|^2dx\\&\leq C\int_\Omega(1+|u|^{2(p-2)}+|v|^{2(p-2)})|u-v|^2dx\\&\leq C\Big(1+|u|^{2(p-2)}_{L^{\frac{2n}{n-2}}}+|v|^{2(p-2)}_{L^{\frac{2n}{n-2}}}\Big)|w|^2_{L^{\frac{2n}{n-2}}}\end{aligned}$$

$$\leq \quad C\Big(1 + \|u\|_1^{2(p-2)} + \|v\|_1^{2(p-2)}\Big)\|w\|_1^2. \tag{4.17}$$

Inserting (4.17) into (4.7) and using (4.5), we have

$$\begin{aligned}&\frac{d}{dt}(|Q_N w(t)|_2^2 + \|Q_N w(t)\|_1^2) + 2\|Q_N w(t)\|_1^2\\&\leq C\lambda_{N+1}^{-1/2}\Big(1 + \|u\|_1^{2(p-2)} + \|v\|_1^{2(p-2)}\Big)\|w\|_1^2 + \lambda_{N+1}^{-1/2}\|Q_N w(t)\|_1^2.\end{aligned} \tag{4.18}$$

Thus,

$$\begin{aligned}&\frac{d}{dt}(|Q_N w(t)|_2^2 + \|Q_N w(t)\|_1^2) + \lambda_1|Q_N w(t)|_2^2 + (1 - \lambda_{N+1}^{-1/2})\|Q_N w(t)\|_1^2\\&\leq C\Big(1 + \|u\|_1^{2(p-2)} + \|v\|_1^{2(p-2)}\Big)\|w\|_1^2\\&\leq C\lambda_{N+1}^{-1/2}(1 + \lambda_1^{-1})\|w(\tau)\|_1^2\Big(1 + \|u\|_1^{2(p-2)} + \|v\|_1^{2(p-2)}\Big)e^{\mu_1(t-\tau)},\ t > \tau.\end{aligned} \tag{4.19}$$

Integrating (4.19), we lead to

$$\begin{aligned}&e^{\alpha_3 t}(|Q_N w(t)|_2^2 + \|Q_N w(t)\|_1^2)\\&\leq e^{\alpha_3\tau}(1 + \lambda_{N+1}^{-1})\|Q_N w(\tau)\|_1^2\\&\quad + C\lambda_{N+1}^{-1/2}(1 + \lambda_1^{-1})\|w(\tau)\|_1^2\int_\tau^t \Big(1 + \|u(s)\|_1^{2(p-2)} + \|v(s)\|_1^{2(p-2)}\Big)e^{\alpha_3 s}e^{\mu_1(s-\tau)}ds\\&\qquad t > \tau,\end{aligned}$$

i.e.,

$$\begin{aligned}&(|Q_N w(t)|_2^2 + \|Q_N w(t)\|_1^2)\\&\leq e^{-\alpha_3(t-\tau)}(1 + \lambda_{N+1}^{-1})\|w(\tau)\|_1^2\\&\quad + C\lambda_{N+1}^{-1/2}(1 + \lambda_1^{-1})\|w(\tau)\|_1^2 e^{-\alpha_3 t}\int_\tau^t \Big(1 + \|u(s)\|_1^{2(p-2)} + \|v(s)\|_1^{2(p-2)}\Big)e^{\alpha_3 s}e^{\mu_1(s-\tau)}ds\\&\qquad t > \tau.\end{aligned} \tag{4.20}$$

We now deal with the right-side terms in (4.20) as follows. For the first term, taking $t - \tau = T_0 = \frac{ln2(1+\lambda_{N+1}^{-1})}{\alpha_3}$, we have

$$e^{-\alpha_3 T_0}(1 + \lambda_{N+1}^{-1})\|w(\tau)\|_1^2 \leq \frac{1}{2}\|w(\tau)\|_1^2. \tag{4.21}$$

For the second term, let us choose N large enough such that

$$\begin{aligned}&C\lambda_{N+1}^{-1/2}(1 + \lambda_1^{-1})\|w(\tau)\|_1^2 e^{-\alpha_3 t}\int_\tau^t e^{\alpha_3 s}e^{\mu_1(s-\tau)}ds\\&\leq C\lambda_{N+1}^{-1/2}(1 + \lambda_1^{-1})\|w(\tau)\|_1^2(\alpha_3 + \mu_1)^{-1}e^{\mu_1 T_0}\\&\leq C\lambda_{N+1}^{-1/2}(1 + \lambda_{N+1}^{-1})^{\frac{\mu_1}{\alpha_3}}\|w(\tau)\|_1^2.\end{aligned} \tag{4.22}$$

For the remaining two terms, we only build the estimate to the third term and choose N large enough such that

$$C\lambda_{N+1}^{-1/2}(1 + \lambda_1^{-1})\|w(\tau)\|_1^2 e^{-\alpha_3 t}\int_\tau^t \|u(s)\|_1^{2(p-2)}e^{\alpha_3 s}e^{\mu_1(s-\tau)}ds$$

$$
\begin{aligned}
&\leq C\lambda_{N+1}^{-1/2}(1+\lambda_1^{-1})\|w(\tau)\|_1^2 e^{-\alpha_3 t}\int_\tau^t \Big(1+e^{-\alpha_1 s}\int_\tau^s e^{\alpha_1 r}|f(r)|_2^2 dr\Big)^{(p-2)} e^{\alpha_3 s}e^{\mu_1(s-\tau)}ds\\
&\leq C\lambda_{N+1}^{-1/2}(1+\lambda_1^{-1})\|w(\tau)\|_1^2 e^{-\alpha_3 t}\int_\tau^t \Big(1+\int_\tau^s |f(r)|_2^2 dr\Big)^{(p-2)} e^{\alpha_3 s}e^{\mu_1(s-\tau)}ds\\
&\leq C\lambda_{N+1}^{-1/2}(1+\lambda_1^{-1})\|w(\tau)\|_1^2 e^{-\alpha_3 t}\int_\tau^t e^{\alpha_3 s}e^{\mu_1(s-\tau)}ds\\
&\leq C\lambda_{N+1}^{-1/2}(1+\lambda_1^{-1})\|w(\tau)\|_1^2(\alpha_3+\mu_1)^{-1}e^{\mu_1 T_0}\\
&\leq C\lambda_{N+1}^{-1/2}(1+\lambda_{N+1}^{-1})^{\frac{\mu_1}{\alpha_3}}\|w(\tau)\|_1^2. \qquad (4.23)
\end{aligned}
$$

Thus, let us choose N large enough and $t-\tau = T_0 = \frac{ln2(1+\lambda_{N+1}^{-1})}{\alpha_3}$ such that

$$
\|Q_N w(t)\|_1^2 \;\leq\; \Big(\frac{1}{2}+C\lambda_{N+1}^{-1/2}(1+\lambda_{N+1}^{-1})^{\frac{\mu_1}{\alpha_3}}\Big)\|w(\tau)\|_1^2. \qquad (4.24)
$$

Since $\lambda_{N+1}\to+\infty$, $\frac{1}{2}+C\lambda_{N+1}^{-1/2}(1+\lambda_{N+1}^{-1})^{\frac{\mu_1}{\alpha_3}}<1$ when N is sufficiently large.
Obviously,

$$
\|P_N w(t)\|_1^2 \leq L\|w(\tau)\|_1^2 \text{ and } \|Q_N w(t)\|_1^2 \leq \delta_2\|w(\tau)\|_1^2, \qquad (4.25)
$$

where $L=(1+\lambda_1^{-1})e^{\mu_1 T_0}$ and $\delta_2=\frac{1}{2}+C\lambda_{N+1}^{-1/2}(1+\lambda_{N+1}^{-1})^{\frac{\mu_1}{\alpha_3}}$.
Consequently, we conclude the estimate of the fractal dimension of the pullback attractor $\mathcal{A}$ by Lemma 2.4 for $2\leq p\leq\frac{2n-2}{n-2}$, $n\geq 3$

$$
dim_F(A(t)) \leq dimP\cdot log\Big(1+\frac{8L}{1-\delta_2}\Big)\Big[log\Big(\frac{2}{1+\delta_2}\Big)\Big]^{-1}.
$$

The proof is finally complete. □

5. Conclusions

This paper mainly study the pullback attractors for the nonclassical diffusion equations, including the following two results: (i) the pullback attractors for the nonclassical diffusion equations with arbitrary polynomial growth nonlinearity is obtained by operator decomposition method. (ii) the fractal dimensional of pullback attractors is finite. Besides, the method in this article can also be used to investigate other evolution equations.

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Conflict of interest

This work does not have any conflicts of interest.

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