Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 8035–8063.
DOI: 10.3934/math.2022448
Received: 07 August 2021
Revised: 03 January 2022
Accepted: 10 January 2022
Published: 23 February 2022

Research article

Post-quantum Ostrowski type integral inequalities for functions of two variables

Miguel Vivas-Cortez[1], **Muhammad Aamir Ali**[2,*], **Hüseyin Budak**[3] **and Ifra Bashir Sial**[4]

[1] Pontificia Universidad Católica del Ecuador, Facultad de Ciencias Naturales y Exactas, Escuela de Ciencias Físicas y Matemáticas, Sede Quito, Ecuador

[2] Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China

[3] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Turkey

[4] School of Sciences, Jiangsu University, Zhenjiang 212013, China

* **Correspondence:** Email: mahr.muhammad.aamir@gmail.com.

Abstract: In this study, we give the notions about some new post-quantum partial derivatives and then use these derivatives to prove an integral equality via post-quantum double integrals. We establish some new post-quantum Ostrowski type inequalities for differentiable coordinated functions using the newly established equality. We also show that the results presented in this paper are the extensions of some existing results.

Keywords: Ostrowski inequality; (p, q)-integrals; post-quantum calculus; co-ordinated convex function
Mathematics Subject Classification: 26D07, 26D10, 26D15

1. Introduction

A. M. Ostowski established the following intriguing integral inequality in 1938, which is known in the literature as the Ostrowski inequality.

Theorem 1.1. *[47] Let $F : [\pi_1, \pi_2] \to \mathbb{R}$ be a differentiable function on (π_1, π_2) whose derivative is bounded on (π_1, π_2), i.e., $\left\|F'(\tau)\right\|_\infty := \sup\left|F'(\tau)\right| < \infty$, for all $\tau \in (\pi_1, \pi_2)$. Then we have the following integral inequality:*

$$\left|F(x) - \frac{1}{\pi_2 - \pi_1}\int_{\pi_1}^{\pi_2} F(x)dx\right| \le \left[\frac{1}{4} + \frac{(x - \frac{\pi_1+\pi_2}{2})}{(\pi_2 - \pi_1)^2}\right](\pi_2 - \pi_1)\left\|F'\right\|_\infty, \tag{1.1}$$

for all $x \in [\pi_1, \pi_2]$. The $\frac{1}{4}$ is the best possible.

Inequality (1.1) can be rewritten in the following way:

$$\left| F(x) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(x)dx \right| \leq \left[\frac{(x-\pi_1)^2 + (\pi_2 - x)^2}{2(\pi_2 - \pi_1)} \right] \left\| F' \right\|_\infty . \tag{1.2}$$

Since 1938, numerous mathematicians have worked on and around the Ostrowski inequality, in a variety of ways and with a variety of applications in Numerical Analysis and Probability, etc.

Many authors investigate several versions of the Ostrowski integral inequality for bounded variation mappings, Lipschitzian mappings, monotonic mappings, absolutely continuous mappings, convex mappings, and n-times differentiable mappings with error estimates for various particular means and numerical quadrature techniques. For recent results and generalizations concerning Ostrowski's inequality, one can consult [8, 9, 15, 24, 26, 27, 42, 48–50, 52] and the references therein.

The following is a formal definition of co-ordinated convex (concave) functions:

Definition 1.2. *A function $F : \Delta \to \mathbb{R}$ is called co-ordinated convex on Δ, for all $(x, u), (y, v) \in \Delta$ and $\tau, s \in [0, 1]$, if it satisfies the following inequality:*

$$\begin{aligned} & F(\tau x + (1-\tau)\, y, su + (1-s)\, v) \\ \leq & \tau s\, F(x, u) + \tau(1-s)F(x, v) + s(1-\tau)F(y, u) + (1-\tau)(1-s)F(y, v). \end{aligned} \tag{1.3}$$

The mapping F is a co-ordinated concave on Δ if the inequality (1.3) holds in reversed direction for all $\tau, s \in [0, 1]$ and $(x, u), (y, v) \in \Delta$.

For co-ordinated convex functions, M. A. Latif et al. established the following Ostrowski type inequalities in [41]:

Theorem 1.3. *[41] Let $F : \Delta := [\pi_1, \pi_2] \times [\pi_3, \pi_4] \to \mathbb{R}$ be a twice partial differentiable mapping on Δ° with $\pi_1 < \pi_2,\ \pi_3 < \pi_4,\ \pi_1, \pi_3 \geq 0$ such that $\frac{\partial^2 F}{\partial s \partial \tau} \in L(\Delta)$. If $\left|\frac{\partial^2 F}{\partial s \partial \tau}\right|$ is co-ordinated convex on Δ and $\left|\frac{\partial^2 F}{\partial s \partial \tau}\right| \leq M,\ (x, y) \in \Delta$, then the following inequality holds:*

$$\begin{aligned} & \left| F(x, y) + \frac{1}{(\pi_2 - \pi_1)(\pi_4 - \pi_3)} \int_{\pi_1}^{\pi_2} \int_{\pi_3}^{\pi_4} F(u, v)dvdu - \pi_{11} \right| \\ \leq \;& M \left[\frac{(x-\pi_1)^2 + (\pi_2 - x)^2}{2(\pi_2 - \pi_1)} \right] \left[\frac{(y-\pi_3)^2 + (\pi_4 - y)^2}{2(\pi_4 - \pi_3)} \right], \end{aligned} \tag{1.4}$$

where

$$\pi_{11} = \frac{1}{\pi_4 - \pi_3} \int_{\pi_3}^{\pi_4} F(x, v)dv + \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(u, y)du.$$

Theorem 1.4. *[41] Let $F : \Delta := [\pi_1, \pi_2] \times [\pi_3, \pi_4] \to \mathbb{R}$ be a twice partial differentiable mapping on Δ° with $\pi_1 < \pi_2,\ \pi_3 < \pi_4,\ \pi_1, \pi_3 \geq 0$ such that $\frac{\partial^2 F}{\partial s \partial \tau} \in L(\Delta)$. If $\left|\frac{\partial^2 F}{\partial s \partial \tau}\right|^s$ is co-ordinated convex on Δ, $s > 1,\ \frac{1}{s} + \frac{1}{r} = 1$ and $\left|\frac{\partial^2 F}{\partial s \partial \tau}(x, y)\right| \leq M,\ (x, y) \in \Delta$, then the following inequality holds:*

$$\left|F(x,y)+\frac{1}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\int_{\pi_1}^{\pi_2}\int_{\pi_3}^{\pi_4}F(u,v)dvdu-\pi_{11}\right| \tag{1.5}$$
$$\leq \frac{M}{(1+r)^{\frac{2}{r}}}\left[\frac{(x-\pi_1)^2+(\pi_2-x)^2}{2(\pi_2-\pi_1)}\right]\left[\frac{(y-\pi_3)^2+(\pi_4-y)^2}{2(\pi_4-\pi_3)}\right],$$

where π_{11} is defined in Theorem 1.3.

Theorem 1.5. *[41] Let $F:\Delta:=[\pi_1,\pi_2]\times[\pi_3,\pi_4]\to\mathbb{R}$ be a twice partial differentiable mapping on Δ° with $\pi_1<\pi_2,\ \pi_3<\pi_4,\ \pi_1,\pi_3\geq 0$ such that $\frac{\partial^2F}{\partial s\partial\tau}\in L(\Delta)$. If $\left|\frac{\partial^2F}{\partial s\partial\tau}\right|^p$ is co-ordinated convex on Δ, $p\geq 1$ and $\left|\frac{\partial^2F}{\partial s\partial\tau}(x,y)\right|\leq M,\ (x,y)\in\Delta$, then the following inequality holds:*

$$\left|F(x,y)+\frac{1}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\int_{\pi_1}^{\pi_2}\int_{\pi_3}^{\pi_4}F(u,v)dvdu-\pi_{11}\right| \tag{1.6}$$
$$\leq \frac{M}{4}\left[\frac{(x-\pi_1)^2+(\pi_2-x)^2}{2(\pi_2-\pi_1)}\right]\left[\frac{(y-\pi_3)^2+(\pi_4-y)^2}{2(\pi_4-\pi_3)}\right],$$

where π_{11} is defined in Theorem 1.3.

On the other side, in the domain of q-analysis, many works are being carried out initiating from Euler in order to attain adeptness in mathematics that constructs quantum computing q-calculus considered as a relationship between physics and mathematics. In different areas of mathematics, it has numerous applications such as combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and other sciences, mechanics, the theory of relativity, and quantum theory [31,35]. Quantum calculus also has many applications in quantum information theory which is an interdisciplinary area that encompasses computer science, information theory, philosophy, and cryptography, among other areas [16, 17]. Apparently, Euler invented this important mathematics branch. He used the q parameter in Newton's work on infinite series. Later, in a methodical manner, the q-calculus that knew without limits calculus was firstly given by F. H. Jackson [30,33]. In 1966, W. Al-Salam [12] introduced a q-analogue of the q-fractional integral and q-Riemann-Liouville fractional. Since then, the related research has gradually increased. In particular, in 2013, J. Tariboon and S. K. Ntouyas introduced ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral in [54]. In 2020, S. Bermudo et al. introduced the notion of ${}^{\pi_2}D_q$ derivative and q^{π_2}-integral in [14]. T. Acar et al. generalized to quantum calculus and introduced the notions of post-quantum calculus or shortly (p,q)-calculus in [1]. In [53], M. Tunç and E. Göv gave the post-quantum variant of ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral. Recently, in 2021, Y. M. Chu et al. introduced the notions of ${}^{\pi_2}D_{p,q}$ derivative and $(p,q)^{\pi_2}$-integral in [25].

Many integral inequalities have been studied using quantum and post-quantum integrals for various types of functions. For example, in [3, 6, 10, 11, 14, 18, 19, 34, 43, 44], the authors used ${}_{\pi_1}D_q, {}^{\pi_2}D_q$-derivatives and q_{π_1}, q^{π_2}-integrals to prove Hermite-Hadamard integral inequalities and their left-right estimates for convex and coordinated convex functions. In [45], M. A. Noor et al. presented a generalized version of quantum integral inequalities. For generalized quasi-convex functions, E. R. Nwaeze et al. proved certain parameterized quantum integral inequalities in [46]. M. A. Khan et al. proved quantum Hermite-Hadamard inequality using the green function in [37]. H. Budak et al. [20],

M. A. Ali et al. [2,4] and M. Vivas-Cortez et al. [55] developed new quantum Simpson's and quantum Newton's type inequalities for convex and coordinated convex functions. For quantum Ostrowski's inequalities for convex and co-ordinated convex functions on can consult [5,7,23]. M. Kunt et al. [38] generalized the results of [10] and proved Hermite-Hadamard type inequalities and their left estimates using ${}_{\pi_1}D_{p,q}$-difference operator and $(p,q)_{\pi_1}$-integral. Recently, M. A. Latif et al. [39] found the right estimates of Hermite-Hadamard type inequalities proved by M. Kunt et al. [38]. To prove Ostrowski's inequalities, Y.-M. Chu et al. [25] used the concepts of ${}^{\pi_2}D_{p,q}$-difference operator and $(p,q)^{\pi_2}$-integral.

The following quantum variants of inequalities (1.4)–(1.6) proved my H. Budak et al. in [21].

Theorem 1.6. *[21] Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° and partial q_1q_2-derivatives $\frac{{}^{\pi_2,\ \pi_4}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}$, $\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}$, $\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}$, $\frac{{}_{\pi_1,\ \pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}$ be continuous and integrable on $[\pi_1,\pi_2]\times[\pi_3,\pi_4]\subseteq\Delta^\circ$. If $\left|\frac{{}^{\pi_2,\ \pi_4}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}\right|$, $\left|\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}\right|$, $\left|\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}\right|$, $\left|\frac{{}_{\pi_1,\ \pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}\right|\le M$ for all $(\tau,s)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we have the following quantum Ostrowski's type inequality:*

$$\begin{aligned}
&\left|\frac{1}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\left[\int_x^{\pi_2}\int_y^{\pi_4}F(\tau,s)\ {}^{\pi_2}d_{q_1}\tau\ {}^{\pi_4}d_{q_2}s+\int_x^{\pi_2}\int_{\pi_3}^{y}F(\tau,s)\ {}^{\pi_2}d_{q_1}\tau\ {}_{\pi_3}d_{q_2}s\right.\right. \qquad (1.7)\\
&\left.+\int_{\pi_1}^{x}\int_y^{\pi_4}F(\tau,s)\ {}_{\pi_1}d_{q_1}\tau\ {}^{\pi_4}d_{q_2}s+\int_{\pi_1}^{x}\int_{\pi_3}^{y}F(\tau,s)\ {}_{\pi_1}d_{q_1}\tau\ {}_{\pi_3}d_{q_2}s\right]\\
&-\frac{1}{\pi_4-\pi_3}\left[\int_y^{\pi_4}F(x,s)\ {}^{\pi_4}d_{q_2}s+\int_{\pi_3}^{y}F(x,s)\ {}_{\pi_3}d_{q_2}s\right]\\
&\left.-\frac{1}{\pi_2-\pi_1}\left[\int_x^{\pi_2}F(\tau,y)\ {}^{\pi_2}d_{q_1}\tau+\int_{\pi_1}^{x}F(\tau,y)\ {}_{\pi_1}d_{q_1}\tau\right]+F(x,y)\right|\\
\le\ &\frac{M}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\frac{q_1q_2\left(1+[2]_{q_1}\right)\left(1+[2]_{q_2}\right)}{[3]_{q_1}[3]_{q_2}}\\
&\left[\frac{(\pi_2-x)^2+(x-\pi_1)^2}{[2]_{q_1}}\right]\left[\frac{(\pi_4-y)^2+(y-\pi_3)^2}{[2]_{q_2}}\right]
\end{aligned}$$

for all $(x,y)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$ where $q_1,q_2\in(0,1)$.

Theorem 1.7. *[21] Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° and partial q_1q_2-derivatives $\frac{{}^{\pi_2,\ \pi_4}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}$, $\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}$, $\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}$, $\frac{{}_{\pi_1,\ \pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}$ be continuous and integrable on $[\pi_1,\pi_2]\times[\pi_3,\pi_4]\subseteq\Delta^\circ$. If $\left|\frac{{}^{\pi_2,\ \pi_4}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}\right|$, $\left|\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}\right|$, $\left|\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}\right|$, $\left|\frac{{}_{\pi_1,\ \pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}\right|\le M$ for all $(\tau,s)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we have the following quantum Ostrowski's type inequality:*

$$\begin{aligned}
&\left|\frac{1}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\left[\int_x^{\pi_2}\int_y^{\pi_4}F(\tau,s)\ {}^{\pi_2}d_{q_1}\tau\ {}^{\pi_4}d_{q_2}s+\int_x^{\pi_2}\int_{\pi_3}^{y}F(\tau,s)\ {}^{\pi_2}d_{q_1}\tau\ {}_{\pi_3}d_{q_2}s\right.\right. \qquad (1.8)\\
&\left.+\int_{\pi_1}^{x}\int_y^{\pi_4}F(\tau,s)\ {}_{\pi_1}d_{q_1}\tau\ {}^{\pi_4}d_{q_2}s+\int_{\pi_1}^{x}\int_{\pi_3}^{y}F(\tau,s)\ {}_{\pi_1}d_{q_1}\tau\ {}_{\pi_3}d_{q_2}s\right]\\
&-\frac{1}{\pi_4-\pi_3}\left[\int_y^{\pi_4}F(x,s)\ {}^{\pi_4}d_{q_2}s+\int_{\pi_3}^{y}F(x,s)\ {}_{\pi_3}d_{q_2}s\right]
\end{aligned}$$

$$- \frac{1}{\pi_2 - \pi_1} \left[\int_x^{\pi_2} F(\tau, y) \ {}^{\pi_2}d_{q_1}\tau + \int_{\pi_1}^{x} F(\tau, y) \ {}_{\pi_1}d_{q_1}\tau \right] + F(x,y) \Bigg|$$

$$\leq \frac{q_1 q_2 M}{(\pi_2 - \pi_1)(\pi_4 - \pi_3)} \left(\frac{1}{[r+1]_{q_1}} \frac{1}{[r+1]_{q_2}} \right)^{\frac{1}{r}} \left[(\pi_2 - x)^2 + (x - \pi_1)^2 \right] \left[(\pi_4 - y)^2 + (y - \pi_3)^2 \right]$$

for all $(x, y) \in [\pi_1, \pi_2] \times [\pi_3, \pi_4]$ *where* $q_1, q_2 \in (0, 1)$ *and* $\frac{1}{r} + \frac{1}{s} = 1$, $s > 1$.

Theorem 1.8. *[21] Let* $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a twice partially* q_1q_2*-differentiable function on* Δ° *and partial* q_1q_2*-derivatives* $\frac{{}^{\pi_2,\ \pi_4}\partial^2_{q_1,q_2} F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}$, $\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2} F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}$, $\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2} F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}$, $\frac{{}_{\pi_1,\ \pi_3}\partial^2_{q_1,q_2} F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}$ *be continuous and integrable on* $[\pi_1, \pi_2] \times [\pi_3, \pi_4] \subseteq \Delta^\circ$. *If* $\left|\frac{{}^{\pi_2,\ \pi_4}\partial^2_{q_1,q_2} F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}\right|$, $\left|\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2} F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}\right|$, $\left|\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2} F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}\right|$, $\left|\frac{{}_{\pi_1,\ \pi_3}\partial^2_{q_1,q_2} F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}\right| \leq M$ *for all* $(\tau, s) \in [\pi_1, \pi_2] \times [\pi_3, \pi_4]$, *then we have the following quantum Ostrowski's type inequality:*

$$\Bigg| \frac{1}{(\pi_2 - \pi_1)(\pi_4 - \pi_3)} \left[\int_x^{\pi_2} \int_y^{\pi_4} F(\tau, s) \ {}^{\pi_2}d_{q_1}\tau \ {}^{\pi_4}d_{q_2}s + \int_x^{\pi_2} \int_{\pi_3}^{y} F(\tau, s) \ {}^{\pi_2}d_{q_1}\tau \ {}_{\pi_3}d_{q_2}s \right. \tag{1.9}$$

$$\left. + \int_{\pi_1}^{x} \int_y^{\pi_4} F(\tau, s) \ {}_{\pi_1}d_{q_1}\tau \ {}^{\pi_4}d_{q_2}s + \int_{\pi_1}^{x} \int_{\pi_3}^{y} F(\tau, s) \ {}_{\pi_1}d_{q_1}\tau \ {}_{\pi_3}d_{q_2}s \right]$$

$$- \frac{1}{\pi_4 - \pi_3} \left[\int_y^{\pi_4} F(x, s) \ {}^{\pi_4}d_{q_2}s + \int_{\pi_3}^{y} F(x, s) \ {}_{\pi_3}d_{q_2}s \right]$$

$$- \frac{1}{\pi_2 - \pi_1} \left[\int_x^{\pi_2} F(\tau, y) \ {}^{\pi_2}d_{q_1}\tau + \int_{\pi_1}^{x} F(\tau, y) \ {}_{\pi_1}d_{q_1}\tau \right] + F(x,y) \Bigg|$$

$$\leq \frac{M q_1 q_2}{(\pi_2 - \pi_1)(\pi_4 - \pi_3)} \left(\frac{\left(1 + [2]_{q_1}\right)\left(1 + [2]_{q_2}\right)}{[3]_{q_1}\,[3]_{q_2}} \right)^{\frac{1}{s}}$$

$$\times \left[\frac{(\pi_2 - x)^2 + (x - \pi_1)^2}{[2]_{q_1}} \right] \left[\frac{(\pi_4 - y)^2 + (y - \pi_3)^2}{[2]_{q_2}} \right]$$

for all $(x, y) \in [\pi_1, \pi_2] \times [\pi_3, \pi_4]$ *where* $q_1, q_2 \in (0, 1)$ *and* $s \geq 1$.

Inspired by this ongoing studies, we introduce some new notions of post-quantum partial derivatives and prove some new ostrowski type inequalities for the functions of two variables by using the post-quantum double integrals and newly introduced post-quantum partial derivatives. Moreover, we show that the results presented in this paper are the extensions of results proved in [21, 40].

The following is the structure of this paper: A brief overview of the concepts of q-calculus, as well as some related works, is given in Section 2. In Section 3, we recall the notions of (p, q)-calculus and give some realted works. In Section 4, we show the relationship between the results presented here and comparable results in the literature by proving some new post-quantum Ostrowski type inequalities for the functions of two variables. Section 5 concludes with some recommendations for future studies.

2. Quantum calculus and some inequalities

In this section, we present some required definitions and inequalities.

In [33], F. H. Jackson gave the q-Jackson integral from 0 to π_2 for $0 < q < 1$ as follows:

$$\int_0^{\pi_2} F(x)\ d_q x = (1-q)\pi_2 \sum_{n=0}^{\infty} q^n F(\pi_2 q^n) \tag{2.1}$$

provided the sum converge absolutely. Moreover, he gave the q-Jackson integral in an arbitrary interval $[\pi_1, \pi_2]$ as:

$$\int_{\pi_1}^{\pi_2} F(x)\ d_q x = \int_0^{\pi_2} F(x)\ d_q x - \int_0^{\pi_1} F(x)\ d_q x.$$

Definition 2.1. *[54] For a continuous function $F : [\pi_1, \pi_2] \to \mathbb{R}$, then q_{π_1}-derivative of F at $x \in [\pi_1, \pi_2]$ is characterized by the expression:*

$$_{\pi_1}D_q F(x) = \frac{F(x) - F(qx + (1-q)\pi_1)}{(1-q)(x - \pi_1)},\ x \neq \pi_1. \tag{2.2}$$

For $x = \pi_1$, we state $_{\pi_1}D_q F(\pi_1) = \lim_{x \to \pi_1}\ _{\pi_1}D_q F(x)$ if it exists and it is finite.

Definition 2.2. *[14] For a continuous function $F : [\pi_1, \pi_2] \to \mathbb{R}$, then q^{π_2}-derivative of F at $x \in [\pi_1, \pi_2]$ is characterized by the expression:*

$$^{\pi_2}D_q F(x) = \frac{F(qx + (1-q)\pi_2) - F(x)}{(1-q)(\pi_2 - x)},\ x \neq \pi_2. \tag{2.3}$$

For $x = \pi_2$, we state $^{\pi_2}D_q F(\pi_2) = \lim_{x \to \pi_2}\ ^{\pi_2}D_q F(x)$ if it exists and it is finite.

Definition 2.3. *[54] Let $F : [\pi_1, \pi_2] \to \mathbb{R}$ be a continuous function. Then, the q_{π_1}-definite integral on $[\pi_1, \pi_2]$ is defined as:*

$$\begin{aligned} \int_{\pi_1}^{\pi_2} F(x)\ _{\pi_1}d_q x &= (1-q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} q^n F(q^n \pi_2 + (1-q^n)\pi_1) \\ &= (\pi_2 - \pi_1) \int_0^1 F((1-\tau)\pi_1 + \tau\pi_2)\ d_q\tau\ . \end{aligned} \tag{2.4}$$

On the other hand, S. Bermudo et al. gave the following new definition:

Definition 2.4. *[14] Let $F : [\pi_1, \pi_2] \to \mathbb{R}$ be a continuous function. Then, the q^{π_2}-definite integral on $[\pi_1, \pi_2]$ is defined as:*

$$\begin{aligned} \int_{\pi_1}^{\pi_2} F(x)\ ^{\pi_2}d_q x &= (1-q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} q^n F(q^n \pi_1 + (1-q^n)\pi_2) \\ &= (\pi_2 - \pi_1) \int_0^1 F(\tau\pi_1 + (1-\tau)\pi_2)\ d_q\tau\ . \end{aligned} \tag{2.5}$$

For more details about q^{π_2}-integrals and corresponding inequalities one can see [14].

Now, let's give the following notation which will be used many times in the next sections (see, [35]):

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

Moreover, we give the following Lemma for our main results:

Lemma 2.5. *[54] We have the equality*

$$\int_{\pi_1}^{\pi_2} (x - \pi_1)^{\alpha} \ {}_{\pi_1}d_q x \ = \frac{(\pi_2 - \pi_1)^{\alpha+1}}{[\alpha + 1]_q}$$

for $\alpha \in \mathbb{R} \backslash \{-1\}$.

In [23], H. Budak et al. proved the following variant of quantum Ostrowski inequality using the q_{π_1} and q^{π_2}-integrals:

Theorem 2.6. *[23] Let* $F : [\pi_1, \pi_2] \subset \mathbb{R} \to \mathbb{R}$ *be a function and* ${}^{\pi_2}D_q F$, ${}_{\pi_1}D_q F$ *be two continuous and integrable functions on* $[\pi_1, \pi_2]$. *If* $\left| {}^{\pi_2}D_q \ F(\tau) \right|, \left| {}_{\pi_1}D_q \ F(\tau) \right| \leq M$ *for all* $\tau \in [\pi_1, \pi_2]$, *then we have the following quantum Ostrowski type inequality:*

$$\left| F(x) - \frac{1}{\pi_2 - \pi_1} \left[\int_{\pi_1}^{x} F(\tau) \ {}_{\pi_1}d_q \tau \ + \int_{x}^{\pi_2} F(\tau) \ {}^{\pi_2}d_q \tau \ \right] \right| \tag{2.6}$$

$$\leq \ \frac{qM}{(\pi_2 - \pi_1)} \left[\frac{(x - \pi_1)^2 + (\pi_2 - x)^2}{[2]_q} \right]$$

for all $x \in [\pi_1, \pi_2]$ *where* $0 < q < 1$.

On the other hand, the authors gave the following definitions of $q_{\pi_1\pi_3}$, $q_{\pi_1}^{\pi_4}$, $q_{\pi_2}^{\pi_3}$ and $q^{\pi_2\pi_4}$ integrals and related inequalities of Hermite-Hadamard type:

Definition 2.7. *[19, 40] Suppose that* $F : [\pi_1, \pi_2] \times [\pi_3, \pi_4] \subset \mathbb{R}^2 \to \mathbb{R}$ *is a continuous function. Then, the following* $q_{\pi_1\pi_3}$, $q_{\pi_1}^{\pi_4}$, $q_{\pi_3}^{\pi_2}$ *and* $q^{\pi_2\pi_4}$ *integrals on* $[\pi_1, \pi_2] \times [\pi_3, \pi_4]$ *are defined by*

$$\begin{aligned}
\int_{\pi_1}^{x} \int_{\pi_3}^{y} F(\tau, s) \ {}_{\pi_3}d_{q_2} s \ \ {}_{\pi_1}d_{q_1} \tau \ &= (1 - q_1)(1 - q_2)(x - \pi_1)(y - \pi_3) \\
&\quad \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x + (1 - q_1^n)\pi_1, q_2^m y + (1 - q_2^m)\pi_3) \\
\int_{\pi_1}^{x} \int_{y}^{\pi_4} F(\tau, s) \ {}^{\pi_4}d_{q_2} s \ \ {}_{\pi_1}d_{q_1} \tau \ &= (1 - q_1)(1 - q_2)(x - \pi_1)(\pi_4 - y) \\
&\quad \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x + (1 - q_1^n)\pi_1, q_2^m y + (1 - q_2^m)\pi_4)
\end{aligned} \tag{2.7}$$

$$\int_{x}^{\pi_2}\int_{\pi_3}^{y} F(\tau,s)\ {}_{\pi_3}d_{q_2}s\ {}^{\pi_2}d_{q_1}\tau = (1-q_1)(1-q_2)(\pi_2-x)(y-\pi_3) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x+(1-q_1^n)\pi_2, q_2^m y+(1-q_2^m)\pi_3) \tag{2.8}$$

and

$$\int_{x}^{\pi_2}\int_{y}^{\pi_4} F(\tau,s)\ {}^{\pi_4}d_{q_2}s\ {}^{\pi_2}d_{q_1}\tau = (1-q_1)(1-q_2)(\pi_2-x)(\pi_4-y) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x+(1-q_1^n)\pi_2, q_2^m y+(1-q_2^m)\pi_4) \tag{2.9}$$

respectively, for $(x,y)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$.

Definition 2.8. *[40,57] Let* $F:[\pi_1,\pi_2]\times[\pi_3,\pi_4]\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a continuous function of two variables. Then, the partial* q_1*-derivatives,* q_2*-derivatives and* q_1q_2*-derivatives at* $(x,y)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$ *can be given as follows:*

$$\frac{{}_{\pi_1}\partial_{q_1}F(x,y)}{{}_{\pi_1}\partial_{q_1}x} = \frac{F(q_1x+(1-q_1)\pi_1,y)-F(x,y)}{(1-q_1)(x-\pi_1)},\ x\neq\pi_1$$

$$\frac{{}_{\pi_3}\partial_{q_2}F(x,y)}{{}_{\pi_3}\partial_{q_2}y} = \frac{F(x,q_2y+(1-q_2)\pi_3)-F(x,y)}{(1-q_2)(y-\pi_3)},\ y\neq\pi_3$$

$$\frac{{}_{\pi_1,\ \pi_3}\partial^2_{q_1,q_2}F(x,y)}{{}_{\pi_1}\partial_{q_1}x\ {}_{\pi_3}\partial_{q_2}y} = \frac{1}{(x-\pi_1)(y-\pi_3)(1-q_1)(1-q_2)}\left[F(q_1x+(1-q_1)\pi_1,q_2y+(1-q_2)\pi_3)\right.$$
$$\left.-F(q_1x+(1-q_1)\pi_1,y)-F(x,q_2y+(1-q_2)\pi_3)+F(x,y)\right],\ x\neq\pi_1,\ y\neq\pi_3$$

$$\frac{{}^{\pi_2}\partial_{q_1}F(x,y)}{{}^{\pi_2}\partial_{q_1}x} = \frac{F(q_1x+(1-q_1)\pi_2,y)-F(x,y)}{(1-q_1)(\pi_2-x)},\ x\neq\pi_2$$

$$\frac{{}^{\pi_4}\partial_{q_2}F(x,y)}{{}^{\pi_4}\partial_{q_2}y} = \frac{F(x,q_2y+(1-q_2)\pi_4)-F(x,y)}{(1-q_2)(\pi_4-y)},\ y\neq\pi_4$$

$$\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2}F(x,y)}{{}_{\pi_1}\partial_{q_1}x\ {}^{\pi_4}\partial_{q_2}y} = \frac{1}{(x-\pi_1)(\pi_4-y)(1-q_1)(1-q_2)}\left[F(q_1x+(1-q_1)\pi_1,q_2y+(1-q_2)\pi_4)\right.$$
$$\left.-F(q_1x+(1-q_1)\pi_1,y)-F(x,q_2y+(1-q_2)\pi_4)+F(x,y)\right],\ x\neq\pi_1,\ y\neq\pi_4,$$

$$\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2}F(x,y)}{{}^{\pi_2}\partial_{q_1}x\ {}_{\pi_3}\partial_{q_2}y} = \frac{1}{(\pi_2-x)(y-\pi_3)(1-q_1)(1-q_2)}\left[F(q_1x+(1-q_1)\pi_2,q_2y+(1-q_2)\pi_3)\right.$$
$$\left.-F(q_1x+(1-q_1)\pi_2,y)-F(x,q_2y+(1-q_2)\pi_3)+F(x,y)\right],\ x\neq\pi_2,\ y\neq\pi_3,$$

$$\frac{{}^{\pi_2,\,\pi_4}\partial^2_{q_1,q_2}F(x,y)}{{}^{\pi_2}\partial_{q_1}x\,{}^{\pi_4}\partial_{q_2}y} = \frac{1}{(\pi_2-x)(\pi_4-y)(1-q_1)(1-q_2)}\left[F(q_1x+(1-q_1)\pi_2, q_2y+(1-q_2)\pi_4)\right.$$
$$\left.-F(q_1x+(1-q_1)\pi_2,y)-F(x,q_2y+(1-q_2)\pi_4)+F(x,y)\right],\ x\neq\pi_2,\ y\neq\pi_4.$$

3. Post-quantum calculus and some inequalities

In this section, we review some fundamental notions and notations of (p,q)-calculus.

The $[n]_{p,q}$ is said to be (p,q)-integers and expressed as [13,28,29]:

$$[n]_{p,q}=\frac{p^n-q^n}{p-q}$$

with $0<q<p\leq 1$. The $[n]_{p,q}!$ and $\begin{bmatrix} n \\ k \end{bmatrix}!$ are called (p,q)-factorial and (p,q)-binomial, respectively, and expressed as:

$$[n]_{p,q}! = \prod_{k=1}^{n}[k]_{p,q},\ n\geq 1,\ [0]_{p,q}!=1,$$
$$\begin{bmatrix} n \\ k \end{bmatrix}! = \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}.$$

Definition 3.1. *[1] The (p,q)-derivative of mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ is given as:*

$$D_{p,q}F(x)=\frac{F(px)-F(qx)}{(p-q)x},\ x\neq 0$$

with $0<q<p\leq 1$.

Definition 3.2. *[53] The $(p,q)_{\pi_1}$-derivative of mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ is given as:*

$$_{\pi_1}D_{p,q}F(x)=\frac{F(px+(1-p)\pi_1)-F(qx+(1-q)\pi_1)}{(p-q)(x-\pi_1)},\ x\neq\pi_1 \tag{3.1}$$

with $0<q<p\leq 1$. For $x=\pi_1$, we state $_{\pi_1}D_{p,q}F(\pi_1)=\lim_{x\to\pi_1}\ _{\pi_1}D_{p,q}F(x)$ if it exists and it is finite.

Definition 3.3. *[25] The $(p,q)^{\pi_2}$-derivative of mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ is given as:*

$$^{\pi_2}D_{p,q}F(x)=\frac{F(qx+(1-q)\pi_2)-F(px+(1-p)\pi_2)}{(p-q)(\pi_2-x)},\ x\neq\pi_2. \tag{3.2}$$

with $0<q<p\leq 1$. For $x=\pi_2$, we state $^{\pi_2}D_{p,q}F(\pi_2)=\lim_{x\to\pi_2}\ ^{\pi_2}D_{p,q}F(x)$ if it exists and it is finite.

Remark 3.4. *It is clear that if we use $p=1$ in (3.1) and (3.2), then the equalities (3.1) and (3.2) reduce to (2.2) and (2.3), respectively.*

Definition 3.5. *[53] The definite $(p,q)_{\pi_1}$-integral of mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ on $[\pi_1,\pi_2]$ is stated as:*

$$\int_{\pi_1}^{x}F(\tau)\ _{\pi_1}d_{p,q}\tau=(p-q)(x-\pi_1)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)\pi_1\right) \tag{3.3}$$

with $0<q<p\leq 1$.

Definition 3.6. *[25] The definite $(p,q)^{\pi_2}$-integral of mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ on $[\pi_1,\pi_2]$ is stated as:*

$$\int_x^{\pi_2} F(\tau)\ {}^{\pi_2}d_{p,q}\tau = (p-q)(\pi_2-x)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)\pi_2\right) \tag{3.4}$$

with $0<q<p\le 1$.

Remark 3.7. *It is evident that if we pick $p=1$ in (3.3) and (3.4), then the equalities (3.3) and (3.4) change into (2.4) and (2.5), respectively.*

Remark 3.8. *If we take $\pi_1=0$ and $x=\pi_2=1$ in (3.3), then we have*

$$\int_0^1 F(\tau)\ {}_0d_{p,q}\tau = (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F\left(\frac{q^n}{p^{n+1}}\right).$$

Similarly, by taking $x=\pi_1=0$ and $\pi_2=1$ in (3.4), then we obtain that

$$\int_0^1 F(\tau)\ {}^1d_{p,q}\tau = (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F\left(1-\frac{q^n}{p^{n+1}}\right).$$

Lemma 3.9. *[56] We have the following equalities*

$$\int_{\pi_1}^{\pi_2}(\pi_2-x)^{\alpha}\ {}^{\pi_2}d_{p,q}x = \frac{(\pi_2-\pi_1)^{\alpha+1}}{[\alpha+1]_{p,q}}$$

$$\int_{\pi_1}^{\pi_2}(x-\pi_1)^{\alpha}\ {}_{\pi_1}d_{p,q}x = \frac{(\pi_2-\pi_1)^{\alpha+1}}{[\alpha+1]_{p,q}},$$

where $\alpha\in\mathbb{R}\setminus\{-1\}$.

For more details in (p,q)-calculus, one can consult [22, 32, 51].

In [38], M. Kunt et al. proved the following HH type inequalities for convex functions via $(p,q)_{\pi_1}$-integral:

Theorem 3.10. *[38] For a convex mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ which is differentiable on $[\pi_1,\pi_2]$, the following inequalities hold for $(p,q)_{\pi_1}$-integral:*

$$F\left(\frac{q\pi_1+p\pi_2}{[2]_{p,q}}\right)\le\frac{1}{p(\pi_2-\pi_1)}\int_{\pi_1}^{p\pi_2+(1-p)\pi_1}F(x)\ {}_{\pi_1}d_{p,q}x\le\frac{qF(\pi_1)+pF(\pi_2)}{[2]_{p,q}}, \tag{3.5}$$

where $0<q<p\le 1$.

Recently, M. Vivas-Cortez et al. [56] proved the following HH type inequalities for convex functions using the $(p,q)^{\pi_2}$-integral:

Theorem 3.11. *[56] For a convex mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ which is differentiable on $[\pi_1,\pi_2]$, the following inequalities hold for $(p,q)^{\pi_2}$-integral:*

$$F\left(\frac{p\pi_1+q\pi_2}{[2]_{p,q}}\right)\le\frac{1}{p(\pi_2-\pi_1)}\int_{p\pi_1+(1-p)\pi_2}^{\pi_2}F(x)\ {}^{\pi_2}d_{p,q}x\le\frac{pF(\pi_1)+qF(\pi_2)}{[2]_{p,q}}, \tag{3.6}$$

where $0<q<p\le 1$.

In [36] and [58], the authors gave the following notions of post-quantum integrals for the functions of two variables.

Definition 3.12. *[36, 58] For a function* $F : [\pi_1, \pi_2] \times [\pi_3, \pi_4] \to \mathbb{R}$*:*

(1) The $(p,q)_{\pi_1}^{\pi_4}$ *integral of* F *is given as:*

$$\int_{\pi_1}^{x}\int_{y}^{\pi_4} F(\tau, s)\ {}^{\pi_4}d_{p_2,q_2}s\ {}_{\pi_1}d_{p_1,q_1}\tau = (p_1-q_1)(p_2-q_2)(x-\pi_1)(\pi_4-y)$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)\pi_1, \frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)\pi_4\right),$$

where $x, y \in [\pi_1, p_1\pi_2 + (1-p_1)\pi_1] \times [p_2\pi_3 + (1-p_2)\pi_4, \pi_4]$.

(2) The $(p,q)_{\pi_3}^{\pi_2}$ *integral of* F *is given as:*

$$\int_{x}^{\pi_2}\int_{\pi_3}^{y} F(\tau, s)\ {}_{\pi_3}d_{p_2,q_2}s\ {}^{\pi_2}d_{p_1,q_1}\tau = (p_1-q_1)(p_2-q_2)(\pi_2-x)(y-\pi_3)$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)\pi_2, \frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)\pi_3\right)$$

where $x, y \in [p_1\pi_1 + (1-p_1)\pi_1, \pi_2] \times [\pi_3, p_2\pi_4 + (1-p_2)\pi_3]$.

(3) The $(p,q)^{\pi_2\pi_4}$ *integral of* F *is given as:*

$$\int_{x}^{\pi_2}\int_{y}^{\pi_4} F(\tau, s)\ {}^{\pi_4}d_{p_2,q_2}s\ {}^{\pi_2}d_{p_1,q_1}\tau = (p_1-q_1)(p_2-q_2)(\pi_2-x)(\pi_4-y)$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)\pi_2, \frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)\pi_4\right),$$

where $x, y \in [p_1\pi_1 + (1-p_1)\pi_2, \pi_2] \times [p_2\pi_3 + (1-p_2)\pi_4, \pi_4]$.

(4) The $(p,q)_{\pi_1\pi_3}$ *integral of* F *is given as:*

$$\int_{\pi_1}^{x}\int_{\pi_3}^{y} F(\tau, s)\ {}_{\pi_3}d_{p_2,q_2}s\ {}_{\pi_1}d_{p_1,q_1}\tau = (p_1-q_1)(p_2-q_2)(x-\pi_1)(y-\pi_3)$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)\pi_1, \frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)\pi_3\right)$$

where $x, y \in [\pi_1, p_1\pi_2 + (1-p_1)\pi_3] \times [\pi_3, p_2\pi_4 + (1-p_2)\pi_3]$.

Remark 3.13. *It is obvious that if we use* $p_1 = p_2 = 1$*, then Definition 3.12 transforms into Definition 2.7.*

In [36], H. Kalsoom et al. introduced the following notions of post-quantum partial derivatives.

Definition 3.14. *[36] Let* $F : [\pi_1, \pi_2] \times [\pi_3, \pi_4] \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a continuous function of two variables. Then the partial* p_1q_1*-derivatives,* p_2q_2*-derivatives and* $p_1q_1p_2q_2$*-derivatives at* $(x, y) \in [\pi_1, \pi_2] \times [\pi_3, \pi_4]$ *can be given as follows:*

$$\frac{{}_{\pi_1}\partial_{p_1,q_1}F(x,y)}{{}_{\pi_1}\partial_{p_1,q_1}x} = \frac{F(q_1x+(1-q_1)\pi_1, y) - F(p_1x+(1-p_1)\pi_1, y)}{(p_1-q_1)(x-\pi_1)},\ x \neq \pi_1$$

$$\frac{{}_{\pi_3}\partial_{p_2,q_2}F(x,y)}{{}_{\pi_3}\partial_{p_2,q_2}y} = \frac{F(x,q_2y+(1-q_2)\pi_3)-F(x,p_2y+(1-p_2)\pi_3)}{(p_2-q_2)(y-\pi_3)},\ y\neq\pi_3$$

$$\frac{{}_{\pi_1,\,\pi_3}\partial^2_{p_1,q_1,p_2,q_2}F(x,y)}{{}_{\pi_1}\partial_{p_1,q_1}x\ {}_{\pi_3}\partial_{p_2,q_2}y} = \frac{1}{(x-\pi_1)(y-\pi_3)(p_1-q_1)(p_2-q_2)}\left[F(q_1x+(1-q_1)\pi_1,q_2y+(1-q_2)\pi_3)\right.$$

$$-F(q_1x+(1-q_1)\pi_1,p_2y+(1-p_2)\pi_3)-F(p_1x+(1-p_1)\pi_1,q_2y+(1-q_2)\pi_3)$$

$$\left.+F(p_1x+(1-p_1)\pi_1,p_2y+(1-p_2)\pi_3)\right],\ x\neq\pi_1,\ y\neq\pi_3.$$

Now, from the above given concepts, we give the following new post-quantum partial derivatives.

Definition 3.15. *Let $F:[\pi_1,\pi_2]\times[\pi_3,\pi_4]\subseteq\mathbb{R}^2\to\mathbb{R}$ be a continuous function of two variables. Then the partial p_1q_1-derivatives, p_2q_2-derivatives and $p_1q_1p_2q_2$-derivatives at $(x,y)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$ can be given as follows:*

$$\begin{aligned}
\frac{{}^{\pi_2}\partial_{p_1,q_1}F(x,y)}{{}^{\pi_2}\partial_{p_1,q_1}x} &= \frac{F(q_1x+(1-q_1)\pi_2,y)-F(p_1x+(1-p_1)\pi_2,y)}{(p_1-q_1)(\pi_2-x)},\ x\neq\pi_2\\
\frac{{}^{\pi_4}\partial_{p_2,q_2}F(x,y)}{{}^{\pi_4}\partial_{p_2,q_2}y} &= \frac{F(x,q_2y+(1-q_2)\pi_4)-F(x,p_2y+(1-p_2)\pi_4)}{(p_2-q_2)(\pi_4-y)},\ y\neq\pi_4\\
\frac{{}^{\pi_4}_{\pi_1}\partial^2_{p_1,q_1,p_2,q_2}F(x,y)}{{}_{\pi_1}\partial_{p_1,q_1}x\ {}^{\pi_4}\partial_{p_2,q_2}y} &= \frac{1}{(x-\pi_1)(\pi_4-y)(p_1-q_1)(p_2-q_2)}\\
&\quad\times\left[F(q_1x+(1-q_1)\pi_1,q_2y+(1-q_2)\pi_4)\right.\\
&\quad-F(q_1x+(1-q_1)\pi_1,p_2y+(1-p_2)\pi_4)\\
&\quad-F(p_1x+(1-p_1)\pi_1,q_2y+(1-q_2)\pi_4)\\
&\quad\left.+F(p_1x+(1-p_1)\pi_1,p_2y+(1-p_2)\pi_4)\right],\ x\neq\pi_1,\ y\neq\pi_4,\\
\frac{{}^{\pi_2}_{\pi_3}\partial^2_{p_1,q_1,p_2,q_2}F(x,y)}{{}^{\pi_2}\partial_{p_1,q_1}x\ {}_{\pi_3}\partial_{p_2,q_2}y} &= \frac{1}{(\pi_2-x)(y-\pi_3)(p_1-q_1)(p_2-q_2)}\\
&\quad\times\left[F(q_1x+(1-q_1)\pi_2,q_2y+(1-q_2)\pi_3)\right.\\
&\quad-F(q_1x+(1-q_1)\pi_2,p_2y+(1-p_2)\pi_3)\\
&\quad-F(p_1x+(1-p_1)\pi_2,q_2y+(1-q_2)\pi_3)\\
&\quad\left.+F(p_1x+(1-p_1)\pi_2,p_2y+(1-p_2)\pi_3)\right],\ x\neq\pi_2,\ y\neq\pi_3,\\
\frac{{}^{\pi_2,\,\pi_4}\partial^2_{p_1,q_1,p_2,q_2}F(x,y)}{{}^{\pi_2}\partial_{p_1,q_1}x\ {}^{\pi_4}\partial_{p_2,q_2}y} &= \frac{1}{(\pi_2-x)(\pi_4-y)(p_1-q_1)(p_2-q_2)}\\
&\quad\times\left[F(q_1x+(1-q_1)\pi_2,q_2y+(1-q_2)\pi_4)\right.\\
&\quad-F(q_1x+(1-q_1)\pi_2,p_2y+(1-p_2)\pi_4)\\
&\quad-F(p_1x+(1-p_1)\pi_2,q_2y+(1-q_2)\pi_4)\\
&\quad\left.+F(p_1x+(1-p_1)\pi_2,p_2y+(1-p_2)\pi_4)\right],\ x\neq\pi_2,\ y\neq\pi_4.
\end{aligned}$$

Remark 3.16. *It is obvious that if we set $p_1 = p_2 = 1$ in Definitions 3.14 and 3.15, then we obtain the Definition 2.8.*

4. Quantum Ostrowski type inequalities for function of two variables

In this section, we prove some new post-quantum Ostrowski type inequalities for the functions of two variables.

Lemma 4.1. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially $p_1q_1p_2q_2$-differentiable function on Δ°. If partial $p_1q_1p_2q_2$-derivatives $\frac{{}^{\pi_2,\,\pi_4}\partial^2_{p_1,q_1,p_2,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\,{}^{\pi_4}\partial_{p_2,q_2}s}$, $\frac{{}^{\pi_4}_{\pi_1}\partial^2_{p_1,q_1,p_2,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\,{}^{\pi_4}\partial_{p_2,q_2}s}$, $\frac{{}^{\pi_2}_{\pi_3}\partial^2_{p_1,q_1,p_2,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\,{}_{\pi_3}\partial_{p_2,q_2}s}$ and $\frac{{}_{\pi_1,\,\pi_3}\partial^2_{p_1,q_1,p_2,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\,{}_{\pi_3}\partial_{p_2,q_2}s}$ are continuous and integrable on $[\pi_1,\pi_2]\times[\pi_3,\pi_4]\subseteq\Delta^\circ$, then following identity holds for $p_1q_1p_2q_2$-integrals:*

$$
\begin{aligned}
&{}^{\pi_3\pi_4}_{\pi_1\pi_2}\mathcal{J}_{p_1,q_1,p_2,q_2}\left(F(\tau,s)\right)\\
=&\frac{q_1q_2}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\\
&\times\Bigg[(\pi_2-x)^2(\pi_4-y)^2\int_0^1\int_0^1\tau s\frac{{}^{\pi_2,\,\pi_4}\partial^2_{p_1,q_1,p_2,q_2}F(\tau x+(1-\tau)\pi_2,\,sy+(1-s)\pi_4)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\,{}^{\pi_4}\partial_{p_2,q_2}s}\,d_{p_1,q_1}\tau\,d_{p_2,q_2}s\\
&+(\pi_2-x)^2(y-\pi_3)^2\int_0^1\int_0^1\tau s\frac{{}^{\pi_2}_{\pi_3}\partial^2_{p_1,q_1,p_2,q_2}F(\tau x+(1-\tau)\pi_2,\,sy+(1-s)\pi_3)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\,{}_{\pi_3}\partial_{p_2,q_2}s}\,d_{p_1,q_1}\tau\,d_{p_2,q_2}s\\
&+(x-\pi_1)^2(\pi_4-y)^2\int_0^1\int_0^1\tau s\frac{{}^{\pi_4}_{\pi_1}\partial^2_{p_1,q_1,p_2,q_2}F(\tau x+(1-\tau)\pi_1,\,sy+(1-s)\pi_4)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\,{}^{\pi_4}\partial_{p_2,q_2}s}\,d_{p_1,q_1}\tau\,d_{p_2,q_2}s\\
&+(x-\pi_1)^2(y-\pi_3)^2\int_0^1\int_0^1\tau s\frac{{}_{\pi_1,\pi_3}\partial^2_{p_1,q_1,p_2,q_2}F(\tau x+(1-\tau)\pi_1,\,sy+(1-s)\pi_3)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\,{}_{\pi_3}\partial_{p_2,q_2}s}\,d_{p_1,q_1}\tau\,d_{p_2,q_2}s\Bigg]
\end{aligned}
$$

where

$$
\begin{aligned}
&{}^{\pi_3\pi_4}_{\pi_1\pi_2}\mathcal{J}_{p_1,q_1,p_2,q_2}\left(F(\tau,s)\right) \qquad (4.1)\\
=&\frac{1}{p_1p_2(\pi_2-\pi_1)(\pi_4-\pi_3)}\Bigg[\int_{p_1x+(1-p_1)\pi_2}^{\pi_2}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4}F(\tau,s)\;{}^{\pi_2}d_{p_1,q_1}\tau\,{}^{\pi_4}d_{p_2,q_2}s\\
&+\int_{p_1x+(1-p_1\pi_2)}^{\pi_2}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3}F(\tau,s)\;{}^{\pi_2}d_{p_1,q_1}\tau\,{}_{\pi_3}d_{p_2,q_2}s\\
&+\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4}F(\tau,s)\;{}_{\pi_1}d_{p_1,q_1}\tau\,{}^{\pi_4}d_{p_2,q_2}s\\
&+\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3}F(\tau,s)\;{}_{\pi_1}d_{p_1,q_1}\tau\,{}_{\pi_3}d_{p_2,q_2}s\Bigg]\\
&-\frac{1}{p_2(\pi_4-\pi_3)}\Bigg[\int_{p_2y+(1-p_2)\pi_4}^{\pi_4}F(x,s)\;{}^{\pi_4}d_{p_2,q_2}s+\int_{\pi_3}^{p_2y+(1-p_2)\pi_3}F(x,s)\;{}_{\pi_3}d_{p_2,q_2}s\Bigg]\\
&-\frac{1}{\pi_2-\pi_1}\Bigg[\int_{p_1x+(1-p_1)\pi_2}^{\pi_2}F(\tau,y)\;{}^{\pi_2}d_{p_1,q_1}\tau+\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}F(\tau,y)\;{}_{\pi_1}d_{p_1,q_1}\tau\Bigg]+F(x,y)
\end{aligned}
$$

for all $(x,y)\in[\pi_1,\pi_2]\times[\pi_3,d]$ and $0<q_i<p_i\leq 1$.

Proof. From Definitions 3.14 and 3.15, we have

$$\frac{{}^{\pi_2,\,\pi_4}\partial^2_{p_1,q_1,p_2,q_2}F\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_4\right)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\ {}^{\pi_4}\partial_{p_2,q_2}s} \tag{4.2}$$
$$= \frac{1}{(p_1-q_1)(p_2-q_2)(\pi_2-x)(\pi_4-y)\tau s}\left[F\left(\tau q_1 x+(1-\tau q_1)\pi_2, sq_2 y+(1-sq_2)\pi_4\right)\right.$$
$$-F\left(\tau q_1 x+(1-\tau q_1)\pi_2, sp_2 y+(1-sp_2)\pi_4\right)-F\left(\tau p_1 x+(1-\tau p_1)\pi_2, sq_2 y+(1-sq_2)\pi_4\right)$$
$$\left.+F\left(\tau p_1 x+(1-\tau p_1)\pi_2, sp_2 y+(1-sp_2)\pi_4\right)\right],$$

$$\frac{{}^{\pi_2}_{\pi_3}\partial^2_{p_1,q_1,p_2,q_2}F\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_3\right)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\ {}_{\pi_3}\partial_{p_2,q_2}s} \tag{4.3}$$
$$= \frac{1}{(p_1-q_1)(p_2-q_2)(\pi_2-x)(y-\pi_3)\tau s}\left[F\left(\tau q_1 x+(1-\tau q_1)\pi_2, sq_2 y+(1-sq_2)\pi_3\right)\right.$$
$$-F\left(\tau q_1 x+(1-\tau q_1)\pi_2, sp_2 y+(1-sp_2)\pi_4\right)-F\left(\tau p_1 x+(1-\tau p_1)\pi_2, sq_2 y+(1-sq_2)\pi_3\right)$$
$$\left.+F\left(\tau p_1 x+(1-\tau p_1)\pi_2, sp_2 y+(1-sp_2)\pi_4\right)\right],$$

$$\frac{{}^{\pi_4}_{\pi_1}\partial^2_{p_1,q_1,p_2,q_2}F\left(\tau x+(1-\tau)\pi_1, sy+(1-s)\pi_4\right)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\ {}^{\pi_4}\partial_{p_2,q_2}s} \tag{4.4}$$
$$= \frac{1}{(p_1-q_1)(p_2-q_2)(x-\pi_1)(\pi_4-y)\tau s}\left[F\left(\tau q_1 x+(1-\tau q_1)\pi_1, sq_2 y+(1-sq_2)\pi_4\right)\right.$$
$$-F\left(\tau q_1 x+(1-\tau q_1)\pi_1, sp_2 y+(1-sp_2)\pi_4\right)-F\left(\tau p_1 x+(1-\tau p_1)\pi_1, sq_2 y+(1-sq_2)\pi_4\right)$$
$$\left.+F\left(\tau p_1 x+(1-\tau p_1)\pi_1, sp_2 y+(1-sp_2)\pi_4\right)\right],$$

$$\frac{{}_{\pi_1,\pi_3}\partial^2_{p_1,q_1,p_2,q_2}F\left(\tau x+(1-\tau)\pi_1, sy+(1-s)\pi_3\right)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\ {}_{\pi_3}\partial_{p_2,q_2}s} \tag{4.5}$$
$$= \frac{1}{(p_1-q_1)(p_2-q_2)(x-\pi_1)(y-\pi_3)\tau s}\left[F\left(\tau q_1 x+(1-\tau q_1)\pi_1, sq_2 y+(1-sq_2)\pi_3\right)\right.$$
$$-F\left(\tau q_1 x+(1-\tau q_1)\pi_1, sp_2 y+(1-sp_2)\pi_3\right)-F\left(\tau p_1 x+(1-\tau p_1)\pi_1, sq_2 y+(1-sq_2)\pi_3\right)$$
$$\left.+F\left(\tau p_1 x+(1-\tau p_1)\pi_1, sp_2 y+(1-sp_2)\pi_3\right)\right].$$

By the equality (4.2) and Definition 3.12, we have

$$
\begin{aligned}
I_1 &= \int_0^1 \int_0^1 \tau s \frac{{}^{\pi_2,\,\pi_4}\partial^2_{p_1,q_1,p_2,q_2} F\left(\tau x + (1-\tau)\pi_2, sy + (1-s)\pi_4\right)}{{}^{\pi_2}\partial_{p_1,q_1}\tau \; {}^{\pi_4}\partial_{p_2,q_2} s} \, d_{p_1,q_1}\tau \, d_{p_2,q_2} s \\
&= \frac{1}{(p_1-q_1)(p_2-q_2)(\pi_2-x)(\pi_4-y)} \int_0^1 \int_0^1 \left[F\left(\tau q_1 x + (1-\tau q_1)\pi_2, s q_2 y + (1-s q_2)\pi_4\right)\right. \\
&\quad -F\left(\tau q_1 x + (1-\tau q_1)\pi_2, s p_2 y + (1-s p_2)\pi_4\right) - F\left(\tau p_1 x + (1-\tau p_1)\pi_2, s q_2 y + (1-s q_2)\pi_4\right) \\
&\quad \left. +F\left(\tau p_1 x + (1-\tau p_1)\pi_2, s p_2 y + (1-s p_2)\pi_4\right)\right] \, d_{p_1,q_1}\tau \, d_{p_2,q_2} s \\
&= \frac{1}{(\pi_2-x)(\pi_4-y)} \left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n q_2^m}{p_1^{n+1} p_2^{m+1}} F\left(\frac{q_1^{n+1}}{p_1^{n+1}} x + \left(1-\frac{q_1^{n+1}}{p_1^{n+1}}\right)\pi_2, \frac{q_2^{m+1}}{p_2^{m+1}} y + \left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)\pi_4\right)\right. \\
&\quad -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n q_2^m}{p_1^{n+1} p_2^{m+1}} F\left(\frac{q_1^{n+1}}{p_1^{n+1}} x + \left(1-\frac{q_1^{n+1}}{p_1^{n+1}}\right)\pi_2, \frac{q_2^{m}}{p_2^{m}} y + \left(1-\frac{q_2^{m}}{p_2^{m}}\right)\pi_4\right) \\
&\quad -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n q_2^m}{p_1^{n+1} p_2^{m+1}} F\left(\frac{q_1^{n}}{p_1^{n}} x + \left(1-\frac{q_1^{n}}{p_1^{n}}\right)\pi_2, \frac{q_2^{m+1}}{p_2^{m+1}} y + \left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)\pi_4\right) \\
&\quad \left. +\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n q_2^m}{p_1^{n+1} p_2^{m+1}} F\left(\frac{q_1^{n}}{p_1^{n}} x + \left(1-\frac{q_1^{n}}{p_1^{n}}\right)\pi_2, \frac{q_2^{m}}{p_2^{m}} y + \left(1-\frac{q_2^{m}}{p_2^{m}}\right)\pi_4\right)\right] \\
&= \frac{1}{(\pi_2-x)(\pi_4-y)} \\
&\quad \times\left[\frac{p_1 p_2}{q_1 q_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n q_2^m}{p_1^{n+1} p_2^{m+1}} F\left(\frac{q_1^{n}}{p_1^{n+1}} p_1 x + \left(1-\frac{q_1^{n}}{p_1^{n+1}} p_1\right)\pi_2, \frac{q_2^{m}}{p_2^{m+1}} p_2 y + \left(1-\frac{q_2^{m}}{p_2^{m+1}} p_2\right)\pi_4\right)\right. \\
&\quad -\frac{p_2}{q_1 q_2}\sum_{m=0}^{\infty} \frac{q_2^m}{p_2^{m+1}} F\left(x, \frac{q_2^{m}}{p_2^{m+1}} p_2 y + \left(1-\frac{q_2^{m}}{p_2^{m+1}} p_2\right)\pi_4\right) \\
&\quad -\frac{p_1}{q_1 q_2}\sum_{n=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} F\left(\frac{q_1^{n}}{p_1^{n+1}} p_1 x + \left(1-\frac{q_1^{n}}{p_1^{n+1}} p_1\right)\pi_2, y\right) + \frac{1}{q_1 q_2} F(x,y) \\
&\quad -\frac{p_1}{q_1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n q_2^m}{p_1^{n+1} p_2^{m+1}} F\left(\frac{q_1^{n}}{p_1^{n+1}} p_1 x + \left(1-\frac{q_1^{n}}{p_1^{n+1}} p_1\right)\pi_2, \frac{q_2^{m}}{p_2^{m+1}} p_2 y + \left(1-\frac{q_2^{m}}{p_2^{m+1}} p_2\right)\pi_4\right) \\
&\quad +\frac{1}{q_1}\sum_{m=0}^{\infty} \frac{q_2^m}{p_2^{m+1}} F\left(x, \frac{q_2^{m}}{p_2^{m+1}} p_2 y + \left(1-\frac{q_2^{m}}{p_2^{m+1}} p_2\right)\pi_4\right) \\
&\quad -\frac{p_2}{q_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n q_2^m}{p_1^{n+1} p_2^{m+1}} F\left(\frac{q_1^{n}}{p_1^{n+1}} p_1 x + \left(1-\frac{q_1^{n}}{p_1^{n+1}} p_1\right)\pi_2, \frac{q_2^{m}}{p_2^{m+1}} p_2 y + \left(1-\frac{q_2^{m}}{p_2^{m+1}} p_2\right)\pi_4\right) \\
&\quad +\frac{1}{q_2}\sum_{n=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} F\left(\frac{q_1^{n}}{p_1^{n+1}} p_1 x + \left(1-\frac{q_1^{n}}{p_1^{n+1}} p_1\right)\pi_2, y\right) \\
&\quad \left. +\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n q_2^m}{p_1^{n+1} p_2^{m+1}} F\left(\frac{q_1^{n}}{p_1^{n+1}} p_1 x + \left(1-\frac{q_1^{n}}{p_1^{n+1}} p_1\right)\pi_2, \frac{q_2^{m}}{p_2^{m+1}} p_2 y + \left(1-\frac{q_2^{m}}{p_2^{m+1}} p_2\right)\pi_4\right)\right] \\
&= \frac{1}{(\pi_2-x)(\pi_4-y)}
\end{aligned}
$$

$$\times\left[\frac{(p_1-q_1)(p_2-q_2)}{q_1q_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^mF\left(\frac{q_1^n}{p_1^{n+1}}p_1x+\left(1-\frac{q_1^n}{p_1^{n+1}}p_1\right)\pi_2,\frac{q_2^m}{p_2^{m+1}}p_2y+\left(1-\frac{q_2^m}{p_2^{m+1}}p_2\right)\pi_4\right)\right.$$
$$-\frac{(p_2-q_2)}{q_1q_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}F\left(x,\frac{q_2^m}{p_2^{m+1}}p_2y+\left(1-\frac{q_2^m}{p_2^{m+1}}p_2\right)\pi_4\right)$$
$$\left.-\frac{(p_1-q_1)}{q_1q_2}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}F\left(\frac{q_1^n}{p_1^{n+1}}p_1x+\left(1-\frac{q_1^n}{p_1^{n+1}}p_1\right)\pi_2,y\right)+\frac{1}{q_1q_2}F(x,y)\right]$$
$$=\frac{1}{q_1q_2}\left[\frac{1}{p_1p_2(\pi_2-x)^2(\pi_4-y)^2}\int_{p_1x+(1-p_1)\pi_2}^{\pi_2}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4}F(\tau,s)\ {}^{\pi_2}d_{p_1,q_1}\tau\ {}^{\pi_4}d_{p_2,q_2}s\right.$$
$$-\frac{1}{p_2(\pi_2-x)(\pi_4-y)^2}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4}F(x,s)\ {}^{\pi_4}d_{p_2,q_2}s$$
$$\left.-\frac{1}{p_1(\pi_2-x)^2(\pi_4-y)}\int_{p_1x+(1-p_1)\pi_2}^{\pi_2}F(\tau,y)\ {}^{\pi_2}d_{p_1,q_1}\tau+\frac{1}{(\pi_2-x)(d-y)}F(x,y)\right].$$

Similarly, by the equalities (4.3)–(4.5) we obtain the identities

$$I_2=\int_0^1\int_0^1\tau s\frac{{}^{\pi_2}_{\pi_3}\partial^2_{p_1,q_1,p_2,q_2}F(\tau x+(1-\tau)\pi_2,sy+(1-s)\pi_3)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\ {}_{\pi_3}\partial_{p_2,q_2}s}\,d_{p_1,q_1}\tau\ d_{p_2,q_2}s \tag{4.6}$$
$$=\frac{1}{q_1q_2}\left[\frac{1}{p_1p_2(\pi_2-x)^2(y-\pi_3)^2}\int_{p_1x+(1-p_1)\pi_2}^{\pi_2}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3}F(\tau,s)\ {}^{\pi_2}d_{p_1,q_1}\tau\ {}_{\pi_3}d_{p_2,q_2}s\right.$$
$$-\frac{1}{p_2(\pi_2-x)(y-\pi_3)^2}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3}F(x,s)\ {}_{\pi_3}d_{p_2,q_2}s$$
$$\left.-\frac{1}{p_1(\pi_2-x)^2(y-\pi_3)}\int_{p_1x+(1-p_1)\pi_2}^{\pi_2}F(\tau,y)\ {}^{\pi_2}d_{p_1,q_1}\tau+\frac{1}{(\pi_2-x)(y-\pi_3)}F(x,y)\right],$$

$$I_3=\int_0^1\int_0^1\tau s\frac{{}^{\pi_4}_{\pi_1}\partial^2_{p_1,q_1,p_2,q_2}F(\tau x+(1-\tau)\pi_1,sy+(1-s)\pi_4)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\ {}^{\pi_4}\partial_{p_2,q_2}s}\,d_{p_1,q_1}\tau\ d_{p_2,q_2}s \tag{4.7}$$
$$=\frac{1}{q_1q_2}\left[\frac{1}{p_1p_2(x-\pi_1)^2(\pi_4-y)^2}\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4}F(\tau,s)\ {}_{\pi_1}d_{p_1,q_1}\tau\ {}^{\pi_4}d_{p_2,q_2}s\right.$$
$$-\frac{1}{p_2(x-\pi_1)(\pi_4-y)^2}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4}F(x,s)\ {}^{\pi_4}d_{p_2,q_2}s$$
$$\left.-\frac{1}{p_1(x-\pi_1)^2(\pi_4-y)}\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}F(\tau,y)\ {}_{\pi_1}d_{p_1,q_1}\tau+\frac{1}{(x-\pi_1)(\pi_4-y)}F(x,y)\right],$$

and

$$I_4=\int_0^1\int_0^1\tau s\frac{{}_{\pi_1,\pi_3}\partial^2_{p_1,q_1,p_2,q_2}F(\tau x+(1-\tau)\pi_1,sy+(1-s)\pi_3)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\ {}_{\pi_3}\partial_{p_2,q_2}s}\,d_{p_1,q_1}\tau\ d_{p_2,q_2}s$$
$$=\frac{1}{q_1q_2}\left[\frac{1}{p_1p_2(x-\pi_1)^2(y-\pi_3)^2}\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3}F(\tau,s)\ {}_{\pi_1}d_{p_1,q_1}\tau\ {}_{\pi_3}d_{p_2,q_2}s\right.$$

$$-\frac{1}{p_2(x-\pi_1)(y-\pi_3)^2}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3} F(x,s)\ {}_{\pi_3}d_{p_2,q_2}s$$
$$\left.-\frac{1}{p_1(x-\pi_1)^2(y-\pi_3)}\int_{\pi_1}^{p_1x+(1-p_1)\pi_1} F(\tau,y)\ {}_{\pi_1}d_{p_1,q_1}\tau+\frac{1}{(x-\pi_1)(y-\pi_3)}F(x,y)\right].$$

Thus, we have

$$\begin{aligned}
& \frac{q_1q_2(\pi_2-x)^2(\pi_4-y)^2}{(\pi_2-\pi_1)(\pi_4-\pi_3)}I_1+\frac{q_1q_2(\pi_2-x)^2(y-\pi_3)^2}{(\pi_2-\pi_1)(\pi_4-\pi_3)}I_2 \\
& +\frac{q_1q_2(x-\pi_1)^2(\pi_4-y)^2}{(\pi_2-\pi_1)(\pi_4-\pi_3)}I_3+\frac{q_1q_2(x-\pi_1)^2(y-\pi_3)^2}{(\pi_2-\pi_1)(\pi_4-\pi_3)}I_4 \\
= & \frac{1}{p_1p_2(\pi_2-\pi_1)(\pi_4-\pi_3)}\left[\int_{p_1x+(1-p_1)\pi_2}^{\pi_2}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4} F(\tau,s)\ {}^{\pi_2}d_{p_1,q_1}\tau\ {}^{\pi_4}d_{p_2,q_2}s\right. \\
& +\int_{p_1x+(1-p_1\pi_2)}^{\pi_2}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3} F(\tau,s)\ {}^{\pi_2}d_{p_1,q_1}\tau\ {}_{\pi_3}d_{p_2,q_2}s \\
& +\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4} F(\tau,s)\ {}_{\pi_1}d_{p_1,q_1}\tau\ {}^{\pi_4}d_{p_2,q_2}s \\
& \left.+\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3} F(\tau,s)\ {}_{\pi_1}d_{p_1,q_1}\tau\ {}_{\pi_3}d_{p_2,q_2}s\right] \\
& -\frac{1}{p_2(\pi_4-\pi_3)}\left[\int_{p_2y+(1-p_2)\pi_4}^{\pi_4} F(x,s)\ {}^{\pi_4}d_{p_2,q_2}s+\int_{\pi_3}^{p_2y+(1-p_2)\pi_3} F(x,s)\ {}_{\pi_3}d_{p_2,q_2}s\right] \\
& -\frac{1}{\pi_2-\pi_1}\left[\int_{p_1x+(1-p_1)\pi_2}^{\pi_2} F(\tau,y)\ {}^{\pi_2}d_{p_1,q_1}\tau+\int_{\pi_1}^{p_1x+(1-p_1)\pi_1} F(\tau,y)\ {}_{\pi_1}d_{p_1,q_1}\tau\right]+F(x,y) \\
= & {}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}(F(\tau,s))
\end{aligned}$$

which completes the proof. □

Remark 4.2. *In Lemma 4.1, if we set $p_1 = p_2 = 1$, then the Lemma 4.1 reduces to [21, Lemma 2].*

Remark 4.3. *In Lemma 4.1, if we set $p_1 = p_2 = 1$ and $q_1, q_2 \to 1^-$, then Lemma 4.1 reduces to [41, Lemma 1].*

In terms of brevity, we will use the following notations

$$\begin{aligned}
\Phi(\tau,s) &= \frac{{}^{\pi_2,\ \pi_4}\partial^2_{p_1,q_1,p_2,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\ {}^{\pi_4}\partial_{p_2,q_2}s},\ \Psi(\tau,s)=\frac{{}_{\pi_1}^{\pi_4}\partial^2_{p_1,q_1,p_2,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\ {}^{\pi_4}\partial_{p_2,q_2}s}, \\
\Theta(\tau,s) &= \frac{{}_{\pi_3}^{\pi_2}\partial^2_{p_1,q_1,p_2,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\ {}_{\pi_3}\partial_{p_2,q_2}s} \text{ and } \Omega(\tau,s)=\frac{{}_{\pi_1,\ \pi_3}\partial^2_{p_1,q_1,p_2,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\ {}_{\pi_3}\partial_{p_2,q_2}s}.
\end{aligned}$$

Theorem 4.4. *Suppose that the assumptions of Lemma 4.1 hold. If $|\Phi(\tau,s)|$, $|\Theta(\tau,s)|$, $|\Psi(\tau,s)|$ and $|\Omega(\tau,s)|$ are co-ordinated convex on $[\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we have the inequality*

$$
\begin{aligned}
&\left|{}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}\left(F\left(\tau,s\right)\right)\right| \\
\leq\ & \frac{1}{\left(\pi_2-\pi_1\right)\left(\pi_4-\pi_3\right)}\frac{q_1q_2}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}} \\
&\times\Big[\left(\pi_2-x\right)^2\left(\pi_4-y\right)^2 \\
&\times\Big([2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Phi\left(x,y\right)\right|+[2]_{p_1,q_1}\Big([3]_{p_2,q_2}-[2]_{p_2,q_2}\Big)\left|\Phi\left(x,\pi_4\right)\right| \\
&+[2]_{p_2,q_2}\Big([3]_{p_1,q_1}-[2]_{p_1,q_1}\Big)\left|\Phi\left(\pi_2,y\right)\right|+\left|\Phi\left(\pi_2,\pi_4\right)\right|\Big) \\
&+\left(\pi_2-x\right)^2\left(y-\pi_3\right)^2 \\
&\times\Big([2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Theta\left(x,y\right)\right|+[2]_{p_1,q_1}\Big([3]_{p_2,q_2}-[2]_{p_2,q_2}\Big)\left|\Theta\left(x,\pi_3\right)\right| \\
&+[2]_{p_2,q_2}\Big([3]_{p_1,q_1}-[2]_{p_1,q_1}\Big)\left|\Theta\left(\pi_2,y\right)\right|+\left|\Theta\left(\pi_2,\pi_3\right)\right|\Big) \\
&\left(x-\pi_1\right)^2\left(\pi_4-y\right)^2 \\
&\times\Big([2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Psi\left(x,y\right)\right|+[2]_{p_1,q_1}\Big([3]_{p_2,q_2}-[2]_{p_2,q_2}\Big)\left|\Psi\left(x,\pi_4\right)\right| \\
&+[2]_{p_2,q_2}\Big([3]_{p_1,q_1}-[2]_{p_1,q_1}\Big)\left|\Psi\left(\pi_1,y\right)\right|+\left|\Psi\left(\pi_1,\pi_4\right)\right|\Big) \\
&+\left(x-\pi_1\right)^2\left(y-\pi_3\right)^2 \\
&\times\Big([2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Omega\left(x,y\right)\right|+[2]_{p_1,q_1}\Big([3]_{p_2,q_2}-[2]_{p_2,q_2}\Big)\left|\Omega\left(x,\pi_3\right)\right| \\
&+[2]_{p_2,q_2}\Big([3]_{p_1,q_1}-[2]_{p_1,q_1}\Big)\left|\Omega\left(\pi_1,y\right)\right|+\left|\Omega\left(\pi_1,\pi_3\right)\right|\Big)\Big].
\end{aligned}
$$

Proof. Taking modulus in (4.1), we have

$$
\begin{aligned}
&\left|{}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}\left(F\left(\tau,s\right)\right)\right| \\
\leq\ & \frac{q_1q_2}{\left(\pi_2-\pi_1\right)\left(\pi_4-\pi_3\right)} \\
&\times\Bigg[\left(\pi_2-x\right)^2\left(\pi_4-y\right)^2\int_0^1\int_0^1\tau s\left|\Phi\left(\tau x+\left(1-\tau\right)\pi_2,sy+\left(1-s\right)\pi_4\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \\
&+\left(\pi_2-x\right)^2\left(y-\pi_3\right)^2\int_0^1\int_0^1\tau s\left|\Theta\left(\tau x+\left(1-\tau\right)\pi_2,sy+\left(1-s\right)\pi_3\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \\
&+\left(x-\pi_1\right)^2\left(\pi_4-y\right)^2\int_0^1\int_0^1\tau s\left|\Psi\left(\tau x+\left(1-\tau\right)\pi_1,sy+\left(1-s\right)\pi_4\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \\
&+\left(x-\pi_1\right)^2\left(y-\pi_3\right)^2\int_0^1\int_0^1\tau s\left|\Omega\left(\tau x+\left(1-\tau\right)\pi_1,sy+\left(1-s\right)\pi_3\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s\Bigg].
\end{aligned} \tag{4.8}
$$

Since $|\Phi(\tau,s)|$ is co-ordinated convex, we obtain

$$
\begin{aligned}
&\int_0^1\int_0^1\tau s\left|\Phi\left(\tau x+\left(1-\tau\right)\pi_2,sy+\left(1-s\right)\pi_4\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \\
\leq\ & \int_0^1\int_0^1\tau s\left[\begin{array}{c}\tau s\left|\Phi\left(x,y\right)\right|+\tau\left(1-s\right)\left|\Phi\left(x,\pi_4\right)\right|+\left(1-\tau\right)s\left|\Phi\left(\pi_2,y\right)\right| \\ +\left(1-\tau\right)\left(1-s\right)\left|\Phi\left(\pi_2,\pi_4\right)\right|\end{array}\right]d_{p_1,q_1}\tau d_{p_2,q_2}s
\end{aligned} \tag{4.9}
$$

$$= \frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}|\Phi(x,y)| + [2]_{p_1,q_1}\left([3]_{p_2,q_2} - [2]_{p_2,q_2}\right)|\Phi(x,\pi_4)| \\ + [2]_{p_2,q_2}\left([3]_{p_1,q_1} - [2]_{p_1,q_1}\right)|\Phi(\pi_2,y)| + |\Phi(\pi_2,\pi_4)|\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}.$$

By the similar way, as $|\Theta(\tau,s)|$, $|\Psi(\tau,s)|$ and $|\Omega(\tau,s)|$ are co-ordinated convex, we establish

$$\int_0^1\int_0^1 \tau s\left|\Theta(\tau x + (1-\tau)\pi_2, sy + (1-s)\pi_3)\right| \, d_{p_1,q_1}\tau d_{p_2,q_2}s \tag{4.10}$$

$$\leq \frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}|\Theta(x,y)| + [2]_{p_1,q_1}\left([3]_{p_2,q_2} - [2]_{p_2,q_2}\right)|\Theta(x,\pi_3)| \\ + [2]_{p_2,q_2}\left([3]_{p_1,q_1} - [2]_{p_1,q_1}\right)|\Theta(\pi_2,y)| + |\Theta(\pi_2,\pi_3)|\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}},$$

$$\int_0^1\int_0^1 \tau s\left|\Psi(\tau x + (1-\tau)\pi_1, sy + (1-s)\pi_4)\right| \, d_{p_1,q_1}\tau d_{p_2,q_2}s \tag{4.11}$$

$$\leq \frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}|\Psi(x,y)| + [2]_{p_1,q_1}\left([3]_{p_2,q_2} - [2]_{p_2,q_2}\right)|\Psi(x,\pi_4)| \\ + [2]_{p_2,q_2}\left([3]_{p_1,q_1} - [2]_{p_1,q_1}\right)|\Psi(\pi_1,y)| + |\Psi(\pi_1,\pi_4)|\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}$$

and

$$\int_0^1\int_0^1 \tau s\left|\Omega(\tau x + (1-\tau)\pi_1, sy + (1-s)\pi_3)\right| \, d_{p_1,q_1}\tau d_{p_2,q_2}s \tag{4.12}$$

$$\leq \frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}|\Omega(x,y)| + [2]_{p_1,q_1}\left([3]_{p_2,q_2} - [2]_{p_2,q_2}\right)|\Omega(x,\pi_3)| \\ + [2]_{p_2,q_2}\left([3]_{p_1,q_1} - [2]_{p_1,q_1}\right)|\Omega(\pi_1,y)| + |\Omega(\pi_1,\pi_3)|\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}.$$

If we substitute the inequalities (4.9)–(4.12) in (4.8), then we obtain the desired result. □

Remark 4.5. *In Theorem 4.4, if we set $p_1 = p_2 = 1$, then Theorem 4.4 reduces to [21, Theorem 5].*

Corollary 4.6. *In Theorem 4.4, if we choose $|\Phi(\tau,s)|$, $|\Theta(\tau,s)|$, $|\Psi(\tau,s)|$, $|\Omega(\tau,s)| \leq M$ for all $(\tau,s) \in [\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we obtain the following post-quantum Ostrowski type inequality*

$$\begin{aligned}&\left|{}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}(F(\tau,s))\right| \\ \leq\ & \frac{M}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\frac{q_1q_2[2]_{p_1,q_1}\left([3]_{p_2,q_2} - [2]_{p_2,q_2}\right) + [2]_{p_2,q_2}[3]_{p_1,q_1} + 1}{[3]_{p_1,q_1}[3]_{p_2,q_2}} \\ & \times\left[\frac{(\pi_2-x)^2 + (x-\pi_1)^2}{[2]_{p_1,q_1}}\right]\left[\frac{(\pi_4-y)^2 + (y-\pi_3)^2}{[2]_{p_2,q_2}}\right].\end{aligned}$$

Remark 4.7. *In Corollary 4.6, if we put $p_1 = p_2 = 1$, then we recapture the inequality (1.7).*

Remark 4.8. *In Corollary 4.6, if we set $p_1 = p_2 = 1$ and $q_1, q_2 \to 1^-$, then Corollary 4.6 reduces to Theorem 1.3.*

Theorem 4.9. *Suppose that the assumptions of Lemma 4.1 are hold. If $|\Phi(\tau, s)|^s$, $|\Theta(\tau, s)|^s$, $|\Psi(\tau, s)|^s$ and $|\Omega(\tau, s)|^s$ are co-ordinated convex on $[\pi_1, \pi_2] \times [\pi_3, \pi_4]$, then we have the inequality*

$$\begin{aligned}
&\left| {}^{\pi_3\pi_4}_{\pi_1\pi_2}\mathcal{J}_{p_1,q_1,p_2,q_2}\left(F\left(\tau, s\right)\right)\right| \\
\leq\ & \frac{q_1q_2}{\left(\pi_2-\pi_1\right)\left(\pi_4-\pi_3\right)}\left(\frac{1}{[r+1]_{p_1,q_1}}\frac{1}{[r+1]_{p_2,q_2}}\right)^{\frac{1}{r}} \\
&\times\left[\left(\pi_2-x\right)^2\left(\pi_4-y\right)^2\left(\frac{\begin{array}{c}|\Phi(x,y)|^s+\left([2]_{p_2,q_2}-1\right)|\Phi(x,\pi_4)|^s+\left([2]_{p_1,q_1}-1\right)|\Phi(\pi_2,y)|^s\\ +\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)|\Phi(\pi_2,\pi_4)|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}}\right. \\
&+\left(\pi_2-x\right)^2\left(y-\pi_3\right)^2\left(\frac{\begin{array}{c}|\Theta(x,y)|^s+\left([2]_{p_2,q_2}-1\right)|\Theta(x,\pi_3)|^s+\left([2]_{p_1,q_1}-1\right)|\Theta(\pi_2,y)|^s\\ +\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)|\Theta(\pi_2,\pi_3)|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}} \\
&+\left(x-\pi_1\right)^2\left(\pi_4-y\right)^2\left(\frac{\begin{array}{c}|\Psi(x,y)|^s+\left([2]_{p_2,q_2}-1\right)|\Psi(x,\pi_4)|^s+\left([2]_{p_1,q_1}-1\right)|\Psi(\pi_1,y)|^s\\ +\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)|\Psi(\pi_1,\pi_4)|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}} \\
&\left.+\left(x-\pi_1\right)^2\left(y-\pi_3\right)^2\left(\frac{\begin{array}{c}|\Omega(x,y)|^s+\left([2]_{p_2,q_2}-1\right)|\Omega(x,\pi_3)|^s+\left([2]_{p_1,q_1}-1\right)|\Omega(\pi_1,y)|^s\\ +\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)|\Omega(\pi_1,\pi_3)|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}}\right]
\end{aligned}$$

where $\frac{1}{r} + \frac{1}{s} = 1$, $s > 1$.

Proof. From the Lemma 4.1, we have

$$\begin{aligned}
&\left| {}^{\pi_3\pi_4}_{\pi_1\pi_2}\mathcal{J}_{p_1,q_1,p_2,q_2}\left(F\left(\tau, s\right)\right)\right| \\
\leq\ & \frac{q_1q_2}{\left(\pi_2-\pi_1\right)\left(\pi_4-\pi_3\right)} \\
&\times\left[\left(\pi_2-x\right)^2\left(\pi_4-y\right)^2\int_0^1\int_0^1 \tau s\left|\Phi\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_4\right)\right|\, d_{p_1,q_1}\tau\, d_{p_2,q_2}s\right. \\
&+\left(\pi_2-x\right)^2\left(y-\pi_3\right)^2\int_0^1\int_0^1 \tau s\left|\Theta\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_3\right)\right|\, d_{p_1,q_1}\tau\, d_{p_2,q_2}s
\end{aligned} \tag{4.13}$$

$$+(x-\pi_1)^2(\pi_4-y)^2\int_0^1\int_0^1 \tau s\left|\Psi\left(\tau x+(1-\tau)\pi_1, sy+(1-s)\pi_4\right)\right|\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s$$
$$\left.+(x-\pi_1)^2(y-\pi_3)^2\int_0^1\int_0^1 \tau s\left|\Omega\left(\tau x+(1-\tau)\pi_1, sy+(1-s)\pi_3\right)\right|\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s\right].$$

By using the well-known Hölder inequality and the co-ordinated convexity of $|\Phi(\tau,s)|^s$, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1 \tau s\left|\Phi\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_4\right)\right|\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s \\
\leq\ & \left(\int_0^1\int_0^1 \tau^r s^r\ d_{p_1,q_1}\tau d_{p_2,q_2}s\right)^{\frac{1}{r}} \\
\times\ & \left(\int_0^1\int_0^1 \left|\Phi\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_4\right)\right|^s\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s\right)^{\frac{1}{s}} \\
\leq\ & \left(\frac{1}{[r+1]_{p_1,q_1}}\frac{1}{[r+1]_{p_2,q_2}}\right)^{\frac{1}{r}} \\
\times\ & \left(\int_0^1\int_0^1 \left[\tau s\left|\Phi(x,y)\right|^s+\tau(1-s)\left|\Phi(x,\pi_4)\right|^s\right.\right. \\
+\ & \left.\left.(1-\tau)s\left|\Phi(\pi_2,y)\right|^s+\left(1-\tau\right)(1-s)\left|\Phi(\pi_2,\pi_4)\right|^s\right]d_{p_1,q_1}\tau\ d_{p_2,q_2}s\right)^{\frac{1}{s}} \\
=\ & \left(\frac{1}{[r+1]_{p_1,q_1}}\frac{1}{[r+1]_{p_2,q_2}}\right)^{\frac{1}{r}} \\
\times\ & \left(\frac{\begin{array}{c}\left|\Phi(x,y)\right|^s+\left([2]_{p_2,q_2}-1\right)\left|\Phi(x,\pi_4)\right|^s+\left([2]_{p_1,q_1}-1\right)\left|\Phi(\pi_2,y)\right|^s \\ +\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)\left|\Phi(\pi_2,\pi_4)\right|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}}.
\end{aligned} \tag{4.14}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1\int_0^1 \tau s\left|\Theta\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_3\right)\right|\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s \\
\leq\ & \left(\frac{1}{[r+1]_{p_1,q_1}}\frac{1}{[r+1]_{p_2,q_2}}\right)^{\frac{1}{r}} \\
\times\ & \left(\frac{\begin{array}{c}\left|\Theta(x,y)\right|^s+\left([2]_{p_2,q_2}-1\right)\left|\Theta(x,\pi_3)\right|^s+\left([2]_{p_1,q_1}-1\right)\left|\Theta(\pi_2,y)\right|^s \\ +\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)\left|\Theta(\pi_2,\pi_3)\right|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}},
\end{aligned} \tag{4.15}$$

$$\int_0^1\int_0^1 \tau s\left|\Psi\left(\tau x+(1-\tau)\pi_1, sy+(1-s)\pi_4\right)\right|\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s \tag{4.16}$$

$$\leq \left(\frac{1}{[r+1]_{p_1,q_1}}\frac{1}{[r+1]_{p_2,q_2}}\right)^{\frac{1}{r}}$$

$$\times\left(\frac{\begin{array}{c}\left|\Psi(x,y)\right|^s+\left([2]_{p_2,q_2}-1\right)\left|\Psi(x,\pi_4)\right|^s+\left([2]_{p_1,q_1}-1\right)\left|\Psi(\pi_1,y)\right|^s\\+\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)\left|\Psi(\pi_1,\pi_4)\right|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}}$$

and

$$\int_0^1\int_0^1 \tau s\left|\Omega\left(\tau x+(1-\tau)\pi_1, sy+(1-s)\pi_3\right)\right|\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s \tag{4.17}$$

$$\leq \left(\frac{1}{[r+1]_{p_1,q_1}}\frac{1}{[r+1]_{p_2,q_2}}\right)^{\frac{1}{r}}$$

$$\times\left(\frac{\begin{array}{c}\left|\Omega(x,y)\right|^s+\left([2]_{p_2,q_2}-1\right)\left|\Omega(x,\pi_3)\right|^s+\left([2]_{p_1,q_1}-1\right)\left|\Omega(\pi_1,y)\right|^s\\+\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)\left|\Omega(\pi_1,\pi_3)\right|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}}.$$

By substituting the inequalities (4.14)–(4.17) in (4.13), then we obtain the required result. □

Remark 4.10. *In Theorem 4.9, if we use $p_1 = p_2 = 1$, then Theorem 4.9 reduces to [21, Theorem 6].*

Corollary 4.11. *In Theorem 4.9, if we choose $|\Phi(\tau,s)|, |\Theta(\tau,s)|, |\Psi(\tau,s)|, |\Omega(\tau,s)| \leq M$ for all $(\tau,s)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we obtain the following post-quantum Ostrowski type inequality*

$$\begin{aligned}&\left|{}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}\left(F(\tau,s)\right)\right|\\ \leq\ &\frac{q_1q_2M}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\left(\frac{1}{[r+1]_{p_1,q_1}}\frac{1}{[r+1]_{p_2,q_2}}\right)^{\frac{1}{r}}\left[(\pi_2-x)^2+(x-\pi_1)^2\right]\left[(\pi_4-y)^2+(y-\pi_3)^2\right].\end{aligned}$$

Remark 4.12. *In Corollary 4.11, if we set $p_1 = p_2 = 1$, then we recpature the inequality (1.8).*

Remark 4.13. *In Corollary 4.11, if we set $p_1 = p_2 = 1$ and $q_1, q_2 \to 1^-$, then Corollary 4.11 reduces to Theorem 1.4.*

Theorem 4.14. *Suppose that the assumptions of Lemma 4.1 hold. If $|\Phi(\tau,s)|^s, |\Theta(\tau,s)|^s, |\Psi(\tau,s)|^s$ and $|\Omega(\tau,s)|^s$, $s\geq 1$ are co-ordinated convex on $[\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we have the inequality*

$$
\begin{aligned}
&\left|{}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}\left(F\left(\tau,s\right)\right)\right| \\
\leq\ & \frac{1}{\left(\pi_2-\pi_1\right)\left(\pi_4-\pi_3\right)}\frac{q_1q_2}{[2]_{p_1,q_1}[2]_{p_2,q_2}} \\
&\times\Bigg[\left(\pi_2-x\right)^2\left(\pi_4-y\right)^2 \\
&\times\left(\frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Phi\left(x,y\right)\right|^s+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)\left|\Phi\left(x,\pi_4\right)\right|^s\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left|\Phi\left(\pi_2,y\right)\right|^s+\left|\Phi\left(\pi_2,\pi_4\right)\right|^s\end{array}}{[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)^{\frac{1}{s}} \\
&+\left(\pi_2-x\right)^2\left(y-\pi_3\right)^2 \\
&\times\left(\frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Theta\left(x,y\right)\right|^s+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)\left|\Theta\left(x,\pi_3\right)\right|^s\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left|\Theta\left(\pi_2,y\right)\right|^s+\left|\Theta\left(\pi_2,\pi_3\right)\right|^s\end{array}}{[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)^{\frac{1}{s}} \\
&+\left(x-\pi_1\right)^2\left(\pi_4-y\right)^2 \\
&\times\left(\frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Psi\left(x,y\right)\right|^s+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)\left|\Psi\left(x,\pi_4\right)\right|^s\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left|\Psi\left(\pi_1,y\right)\right|^s+\left|\Psi\left(\pi_1,\pi_4\right)\right|^s\end{array}}{[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)^{\frac{1}{s}} \\
&+\left(x-\pi_1\right)^2\left(y-\pi_3\right)^2 \\
&\times\left(\frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Omega\left(x,y\right)\right|+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)\left|\Omega\left(x,\pi_3\right)\right|\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left|\Omega\left(\pi_1,y\right)\right|+\left|\Omega\left(\pi_1,\pi_3\right)\right|\end{array}}{[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)\Bigg].
\end{aligned}
$$

Proof. By using the power mean inequality and the co-ordinated convexity of $|\Phi(\tau, s)|^s$, we obtain

$$
\int_0^1\int_0^1 \tau s\left|\Phi\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_4\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \tag{4.18}
$$

$$
\begin{aligned}
\leq\ & \left(\int_0^1\int_0^1 \tau s\ d_{p_1,q_1}\tau d_{p_2,q_2}s\right)^{1-\frac{1}{s}} \\
&\times\left(\int_0^1\int_0^1 \tau s\left|\Phi\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_4\right)\right|^s\ d_{p_1,q_1}\tau d_{p_2,q_2}s\right)^{\frac{1}{s}} \\
\leq\ & \left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{1-\frac{1}{s}}
\end{aligned}
$$

$$\times\left(\int_0^1\int_0^1 \tau s\left[\tau s\left|\Phi\left(x,y\right)\right|^s+\tau\left(1-s\right)\left|\Phi\left(x,\pi_4\right)\right|^s\right.\right.$$
$$\left.\left.+\left(1-\tau\right)s\left|\Phi\left(\pi_2,y\right)\right|^s+\left(1-\tau\right)\left(1-s\right)\left|\Phi\left(\pi_2,\pi_4\right)\right|^s\right]d_{p_1,q_1}\tau d_{p_2,q_2}s\right)^{\frac{1}{s}}$$
$$=\left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{1-\frac{1}{s}}$$
$$\times\left(\frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Phi\left(x,y\right)\right|^s+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)\left|\Phi\left(x,\pi_4\right)\right|^s\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left|\Phi\left(\pi_2,y\right)\right|^s+\left|\Phi\left(\pi_2,\pi_4\right)\right|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)^{\frac{1}{s}}.$$

Similarly, since $|\Theta(\tau,s)|^s$, $|\Psi(\tau,s)|^s$ and $|\Omega(\tau,s)|^s$ are co-ordinated convex, we establish

$$\int_0^1\int_0^1 \tau s\left|\Theta\left(\tau x+\left(1-\tau\right)\pi_2,sy+\left(1-s\right)\pi_3\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \tag{4.19}$$
$$\leq\left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{1-\frac{1}{s}}$$
$$\times\left(\frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Theta\left(x,y\right)\right|^s+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)\left|\Theta\left(x,\pi_3\right)\right|^s\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left|\Theta\left(\pi_2,y\right)\right|^s+\left|\Theta\left(\pi_2,\pi_3\right)\right|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)^{\frac{1}{s}}$$

$$\int_0^1\int_0^1 \tau s\left|\Psi\left(\tau x+\left(1-\tau\right)\pi_1,sy+\left(1-s\right)\pi_4\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \tag{4.20}$$
$$\leq\left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{1-\frac{1}{s}}$$
$$\times\left(\frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Psi\left(x,y\right)\right|^s+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)\left|\Psi\left(x,\pi_4\right)\right|^s\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left|\Psi\left(\pi_1,y\right)\right|^s+\left|\Psi\left(\pi_1,\pi_4\right)\right|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)^{\frac{1}{s}}$$

and

$$\int_0^1\int_0^1 \tau s\left|\Omega\left(\tau x+\left(1-\tau\right)\pi_1,sy+\left(1-s\right)\pi_3\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \tag{4.21}$$
$$\leq\left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{1-\frac{1}{s}}$$

$$\times\left(\frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}|\Omega(x,y)|^s+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)|\Omega(x,\pi_3)|^s\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)|\Omega(\pi_1,y)|^s+|\Omega(\pi_1,\pi_3)|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)^{\frac{1}{s}}.$$

If we substitute the inequalities (4.18)–(4.21) in (4.13), then we obtain the desired result. □

Remark 4.15. *In Theorem 4.14, if we assume $p_1 = p_2 = 1$, then Theorem 4.14 becomes [21, Theorem 7].*

Corollary 4.16. *In Theorem 4.14, if we choose $|\Phi(\tau,s)|, |\Theta(\tau,s)|, |\Psi(\tau,s)|, |\Omega(\tau,s)| \le M$ for all $(\tau,s)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we obtain the following post-quantum Ostrowski type inequality*

$$\begin{aligned}&\left|{}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}(F(\tau,s))\right|\\ \le\ &\frac{Mq_1q_2}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\left(\frac{[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)+[2]_{p_2,q_2}[3]_{p_1,q_1}+1}{[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)^{\frac{1}{s}}\\ &\times\left[\frac{(\pi_2-x)^2+(x-\pi_1)^2}{[2]_{p_1,q_1}}\right]\left[\frac{(\pi_4-y)^2+(y-\pi_3)^2}{[2]_{p_2,q_2}}\right].\end{aligned}$$

Remark 4.17. *In Corollary 4.16, if we consider $p_1 = p_2 = 1$, then we recapture the inequality (1.9).*

Remark 4.18. *In Corollary 4.16, if we consider $p_1 = p_2 = 1$ and $q_1, q_2 \to 1^-$, then Corollary 4.16 reduces to Theorem 1.5.*

5. Conclusions

In this study, we proved some new post-quantum variants of Ostrowski type inequalities for the differentiable functions of two variables. We also proved that the results proved in this study are the refinements of some existing results in the field of integral inequalities. It is an interesting and new problem that the upcoming researchers can obtain the similar inequalities for the other kind of convexity in their future work.

Acknowledgments

We want to give thanks to the Dirección de investigación from Pontificia Universidad Católica del Ecuador for technical support to our research project entitled: “Algunas desigualdades integrales para funciones convexas generalizadas y aplicaciones”.

Conflict of interest

The authors declare no conflicts of interest.

References

1. T. Acar, A. Aral, S. A. Mohiuddine, On Kantorovich modification of (p,q)-Baskakov operators, *Iran. J. Sci. Technol. Trans. Sci.*, **42** (2018), 1459–1464. https://doi.org/10.1007/s40995-017-0154-8
2. M. A. Ali, H. Budak, Z. Zhang, H. Yıldırım, Some new Simpson's type inequalities for co-ordinated convex functions in quantum calculus, *Math. Meth. Appl. Sci.*, **44** (2021), 4515–4540. https://doi.org/10.1002/mma.7048
3. M. A. Ali, H. Budak, M. Abbas, Y. M. Chu, Quantum Hermite-Hadamard-type inequalities for functions with convex absolute values of second q^b-derivatives, *Adv. Differ. Equ.*, **2021** (2021), 7. https://doi.org/10.1186/s13662-020-03163-1
4. M. A. Ali, M. Abbas, H. Budak, P. Agarwal, G. Murtaza, Y. M. Chu, New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions, *Adv. Differ. Equ.*, **2021** (2021), 64. https://doi.org/10.1186/s13662-021-03226-x
5. M. A. Ali, Y. M. Chu, H. Budak, A. Akkurt, H. Yıldırım, Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables, *Adv. Differ. Equ.*, **2021** (2021), 25. https://doi.org/10.1186/s13662-020-03195-7
6. M. A. Ali, N. Alp, H. Budak, Y. M. Chu, Z. Zhang, On some new quantum midpoint type inequalities for twice quantum differentiable convex functions, *Open Math.*, **19** (2021), 427–439. https://doi.org/10.1515/math-2021-0015
7. M. A. Ali, H. Budak, A. Akkurt, Y. M. Chu, Quantum Ostrowski type inequalities for twice quantum differentiable functions in quantum calculus, *Open Math.*, **19** (2021), 440–449. https://doi.org/10.1515/math-2021-0020
8. M. Alomari, M. Darus, S. S. Dragomir, P. Cerone, Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, *Appl. Math. Lett.*, **23** (2010), 1071–1076. https://doi.org/10.1016/j.aml.2010.04.038
9. M. Alomari, M. Darus, Some Ostrowski type inequalities for quasi-convex functions with applications to special means, *RGMIA Res. Rep. Coll.*, **13** (2010).
10. N. Alp, M. Z. Sarikaya, M. Kunt, İ. İşcan, q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions, *J. King Saud Univ.-Sci.*, **30** (2018), 193–203. https://doi.org/10.1016/j.jksus.2016.09.007
11. N. Alp, M. Z. Sarikaya, Hermite Hadamard's type inequalities for co-ordinated convex functions on quantum integral, *Appl. Math. E-Notes*, **20** (2020), 341–356.
12. W. A. Al-Salam, Some fractional q-integrals and q-derivatives, *Proc. Edinburgh Math. Soc.*, **15** (1966), 135–140. https://doi.org/10.1017/S0013091500011469
13. S. Araci, U. Duran, M. Acikgoz, H. M. Srivastava, A certain (p,q)-derivative operator and associated divided differences, *J. Inequal. Appl.*, **2016** (2016), 301. https://doi.org/10.1186/s13660-016-1240-8
14. S. Bermudo, P. Kórus, J. N. Valdés, On q-Hermite-Hadamard inequalities for general convex functions, *Acta Math. Hungar.*, **162** (2020), 364–374. https://doi.org/10.1007/s10474-020-01025-6

15. N. S. Barnett, S. S. Dragomir, An Ostrowski type inequality for double integrals and applications for cubature formulae, *Res. Rep. Coll.*, **1** (1998), 13–23.

16. F. Benatti, M. Fannes, R. Floreanini, D. Petritis, *Quantum information, computation and cryptography: An introductory survey of theory, technology and experiments*, Springer, 2010. https://doi.org/10.1007/978-3-642-11914-9

17. A. Bokulich, G. Jaeger, *Philosophy of quantum information theory and entaglement*, Cambridge Uniersity Press, 2010.

18. H. Budak, Some trapezoid and midpoint type inequalities for newly defined quantum integrals, *Proyecciones*, **40** (2021), 199–215. http://dx.doi.org/10.22199/issn.0717-6279-2021-01-0013

19. H. Budak, M. A. Ali, M. Tarhanaci, Some new quantum Hermite-Hadamard-like inequalities for coordinated convex functions, *J. Optim. Theory Appl.*, **186** (2020), 899–910. https://doi.org/10.1007/s10957-020-01726-6

20. H. Budak, S. Erden, M. A. Ali, Simpson and Newton type inequalities for convex functions via newly defined quantum integrals, *Math. Meth. Appl. Sci.*, **44** (2020), 378–390. https://doi.org/10.1002/mma.6742

21. H. Budak, M. A. Ali, T. Tunç, Quantum Ostrowski-type integral inequalities for functions of two variables, *Math. Meth. Appl. Sci.*, **44** (2021), 5857–5872. https://doi.org/10.1002/mma.7153

22. I. M. Burban, A. U. Klimyk, P,Q-differentiation, P,Q-integration and P,Q-hypergeometric functions related to quantum groups, *Integral Transf. Spec. F.*, **2** (1994), 15–36. https://doi.org/10.1080/10652469408819035

23. H. Budak, M. A. Ali, N. Alp, Y. M. Chu, Quantum Ostrowski type integral inequalities, *J. Math. Inequal.*, unpublished work.

24. P. Cerone, S. S. Dragomir, Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions, *Demonstratio Math.*, **37** (2004), 299–308. https://doi.org/10.1515/dema-2004-0208

25. Y. M. Chu, M. U. Awan, S. Talib, M. A. Noor, K. I Noor, New post quantum analogues of Ostrowski-type inequalities using new definitions of left-right (p, q)-derivatives and definite integrals, *Adv. Differ. Equ.*, **2020** (2020), 634. https://doi.org/10.1186/s13662-020-03094-x

26. S. S. Dragomir, A. Sofo, Ostrowski type inequalities for functions whose derivatives are convex, *Proc. Int. Conf. Modell. Simul.*, Victoria University, Melbourne, Australia, 2002.

27. S. S. Dragomir, N. S. Barnett, P. Cerone, An n-dimensional version of Ostrowski's inequality for mappings of Hölder type, *RGMIA Res. Pep. Coll.*, **2** (1999), 169–180.

28. U. Duran, *Post quantum calculus*, Master Thesis, University of Gaziantep, 2016.

29. U. Duran, M. Acikgoz, S. Araci, A study on some new results arising from (p, q)-calculus, *TWMS J. Pure Appl. Math.*, **11** (2020), 57–71.

30. T. Ernst, *The history of q-calculus and new method*, Sweden: Department of Mathematics, Uppsala University, 2000.

31. T. Ernst, *A comprehensive treatment of q-calculus*, Springer, 2012. https://doi.org/10.1007/978-3-0348-0431-8

32. R. Jagannathan, K. S. Rao, Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series, *Proc. Int. Conf. Number Theory Math. Phys.*, 2005.

33. F. H. Jackson, On a q-definite integrals, *Quart. J. Pure Appl. Math.*, **41** (1910), 193–203.

34. S. Jhanthanam, J. Tariboon, S. K. Ntouyas, K. Nonlapon, On q-Hermite-Hadamard inequalities for differentiable convex functions, *Mathematics*, **7** (2019), 632. https://doi.org/10.3390/math7070632

35. V. Kac, P. Cheung, *Quantum calculus*, Springer, 2002. https://doi.org/10.1007/978-1-4613-0071-7

36. H. Kalsoom, S. Rashid, M. Idrees, F. Safdar, S. Akram, D. Baleanu, et al., Post quantum inequalities of Hermite-Hadamard type associated with co-ordinated higher-order generalized strongly pre-invex and quasi-pre-invex mappings, *Symmetry*, **12** (2020), 443. https://doi.org/10.3390/sym12030443

37. M. A. Khan, M. Noor, E. R. Nwaeze, Y. M. Chu, Quantum Hermite-Hadamard inequality by means of a Green function, *Adv. Differ. Equ.*, **2020** (2020), 99. https://doi.org/10.1186/s13662-020-02559-3

38. M. Kunt, İ. İşcan, N. Alp, M. Z. Sarikaya, (p, q)-Hermite-Hadamard inequalities and (p, q)-estimates for midpoint inequalities via convex quasi-convex functions, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.*, **112** (2018), 969–992. http://doi.org/10.1007/s13398-017-0402-y

39. M. A. Latif, M. Kunt, S. S. Dragomir, İ. İşcan, Post-quantum trapezoid type inequalities, *AIMS Math.*, **5** (2020), 4011–4026. http://dx.doi.org/10.3934/math.2020258

40. M. A. Latif, S. S. Dragomir, E. Momoniat, Some q-analogues of Hermite-Hadamard inequality of functions of two variables on finite rectangles in the plane, *J. King Saud Univ.-Sci.*, **29** (2017), 263–273. https://doi.org/10.1016/j.jksus.2016.07.001

41. M. A Latif, S. Hussain, S. S. Dragomir, New Ostrowski type inequalities for co-ordinated convex functions, *TJMM*, **4** (2012), 125–136.

42. M. A. Latif, S. Hussain, New inequalities of Ostowski type for co-ordinated convex functions via fractional integral, *J. Fract. Calc. Appl.*, **2** (2012), 1–15.

43. W. Liu, H. Zhuang, Some quantum estimates of Hermite-Hadamard inequalities for convex functions, *J. Appl. Anal. Comput.*, **7** (2017), 501–522. https://doi.org/10.11948/2017031

44. M. A. Noor, K. I. Noor, M. U. Awan, Some quantum estimates for Hermite-Hadamard inequalities, *Appl. Math. Comput.*, **251** (2015), 675–679. https://doi.org/10.1016/j.amc.2014.11.090

45. M. A. Noor, K. I. Noor, M. U. Awan, Some quantum integral inequalities via preinvex functions, *Appl. Math. Comput.*, **269** (2015), 242–251. https://doi.org/10.1016/j.amc.2015.07.078

46. E. R. Nwaeze, A. M. Tameru, New parameterized quantum integral inequalities via η-quasiconvexity, *Adv. Differ. Equ.*, **2019** (2019), 425. https://doi.org/10.1186/s13662-019-2358-z

47. A. M. Ostrowski, Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert, *Comment. Math. Helv.*, **10** (1938), 226–227. https://doi.org/10.1007/BF01214290

48. B. G. Pachpatte, On an inequality of Ostrowski type in three independent variables, *J. Math. Anal. Appl.*, **249** (2000), 583–591. https://doi.org/10.1006/jmaa.2000.6913

49. B. G. Pachpatte, On a new Ostrowski type inequality in two independent variables, *Tamkang J. Math.*, **32** (2001), 45–49. http://dx.doi.org/10.5556/j.tkjm.32.2001.45-49
50. B. G. Pachpatte, A new Ostrowski type inequality for double integrals, *Soochow J. Math.*, **32** (2006), 317–322.
51. V. Sahai, S. Yadav, Representations of two parameter quantum algebras and p,q-special functions, *J. Math. Anal. Appl.*, **335** (2007), 268–279. https://doi.org/10.1016/j.jmaa.2007.01.072
52. M. Z. Sarikaya, On the Ostrowski type integral inequality, *Acta Math. Univ. Comenianae*, **79** (2010), 129–134.
53. M. Tunç, E. Göv, Some integral inequalities via (p, q)-calculus on finite intervals, *Filomat*, **35** (2021), 1421–1430. https://doi.org/10.2298/FIL2105421T
54. J. Tariboon, S. K. Ntouyas, Quantum calculus on finite intervals and applications to impulsive difference equations, *Adv. Differ. Equ.*, **2013** (2013), 282. https://doi.org/10.1186/1687-1847-2013-282
55. M. Vivas-Cortez, M. A. Ali, A. Kashuri, I. B. Sial, Z. Zhang, Some new Newton's type integral inequalities for co-ordinated convex functions in quantum calculus, *Symmetry*, **12** (2020), 1476. https://doi.org/10.3390/sym12091476
56. M. Vivas-Cortez, M. A. Ali, H. Budak, H. Kalsoom, P. Agarwal, Some new Hermite-Hadamard and related inequalities for convex functions via (p, q)-integral, *Entropy*, **23** (2021), 828. https://doi.org/10.3390/e23070828
57. M. Vivas-Cortez, M. A. Ali, H. Kalsoom, H. Budak, M. Z. Sarikaya, H. Benish, Trapezoidal type inequalities for co-ordinated convex functions via quantum calculus, *Math. Probl. Eng.*, unpublished work.
58. F. Wannalookkhee, K. Nonlaopon, J. Tariboon, S. K. Ntouyas, On Hermite-Hadamard type inequalities for coordinated convex functions via (p, q)-calculus, *Mathematics*, **9** (2021), 698. https://doi.org/10.3390/math9070698

AIMS Press