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Research article

Commuting H-Toeplitz operators with quasihomogeneous symbols

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Abstract: In this paper, we characterize the commutativity of H-Toeplitz operators with quasihomogeneous symbols on the Bergman space, which is different from the case of Toeplitz operators with same symbols on the Bergman space.

Keywords: H-Toeplitz operator; Bergman space; commutativity; quasihomogeneous function
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1. Introduction

Let $\mathbb{D}$ be unit disk in the complex plane $\mathbb{C}$ and $dA(z) = \frac{1}{\pi}rdrd\theta$ be normalized Lebesgue area measure on $\mathbb{D}$. Let $L^2(\mathbb{D}, dA)$ denote the Hilbert space of all square integrable functions on $\mathbb{D}$ with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z), \quad f, g \in L^2(\mathbb{D}, dA).$$

Let $H(\mathbb{D})$ be the set of analytic functions on $\mathbb{D}$. The Bergman space $L_a^2(\mathbb{D}) = L^2(\mathbb{D}, dA) \cap H(\mathbb{D})$ is the closed subspace of $L^2(\mathbb{D}, dA)$. For nonnegative integer n, set $e_n(z) = \sqrt{n+1}z^n$, $z \in \mathbb{D}$. Then $\{e_n\}_{n\geq 0}$ is an orthonormal basis of $L_a^2(\mathbb{D})$. The Bergman space is a reproducing Hilbert space with the reproducing kernel $K_z(w) = \frac{1}{(1-\bar{z}w)^2}$, $z, w \in \mathbb{D}$. Let P be orthogonal projection from $L^2(\mathbb{D}, dA)$ onto Bergman space $L_a^2(\mathbb{D})$. For $f \in L^2(\mathbb{D}, dA)$, it has the reproducing formula

$$P(f)(z) = \langle f, K_z\rangle = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^2} dA(w). \tag{1.1}$$

For $\phi \in L^\infty(\mathbb{D})$, the multiplication operator M_ϕ is defined by $M_\phi(f) = \phi f$. The Toeplitz operator $T_\phi : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ and the Hankel operator $H_\phi : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ are defined respectively by

$$T_\phi = PM_\phi, \quad H_\phi = PM_\phi J,$$

where $J : L_a^2(\mathbb{D}) \to \overline{L_a^2(\mathbb{D})}$ is defined by $J(e_n) = \overline{e_{n+1}}$ $(n \geq 0)$. It is clear that T_ϕ and H_ϕ are bounded operators on the Bergman space $L_a^2(\mathbb{D})$.

Let $L_h^2(\mathbb{D})$ be a set of all harmonic functions in $L^2(\mathbb{D}, dA)$. The operator $K : L_a^2(\mathbb{D}) \to L_h^2(\mathbb{D})$ is defined by

$$K(e_{2n}) = e_n, \quad K(e_{2n+1}) = \overline{e_{n+1}}, \quad n = 0, 1, 2, \cdots .$$

Obviously K is bounded on $L_a^2(\mathbb{D})$, $\|K\| = 1$ and its adjoint operator $K^* : L_h^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ is given by

$$K^*(e_n) = e_{2n}, \quad K^*(\overline{e_{n+1}}) = e_{2n+1}, \quad n \geq 0.$$

For $\phi \in L^\infty(\mathbb{D})$, the H-Toeplitz operator $B_\phi : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ is defined by

$$B_\phi = PM_\phi K.$$

It is easy to see that

$$B_\phi^* = K^* P_h M_{\bar{\phi}},$$

where P_h is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L_h^2(\mathbb{D})$.

It is noted that the H-Toeplitz operator is closely related to the Toeplitz and Hankel operators. In fact, for each nonnegative integer n, we have

$$B_\phi(e_{2n}) = PM_\phi K(e_{2n}) = PM_\phi(e_n) = T_\phi(e_n) \tag{1.2}$$

and

$$B_\phi(e_{2n+1}) = PM_\phi K(e_{2n+1}) = PM_\phi J(e_n) = H_\phi(e_n). \tag{1.3}$$

It is known that $T_\phi = 0$ if and only if $\phi = 0$, whence $B_\phi = 0$ if and only if $\phi = 0$.

Recently, lots of study about Toeplitz and Hankel operators have been done on the Bergman space (see [1, 6, 7, 9, 11, 13–15]). Various generalizations of Toeplitz and Hankel operators on spaces of analytic functions have been studied by many mathematicians. In 2007, Arora and Paliwal [2] have introduced and studied H-Toeplitz operators on the Hardy space, where they have clubbed the notion of Toeplitz and Hankel operators together. The importance of this notion is that it is associated with a class of Toeplitz operators and a class of Hankel operators on the Hardy space where the original operators are neither Toeplitz nor Hankel. Moreover, it can also be observed that an $n \times n$ H-Toeplitz matrix has $2n - 1$ degree of freedom rather than n^2 and therefore for large n, it is comparatively easy to solve the system of linear equations where the coefficient matrix is an H-Toeplitz matrix.

In 1964, Brown and Halmos [4] showed that on the Hardy space, two bounded Toeplitz operators T_ϕ and T_ψ commute if and only if: (i) Both ϕ and ψ are analytic, or (ii) both $\bar{\phi}$ and $\bar{\psi}$ are analytic, or (iii) one is a linear function of the other. In [3] Axler and Cuckovic proved that if the two symbols are bounded harmonic functions, then the same result is also true for Toeplitz operators on the Bergman space. The situation with a general symbol is rather more complicated. Let $\phi, \psi \in L^\infty(\mathbb{D})$ be radial functions, i.e., $\phi(z) = \phi(|z|)$, $z \in \mathbb{D}$. It is well known and easy to see that two Toeplitz operators with radial symbols commute. In [10] Louhichi and Zakariasy showed that if p and s are integers such that $ps \leq 0$, then the Toeplitz operators with symbols $e^{ip\theta}\phi$ and $e^{is\theta}\psi$ commute only in certain trivial cases. This result is not true if both of the integers p and s satisfy $ps > 0$. There are lots of examples of functions of positive quasihomogeneous degree which are the symbols of commuting Toeplitz operators (see [5]).

The H-Toeplitz operator on the Bergman space was first studied recentely by [8], where the commutativity of H-Toeplitz operators with analytic or harmonic symbols is discussed. Motivated by these works, in this paper we will characterize the commuting H-Toeplitz operators with quasihomogeneous symbols, nonharmonic ones.

The organisation of paper is as follows. In Section 2, we shall collect some notations and results as preliminaries. In Section 3, we will first discuss when the product of two H-Toeplitz operators with quasihomogeneous symbols is still an H-Toeplitz operator (see Theorem 3.1), as a byproduct, we get the characterization of semi-commuting H-Toeplitz operators with quasihomogeneous symbols (see Corollary 3.2). The remaining of Section 3 will characterize the commuting H-Toeplitz operators with quasihomogeneous symbols in terms of different degrees and same degrees respectively (see Theorems 3.4 and 3.6 respectively).

2. Preliminaries

Let $\mathcal{R}$ be the space of functions which are square integrable in $[0,1]$ with respect to the measure rdr. By using the fact that the trigonometric polynomials are dense in $L^2(\mathbb{D}, dA)$ and that for $k_1 \neq k_2$, $e^{ik_1\theta}\mathcal{R}$ is orthogonal to $e^{ik_2\theta}\mathcal{R}$, one sees that

$$L^2(\mathbb{D}, dA) = \sum_{k\in\mathbb{Z}} e^{ik\theta}\mathcal{R}.$$

Thus, each function $\phi \in L^2(\mathbb{D}, dA)$ can be written as (see [5])

$$\phi(re^{i\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta}\varphi_k(r), \quad \varphi_k \in \mathcal{R}.$$

Moreover, if $\phi \in L^\infty(\mathbb{D}) \subset L^2(\mathbb{D}, dA)$, then for each $r \in [0,1)$,

$$|\varphi_k(r)| = \left|\frac{1}{2\pi}\int_0^{2\pi} \phi(re^{i\theta})e^{-ik\theta}d\theta\right| \leq \sup_{z\in\mathbb{D}} |\phi(z)|, \quad k \in \mathbb{Z}.$$

Hence, the functions φ_k are bounded in the disk. We call every function in $e^{ik\theta}\mathcal{R}$ to be *quasihomogeneous function* of degree k.

The following lemma will be used frequently.

Lemma 2.1. *Let p be an integer and φ a bounded radial function. Then for each nonnegative integer n,*

$$B_{e^{ip\theta}\varphi}(z^{2n}) = \begin{cases} 2\sqrt{\frac{n+1}{2n+1}}(n+p+1)\widehat{\varphi}(2n+p+2)z^{n+p}, & n+p \geq 0, \\ 0, & n+p < 0 \end{cases}$$

and

$$B_{e^{ip\theta}\varphi}(z^{2n+1}) = \begin{cases} 2\sqrt{\frac{n+2}{2n+2}}(p-n)\widehat{\varphi}(p+2)z^{p-1-n}, & n+1 \leq p, \\ 0, & n+1 > p. \end{cases}$$

Proof. Note that $K_z(w) = \sum_{j=0}^{\infty}(1+j)\bar{z}^j w^j$, it follows from the reproducing formula (1.1) that, when $n+p \geq 0$,

$$\begin{aligned}
B_{e^{ip\theta}\varphi}(z^{2n}) = PM_{e^{ip\theta}\varphi}K(z^{2n}) &= \sqrt{\frac{2n+2}{4n+2}}PM_{e^{ip\theta}\varphi}(z^n) \\
&= \sqrt{\frac{n+1}{2n+1}}\int_{\mathbb{D}} e^{ip\theta}\varphi(w)w^n\sum_{j=0}^{\infty}(1+j)\overline{w}^j z^j dA(w) \\
&= \sqrt{\frac{n+1}{2n+1}}\sum_{j=0}^{\infty}\int_0^1\int_0^{2\pi} e^{i(p+n-j)\theta}\varphi(r)r^{n+j+1}(1+j)z^j\frac{1}{\pi}d\theta dr \\
&= \sqrt{\frac{n+1}{2n+1}}(2n+2p+2)z^{n+p}\int_0^1 \varphi(r)r^{2n+p+1}dr \\
&= 2\sqrt{\frac{n+1}{2n+1}}(n+p+1)\widehat{\varphi}(2n+p+2)z^{n+p}.
\end{aligned}$$

When $n+p<0$, the above calculation also shows $B_{e^{ip\theta}\varphi}(z^{2n}) = 0$. Similarly, when $n+1 \leq p$,

$$\begin{aligned}
B_{e^{ip\theta}\varphi}(z^{2n+1}) = PM_{e^{ip\theta}\varphi}K(z^{2n+1}) &= \sqrt{\frac{2n+4}{4n+4}}PM_{e^{ip\theta}\varphi}(\overline{z}^{n+1}). \\
&= \sqrt{\frac{n+2}{2n+2}}\int_{\mathbb{D}} e^{ip\theta}\varphi(w)\overline{w}^{n+1}\sum_{j=0}^{\infty}(1+j)\overline{w}^j z^j dA(w) \\
&= \sqrt{\frac{n+2}{2n+2}}\sum_{j=0}^{\infty}\int_0^1\int_0^{2\pi} e^{i(p-n-1-j)\theta}\varphi(r)r^{n+2+j}(1+j)z^j\frac{d\theta dr}{\pi} \\
&= \sqrt{\frac{n+2}{2n+2}}(2p-2n)z^{p-1-n}\int_0^1 \varphi(r)r^{p+1}dr \\
&= 2\sqrt{\frac{n+2}{2n+2}}(p-n)\widehat{\varphi}(p+2)z^{p-1-n}.
\end{aligned}$$

When $n+1>p$, the above computation also gives $B_{e^{ip\theta}\varphi}(z^{2n+1}) = 0$. The proof is complete. □

An operator that will arise in our study of H-Toeplitz operators is the Mellin transform, which is defined for any function $\varphi \in L^1([0,1], rdr)$, by the formula

$$\widehat{\varphi}(z) = \int_0^1 \varphi(r)r^{z-1}dr.$$

It is clear that $\widehat{\varphi}$ is analytic in the half right plane $\{z : \operatorname{Re} z > 2\}$. It is important and helpful to know that the Mellin transform is uniquely determined by its value on an arithmetic sequence of integers. In fact, we have the following classical theorem (see [12], p. 102).

Lemma 2.2. *Suppose that φ is a bounded analytic function on $\{z : \operatorname{Re} z > 0\}$. If φ vanishes at the pairwise distinct points $\{z_k : k = 1, 2, \ldots\}$, where $\inf\{|z_k|\} > 0$ and $\sum_{k=1}^{\infty}(\frac{1}{z_k}) = \infty$, then $\varphi = 0$.*

As a simple application of the above lemma, we have the following fact which we will use frequently to prove our main results.

Corollary 2.3. *For $\varphi \in L^1([0,1], rdr)$, if there exists a sequence of positive integers $\{n_k\}$, such that $\sum_k 1/n_k = \infty$ and $\widehat{\varphi}(n_k) = 0$ for all k, then $\varphi = 0$.*

3. The main results and the proofs

In this section, we will characterize the commutativity of two H-Toeplitz operators with quasihomogeneous symbols.

In order to prove the semi-commuting H-Toeplitz operators, we first give the result for when the product of two H-Toeplitz operators with quasihomogeneous symbols is still an H-Toeplitz operator.

Theorem 3.1. *Let p, s be integers and ϕ_1, ϕ_2 two bounded radial functions. If there is $\phi \in L^\infty(\mathbb{D})$ such that $B_{e^{ip\theta}\phi_1} B_{e^{is\theta}\phi_2} = B_\phi$, then either $\phi_1 = \phi = 0$ or $\phi_2 = \phi = 0$.*

Proof. Now for $\phi \in L^\infty(\mathbb{D})$, write

$$\phi(re^{i\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta}\varphi_k(r),$$

where each φ_k is bounded radial function. We show the conclusion by considering two cases.

Case 1. We assume $s = 2\ell$ for some integer ℓ. Then using Lemma 2.1, direct calculations give that for nonnegative integer n, when $n + \ell \geq 0$ and $n + \ell + p \geq 0$,

$$B_{e^{ip\theta}\phi_1} B_{e^{is\theta}\phi_2}(z^{4n}) = A'_{n,\ell,p}\widehat{\phi_2}(4n + 2\ell + 2)\widehat{\phi_1}(2n + 2\ell + p + 2)z^{n+\ell+p}, \tag{3.1}$$

where

$$A'_{n,\ell,p} = 4\sqrt{\frac{(2n+1)(n+\ell+1)}{(4n+1)(2n+2\ell+1)}}(2n + 2\ell + 1)(n + \ell + p + 1),$$

and

$$B_\phi(z^{4n}) = \sum_{k=0}^{\infty} 2\sqrt{\frac{2n+1}{4n+1}}(k + 1)\widehat{\varphi}_{k-2n}(k + 2n + 2)z^k. \tag{3.2}$$

Since $B_{e^{ip\theta}\phi_1} B_{e^{is\theta}\phi_2}(z^{4n}) = B_\phi(z^{4n})$, then (3.1) and (3.2) give that

$$\widehat{\varphi}_{k-2n}(k + 2n + 2) = 0, \quad k \neq n + \ell + p. \tag{3.3}$$

Let $j = k-2n$, so $k = j+2n$. For each fixed interger j, when $n > \ell+p-j$, then $k = j+2n > n+\ell+p$. Hence, for each integer j, there is N_j, when $n \geq N_j$, the Eq (3.3) implies

$$\widehat{\varphi}_j(j + 4n + 2) = 0.$$

Obviously $\sum_{n=N_j}^{\infty} 1/(j + 4n + 2) = \infty$, so by Corollary 2.3, we get $\varphi_j = 0$ for every integer j, to obtain that $\phi = 0$.

The above has shown that $B_{e^{ip\theta}\phi_1} B_{e^{is\theta}\phi_2} = 0$. Now the Eq (3.1) gives that there is an integer $N_0 > 0$,

$$\widehat{\phi_2}(4n + 2\ell + 2)\widehat{\phi_1}(2n + 2\ell + p + 2) = 0, \quad n \geq N_0. \tag{3.4}$$

Set

$$E_1 = \{n \geq N_0 : \widehat{\phi_1}(2n + 2\ell + p + 2) = 0\},$$

$$E_2 = \{n \geq N_0 : \widehat{\phi_2}(4n + 2\ell + 2) = 0\}.$$

If $\sum_{n\in E_1} 1/n < \infty$, then $\sum_{n\in E_2} 1/n = \infty$, thus by Corollary 2.3, we get $\phi_2 = 0$; similarly, if $\sum_{n\in E_2} 1/n < \infty$, then it must be $\phi_1 = 0$.

Case 2. We suppose $s = 2\ell + 1$ for some integer ℓ. Then making use of Lemma 2.1, we get that for nonnegative integer n, when $n + \ell + 1 \geq 0$ and $n + \ell + 1 + p \geq 0$,

$$B_{e^{ip\theta}\phi_1} B_{e^{is\theta}\phi_2}(z^{4n+2}) = A''_{n,\ell,p}\widehat{\phi_2}(4n + 2\ell + 5)\widehat{\phi_1}(2n + 2\ell + p + 4)z^{n+\ell+1+p},$$

where

$$A''_{n,\ell,p} = 4\sqrt{\frac{(2n + 2)(n + \ell + 2)}{(4n + 3)(2n + 2\ell + 3)}}(2n + 2\ell + 3)(n + \ell + p + 2),$$

and

$$B_\phi(z^{4n+2}) = \sum_{k=0}^{\infty} 2\sqrt{\frac{2n + 2}{4n + 3}}(k + 1)\widehat{\varphi}_{k-2n-1}(k + 2n + 3)z^k.$$

Using same arguments as done in Case 1, it follows from $B_{e^{ip\theta}\phi_1} B_{e^{is\theta}\phi_2}(z^{4n}) = B_\phi(z^{4n})$ that the conclusion holds too. The proof is complete. □

Note that the authors showed an example in [8] that $B_z B_z \neq B_{z^2}$. The following corollary is quickly, which answers that two H-Toeplitz operators with quasihomogeneous symbols semi-commute only in trivial case.

Corollary 3.2. *Let p, s be integers and ϕ_1, ϕ_2 two bounded radial functions. Then the following are equivalent:*

(a) $B_{e^{ip\theta}\phi_1} B_{e^{is\theta}\phi_2} = B_{e^{i(p+s)\theta}\phi_1\phi_2}$;
(b) $B_{e^{ip\theta}\phi_1} B_{e^{is\theta}\phi_2} = 0$;
(c) $\phi_1 = 0$ *or* $\phi_2 = 0$.

Theorem 3.1 also helps us to identify the H-Toeplitz operators which are idempotents. The following corollary is an immediate consequence.

Corollary 3.3. *Let ϕ be a bounded quasihomogeneous function. Then $B_\phi^2 = B_\phi$ if and only if $\phi = 0$.*

Now we start to characterize when two H-Toeplitz operators with quasihomogeneous symbols commute. The first coming theorem tells that two H-Toeplitz operators with quasihomogeneous symbols such that the signs of their quasihomogeneous degrees are different commute only in the trivial case. Notice that in [8], the authors ever presented an example that $B_z B_{\bar{z}} \neq B_{\bar{z}} B_z$.

Theorem 3.4. *Let p, s be two distinct integers and ϕ_1, ϕ_2 two bounded radial functions. Then $B_{e^{ip\theta}\phi_1} B_{e^{is\theta}\phi_2} = B_{e^{is\theta}\phi_2} B_{e^{ip\theta}\phi_1}$ if and only if $\phi_1 = 0$ or $\phi_2 = 0$.*

Proof. The sufficiency is obvious. We now show the necessity. Suppose that $B_{e^{ip\theta}\phi} B_{e^{is\theta}\psi} = B_{e^{is\theta}\psi} B_{e^{ip\theta}\phi}$ and we deduce the conclusion by the following cases.

Case 1. $p = 2q$ and $s = 2\ell$ for some integers q and ℓ, where $q \neq \ell$. Then when $n + \ell \geq 0$ and $n + \ell + 2q \geq 0$, we have the Eq (3.1); and similarly, when $n + q \geq 0$ and $n + q + 2\ell \geq 0$, we have

$$B_{e^{is\theta}\phi_2} B_{e^{ip\theta}\phi_1}(z^{4n}) = A'_{n,q,s}\widehat{\phi}_1(4n + 2q + 2)\widehat{\phi}_2(2n + 2q + s + 2)z^{n+q+s}, \tag{3.5}$$

where

$$A'_{n,q,s} = 4\sqrt{\frac{(2n+1)(n+q+1)}{(4n+1)(2n+2q+1)}}(2n + 2q + 1)(n + q + s + 1),$$

Note that $n + \ell + p \neq n + q + s$, thus by the Eqs (3.1) and (3.5), we conclude that there is an integer $N_0 > 0$, the Eq (3.4) holds for all $n \geq N_0$. Thus done as the argument in Theorem 3.1 we may conclude that $\phi_1 = 0$ or $\phi_2 = 0$.

Case 2. $p = 2q + 1$ and $s = 2\ell$ for some integers q and ℓ. For any nonnegative integer n,

$$B_{e^{ip\theta}\phi_1} B_{e^{is\theta}\phi_2}(z^{4n}) = B_{e^{is\theta}\phi_2} B_{e^{ip\theta}\phi_1}(z^{4n}).$$

The left side of the above is (3.1) when $n+\ell \geq 0$ and $n+\ell+p \geq 0$, while the right side of the above is zero when $n \geq N$ for large enough positive integer N. It follows that (3.4) holds for $N_0 = \max(N, |\ell| + |p|)$. So as discussed in Case 1, we see that $\phi_1 = 0$ or $\phi_2 = 0$.

Case 3. $p = 2q$ and $s = 2\ell + 1$ for some integers q and ℓ. This case is similar to Case 2.

Case 4. $p = 2q + 1$ and $s = 2\ell + 1$ for some integers q and ℓ with $q \neq \ell$. This case is similar to Case 1 when applying the equality

$$B_{e^{ip\theta}\phi_1} B_{e^{is\theta}\phi_2}(z^{4n+2}) = B_{e^{is\theta}\phi_2} B_{e^{ip\theta}\phi_1}(z^{4n+2})$$

for nonnegtive integer n. The detail is omitted and we finish the proof. □

For the commuting of two H-Toeplitz operators with same degree quasihomogenous symbols, the situation becomes quite hard. We first give the following lemma.

Lemma 3.5. *Let p be an integer and ϕ_1, ϕ_2 two bounded radial functions. Assume $B_{e^{ip\theta}\phi_1} B_{e^{ip\theta}\phi_2} = B_{e^{ip\theta}\phi_2} B_{e^{ip\theta}\phi_1}$.*

(a) If $p = 2q$ for some integer q, then for any $\operatorname{Re} z > \max(0, -2p)$*,*

$$\widehat{r^{p+2}\phi_1}(2z)\widehat{r^{p+2}\phi_2}(z + p) = \widehat{r^{p+2}\phi_2}(2z)\widehat{r^{p+2}\phi_1}(z + p). \tag{3.6}$$

(b) If $p = 2q + 1$ for some integer q, then for any $\operatorname{Re} z > \max(0, -2p - 1)$*,*

$$\widehat{r^{p+4}\phi_1}(2z)\widehat{r^{p+4}\phi_2}(z + p - 1) = \widehat{r^{p+4}\phi_2}(2z)\widehat{r^{p+4}\phi_1}(z + p - 1). \tag{3.7}$$

Proof. (a) Suppose $p = 2q$ for some integer q. So using

$$B_{e^{ip\theta}\phi_1} B_{e^{ip\theta}\phi_2}(z^{4n}) = B_{e^{ip\theta}\phi_2} B_{e^{ip\theta}\phi_1}(z^{4n}),$$

together with (3.1) and (3.5) where $s = p$ and integer $n \geq N = \max(0, -3q)$, it follows that

$$\widehat{\phi}_1(4n + p + 2)\widehat{\phi}_2(2n + 2p + 2) = \widehat{\phi}_2(4n + p + 2)\widehat{\phi}_1(2n + 2p + 2) \tag{3.8}$$

when $n \geq N$, or equivalently,

$$\widehat{r^{p+2}\phi}_1(4n)\widehat{r^{p+2}\phi}_2(2n+p) = \widehat{r^{p+2}\phi}_2(4n)\widehat{r^{p+2}\phi}_1(2n+p)$$

when $n \geq N$. Set

$$\Phi(z) = \widehat{r^{p+2}\phi}_1(2z)\widehat{r^{p+2}\phi}_2(z+p) - \widehat{r^{p+2}\phi}_2(2z)\widehat{r^{p+2}\phi}_1(z+p).$$

It is easy to see that Φ is a bounded analytic function in the right half plane $\{z : \operatorname{Re} z > \max(0, -2p)\}$. The Eq (3.8) tells that $\Phi(2n) = 0$ when $n \geq N$. Thus by Lemma 2.2 it concludes that $\Phi \equiv 0$ in the right half plane $\{z : \operatorname{Re} z > \max(0, -2p)\}$, which gives (3.6).

(b) Suppose $p = 2q + 1$ for some integer q. Similar to the previous case, it follows from

$$B_{e^{ip\theta}\phi_1}B_{e^{ip\theta}\phi_2}(z^{4n+2}) = B_{e^{ip\theta}\phi_2}B_{e^{ip\theta}\phi_1}(z^{4n+2})$$

that for big enough integer N, when $n \geq N$,

$$\widehat{\phi}_1(4n+p+4)\widehat{\phi}_2(2n+2p+3) = \widehat{\phi}_2(4n+p+4)\widehat{\phi}_1(2n+2p+3),$$

or equivalently,

$$\widehat{r^{p+4}\phi}_1(4n)\widehat{r^{p+4}\phi}_2(2n+p-1) = \widehat{r^{p+4}\phi}_2(4n)\widehat{r^{p+4}\phi}_1(2n+p-1)$$

when $n \geq N$. Hence same arguments used in (a) will give (3.7). The proof is complete. □

Now we can characterize the commuting H-Toeplitz operators with same nonnegative degree quasihomogeneous symbols.

Theorem 3.6. *Let p be a nonnegative integer and ϕ_1, ϕ_2 two bounded radial functions. Then $B_{e^{ip\theta}\phi_1}B_{e^{ip\theta}\phi_2} = B_{e^{ip\theta}\phi_2}B_{e^{ip\theta}\phi_1}$ if and only if there exist $\alpha, \beta \in \mathbb{C}$, $|\alpha| + |\beta| \neq 0$ such that $\alpha\phi_1 + \beta\phi_2 = 0$.*

Proof. We first show the sufficiency. If $\alpha\phi_1 + \beta\phi_2 = 0$ for α, β, not all zero, we may assume $\alpha \neq 0$, then $\phi_1 = c\phi_2$, where $c = -\beta/\alpha$. Hence $B_{e^{ip\theta}\phi_1} = cB_{e^{ip\theta}\phi_2}$, and so clearly $B_{e^{ip\theta}\phi_1}B_{e^{ip\theta}\phi_2} = B_{e^{ip\theta}\phi_2}B_{e^{ip\theta}\phi_1}$.

Now we show the necessity using Lemma 3.5. First consider the case p is even. Without loss of generality, we assume $\phi_2 \neq 0$. Put

$$E = \{z : \operatorname{Re} z > 0, \ \widehat{r^{p+2}\phi}_2(z) = 0\}.$$

By (3.6) we get that for $\operatorname{Re} z > 0$,

$$\frac{\widehat{r^{p+2}\phi}_1(2z)}{\widehat{r^{p+2}\phi}_2(2z)} = \frac{\widehat{r^{p+2}\phi}_1(z+p)}{\widehat{r^{p+2}\phi}_2(z+p)}, \quad z+p,\ 2z \notin E. \tag{3.9}$$

Case 1. Suppose p is positive. We claim that, there is $z_0 \in (1+p, 2+p)$ such that for any integer $k \geq 0$,

$$\frac{z_0 - p}{2^k} + 2p \notin E. \tag{3.10}$$

In fact, on the one hand, we note that $\{\frac{z_0-p}{2^k} + 2p : k \geq 0\}$ is a bounded sequence since

$$2p < \frac{z_0 - p}{2^k} + 2p \leq z_0 + p < 2p + 2.$$

On the other hand, it is easy to check that for $z_1, z_2 \in (1+p, 2+p)$ with $z_1 \neq z_2$,

$$\Big\{\frac{z_1 - p}{2^k} + 2p : \ k \geq 0\Big\} \bigcap \Big\{\frac{z_2 - p}{2^k} + 2p : \ k \geq 0\Big\} = \emptyset.$$

Now, if the claim is not true, then for each $z \in (1+p, 2+p)$, there is a nonnegative integer k_z such that $\frac{z-p}{2^{k_z}} + 2p \in E$. It follows that the bounded infinite set

$$\Big\{\frac{z - p}{2^{k_z}} + 2p : \ z \in (1+p, 2+p)\Big\} \subset E,$$

which implies that the analytic function $\widehat{r^{p+2}\phi_2} \equiv 0$, and so $\phi_2 = 0$ by Corollary 2.3, a controdiction. Hence the claim holds.

Now we fix a $z_0 \in (1+p, 2+p)$ such that (3.10) holds for each integer $k \geq 0$. By (3.9), we have

$$\begin{aligned}\frac{\widehat{r^{p+2}\phi_1}}{\widehat{r^{p+2}\phi_2}}\Big(\frac{z_0 - p}{2^k} + 2p\Big) &= \frac{\widehat{r^{p+2}\phi_1}}{\widehat{r^{p+2}\phi_2}}\Big(2 \cdot (\frac{z_0 - p}{2^{k+1}} + p)\Big) \\ &= \frac{\widehat{r^{p+2}\phi_1}}{\widehat{r^{p+2}\phi_2}}\Big(\frac{z_0 - p}{2^{k+1}} + 2p\Big), \quad k \geq 0.\end{aligned}$$

It induces that

$$\frac{\widehat{r^{p+2}\phi_1}}{\widehat{r^{p+2}\phi_2}}\Big(\frac{z_0 - p}{2^k} + 2p\Big) = \frac{\widehat{r^{p+2}\phi_1}}{\widehat{r^{p+2}\phi_2}}\Big(z_0 + p\Big) =: c, \quad k \geq 0.$$

Notice that $\frac{z_0-p}{2^k} + 2p \to 2p$ as $k \to \infty$, so the above implies that the analytic function

$$\frac{\widehat{r^{p+2}\phi_1}}{\widehat{r^{p+2}\phi_2}}(z) \equiv c, \quad \operatorname{Re} z > 0,$$

which means that the Mellin transformation of $r^{p+2}(\phi_1 - c\phi_2)$ is identically zero in the right half plane $\{z : \ \operatorname{Re} z > 0\}$, hence we get that $\phi_1 = c\phi_2$ by Corollary 2.3, as desired.

Case 2. Suppose $p = 0$. In this case, (3.6) becomes

$$\widehat{\phi_1}(2z+2)\widehat{\phi_2}(z+2) = \widehat{\phi_2}(2z+2)\widehat{\phi_1}(z+2), \quad \operatorname{Re} z > 0.$$

Replacing z by $z - 1$ in the above we get

$$\widehat{\phi_1}(2z)\widehat{\phi_2}(z+1) = \widehat{\phi_2}(2z)\widehat{\phi_1}(z+1), \quad \operatorname{Re} z > 1.$$

Applying the same arguments done in Case 1, we will obtain the desired conclusion.

When p is positive and odd, the proof is similar by using (3.7). We omit the detail and finish the proof. □

We don't know whether the previous theorem is true when p is negative. But for special symbols, it is still the case.

Theorem 3.7. *Let j, k, s, t be integers. Then $B_{z^j\bar{z}^k}B_{z^s\bar{z}^t} = B_{z^s\bar{z}^t}B_{z^j\bar{z}^k}$ if and only if $j = s$ and $k = t$.*

Proof. The sufficiency is clear. Now we show the necessity. Let $\phi_1 = z^j\bar{z}^k = r^{j+k}e^{i(j-k)\theta}$ and $\phi_2 = z^s\bar{z}^t = r^{s-t}e^{i(s-t)\theta}$. Since $\phi_1 \neq 0$ and $\phi_2 \neq 0$, so Theorem 3.4 tells that $j-k = s-t := p$. It is left to show that $j+k = s+t$.

We only consider the case when $p = 2q$ for some integer q (the case $p = 2q+1$ is similar). So by (3.6) we have

$$\widehat{r^{p+2}r^{j+k}}(2z)\widehat{r^{p+2}r^{s+t}}(z+p) = \widehat{r^{p+2}r^{s+t}}(2z)\widehat{r^{p+2}r^{j+k}}(z+p)$$

when $\operatorname{Re} z > \max(0, -2p)$. By the definition of the Mellin transformation, the above yields that

$$(p+2+j+k+2z)(2p+2+s+t+z) = (p+2+s+t+2z)(2p+2+j+k+z)$$

when $\operatorname{Re} z > \max(0, -2p)$. Thus it is easy to get that $j+k = s+t$, and which together with $j-k = s-t$ induces $j = s$ and $k = t$. The proof is complete. □

4. Conclusions

In this research, it obtains the following characterizations for the commuting H-Toeplitz operators with quasihomogeneous symbols on the Bergman space.

(1) Let p, s be two distinct integers and ϕ_1, ϕ_2 two bounded radial functions. Then $B_{e^{ip\theta}\phi_1}B_{e^{is\theta}\phi_2} = B_{e^{is\theta}\phi_2}B_{e^{ip\theta}\phi_1}$ if and only if $\phi_1 = 0$ or $\phi_2 = 0$.

(2) Let p be a nonnegative integer and ϕ_1, ϕ_2 two bounded radial functions. Then $B_{e^{ip\theta}\phi_1}B_{e^{ip\theta}\phi_2} = B_{e^{ip\theta}\phi_2}B_{e^{ip\theta}\phi_1}$ if and only if there exist $\alpha, \beta \in \mathbb{C}$, $|\alpha| + |\beta| \neq 0$ such that $\alpha\phi_1 + \beta\phi_2 = 0$.

(3) Let p, s be integers and ϕ_1, ϕ_2 two bounded radial functions. Then $B_{e^{ip\theta}\phi_1}B_{e^{is\theta}\phi_2} = B_{e^{i(p+s)\theta}\phi_1\phi_2}$ if and only if $\phi_1 = 0$ or $\phi_2 = 0$.

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Conflict of interest

The authors declare that there is no conflict of interest.

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