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Research article

On a novel impulsive boundary value pantograph problem under Caputo proportional fractional derivative operator with respect to another function

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Abstract: In this manuscript, we study the existence and Ulam's stability results for impulsive multi-order Caputo proportional fractional pantograph differential equations equipped with boundary and integral conditions with respect to another function. The uniqueness result is proved via Banach's fixed point theorem, and the existence results are based on Schaefer's fixed point theorem. In addition, the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of the proposed problem are obtained by applying the nonlinear functional analysis technique. Finally, numerical examples are provided to supplement the applicability of the acquired theoretical results.

Keywords: existence and uniqueness; fractional differential equations; fixed point theorems; impulsive conditions; Ulam-Hyers stability
Mathematics Subject Classification: 26A33, 34A37, 34A08, 34D20, 38B82

1. Introduction

Fractional calculus (FC) has been more important in pure and applied mathematics in recent decades as a result of its applications in engineering and applied sciences. FC deals with the integral and differential operators of non-integer orders. Fractional differential and integral equations have been confirmed to be powerful equipment to explain various real-world problems such as chemistry, biology, physics, signal processing, electrodynamics, economics, finance, and also many more. For more details, we refer readers to the books in [1–5].

One type of famous differential equation (DEq) involves proportional delay terms called pantograph equations (PEqs) of the form:

$$\begin{cases} x'(t) = ax(t) + bx(\lambda t), \quad t \in [0,T], \quad T > 0, \\ x(0) = x_0, \quad \lambda \in (0,1), \quad a, b, \in \mathbb{R}. \end{cases} \tag{1.1}$$

It is studied by Ockendon and Tayler [6] that has been a wide range of applications in a wide range of applied fields of sciences, economics, medicine, engineering and several problems. The PEs is employed to model some processes and phenomena at the present time and depend on previous states. For some interesting papers on PEs, see [7–19] and the references cited therein. In 2013, Balachandran et al. [20] examined the existence of solutions for nonlinear fractional PEs using the FC and fixed point theorems:

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha}x(t) = f(t, x(t), x(\lambda t)), \quad \alpha \in (0,1], \quad t \in [0,1], \\ x(0) = x_0, \quad x_0 \in \mathbb{R}, \quad \lambda \in (0,1), \end{cases} \tag{1.2}$$

where ${}^{C}\mathfrak{D}^{\alpha}$ denotes the Caputo fractional derivative of order α and $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$.

The ordinary impulsive differential equations (IDEs) have been played a significant role almost in every subject to descript physical phenomena in mathematical modeling. They were used to model some processes with discontinuous jumps and instantaneous moves that cannot be modeled by ordinary differential equations. In addition, they have been great considered in many fields of real-world problems such as earthquakes, a mass-spring-damper system with short-term perturbations, finance and pharmacotherapy, see [21–24].

Recently, a qualitative property is a favorite field to study in the areas of engineering and applied sciences. It has two notable topics that are the existence theory and stability analysis. Stability analysis plays a very important tool to study in many fields such as optimization, numerical analysis, economics, mathematical biology and nonlinear analysis, etc. We encounter situations where finding the exact solution is a very difficult task, so stability analysis comes into a major role. Various types of stability like Exponential stability, Lyapunov stability, Mittag-Lefler stability, and Ulam-Hyers ($\mathbb{UH}$) stability have been applied to examine the stability of functional problems. This paper will be studying the $\mathbb{UH}$ stability concept that has been accepted as an easy way and well-known procedure of examination. Ulam and Hyers have initiated the $\mathbb{UH}$ stability concept of the functional problems in Banach space by Ulam and Hyers during 1941. Thereafter, Rassias provided a notable generalization of the $\mathbb{UH}$ stability of mappings by considering variables in 1978 (is called the Ulam-Hyers-Rassias ($\mathbb{UHR}$) stability). The $\mathbb{UH}$ stability and $\mathbb{UHR}$ stability have been extended to integral and differential equations. For more historical details [25–28]. Then the qualitative property of IDEs is very significant and helpful to realize physical phenomena that are not described as in the non-IDEs. Many modern papers apply fractional calculus on IDEs. The researchers have studied the qualitative properties of impulsive fractional differential equations. There are increasingly researches studying the qualitative property on non-impulsive/impulsive fractional differential equations (FDEs).

For instances, in 2009, Benchohra and Slimani [29], using fixed point theory of Banach's, Schaefer's and Leray-Schauder types, discussed the existence and uniqueness criteria of solutions for the initial value problems (IVPs) with impulses:

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha}x(t) = f(t, x(t)), \quad t \neq t_k, \quad t \in [0, T], \quad k = 1, 2, \ldots, m, \\ \Delta x(t_k) = J_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}, \end{cases} \tag{1.3}$$

where $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$, $J_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, m$, and $0 = t_0 < t_1 < \cdot < t_m < t_{m+1} = T$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)$, $x(t_k^-) = x(t_k)$ represent the right and left hand limits of $x(t)$ at $t = t_k$, respectively. Benchohra and Seba [30], using Mönch's fixed point theorem merged with the technique of measures of noncompactness, examined the existence and uniqueness of solutions for the IVPs with impulses (1.3). In 2012, Wang et al. [31] studied the sufficient conditions for the existence of solutions for IVPs with impulses (1.3) by using a fixed point theorem on topological degree for condensing maps via a priori estimate method. In 2015, Benchohra and Lazreg [32] considered the implicit FDEs in Caputo sense with impulse:

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha}_{t_k^+}x(t) = f(t, x(t), {}^{C}\mathfrak{D}^{\alpha}_{t_k^+}x(t)), \quad t \neq t_k, \quad t \in [0, T], \quad k = 1, 2, \ldots, m, \\ \Delta x(t_k) = J_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}, \end{cases} \tag{1.4}$$

where $f \in C([0, T] \times \mathbb{R}^2, \mathbb{R})$. The existence results of (1.4) are established based on the Banach contraction principle and Schaefer's fixed point theorem. In 2017, Benchohra et al. [33] established the existence, uniqueness, and $\mathbb{UH}$ stability of solutions for the nonlinear FDEs in Caputo-Hadamard sense with impulse of the form:

$$\begin{cases} {}^{CH}\mathfrak{D}^{\alpha}_{t_k^+}x(t) = f(t, x(t), {}^{CH}\mathfrak{D}^{\alpha}_{t_k^+}x(t)), \quad t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \ldots, m, \\ \Delta x(t_k) = J_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ ax(0) + bx(T) = c, \quad a, b, c \in \mathbb{R}, \end{cases} \tag{1.5}$$

where ${}^{CH}\mathfrak{D}^{\alpha}_{t_k^+}$ denotes the Caputo-Hadamard fractional derivative of order $\alpha \in (0, 1]$, $f \in C([0, T] \times \mathbb{R}^2, \mathbb{R})$, and $a + b \neq 0$. The existence results are proved by using the Banach contraction principle and Schaefer's fixed point theorem. In 2021, Ali et al. [34] discussed the IVPs of pantograph implicit FDEs with impulsive conditions. The existence results are derived by applying the Banach contraction principle and Schaefer's fixed point theorem. In addition, they studied the $\mathbb{UH}$ results of the following problem:

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha}_{t_k^+}x(t) = f(t, x(t), x(\lambda t), {}^{C}\mathfrak{D}^{\alpha}_{t_k^+}x(t)), \quad t \in [0, T], \quad t \neq t_k, \quad k = 1, 2, \ldots, m, \\ \Delta x(t_k) = J_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}, \quad \lambda \in (0, 1), \end{cases} \tag{1.6}$$

where $f \in C([0, T] \times \mathbb{R}^3, \mathbb{R})$. For modern researches on impulsive FDEs about the existence, uniqueness and stability, see [35–46] and the references cited therein.

Recently, in [47, 48], the authors formulate the proportional fractional operators of a function f with respect to another function ψ and provide its properties. For $\alpha > 0$, $\rho \in (0, 1]$, $\psi \in C^1([a, b])$, $\psi' > 0$, the proportional fractional integral (PFI) of order α of the function $f \in L^1([a, b])$ with respect to another function ψ is defined by

$$ {}^{\rho}\mathcal{I}_a^{\alpha,\psi} f(t) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(s)\right)} (\psi(t) - \psi(s))^{\alpha-1} f(s)\psi'(s)ds, \tag{1.7} $$

where $\Gamma(\cdot)$ is the (Euler's) gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} s^{\alpha-1}e^{-s}ds,\ s > 0$. The Riemann-Liouville proportional fractional derivative (PFD) of order α of the function $f \in C^n([a, b])$ with respect to another function ψ is defined by

$$ {}^{\rho}\mathfrak{D}_a^{\alpha,\psi} f(t) = {}^{\rho}\mathfrak{D}^{n,\psi\,\rho}{}_a\mathcal{I}^{n-\alpha,\psi} f(t) = \frac{{}^{\rho}\mathfrak{D}_t^{n,\psi}}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(s)\right)} (\psi(t) - \psi(s))^{n-\alpha-1} f(s)\psi'(s)ds, \tag{1.8} $$

where $n = [\alpha] + 1$, $[\alpha]$ is the integer part of α, ${}^{\rho}\mathfrak{D}^{n,\psi} = \underbrace{{}^{\rho}\mathfrak{D}^{\psi}\cdot{}^{\rho}\mathfrak{D}^{\psi}\cdots{}^{\rho}\mathfrak{D}^{\psi}}_{\text{n times}}$, and ${}^{\rho}\mathfrak{D}^{\psi} f(t) = (1-\rho)f(t) + \rho f'(t))/\psi'(t)$. The Caputo PFD type is defined by

$$ {}^{C\rho}\mathfrak{D}_a^{\alpha,\psi} f(t) = {}^{\rho}\mathcal{I}_a^{n-\alpha,\psi\rho}\mathfrak{D}^{n,\psi} f(t) = \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(s)\right)} (\psi(t) - \psi(s))^{n-\alpha-1\,\rho}\mathfrak{D}^{n,\psi} f(s)\psi'(s)ds. \tag{1.9} $$

The relation of PFI and PFD of Caputo type which will be used in this manuscript as

$$ {}^{\rho}\mathcal{I}_a^{\alpha,\psi C\rho}\mathfrak{D}_a^{\alpha,\psi} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{{}^{\rho}\mathfrak{D}^{k,\psi} f(a)}{\rho^k k!} (\psi(t) - \psi(a))^k e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(a)\right)}. \tag{1.10} $$

Morover, for $\alpha, \beta > 0$ and $\rho \in (0, 1]$, we have the following property:

$$ \left({}^{\rho}\mathcal{I}_a^{\alpha,\psi} e^{\frac{\rho-1}{\rho}\psi(s)} (\psi(s) - \psi(a))^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\rho^{\alpha}\Gamma(\beta+\alpha)} e^{\frac{\rho-1}{\rho}\psi(t)} (\psi(t) - \psi(a))^{\beta+\alpha-1}. \tag{1.11} $$

Clearly, if we set $\rho = 1$ in (1.7)–(1.9), then we have the Riemann-Liouville fractional operators [2] with $\psi(t) = t$, the Hadamard fractional operators [2] with $\psi(t) = \log t$, the Katugampola fractional operators [49] with $\psi(t) = t^{\mu}/\mu, \mu > 0$, the conformable fractional operators [50] with $\psi(t) = (t-a)^{\mu}/\mu$, $\mu > 0$, and the generalized conformable fractional operators [51] with $\psi(t) = t^{\mu+\phi}/(\mu + \phi)$, respectively. The previous modern works on proportional fractional operators of a function with respect to another function, see [52–56]. To the best of the author's knowledge, there are some manuscripts that have established either impulsive fractional boundary value problems [57, 58] and few papers focused on impulsive Caputo proportional fractional boundary value problems with respect to another function via proportional delay term.

Motivated by the aforesaid utilization of implicit impulsive pantograph differential equations above and a series of papers were presented, we investigate the qualitative properties (existence, uniqueness and $\mathbb{UH}$ stability) of the solutions for the following nonlinear impulsive boundary value pantograph problem under Caputo PFD operator of the form.

$$ \begin{cases} {}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k} x(t) = f(t, x(t), x(\lambda t), {}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k} x(t)), & t \neq t_k,\ k = 1, 2, \ldots, m, \\ \Delta x(t_k) = J_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ \eta x(0) + \beta x(T) = \sum_{i=0}^{m} \delta_i {}^{\rho_i}\mathcal{I}_{t_i}^{\gamma_i,\psi_i} x(\xi_i), \end{cases} \tag{1.12} $$

where ${}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k}$ denotes the Caputo PFD of order α_k with respect to certain continuously differentiable and increasing function ψ_k with $\psi'(t) > 0$ and $\alpha_k \in (0,1]$, $t \in \mathcal{J}_k = (t_k, t_{k+1}] \subseteq \mathcal{J} = [0,T] = \{a\} \cup (\bigcup_0^m \mathcal{J}_k)$, $k = 0,1,\ldots,m$, $0 = t_0 < t_1 < \cdot < t_m < t_{m+1} = T$, $\rho_k \in (0,1]$, $\lambda \in (0,1)$, $f \in C(\mathcal{J} \times \mathbb{R}^3, \mathbb{R})$, $\varphi_k \in C(\mathbb{R},\mathbb{R})$, $k = 1,2,\ldots,m$, ${}^{\rho_i}\mathcal{I}_{t_i}^{\gamma_i,\psi_i}$ denotes the PFI of order $\gamma_i > 0$ with respect to certain continuously differentiable and increasing function ψ_i, $i = 0,1,\ldots,m$. The given constants η, β, $\delta_i \in \mathbb{R}$, $\xi_i \in (t_i, t_{i+1}]$, $i = 0,1,\ldots,m$. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{\epsilon\to 0^+} x(t_k+\epsilon)$, $x(t_k^-) = x(t_k)$ represent the right and left hand limits of $x(t)$ at $t = t_k$, respectively. Notice that, the significance of this discussion on the manuscript is that the problem (1.12) generates many types, including mixed types of impulsive FDEs with boundary conditions, see [29–34] and references cited therein.

The outline of this paper is as follows: In Section 2, we give some basic concepts, notations, definitions and lemmas that will be used in this manuscript. Further, an auxiliary result useful to convert the impulsive problem (1.12) into an equivalent integral equation is constructed in Section 2. In Section 3, showing the existence results, the uniqueness criteria is verified by Banach's fixed point theorem, and the existence criteria is proved by Schaefer's fixed point theorem. Besides, we investigate the different types of Ulam's stability results for the problem (1.12) in Section 4. Finally, illustrative examples are built in Section 5 to clarify the positiveness of our theoretical results.

2. Preliminaries

Throughout this manuscript, let $\mathbb{PC}(J,\mathbb{R}) := \{x : J \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-) = x(t_k)$, $k = 1,2,\ldots,m\}$ the space of piecewise continuous functions. Obviously, $(\mathbb{PC}(J,\mathbb{R}), \|x\|)$ is a Banach space equipped with the norm $\|x\| := \sup_{t\in J}|x(t)|$. In the following, we set the functional equation $F_x(t) = f(t, x(t), x(\lambda t), F_x(t))$, and represents the PFI operator defined in (1.7) of a nonlinear function F_x by a subscript notation by

$$
\begin{aligned}
{}^{\rho}\mathcal{I}_a^{\alpha,\psi}F_x(t) &= \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(s)\right)}(\psi(t)-\psi(s))^{\alpha-1}F_x(s)\psi'(s)ds \\
&= \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(s)\right)}(\psi(t)-\psi(s))^{\alpha-1}f(s,x(s),x(\lambda s),F_x(s))\psi'(s)ds.
\end{aligned}
$$

Next, let us begin by determining what we propose by a solution of (1.12).

Definition 2.1. *A function $x \in \mathbb{PC}(J,\mathbb{R}) \cap (\cup_{k=0}^m \mathbb{AC}(\mathcal{J}_k,\mathbb{R}))$ is said to be a solution of* (1.12) *if x satisfies ${}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k}x(t) = f(t, x(t), x(\lambda t), {}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k}x(t))$, on $\mathcal{J}_k$ with $\Delta x(t_k) = J_k(x(t_k))$ for $k = 1,2,\ldots,m$ under $\eta x(0) + \beta x(T) = \sum_{i=0}^m \delta_i {}^{\rho_i}\mathcal{I}_{t_i}^{\gamma_i,\psi_i}x(\xi_{i+1})$, for $i = 0,1,\ldots,m$.*

Conveniently, for nonnegative $a < b$, we define the following symbol:

$$
\Psi_a(t_a, t_b) = \psi_a(t_b) - \psi_a(t_a). \tag{2.1}
$$

Proposition 2.1. *[48] Let $Re(\alpha) \geq 0$ and $Re(\beta) > 0$. Then, for any $\rho \in (0,1]$ and $n = [Re(\alpha)] + 1$, we have*

(i) $\left({}^{\rho}\mathfrak{D}_a^{\alpha,\psi} e^{\frac{\rho-1}{\rho}\psi(s)}\left(\psi(s)-\psi(a)\right)^{\beta-1}\right)(t) = \frac{\rho^\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha)}e^{\frac{\rho-1}{\rho}\psi(t)}\left(\psi(t)-\psi(a)\right)^{\beta-\alpha-1}, \quad Re(\alpha) \geq 0.$

(ii) $\left({}^{C\rho}\mathfrak{D}_a^{\alpha,\psi} e^{\frac{\rho-1}{\rho}\psi(s)}\left(\psi(s)-\psi(a)\right)^{\beta-1}\right)(t) = \frac{\rho^{\alpha}\Gamma(\beta)}{\Gamma(\beta-\alpha)} e^{\frac{\rho-1}{\rho}\psi(t)}\left(\psi(t)-\psi(a)\right)^{\beta-\alpha-1}, \quad Re(\beta) > n.$

For $k = 0, 1, \ldots, n-1$, *we have*

$$\left({}^{C\rho}\mathfrak{D}_a^{\alpha,\psi} e^{\frac{\rho-1}{\rho}\psi(s)}\left(\psi(s)-\psi(a)\right)^{k}\right)(t) = 0 \quad \text{and} \quad \left({}^{C\rho}\mathfrak{D}_a^{\alpha,\psi} e^{\frac{\rho-1}{\rho}\psi(s)}\right)(t) = 0.$$

Corollary 2.1. *[57] Let* $0 < Re(\beta) < Re(\alpha)$ *and* $m-1 < Re(\beta) \le m$. *Then we have*

$${}^{C\rho}\mathfrak{D}_a^{\beta,\psi}\,{}_a\mathfrak{I}^{\alpha,\rho,\psi} f(t) = {}^{\rho}\mathcal{I}_a^{\alpha-\beta,\psi} f(t).$$

Next, we provide an essential Lemma 2.1 that is used to prove the main results of (1.12).

Lemma 2.1. *Let* $0 < \alpha_k \le 1$, $0 < \rho_k \le 1$, $F_x \in \mathbb{AC}(\mathcal{J}\times\mathbb{R}^3, \mathbb{R})$ *for any* $x \in C(\mathcal{J}, \mathbb{R})$ *and* $\Lambda \neq 0$. *Then the following problem:*

$$\begin{cases} {}^{C\rho_k}\mathfrak{D}_{t_k}^{\alpha_k,\psi_k} x(t) = F_x(t), \quad t \neq t_k, \quad k = 0, 1, 2, \ldots, m, \\ \Delta x(t_k) = J_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ \eta x(0) + \beta x(T) = \sum_{i=0}^{m} \delta_i {}^{\rho_i}\mathcal{I}_{t_i}^{\gamma_i,\psi_i} x(\xi_i), \end{cases} \tag{2.2}$$

is equivalent to the following integral equation:

$$\begin{aligned} x(t) \;=\; & {}^{\rho_k}\mathcal{I}_{t_k}^{\alpha_k,\psi_k} F_x(t) + e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,t)} \Bigg\{ \frac{1}{\Lambda} \prod_{i=1}^{k} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(t_{i-1},t_i)} \Bigg[\sum_{i=0}^{m} \delta_i {}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i} F_x(\xi_i) - \beta {}^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m} F_x(T) \\ & - \beta e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)} \sum_{i=1}^{m} \left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}} F_x(t_i) + J_i(x(t_i)) \right) \prod_{j=i}^{m-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})} \right) \\ & + \sum_{i=0}^{m} \frac{\delta_i \Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)} e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)} \sum_{j=1}^{i} \left(\left({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}} F_x(t_j) + J_j(x(t_j)) \right) \prod_{l=j}^{i-1} e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})} \right) \Bigg] \\ & + \sum_{i=1}^{k} \left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}} F_x(t_i) + J_i(x(t_i)) \right) \prod_{j=i}^{k-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})} \right) \Bigg\}, \quad t \in \mathcal{J}_k, \end{aligned} \tag{2.3}$$

where

$$\Lambda := \eta + \beta \prod_{i=1}^{m+1} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(t_{i-1},t_i)} - \sum_{i=0}^{m} \frac{\delta_i \Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)} e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)} \prod_{j=1}^{i} e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\Psi_{j-1}(t_{j-1},t_j)}. \tag{2.4}$$

Proof. Firstly, for $t \in \mathcal{J}_0 = [t_0, t_1]$, we convert (2.2) into integral equation by taking the PFI operator ${}^{\rho_0}\mathcal{I}_{t_0}^{\alpha_0,\psi_0}$ to both sides of (2.2) and also applying (1.10), we have

$$x(t) = {}^{\rho_0}\mathcal{I}_{t_0}^{\alpha_0,\psi_0} F_x(t) + c_0\, e^{\frac{\rho_0-1}{\rho_0}(\psi_0(t)-\psi_0(t_0))},$$

where $c_0 = x(t_0^+)$. For $t \in \mathcal{J}_1 = (t_1, t_2]$, by taking ${}^{\rho_1}\mathcal{I}_{t_1}^{\alpha_1,\psi_1}$ to both sides of (2.2) and again using (1.10), we obtain

$$x(t) = x(t_1^+) e^{\frac{\rho_1-1}{\rho_1}(\psi_1(t)-\psi_1(t_1))} + {}^{\rho_1}\mathcal{I}_{t_1}^{\alpha_1,\psi_1} F_x(t).$$

From an impulsive condition, $x(t_1^+) = x(t_1^-) + J_1(x(t_1))$, we get

$$\begin{aligned} x(t) &= \Big[x(t_1^-) + J_1(x(t_1))\Big]e^{\frac{\rho_1-1}{\rho_1}(\psi_1(t)-\psi_1(t_1))} + {}^{\rho_1}\mathcal{I}_{t_1}^{\alpha_1,\psi_1}F_x(t) \\ &= {}^{\rho_1}\mathcal{I}_{t_1}^{\alpha_1,\psi_1}F_x(t) + \Big\{c_0 e^{\frac{\rho_0-1}{\rho_0}(\psi_0(t_1)-\psi_0(t_0))} + \Big[{}^{\rho_0}\mathcal{I}_{t_0}^{\alpha_0,\psi_0}F_x(t_1) + J_1(x(t_1))\Big]\Big\}e^{\frac{\rho_1-1}{\rho_1}(\psi_1(t)-\psi_1(t_1))}. \end{aligned}$$

For $t \in \mathcal{J}_2 = (t_2, t_3]$, by using the operator ${}^{\rho_2}\mathcal{I}_{t_2}^{\alpha_2,\psi_2}$ to both sides of (2.2), we have

$$x(t) = {}^{\rho_2}\mathcal{I}_{t_2}^{\alpha_2,\psi_2}F_x(t) + x(t_2^+)e^{\frac{\rho_2-1}{\rho_2}(\psi_2(t)-\psi_2(t_2))}.$$

In view of the impulsive condition $x(t_2^+) = x(t_2^-) + J_2(x(t_2))$, we obtain

$$\begin{aligned} x(t) &= x(t_2^+)e^{\frac{\rho_2-1}{\rho_2}(\psi_2(t)-\psi_2(t_2))} + {}^{\rho_2}\mathcal{I}_{t_2}^{\alpha_2,\psi_2}F_x(t) \\ &= {}^{\rho_2}\mathcal{I}_{t_2}^{\alpha_2,\psi_2}F_x(t) + \Big\{c_0\, e^{\frac{\rho_0-1}{\rho_0}(\psi_0(t_1)-\psi_0(t_0))}e^{\frac{\rho_1-1}{\rho_1}(\psi_1(t_2)-\psi_1(t_1))} \\ &\quad + \Big[{}^{\rho_0}\mathcal{I}_{t_0}^{\alpha_0,\psi_0}F_x(t_1) + J_1(x(t_1))\Big]e^{\frac{\rho_1-1}{\rho_1}(\psi_1(t_2)-\psi_1(t_1))} + \Big[{}^{\rho_1}\mathcal{I}_{t_1}^{\alpha_1,\psi_1}F_x(t_2) + J_2(x(t_2))\Big]\Big\}e^{\frac{\rho_2-1}{\rho_2}(\psi_2(t)-\psi_2(t_2))}. \end{aligned}$$

By a similar ways repeating the same process, for $t \in \mathcal{J}_k = (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$, we have

$$\begin{aligned} x(t) &= {}^{\rho_k}\mathcal{I}_{t_k}^{\alpha_k,\psi_k}F_x(t) + e^{\frac{\rho_k-1}{\rho_k}(\psi_k(t)-\psi_k(t_k))}\Big\{c_0\prod_{i=1}^{k} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))} \\ &\quad + \sum_{i=1}^{k}\Big(\Big({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}F_x(t_i) + J_i(x(t_i))\Big)\prod_{j=i}^{k-1} e^{\frac{\rho_j-1}{\rho_j}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\Big)\Big\}. \end{aligned} \tag{2.5}$$

Applying the conditions $\eta x(0) + \beta x(T) = \sum_{i=0}^{m}\delta_i{}^{\rho_i}\mathcal{I}_{t_i}^{\gamma_i,\psi_i}x(\xi_i)$ with the symbol (2.1), we obtain

$$\begin{aligned} \eta x(0) + \beta x(T) &= \eta c_0 + \beta{}^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}F_x(T) + c_0\beta\prod_{i=1}^{m+1} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(t_{i-1},t_i)} \\ &\quad + \beta e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\sum_{i=1}^{m}\Big(\Big({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}F_x(t_i) + J_i(x(t_i))\Big)\prod_{j=i}^{m-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\Big), \end{aligned} \tag{2.6}$$

$$\begin{aligned} \sum_{i=0}^{m}\delta_i{}^{\rho_i}\mathcal{I}_{t_i}^{\gamma_i,\psi_i}x(\xi_i) &= \sum_{i=0}^{m}\delta_i{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}F_x(\xi_i) + \sum_{i=0}^{m}\frac{\delta_i\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\Big\{c_0\prod_{j=1}^{i} e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\Psi_{j-1}(t_{j-1},t_j)} \\ &\quad + \sum_{j=1}^{i}\Big(\Big({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}F_x(t_j) + J_j(x(t_j))\Big)\prod_{l=j}^{i-1} e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})}\Big)\Big\}. \end{aligned} \tag{2.7}$$

By solving (2.6) and (2.7), we get that

$$
\begin{aligned}
c_0 &= \frac{1}{\Lambda}\Bigg[\sum_{i=0}^{m}\delta_i{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}F_x(\xi_i)-\beta^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}F_x(T)\\
&\quad -\beta e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\sum_{i=1}^{m}\left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}F_x(t_i)+J_i(x(t_i))\right)\prod_{j=i}^{m-1}e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\right)\\
&\quad +\sum_{i=0}^{m}\frac{\delta_i\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\sum_{j=1}^{i}\left(\left({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}F_x(t_j)+J_j(x(t_j))\right)\prod_{l=j}^{i-1}e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})}\right)\Bigg].
\end{aligned}
$$

Substituting the value of c_0 in (2.5), yields the solution (2.3).

Conversely, suppose that x satisfies (2.3), taking the Caputo PFD ${}^{C\rho_k}\mathfrak{D}_{t_k}^{\alpha_k,\psi_k}$ into both sides of the Volterra integral equation (2.3) and using Proposition 2.1 with Corollary 2.1, we get that

$$
\begin{aligned}
{}^{C\rho_k}\mathfrak{D}_{t_k}^{\alpha_k,\psi_k}x(t) &= {}^{C\rho_k}\mathfrak{D}_{t_k}^{\alpha_k,\psi_k}\,{}^{\rho_k}\mathcal{I}_{t_k}^{\alpha_k,\psi_k}F_x(t)\\
&\quad +{}^{C\rho_k}\mathfrak{D}_{t_k}^{\alpha_k,\psi_k}e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,t)}\Bigg\{\frac{1}{\Lambda}\prod_{i=1}^{k}e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(t_{i-1},t_i)}\Bigg[\sum_{i=0}^{m}\delta_i{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}F_x(\xi_i)-\beta^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}F_x(T)\\
&\quad -\beta e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\sum_{i=1}^{m}\left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}F_x(t_i)+J_i(x(t_i))\right)\prod_{j=i}^{m-1}e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\right)\\
&\quad +\sum_{i=0}^{m}\frac{\delta_i\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\sum_{j=1}^{i}\left(\left({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}F_x(t_j)+J_j(x(t_j))\right)\prod_{l=j}^{i-1}e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})}\right)\Bigg]\\
&\quad +\sum_{i=1}^{k}\left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}F_x(t_i)+J_i(x(t_i))\right)\prod_{j=i}^{k-1}e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\right)\Bigg\}\\
&= F_x(t), \quad t\in\mathcal{J}_k.
\end{aligned}
$$

Next, we show that x satisfies the boundary conditions. Applying the operator ${}^{\rho_i}\mathcal{I}_{t_i}^{\gamma_i,\psi_i}$ to both sides of (2.3) with (1.11), for $i=0,1,\ldots,m$, we obtain

$$
\begin{aligned}
\sum_{i=0}^{m}\delta_i{}^{\rho_i}\mathcal{I}_{t_i}^{\gamma_i,\psi_i}x(\xi_i) &= \sum_{i=0}^{m}\delta_i{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}F_x(\xi_i)+\sum_{i=0}^{m}\frac{\delta_i e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho^{\gamma_i}\Gamma(\gamma_i+1)}\Bigg\{\frac{1}{\Lambda}\prod_{i=1}^{m}e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(t_{i-1},t_i)}\\
&\quad\times\Bigg[\sum_{i=0}^{m}\delta_i{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}F_x(\xi_i)-\beta^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}F_x(T)\\
&\quad -\beta e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\sum_{i=1}^{m}\left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}F_x(t_i)+J_i(x(t_i))\right)\prod_{j=i}^{m-1}e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\right)\\
&\quad +\sum_{i=0}^{m}\frac{\delta_i\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\sum_{j=1}^{i}\left(\left({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}F_x(t_j)+J_j(x(t_j))\right)\prod_{l=j}^{i-1}e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})}\right)\Bigg]\\
&\quad +\sum_{i=1}^{m}\left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}F_x(t_i)+J_i(x(t_i))\right)\prod_{j=i}^{m-1}e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\right)\Bigg\},
\end{aligned}
$$

$$\begin{aligned}\beta x(T) &= \beta^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}F_x(T)+\beta e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\Bigg\{\frac{1}{\Lambda}\prod_{i=1}^{m}e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(t_{i-1},t_i)}\\
&\quad\times\Bigg[\sum_{i=0}^{m}\delta_i{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}F_x(\xi_i)-\beta^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}F_x(T)\\
&\quad-\beta e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\sum_{i=1}^{m}\left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}F_x(t_i)+J_i(x(t_i))\right)\prod_{j=i}^{m-1}e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\right)\\
&\quad+\sum_{i=0}^{m}\frac{\delta_i\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\sum_{j=1}^{i}\left(\left({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}F_x(t_j)+J_j(x(t_j))\right)\prod_{l=j}^{i-1}e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})}\right)\Bigg]\\
&\quad+\sum_{i=1}^{m}\left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}F_x(t_i)+J_i(x(t_i))\right)\prod_{j=i}^{m-1}e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\right)\Bigg\},\\
\eta x(0) &= \frac{\eta}{\Lambda}\Bigg[\sum_{i=0}^{m}\delta_i{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}F_x(\xi_i)-\beta^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}F_x(T)\\
&\quad-\beta e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\sum_{i=1}^{m}\left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}F_x(t_i)+J_i(x(t_i))\right)\prod_{j=i}^{m-1}e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\right)\\
&\quad+\sum_{i=0}^{m}\frac{\delta_i\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\sum_{j=1}^{i}\left(\left({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}F_x(t_j)+J_j(x(t_j))\right)\prod_{l=j}^{i-1}e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})}\right)\Bigg],\end{aligned}$$

where Λ is given by (2.4). Therefore,

$$\eta x(0)+\beta x(T)=\sum_{i=0}^{m}\delta_i{}^{\rho_i}\mathcal{I}_{t_i}^{\gamma_i,\psi_i}x(\xi_i).$$

The proof is finished. □

3. Existence and uniqueness criterias

In this section, we prove the existence and uniqueness results for the problem (1.12) via Banach's and Schaefer's fixed point theorems. Firstly, we convert the problem (1.12) into a fixed point equation $x=\mathcal{Q}x$, we define an operator $\mathcal{Q}:\mathbb{PC}(\mathcal{J},\mathbb{R})\to\mathbb{PC}(\mathcal{J},\mathbb{R})$ according to Lemma 2.1 as follow:

$$\begin{aligned}(\mathcal{Q}x)(t) &= {}^{\rho_k}\mathcal{I}_{t_k}^{\alpha_k,\psi_k}F_x(t)+e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,t)}\Bigg\{\frac{1}{\Lambda}\prod_{i=1}^{k}e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(t_{i-1},t_i)}\Bigg[\sum_{i=0}^{m}\delta_i{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}F_x(\xi_i)\\
&\quad-\beta^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}F_x(T)-\beta e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\sum_{i=1}^{m}\left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}F_x(t_i)+J_i(x(t_i))\right)\prod_{j=i}^{m-1}e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\right)\\
&\quad+\sum_{i=0}^{m}\frac{\delta_i\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\sum_{j=1}^{i}\left(\left({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}F_x(t_j)+J_j(x(t_j))\right)\prod_{l=j}^{i-1}e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})}\right)\Bigg]\\
&\quad+\sum_{i=1}^{k}\left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}F_x(t_i)+J_i(x(t_i))\right)\prod_{j=i}^{k-1}e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\right)\Bigg\}.\end{aligned}\tag{3.1}$$

Clearly, the problem (1.12) has a solution if and only if the operator Q has fixed points. For the sake of convenience, we assume the following notations of constants:

$$\begin{aligned}\Omega_1 &= \left(1+\frac{|\beta|}{|\Lambda|}\right)\sum_{i=1}^{m+1}\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+\frac{1}{|\Lambda|}\left(\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\alpha_i+\gamma_i}(t_i,\xi_i)}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i+1)}\right.\\ &\quad\left.+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\frac{\Psi_{j-1}^{\alpha_{j-1}}(t_{j-1},t_j)}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1}+1)}\right), \qquad (3.2)\\ \Omega_2 &= m\left(1+\frac{|\beta|}{|\Lambda|}\right)+\frac{1}{|\Lambda|}\sum_{i=0}^{m}\frac{i|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}. \qquad (3.3)\end{aligned}$$

3.1. Uniqueness criteria

In the forthcoming first theorem, we will prove the uniqueness of solution for the problem (1.12) by applying Banach's fixed point theorem.

Lemma 3.1. *(Banach's fixed point theorem [59]) Let D be a non-empty closed subset of a Banach space E. Then any contraction mapping Q from D into itself has a unique fixed point.*

Theorem 3.1. *Assume that $\psi_k \in C(\mathcal{J},\mathbb{R})$ with $\psi_k'(t) > 0$ for $t \in \mathcal{J}$, $k = 0, 1, 2, \ldots, m$, $f : \mathcal{J}\times\mathbb{R}^3 \to \mathbb{R}$ and $\varphi_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, m$ are continuous functions, which satisfy the following assumptions:*

($\mathcal{A}_1$) *There exist constants $\mathbb{L}_1 > 0$ and $0 < \mathbb{L}_2 < 1$ such that, for every $t \in \mathcal{J}$ and $x_i, y_i, z_i \in \mathbb{R}$, $i = 1, 2$,*

$$|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)| \le \mathbb{L}_1\left(|x_1-x_2|+|y_1-y_2|\right)+\mathbb{L}_2|z_1-z_2|.$$

($\mathcal{A}_2$) *There exists a constant $\mathbb{M}_1 > 0$, for any $x, y \in \mathbb{R}$, such that*

$$|J_k(x)-J_k(y)| \le \mathbb{M}_1|x-y|, \quad k=1,2,\ldots,m.$$

Then the problem (1.12) *has a unique solution on $\mathcal{J}$ provided that*

$$\frac{2\mathbb{L}_1\Omega_1}{1-\mathbb{L}_2}+\mathbb{M}_1\Omega_2 < 1. \qquad (3.4)$$

Proof. Suppose that $\mathbb{K}_1$ and $\mathbb{K}_2$ are nonnegative constants such that $\mathbb{K}_1 = \sup_{t\in\mathcal{J}}|F_0(t)| < +\infty$, where $F_0(t) = f(t, 0, 0, 0)$ and $\mathbb{K}_2 = \max\{J_k(0) : k = 1, 2, \ldots, m\}$. Define a bounded, closed and convex subset B_{r_1} of $\mathbb{PC}(\mathcal{J},\mathbb{R})$, where $B_{r_1} = \{x \in \mathbb{PC}(\mathcal{J},\mathbb{R}) : \|x\| \le r_1\}$, r_1 is chosen such that

$$r_1 \ge \frac{\frac{\mathbb{K}_1\Omega_1}{1-\mathbb{L}_2}+\mathbb{K}_2\Omega_2}{1-\left(\frac{2\mathbb{L}_1\Omega_1}{1-\mathbb{L}_2}+\mathbb{M}_1\Omega_2\right)}.$$

We split the proof into two steps:
Step I. We show that $QB_{r_1} \subset B_{r_1}$.
For any $x \in B_{r_1}$, we have

$$\begin{aligned}
|(Qx)(t)| \leq\ & {}^{\rho_k}\mathcal{I}_{t_k}^{\alpha_k,\psi_k}|F_x(t)| + e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,t)}\Bigg\{\frac{1}{|\Lambda|}\prod_{i=1}^{k} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(t_{i-1},t_i)}\Bigg[\sum_{i=0}^{m}|\delta_i|^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}|F_x(\xi_i)| \\
& +|\beta|^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}|F_x(T)| + |\beta|e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\sum_{i=1}^{m}\Bigg(\Big({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|F_x(t_i)| + |J_i(x(t_i))|\Big)\prod_{j=i}^{m-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\Bigg) \\
& + \sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\sum_{j=1}^{i}\Bigg(\Big({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}|F_x(t_j)| + |J_j(x(t_j))|\Big)\prod_{l=j}^{i-1} e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})}\Bigg)\Bigg] \\
& + \sum_{i=1}^{k}\Bigg(\Big({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|F_x(t_i)| + |J_i(x(t_i))|\Big)\prod_{j=i}^{k-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\Bigg)\Bigg\}. \qquad (3.5)
\end{aligned}$$

By using $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, we have

$$\begin{aligned}
|F_x(t)| &\leq |F_x(t) - F_0(t)| + |F_0(t)| \\
&\leq |f(t,x(t),x(\lambda t),F_x(t)) - f(t,0,0,0)| + |f(t,0,0,0)| \\
&\leq \mathbb{L}_1(|x(t)| + |x(\lambda t)|) + \mathbb{L}_2|F_x(t)| + \mathbb{K}_1 \\
&\leq \frac{2\mathbb{L}_1 r_1 + \mathbb{K}_1}{1-\mathbb{L}_2}, \qquad (3.6) \\
|J_k(x)| &\leq |J_k(x) - J_k(0)| + |J_k(0)| \leq \mathbb{M}_1 r_1 + \mathbb{K}_2, \quad k = 1,2,\ldots,m. \qquad (3.7)
\end{aligned}$$

Then substituting (3.6) and (3.7) into (3.5) with using (1.7), one has

$$\begin{aligned}
|(Qx)(t)| \leq\ & \frac{2\mathbb{L}_1 r_1 + \mathbb{K}_1}{1-\mathbb{L}_2}{}^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}(1)(T) + e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\Bigg\{\frac{1}{|\Lambda|}\prod_{i=1}^{m} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(t_{i-1},t_i)} \\
& \times\Bigg[\frac{2\mathbb{L}_1 r_1 + \mathbb{K}_1}{1-\mathbb{L}_2}\sum_{i=0}^{m}|\delta_i|^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}(1)(\xi_i) + \frac{2\mathbb{L}_1 r_1 + \mathbb{K}_1}{1-\mathbb{L}_2}|\beta|^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}(1)(T) \\
& +|\beta|e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\sum_{i=1}^{m}\Bigg(\Bigg(\frac{2\mathbb{L}_1 r_1 + \mathbb{K}_1}{1-\mathbb{L}_2}{}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}(1)(t_i) + \mathbb{M}_1 r_1 + \mathbb{K}_2\Bigg)\prod_{j=i}^{m-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\Bigg) \\
& + \sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\sum_{j=1}^{i}\Bigg(\Bigg(\frac{2\mathbb{L}_1 r_1 + \mathbb{K}_1}{1-\mathbb{L}_2}{}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}(1)(t_j) + \mathbb{M}_1 r_1 + \mathbb{K}_2\Bigg) \\
& \times\prod_{l=j}^{i-1} e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})}\Bigg)\Bigg] + \sum_{i=1}^{k}\Bigg(\Bigg(\frac{2\mathbb{L}_1 r_1 + \mathbb{K}_1}{1-\mathbb{L}_2}{}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}(1)(t_i) + \mathbb{M}_1 r_1 + \mathbb{K}_2\Bigg)\prod_{j=i}^{k-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\Bigg)\Bigg\} \\
\leq\ & \frac{2\mathbb{L}_1 r_1 + \mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{1}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T} e^{\frac{\rho_m-1}{\rho_m}\Psi_m(s,T)}\Psi_m^{\alpha_m-1}(s,T)\psi_m'(s)ds \\
& +e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\Bigg\{\frac{1}{|\Lambda|}\prod_{i=1}^{m} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(t_{i-1},t_i)}\Bigg[\frac{2\mathbb{L}_1 r_1 + \mathbb{K}_1}{1-\mathbb{L}_2}\sum_{i=0}^{m}\frac{|\delta_i|}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i)} \\
& \times\int_{t_i}^{\xi_i} e^{\frac{\rho_i-1}{\rho_i}\Psi_i(s,\xi_i)}\Psi_i^{\alpha_i+\gamma_i-1}(s,\xi_i)\psi_i'(s)ds + \frac{2\mathbb{L}_1 r_1 + \mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{|\beta|}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}
\end{aligned}$$

$$\times \int_{t_m}^{T} e^{\frac{\rho_m-1}{\rho_m}\Psi_m(s,T)}\Psi_m^{\alpha_m-1}(s,T)\psi_m'(s)ds + |\beta| e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)} \sum_{i=1}^{m}\Bigg(\bigg(\frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}$$

$$\times \int_{t_{i-1}}^{t_i} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(s,t_i)}\Psi_{i-1}^{\alpha_{i-1}-1}(s,t_i)\psi_{i-1}'(s)ds + \mathbb{M}_1 r_1+\mathbb{K}_2\bigg)\prod_{j=i}^{m-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\Bigg)$$

$$+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)} e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\sum_{j=1}^{i}\Bigg(\bigg(\frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{1}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1})}$$

$$\times \int_{t_{j-1}}^{t_j} e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\Psi_{j-1}(s,t_j)}\Psi_{j-1}^{\alpha_{j-1}-1}(s,t_j)\psi_{j-1}'(s)ds + \mathbb{M}_1 r_1+\mathbb{K}_2\bigg)\prod_{l=j}^{i-1} e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})}\Bigg)\Bigg]$$

$$+\sum_{i=1}^{k}\Bigg(\bigg(\frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(s,t_i)}\Psi_{i-1}^{\alpha_{i-1}-1}(s,t_i)\psi_{i-1}'(s)ds$$

$$+\mathbb{M}_1 r_1+\mathbb{K}_2\bigg)\prod_{j=i}^{k-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\Bigg)\Bigg\}.$$

By using $0 < e^{\frac{\rho_l-1}{\rho_l}\Psi_l(s,u)} \le 1$ for $0 \le s \le u \le T$, $l = 0, 1, \ldots, m$, we get

$$\begin{aligned}
|(Qx)(t)| \le\ & \frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{1}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\Psi_m^{\alpha_m-1}(s,T)\psi_m'(s)ds \\
&+\frac{1}{|\Lambda|}\Bigg[\frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\sum_{i=0}^{m}\frac{|\delta_i|}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i)}\int_{t_i}^{\xi_i}\Psi_i^{\alpha_i+\gamma_i-1}(s,\xi_i)\psi_i'(s)ds \\
&+\frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{|\beta|}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\Psi_m^{\alpha_m-1}(s,T)\psi_m'(s)ds \\
&+|\beta|\sum_{i=1}^{m}\bigg(\frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\Psi_{i-1}^{\alpha_{i-1}-1}(s,t_i)\psi_{i-1}'(s)ds+\mathbb{M}_1 r_1+\mathbb{K}_2\bigg) \\
&+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\bigg(\frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{1}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j}\Psi_{j-1}^{\alpha_{j-1}-1}(s,t_j)\psi_{j-1}'(s)ds+\mathbb{M}_1 r_1+\mathbb{K}_2\bigg)\Bigg] \\
&+\sum_{i=1}^{m}\bigg(\frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\Psi_{i-1}^{\alpha_{i-1}-1}(s,t_i)\psi_{i-1}'(s)ds+\mathbb{M}_1 r_1+\mathbb{K}_2\bigg) \\
=\ & \frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{\Psi_m^{\alpha_m}(t_m,T)}{\rho_m^{\alpha_m}\Gamma(\alpha_m+1)}+\frac{1}{|\Lambda|}\Bigg[\frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\alpha_i+\gamma_i}(t_i,\xi_i)}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i+1)} \\
&+\frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{|\beta|\Psi_m^{\alpha_m}(t_m,T)}{\rho_m^{\alpha_m}\Gamma(\alpha_m+1)}+|\beta|\sum_{i=1}^{m}\bigg(\frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+\mathbb{M}_1 r_1+\mathbb{K}_2\bigg) \\
&+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\bigg(\frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{\Psi_{j-1}^{\alpha_{j-1}}(t_{j-1},t_j)}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1}+1)}+\mathbb{M}_1 r_1+\mathbb{K}_2\bigg)\Bigg] \\
&+\sum_{i=1}^{m}\bigg(\frac{2\mathbb{L}_1 r_1+\mathbb{K}_1}{1-\mathbb{L}_2}\cdot\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+\mathbb{M}_1 r_1+\mathbb{K}_2\bigg)
\end{aligned}$$

$$
\begin{aligned}
&= r_1\left\{\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\left[\left(1+\frac{|\beta|}{|\Lambda|}\right)\sum_{i=1}^{m+1}\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+\frac{1}{|\Lambda|}\left(\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\alpha_i+\gamma_i}(t_i,\xi_i)}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i+1)}\right.\right.\right.\\
&\quad\left.\left.+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\frac{\Psi_{j-1}^{\alpha_{j-1}}(t_{j-1},t_j)}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1}+1)}\right)\right]+\mathbb{M}_1\left[m\left(1+\frac{|\beta|}{|\Lambda|}\right)+\frac{1}{|\Lambda|}\sum_{i=0}^{m}\frac{i|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\right]\Bigg\}\\
&\quad+\frac{\mathbb{K}_1}{1-\mathbb{L}_2}\left[\left(1+\frac{|\beta|}{|\Lambda|}\right)\sum_{i=1}^{m+1}\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+\frac{1}{|\Lambda|}\left(\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\alpha_i+\gamma_i}(t_i,\xi_i)}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i+1)}\right.\right.\\
&\quad\left.\left.+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\frac{\Psi_{j-1}^{\alpha_{j-1}}(t_{j-1},t_j)}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1}+1)}\right)\right]+\mathbb{K}_2\left[m\left(1+\frac{|\beta|}{|\Lambda|}\right)+\frac{1}{|\Lambda|}\sum_{i=0}^{m}\frac{i|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\right]\\
&= r_1\left\{\frac{2\mathbb{L}_1\Omega_1}{1-\mathbb{L}_2}+\mathbb{M}_1\Omega_2\right\}+\frac{\mathbb{K}_1\Omega_1}{1-\mathbb{L}_2}+\mathbb{K}_2\Omega_2\le r_1,
\end{aligned}
$$

which implies that $QB_{r_1}\subset B_{r_1}$.

Step II. We prove that Q is a contraction.

Let $x, y\in B_{r_1}$. Then, for each $t\in\mathcal{J}$, we consider

$$
\begin{aligned}
|(Qx)(t)-(Qy)(t)| &\le {}^{\rho_k}\mathcal{I}_{t_k}^{\alpha_k,\psi_k}|F_x(t)-F_y(t)|+e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,t)}\left\{\frac{1}{|\Lambda|}\prod_{i=1}^{k}e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(t_{i-1},t_i)}\right.\\
&\quad\times\left[\sum_{i=0}^{m}|\delta_i|{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}|F_x(\xi_i)-F_y(\xi_i)|+|\beta|{}^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}|F_x(T)-F_y(T)|\right.\\
&\quad+|\beta|e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\sum_{i=1}^{m}\left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|F_x(t_i)-F_y(t_i)|+|J_i(x(t_i))-J_i(y(t_i))|\right)\right.\\
&\quad\left.\times\prod_{j=i}^{m-1}e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\right)+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\sum_{j=1}^{i}\left(\left({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}|F_x(t_j)-F_y(t_j)|\right.\right.\\
&\quad\left.\left.\left.+|J_j(x(t_j))-J_j(y(t_j))|\right)\prod_{l=j}^{i-1}e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})}\right)\right]+\sum_{i=1}^{k}\left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|F_x(t_i)-F_y(t_i)|\right|\right.\\
&\quad\left.\left.+|J_i(x(t_i))-J_i(y(t_i))|\right)\prod_{j=i}^{k-1}e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\right)\Bigg\}. \qquad (3.8)
\end{aligned}
$$

From $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ with the fact of $0<e^{\frac{\rho_l-1}{\rho_l}\Psi_l(s,u)}\le 1$ for $0\le s\le u\le T$, $l=0,1,\ldots,m$, we compute (3.8) as follow:

$$
\begin{aligned}
&|(Qx)(t)-(Qy)(t)|\\
&\le {}^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}|F_x(t)-F_y(t)|+\frac{1}{|\Lambda|}\left[\sum_{i=0}^{m}|\delta_i|{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}|F_x(\xi_i)-F_y(\xi_i)|+|\beta|{}^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}|F_x(T)-F_y(T)|\right.\\
&\quad+|\beta|\sum_{i=1}^{m}\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|F_x(t_i)-F_y(t_i)|+|J_i(x(t_i))-J_i(y(t_i))|\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\left({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}|F_x(t_j)-F_y(t_j)|+|J_j(x(t_j))-J_j(y(t_j))|\right)\Bigg]\\
&+\sum_{i=1}^{k}\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|F_x(t_i)-F_y(t_i)||+|J_i(x(t_i))-J_i(y(t_i))|\right)\\
\le\ &\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{1}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}e^{\frac{\rho_m-1}{\rho_m}\Psi_m(s,T)}\Psi_m^{\alpha_m-1}(s,T)\psi_m'(s)ds\|x-y\|\\
&+\frac{1}{|\Lambda|}\Bigg[\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\sum_{i=0}^{m}\frac{|\delta_i|}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i)}\int_{t_i}^{\xi_i}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(s,\xi_i)}\Psi_i^{\alpha_i+\gamma_i-1}(s,\xi_i)\psi_i'(s)ds\|x-y\|\\
&+\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{|\beta|}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}e^{\frac{\rho_m-1}{\rho_m}\Psi_m(s,T)}\Psi_m^{\alpha_m-1}(s,T)\psi_m'(s)ds\|x-y\|\\
&+|\beta|\sum_{i=1}^{m}\left(\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(s,t_i)}\Psi_{i-1}^{\alpha_{i-1}-1}(s,t_i)\psi_{i-1}'(s)ds\|x-y\|+\mathbb{M}_1\|x-y\|\right)\\
&+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\left(\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{1}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j}e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\Psi_{j-1}(s,t_j)}\Psi_{j-1}^{\alpha_{j-1}-1}(s,t_j)\psi_{j-1}'(s)ds\|x-y\|\right.\\
&\left.+\mathbb{M}_1\|x-y\|\right)\Bigg]+\sum_{i=1}^{m}\left(\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(s,t_i)}\Psi_{i-1}^{\alpha_{i-1}-1}(s,t_i)\psi_{i-1}'(s)ds\|x-y\|\right.\\
&\left.+\mathbb{M}_1\|x-y\|\right)\\
\le\ &\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{\Psi_m^{\alpha_m}(t_m,T)}{\rho_m^{\alpha_m}\Gamma(\alpha_m+1)}\|x-y\|+\frac{1}{|\Lambda|}\Bigg[\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\alpha_i+\gamma_i}(t_i,\xi_i)}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i+1)}\|x-y\|\\
&+\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{|\beta|\Psi_m^{\alpha_m}(t_m,T)}{\rho_m^{\alpha_m}\Gamma(\alpha_m+1)}\|x-y\|+|\beta|\sum_{i=1}^{m}\left(\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}\|x-y\|+\mathbb{M}_1\|x-y\|\right)\\
&+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\left(\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{\Psi_{j-1}^{\alpha_{j-1}}(t_{j-1},t_j)}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1}+1)}\|x-y\|+\mathbb{M}_1\|x-y\|\right)\Bigg]\\
&+\sum_{i=1}^{m}\left(\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}\|x-y\|+\mathbb{M}_1\|x-y\|\right)\\
=\ &\left(\frac{2\mathbb{L}_1\Omega_1}{1-\mathbb{L}_2}+\mathbb{M}_1\Omega_2\right)\|x-y\|,
\end{aligned}
$$

which implies that

$$
\|\mathcal{Q}x-\mathcal{Q}y\|\le\left(\frac{2\mathbb{L}_1\Omega_1}{1-\mathbb{L}_2}+\mathbb{M}_1\Omega_2\right)\|x-y\|.
$$

Since $[2\mathbb{L}_1\Omega_1/(1-\mathbb{L}_2)+\mathbb{M}_1\Omega_2]<1$, by the conclusion of Banach's fixed point theorem (Lemma 3.1), $\mathcal{Q}$ is a contraction. Hence, $\mathcal{Q}$ has a unique fixed point that is the unique solution of the problem (1.12) on $\mathcal{J}$. The proof is done. □

3.2. Existence criteria

The next result is based on the Schaefer's fixed point theorem.

Lemma 3.2. *(Schaefer's fixed point theorem [59]) Let E be a Banach space and $T : E \to E$ be a completely continuous operator. If the set $D = \{x \in E : x = \sigma T x, 0 < \sigma < 1\}$ is bounded, then T has a fixed point in E.*

Theorem 3.2. *Let $\psi_k \in C(\mathcal{J}, \mathbb{R})$ with $\psi_k'(t) > 0$ for $t \in \mathcal{J}$, $k = 0, 1, 2, \ldots, m$. Assume that $f : \mathcal{J} \times \mathbb{R}^3 \to \mathbb{R}$ and $J_k : \mathbb{R} \to \mathbb{R}$ are continuous functions, $k = 1, 2, \ldots, m$ satisfying the following assumptions:*

($\mathcal{A}_3$) *There exist nonnegative continuous functions q_1, q_2, $q_3 \in C(\mathcal{J}, \mathbb{R}^+)$ such that, for every $t \in \mathcal{J}$ and x, y, $z \in \mathbb{R}$,*

$$|f(t, x, y, z)| \leq g_1(t) + g_2(t)(|x| + |y|) + g_3(t)|z|,$$

with $g_1^* = \sup_{t\in\mathcal{J}}\{g_1(t)\}$, $g_2^* = \sup_{t\in\mathcal{J}}\{g_2(t)\}$ and $g_3^* = \sup_{t\in\mathcal{J}}\{g_3(t)\} < 1$.

($\mathcal{A}_4$) *There exist positive constants $\mathbb{N}_1$, $\mathbb{N}_2$ for any $x \in \mathbb{R}$, such that*

$$|J_k(x)| \leq \mathbb{N}_1|x| + \mathbb{N}_2, \quad k = 1, 2, \ldots, m.$$

Then the problem (1.12) *has at least one solution on $\mathcal{J}$.*

Proof. We will utilize Schaefer's fixed point theorem to show that the operator Q defined as in (3.1) has at least one fixed point. The procedure of the proof is divided into the following four steps.

Step I. We show that Q is continuous.

Let $x_n \in \mathbb{PC}(\mathcal{J}, \mathbb{R})$ be a sequence such that $x_n \to x \in \mathbb{PC}(\mathcal{J}, \mathbb{R})$. Then, for every $t \in \mathcal{J}$, we obtain

$$\begin{aligned}
&|(Qx_n)(t) - (Qx)(t)| \\
\leq\ & {}^{\rho_k}\mathcal{I}_{t_k}^{\alpha_k,\psi_k}|F_{x_n}(t) - F_x(t)| + e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,t)}\Bigg\{\frac{1}{|\Lambda|}\prod_{i=1}^{k} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(t_{i-1},t_i)} \\
&\times\Bigg[\sum_{i=0}^{m}|\delta_i|^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}|F_{x_n}(\xi_i) - F_x(\xi_i)| + |\beta|^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}|F_{x_n}(T) - F_x(T)| \\
&+|\beta|e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\sum_{i=1}^{m}\left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|F_{x_n}(t_i) - F_x(t_i)| + |J_i(x_n(t_i)) - J_i(x(t_i))|\right)\prod_{j=i}^{m-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\right) \\
&+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\sum_{j=1}^{i}\left(\left({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}|F_{x_n}(t_j) - F_x(t_j)| + |J_j(x_n(t_j)) - J_j(x(t_j))|\right)\right. \\
&\left.\times\prod_{l=j}^{i-1} e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})}\right)\Bigg] + \sum_{i=1}^{k}\left(\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|F_{x_n}(t_i) - F_x(t_i)| + |J_i(x_n(t_i)) - J_i(x(t_i))|\right)\prod_{j=i}^{k-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\right)\Bigg\}.
\end{aligned}$$

By using $0 < e^{\frac{\rho_l-1}{\rho_l}\Psi_l(s,u)} \leq 1$ for $0 \leq s \leq u \leq T$, $l = 0, 1, \ldots, m$, we have

$$\begin{aligned}
&|(Qx_n)(t) - (Qx)(t)| \\
\leq\ & \frac{1}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T} e^{\frac{\rho_m-1}{\rho_m}\Psi_m(s,T)}\Psi_m^{\alpha_m-1}(s,T)|F_{x_n}(s) - F_x(s)|\psi_m'(s)ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\Lambda|}\Bigg[\sum_{i=0}^{m}\frac{|\delta_i|}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i)}\int_{t_i}^{\xi_i}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(s,\xi_i)}\Psi_i^{\alpha_i+\gamma_i-1}(s,\xi_i)|F_{x_n}(s)-F_x(s)|\psi_i'(s)ds\\
&+\frac{|\beta|}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}e^{\frac{\rho_m-1}{\rho_m}\Psi_m(s,T)}\Psi_m^{\alpha_m-1}(s,T)|F_{x_n}(s)-F_x(s)|\psi_m'(s)ds\\
&+|\beta|\sum_{i=1}^{m}\Bigg(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(s,t_i)}\Psi_{i-1}^{\alpha_{i-1}-1}(s,t_i)|F_{x_n}(s)-F_x(s)|\psi_{i-1}'(s)ds\\
&+|J_i(x_n(t_i))-J_i(x(t_i))|\Bigg)+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\Bigg(\frac{1}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j}e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\Psi_{j-1}(s,t_j)}\\
&\times\Psi_{j-1}^{\alpha_{j-1}-1}(s,t_j)|F_{x_n}(s)-F_x(s)|\psi_{j-1}'(s)ds+|J_j(x_n(t_j))-J_j(x(t_j))|\Bigg)\Bigg]\\
&+\sum_{i=1}^{m}\Bigg(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(s,t_i)}\Psi_{i-1}^{\alpha_{i-1}-1}(s,t_i)|F_{x_n}(s)-F_x(s)|\psi_{i-1}'(s)ds+|J_i(x_n(t_i))-J_i(x(t_i))|\Bigg)\\
\le\;&\frac{\|F_{x_n}-F_x\|}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\Psi_m^{\alpha_m-1}(s,T)\psi_m'(s)ds+\frac{1}{|\Lambda|}\Bigg[\sum_{i=0}^{m}\frac{|\delta_i|\|F_{x_n}-F_x\|}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i)}\int_{t_i}^{\xi_i}\Psi_i^{\alpha_i+\gamma_i-1}(s,\xi_i)\psi_i'(s)ds\\
&+\frac{|\beta|\|F_{x_n}-F_x\|}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\Psi_m^{\alpha_m-1}(s,T)\psi_m'(s)ds+|\beta|\sum_{i=1}^{m}\Bigg(\frac{\|F_{x_n}-F_x\|}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\Psi_{i-1}^{\alpha_{i-1}-1}(s,t_i)\psi_{i-1}'(s)ds\\
&+\mathbb{M}_1\|x_n-x\|\Bigg)+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\Bigg(\frac{\|F_{x_n}-F_x\|}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j}\Psi_{j-1}^{\alpha_{j-1}-1}(s,t_j)\psi_{j-1}'(s)ds+\mathbb{M}_1\|x_n-x\|\Bigg)\Bigg]\\
&+\sum_{i=1}^{m}\Bigg(\frac{\|F_{x_n}-F_x\|}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\Psi_{i-1}^{\alpha_{i-1}-1}(s,t_i)\psi_{i-1}'(s)ds+\mathbb{M}_1\|x_n-x\|\Bigg)\\
=\;&\Bigg[\Bigg(1+\frac{|\beta|}{|\Lambda|}\Bigg)\sum_{i=1}^{m+1}\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+\frac{1}{|\Lambda|}\Bigg(\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\alpha_i+\gamma_i}(t_i,\xi_i)}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i+1)}+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\\
&\times\sum_{j=1}^{i}\frac{\Psi_{j-1}^{\alpha_{j-1}}(t_{j-1},t_j)}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1}+1)}\Bigg)\Bigg]\|F_{x_n}-F_x\|+\Bigg[m\Bigg(1+\frac{|\beta|}{|\Lambda|}\Bigg)+\frac{1}{|\Lambda|}\sum_{i=0}^{m}\frac{i|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\Bigg]\mathbb{M}_1\|x_n-x\|\\
=\;&\Omega_1\|F_{x_n}-F_x\|+\Omega_2\mathbb{M}_1\|x_n-x\|.
\end{aligned}
$$

By using the continuity of f, we obtain that $\|F_{x_n}-F_x\|\to 0$ and $\|x_n-x\|\to 0$, as $n\to\infty$. Hence, $\|Qx_n-Qx\|\to 0$, which yields that Q is also continuous.

Step II. We show that Q maps a bounded set into a bounded set in $\mathbb{PC}(\mathcal{J},\mathbb{R})$.

Define a ball $B_{r_2}=\{x\in\mathbb{PC}(\mathcal{J},\mathbb{R}):\|x\|\le r_2\}$. From $(\mathcal{A}_3)$ and $(\mathcal{A}_4)$, we have

$$
\begin{aligned}
|F_x(t)| &\le |f(t,x(t),x(\lambda t),F_x(t))|\le g_1(t)+g_2(x(t)+x(\lambda t))+g_3(t)|F_x(t)|\le\frac{g_1^*+2g_2^*r_2}{1-g_3^*}, && (3.9)\\
|J_k(x)| &\le \mathbb{N}_1r_2+\mathbb{N}_2. && (3.10)
\end{aligned}
$$

Then, substituting (3.9) and (3.10) into (3.5) in Theorem 3.1 and applying $0<e^{\frac{\rho_l-1}{\rho_l}\Psi_l(s,u)}\le 1$ for

$0 \le s \le u \le T$, $l = 0, 1, \ldots, m$, we obtain

$$\begin{aligned}
|(Qx)(t)| &\le \frac{g_1^* + 2g_2^* r_2}{1 - g_3^*} {}_{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}(1)(T) + \frac{1}{|\Lambda|}\Bigg[\frac{g_1^* + 2g_2^* r_2}{1 - g_3^*} \sum_{i=0}^{m} |\delta_i|^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}(1)(\xi_i) \\
&\quad + \frac{g_1^* + 2g_2^* r_2}{1 - g_3^*}|\beta|^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}(1)(T) + |\beta|\sum_{i=1}^{m}\left(\frac{g_1^* + 2g_2^* r_2}{1 - g_3^*} {}_{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}(1)(t_i) + \mathbb{N}_1 r_2 + \mathbb{N}_2\right) \\
&\quad + \sum_{i=0}^{m} \frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)} \sum_{j=1}^{i}\left(\frac{g_1^* + 2g_2^* r_2}{1 - g_3^*} {}_{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}(1)(t_j) + \mathbb{N}_1 r_2 + \mathbb{N}_2\right)\Bigg] \\
&\quad + \sum_{i=1}^{m}\left(\frac{g_1^* + 2g_2^* r_2}{1 - g_3^*} {}_{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}(1)(t_i) + \mathbb{N}_1 r_2 + \mathbb{N}_2\right) \\
&\le \frac{g_1^* + 2g_2^* r_2}{1 - g_3^*} \cdot \frac{1}{\rho_m^{\alpha_m}\Gamma(\alpha_m)} \int_{t_m}^{T} \Psi_m^{\alpha_m - 1}(s,T)\psi_m'(s)ds + \frac{1}{|\Lambda|}\Bigg[\frac{g_1^* + 2g_2^* r_2}{1 - g_3^*} \sum_{i=0}^{m} \frac{|\delta_i|}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i)} \\
&\quad \times \int_{t_i}^{\xi_i} \Psi_i^{\alpha_i+\gamma_i-1}(s,\xi_i)\psi_i'(s)ds + \frac{g_1^* + 2g_2^* r_2}{1 - g_3^*} \cdot \frac{|\beta|}{\rho_m^{\alpha_m}\Gamma(\alpha_m)} \int_{t_m}^{T} \Psi_m^{\alpha_m-1}(s,T)\psi_m'(s)ds \\
&\quad + |\beta|\sum_{i=1}^{m}\left(\frac{g_1^* + 2g_2^* r_2}{1 - g_3^*} \cdot \frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} \Psi_{i-1}^{\alpha_{i-1}-1}(s,t_i)\psi_{i-1}'(s)ds + \mathbb{N}_1 r_2 + \mathbb{N}_2\right) \\
&\quad + \sum_{i=0}^{m} \frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)} \sum_{j=1}^{i}\left(\frac{g_1^* + 2g_2^* r_2}{1 - g_3^*} \cdot \frac{1}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1})} \int_{t_{j-1}}^{t_j} \Psi_{j-1}^{\alpha_{j-1}-1}(s,t_j)\psi_{j-1}'(s)ds + \mathbb{N}_1 r_2 + \mathbb{N}_2\right)\Bigg] \\
&\quad + \sum_{i=1}^{m}\left(\frac{g_1^* + 2g_2^* r_2}{1 - g_3^*} \cdot \frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} \Psi_{i-1}^{\alpha_{i-1}-1}(s,t_i)\psi_{i-1}'(s)ds + \mathbb{N}_1 r_2 + \mathbb{N}_2\right) \\
&= \Bigg[\left(1 + \frac{|\beta|}{|\Lambda|}\right)\sum_{i=1}^{m+1} \frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)} + \frac{1}{|\Lambda|}\Bigg(\sum_{i=0}^{m} \frac{|\delta_i|\Psi_i^{\alpha_i+\gamma_i}(t_i,\xi_i)}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i+1)} + \sum_{i=0}^{m} \frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)} \\
&\quad \times \sum_{j=1}^{i} \frac{\Psi_{j-1}^{\alpha_{j-1}}(t_{j-1},t_j)}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1}+1)}\Bigg)\Bigg]\frac{g_1^* + 2g_2^* r_2}{1 - g_3^*} + \left[m\left(1 + \frac{|\beta|}{|\Lambda|}\right) + \frac{1}{|\Lambda|}\sum_{i=0}^{m} \frac{i|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\right](\mathbb{N}_1 r_2 + \mathbb{N}_2).
\end{aligned}$$

It follows that

$$\|Qx\| \le \Omega_1 \frac{g_1^* + 2g_2^* r_2}{1 - g_3^*} + \Omega_2(\mathbb{N}_1 r_2 + \mathbb{N}_2) := \mathbb{H}_1,$$

which implies that $\|Qx\| \le \mathbb{H}_1$. Then the set QB_{r_2} is uniformly bounded.

Step III. We show that Q maps bounded sets into equicontinuous sets of $\mathbb{PC}(\mathcal{J}, \mathbb{R})$.

Let $\tau_1, \tau_2 \in J_k$ for some $k \in \{0, 1, 2, \ldots, m\}$ with $\tau_1 < \tau_2$. Then, for any $x \in B_{r_2}$, where B_{r_2} is as defined as in Step II, by using the property of f is bounded on the compact set $J \times B_{r_2}$, we have

$$\begin{aligned}
&|(Qx)(\tau_2) - (Qx)(\tau_1)| \\
&\le \frac{1}{\rho_k^{\alpha_k}\Gamma(\alpha_k)} \int_{\tau_1}^{\tau_2} e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,\tau_2)}\Psi_k^{\alpha_k-1}(s,\tau_2)|F_x(s)|\psi_k'(s)ds \\
&\quad + \frac{1}{\rho_k^{\alpha_k}\Gamma(\alpha_k)} \int_{t_k}^{\tau_1} \left| e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,\tau_2)}\Psi_k^{\alpha_k-1}(s,\tau_2) - e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,\tau_1)}\Psi_k^{\alpha_k-1}(s,\tau_1)\right| |F_x(s)|\psi_k'(s)ds
\end{aligned}$$

$$
\begin{aligned}
&+\left|e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,\tau_2)}-e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,\tau_1)}\right|\Bigg\{\frac{1}{|\Lambda|}\Bigg[\sum_{i=0}^{m}|\delta_i|^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}|F_x(\xi_i)|+|\beta|^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}|F_x(T)| \\
&+|\beta|\sum_{i=1}^{m}\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|F_x(t_i)|+|J_i(x(t_i))|\right)+\sum_{i=0}^{m}\frac{\delta_i\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\left({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}|F_x(t_j)|+|J_j(x(t_j))|\right)\Bigg] \\
&+\sum_{i=1}^{m}\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|F_x(t_i)|+|J_i(x(t_i))|\right)\Bigg\} \\
\leq\ & \frac{1}{\rho_k^{\alpha_k}\Gamma(\alpha_k+1)}\left(2\Psi_k^{\alpha_k}(\tau_1,\tau_2)+\left|\Psi_k^{\alpha_k}(t_k,\tau_2)-\Psi_k^{\alpha_k}(t_k,\tau_1)\right|\right)\frac{g_1^*+2g_2^*r_2}{1-g_3^*} \\
&+\left|e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,\tau_2)}-e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,\tau_1)}\right|\Bigg\{\frac{1}{|\Lambda|}\Bigg[\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\alpha_i+\gamma_i}(t_i,\xi_i)}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i+1)}\cdot\frac{g_1^*+2g_2^*r_2}{1-g_3^*} \\
&+\frac{|\beta|\Psi_m^{\alpha_m}(t_m,T)}{\rho_m^{\alpha_m}\Gamma(\alpha_m+1)}\cdot\frac{g_1^*+2g_2^*r_2}{1-g_3^*}+|\beta|\sum_{i=1}^{m}\left(\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}\cdot\frac{g_1^*+2g_2^*r_2}{1-g_3^*}+\mathbb{N}_1r_2+\mathbb{N}_2\right) \\
&+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\left(\frac{\Psi_{j-1}^{\alpha_{j-1}}(t_{j-1},t_j)}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1}+1)}\cdot\frac{g_1^*+2g_2^*r_2}{1-g_3^*}+\mathbb{N}_1r_2+\mathbb{N}_2\right)\Bigg] \\
&+\sum_{i=1}^{m}\left(\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}\cdot\frac{g_1^*+2g_2^*r_2}{1-g_3^*}+\mathbb{N}_1r_2+\mathbb{N}_2\right)\Bigg\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
|(Qx)(\tau_2)-(Qx)(\tau_1)| \ \leq\ & \frac{1}{\rho_k^{\alpha_k}\Gamma(\alpha_k+1)}\left(2\Psi_k^{\alpha_k}(\tau_1,\tau_2)+\left|\Psi_k^{\alpha_k}(t_k,\tau_2)-\Psi_k^{\alpha_k}(t_k,\tau_1)\right|\right)\frac{g_1^*+2g_2^*r_2}{1-g_3^*} \\
&+\left|e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,\tau_2)}-e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,\tau_1)}\right|\Bigg\{\frac{g_1^*+2g_2^*r_2}{1-g_3^*}\Bigg[\sum_{i=1}^{m}\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)} \\
&+\frac{|\beta|}{|\Lambda|}\sum_{i=1}^{m+1}\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+\frac{1}{|\Lambda|}\Bigg(\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\alpha_i+\gamma_i}(t_i,\xi_i)}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i+1)} \\
&+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\frac{\Psi_{j-1}^{\alpha_{j-1}}(t_{j-1},t_j)}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1}+1)}\Bigg)\Bigg]+\Omega_2\left(\mathbb{N}_1r_2+\mathbb{N}_2\right)\Bigg\}. \qquad (3.11)
\end{aligned}
$$

From (3.11), we get $\Psi_k^{\alpha_k}(\tau_1,\tau_2)\to 0$, $\left|\Psi_k^{\alpha_k}(t_k,\tau_2)-\Psi_k^{\alpha_k}(t_k,\tau_1)\right|\to 0$ and $\left|e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,\tau_2)}-e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,\tau_1)}\right|\to 0$ as $\tau_2\to\tau_1$. This inequality is independent of unknown variable $x\in B_{r_2}$ and tends to zero as $\tau_2\to\tau_1$, which implies that $\|(Qx)(\tau_2)-(Qx)(\tau_1)\|\to 0$ as $\tau_2\to\tau_1$. Hence by the Arzelá-Ascoli theorem, we can conclude that $Q:\mathbb{PC}(\mathcal{J},\mathbb{R})\to\mathbb{PC}(\mathcal{J},\mathbb{R})$ is completely continuous.

Step IV. We show that the set $\mathbb{D}=\{x\in\mathbb{PC}(\mathcal{J},\mathbb{R}):x=\varrho Qx\}$ is bounded (a priori bounds).

Let $x\in\mathbb{D}$, then $x=\varrho Qx$ for some $0<\varrho<1$. From $(\mathcal{A}_3)$ and $(\mathcal{A}_4)$, for each $t\in J$, we obtain the produce by using the similar process in Step II,

$$
\begin{aligned}
|x(t)| &= |\varrho(Qx)(t)| \\
&\leq \left(1+\frac{|\beta|}{|\Lambda|}\right)\sum_{i=1}^{m+1}\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+\frac{1}{|\Lambda|}\left(\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\alpha_i+\gamma_i}(t_i,\xi_i)}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i+1)}+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\right. \\
&\quad \left.\left.\times\sum_{j=1}^{i}\frac{\Psi_{j-1}^{\alpha_{j-1}}(t_{j-1},t_j)}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1}+1)}\right)\right]\frac{g_1^*+2g_2^*r_2}{1-g_3^*}+\left[m\left(1+\frac{|\beta|}{|\Lambda|}\right)+\frac{1}{|\Lambda|}\sum_{i=0}^{m}\frac{i|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\right](\mathbb{N}_1r_2+\mathbb{N}_2).
\end{aligned}
$$

Then, $\|x\| \leq \Omega_1(g_1^*+2g_2^*r_2)/(1-g_3^*)+\Omega_2(\mathbb{N}_1r_2+\mathbb{N}_2) := \mathbb{H}_1 < \infty$. This implies that the set $\mathbb{D}$ is bounded.

From all the assumptions of Theorem 3.2, we summarize that there exists a positive constant $\mathbb{H}_1$ such that $\|x\| \leq \mathbb{H}_1 < \infty$. By applying Schaefer's fixed point theorem (Lemma 3.2), Q has at least one fixed point which is a solution of the problem (1.12). □

4. Ulam's stability

In this section, we examine the different type of Ulam's stability of the problem (1.12).

First of all, we provide Ulam's stability concepts for the problem (1.12).

Definition 4.1. *If for every $\epsilon > 0$ there is a constant $C_f > 0$ such that, for any solution $z \in \mathbb{PC}(\mathcal{J},\mathbb{R})$ of*

$$
\begin{cases}
\left|{}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k}z(t)-f(t,z(t),z(\lambda t),{}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k}z(t))\right| \leq \epsilon, \\
\left|z(t_k^+)-z(t_k^-)-J_k(z(t_k))\right| \leq \epsilon,
\end{cases} \tag{4.1}
$$

there is a unique solution $x \in \mathbb{PC}(\mathcal{J},\mathbb{R})$ of the problem (1.12) *that satisfies*

$$|z(t)-x(t)| \leq C_f\epsilon, \quad t \in J,$$

then the problem (1.12) *is* $\mathbb{UH}$ *stable.*

Definition 4.2. *If for $\epsilon > 0$ and set of positive real numbers $\mathbb{R}^+$ there exists $\phi \in C(\mathbb{R}^+,\mathbb{R}^+)$, with $\phi(0) = 0$ such that, for any solution $z \in \mathbb{PC}(\mathcal{J},\mathbb{R})$ of*

$$
\begin{cases}
\left|{}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k}z(t)-f(t,z(t),z(\lambda t),{}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k}z(t))\right| \leq \phi(t), \\
\left|z(t_k^+)-z(t_k^-)-J_k(z(t_k))\right| \leq \upsilon,
\end{cases} \tag{4.2}
$$

there exist $\epsilon > 0$ and a unique solution $x \in \mathbb{PC}(\mathcal{J},\mathbb{R})$ of the problem (1.12) *that satisfies*

$$|z(t)-x(t)| \leq \phi(\epsilon), \quad t \in J,$$

then the problem (1.12) *is generalized* $\mathbb{UH}$ *stable.*

Definition 4.3. *If for $\epsilon > 0$ there is a real number $C_f > 0$ such that, for any solution $z \in \mathbb{PC}(\mathcal{J},\mathbb{R})$ of*

$$
\begin{cases}
\left|{}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k}z(t)-f(t,z(t),z(\lambda t),{}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k}z(t))\right| \leq \epsilon\phi(t), \\
\left|z(t_k^+)-z(t_k^-)-J_k(z(t_k))\right| \leq \epsilon\upsilon,
\end{cases} \tag{4.3}
$$

there is a unique solution $x \in \mathbb{PC}(\mathcal{J}, \mathbb{R})$ *of the problem* (1.12) *that satisfies*

$$|z(t) - x(t)| \le C_f \epsilon(\upsilon + \phi(t)), \quad t \in J,$$

then the problem (1.12) *is* $\mathbb{UHR}$ *stable with respect to* (υ, ϕ).

Definition 4.4. *If there exists a real number* $C_f > 0$ *such that, for any solution* $z \in \mathbb{PC}(\mathcal{J}, \mathbb{R})$ *of* (4.2), *there is a unique solution* $x \in \mathbb{PC}(\mathcal{J}, \mathbb{R})$ *of the problem* (1.12) *that satisfies*

$$|z(t) - x(t)| \le C_{f,\omega_\phi}(\upsilon + \phi(t)), \quad t \in \mathcal{J},$$

then the problem (1.12) *is generalized* $\mathbb{UHR}$ *stable with respect to* (υ, ϕ).

Remark 4.1. *It is clear that:*
(*i*) *Definition* 4.1 $\Longrightarrow$ *Definition* 4.2*;*
(*ii*) *Definition* 4.3 $\Longrightarrow$ *Definition* 4.4*;*
(*iii*) *Definition* 4.3 *for* $\upsilon + \phi(t) = 1 \Longrightarrow$ *Definition* 4.1.

Remark 4.2. *The function* $z \in \mathbb{PC}(\mathcal{J}, \mathbb{R})$ *is called a solution for* (4.1) *if there exists a function* $w \in \mathbb{PC}(\mathcal{J}, \mathbb{R})$ *together with a sequence* w_k*,* $k = 1, 2, \ldots, m$ *(which depends on* z*) such that*

(a_1) $|w(t)| \le \epsilon$, $|w_k| \le \epsilon$, $\quad t \in \mathcal{J}$;
(a_2) ${}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k} z(t) = f(t, z(t), z(\lambda t), {}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k} z(t)) + w(t), \quad t \in \mathcal{J}$;
(a_3) $z(t_k^+) - z(t_k^-) = J_k(z(t_k)) + w_k, \quad t \in \mathcal{J}$.

Remark 4.3. *The function* $z \in \mathbb{PC}(\mathcal{J}, \mathbb{R})$ *is called a solution for* (4.2) *if there exists a function* $w \in \mathbb{PC}(\mathcal{J}, \mathbb{R})$ *together with a sequence* w_k*,* $k = 1, 2, \ldots, m$ *(which depends on* z*) such that*

(b_1) $|w(t)| \le \phi(t)$, $|w_k| \le \upsilon, \quad t \in \mathcal{J}$;
(b_2) ${}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k} z(t) = f(t, z(t), z(\lambda t), {}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k} z(t)) + w(t), \quad t \in \mathcal{J}$;
(b_3) $z(t_k^+) - z(t_k^-) = J_k(z(t_k)) + w_k, \quad t \in \mathcal{J}$.

Remark 4.4. *The function* $z \in \mathbb{PC}(\mathcal{J}, \mathbb{R})$ *is called a solution for* (4.3) *if there exists a function* $w \in \mathbb{PC}(\mathcal{J}, \mathbb{R})$ *together with a sequence* w_k*,* $k = 1, 2, \ldots, m$ *(which depends on* z*) such that*

(c_1) $|w(t)| \le \epsilon\phi(t)$, $|w_k| \le \epsilon\upsilon, \quad t \in \mathcal{J}$;
(c_2) ${}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k} z(t) = f(t, z(t), z(\lambda t), {}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k} z(t)) + w(t), \quad t \in \mathcal{J}$;
(c_3) $z(t_k^+) - z(t_k^-) = J_k(z(t_k)) + w_k, \quad t \in \mathcal{J}$.

4.1. The $\mathbb{UH}$ *stability*

Firstly, we construct the results related to $\mathbb{UH}$ stability of impulsive problem (1.12).

Theorem 4.1. *Assume that* $f : \mathcal{J} \times \mathbb{R}^3 \to \mathbb{R}$, $J_k : \mathbb{R} \to \mathbb{R}$ *are continuous. If* ($\mathcal{A}_1$), ($\mathcal{A}_2$) *and* (3.4) *are fulfilled, then the problem* (1.12) *is* $\mathbb{UH}$ *stable.*

Proof. Assume that z is a solution of (4.1). By using (a_2) and (a_3) in Remark 4.2, we obtain

$$\begin{cases} {}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k} z(t) = f(t, z(t), z(\lambda t), {}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k} z(t)) + w(t), \\ z(t_k^+) - z(t_k^-) = J_k(z(t_k)) + w_k, \\ \eta z(0) + \beta z(T) = \displaystyle\sum_{i=0}^{m} \delta_i {}^{\rho_i}\mathcal{I}_{t_i}^{\gamma_i,\psi_i} x(\xi_i). \end{cases} \tag{4.4}$$

From Lemma 2.1, the solution of (4.4) can be written as

$$
\begin{aligned}
z(t) \;=\; & {}^{\rho_k}\mathcal{I}_{t_k}^{\alpha_k,\psi_k}F_x(t) + e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,t)}\Bigg\{\frac{1}{\Lambda}\prod_{i=1}^{k} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(t_{i-1},t_i)}\Bigg[\sum_{i=0}^{m}\delta_i{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}F_x(\xi_i) \\
& -\beta^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}F_x(T) - \beta e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\sum_{i=1}^{m}\Bigg(\Big({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}F_x(t_i) + J_i(x(t_i))\Big)\prod_{j=i}^{m-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\Bigg) \\
& + \sum_{i=0}^{m}\frac{\delta_i\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\sum_{j=1}^{i}\Bigg(\Big({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}F_x(t_j) + J_j(x(t_j))\Big)\prod_{l=j}^{i-1} e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})}\Bigg)\Bigg] \\
& + \sum_{i=1}^{k}\Bigg(\Big({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}F_x(t_i) + J_i(x(t_i))\Big)\prod_{j=i}^{k-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\Bigg)\Bigg\} \\
& + {}^{\rho_k}\mathcal{I}_{t_k}^{\alpha_k,\psi_k}w(t) + e^{\frac{\rho_k-1}{\rho_k}\Psi_k(t_k,t)}\Bigg\{\frac{1}{\Lambda}\prod_{i=1}^{k} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\Psi_{i-1}(t_{i-1},t_i)}\Bigg[\sum_{i=0}^{m}\delta_i{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}w(\xi_i) \\
& -\beta^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}w(T) - \beta e^{\frac{\rho_m-1}{\rho_m}\Psi_m(t_m,T)}\sum_{i=1}^{m}\Bigg(\Big({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}w(t_i) + w_i\Big)\prod_{j=i}^{m-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\Bigg) \\
& + \sum_{i=0}^{m}\frac{\delta_i\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}e^{\frac{\rho_i-1}{\rho_i}\Psi_i(t_i,\xi_i)}\sum_{j=1}^{i}\Bigg(\Big({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}w(t_j) + w_i\Big)\prod_{l=j}^{i-1} e^{\frac{\rho_l-1}{\rho_l}\Psi_l(t_l,t_{l+1})}\Bigg)\Bigg] \\
& + \sum_{i=1}^{k}\Bigg(\Big({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}w(t_i) + w_i\Big)\prod_{j=i}^{k-1} e^{\frac{\rho_j-1}{\rho_j}\Psi_j(t_j,t_{j+1})}\Bigg)\Bigg\}, \quad t\in J_k, \quad k=0,1,2,\ldots,m.
\end{aligned}
$$

By using (a_1) in Remark 4.2 with $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ and the fact of $0 < e^{\frac{\rho_l-1}{\rho_l}(\psi_l(u)-\psi_l(s))} \le 1$ for $0 \le s \le u \le T$, $l = 0,1,\ldots,m$, we estimate

$$
\begin{aligned}
& |z(t)-x(t)| \\
\le\; & {}^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}|F_z(t)-F_x(t)| + \frac{1}{|\Lambda|}\Bigg[\sum_{i=0}^{m}|\delta_i|{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}|F_z(\xi_i)-F_x(\xi_i)| \\
& +|\beta|^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}|F_z(T)-F_x(T)| + |\beta|\sum_{i=1}^{m}\Big({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|F_z(t_i)-F_x(t_i)| + |J_i(z(t_i))-J_i(x(t_i))|\Big) \\
& + \sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\Big({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}|F_z(t_j)-F_x(t_j)| + |J_j(z(t_j))-J_j(x(t_j))|\Big)\Bigg] \\
& + \sum_{i=1}^{m}\Big({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|F_z(t_i)-F_x(t_i)| + |J_i(z(t_i))-J_i(x(t_i))|\Big) + {}^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}|w(t)| \\
& + \frac{1}{|\Lambda|}\Bigg[\sum_{i=0}^{m}|\delta_i|{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}|w(\xi_i)| + |\beta|^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}|w(T)| + |\beta|\sum_{i=1}^{m}\Big({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|w(t_i)| + |w_i|\Big) \\
& + \sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\Big({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}|w(t_j)| + |w_i|\Big)\Bigg] + \sum_{i=1}^{m}\Big({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|w(t_i)| + |w_i|\Big)
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{\Psi_m^{\alpha_m}(t_m,T)}{\rho_m^{\alpha_m}\Gamma(\alpha_m+1)}|z(t)-x(t)| + \frac{1}{|\Lambda|}\Bigg[\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\alpha_i+\gamma_i}(t_i,\xi_i)}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i+1)}|z(t)-x(t)| \\
&\quad + \frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{|\beta|\Psi_m^{\alpha_m}(t_m,T)}{\rho_m^{\alpha_m}\Gamma(\alpha_m+1)}|z(t)-x(t)| + |\beta|\sum_{i=1}^{m}\left(\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}|z(t)-x(t)| \right.\\
&\quad \left. + \mathbb{M}_1|z(t)-x(t)|\right) + \sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\left(\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{\Psi_{j-1}^{\alpha_{j-1}}(t_{j-1},t_j)}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1}+1)}|z(t)-x(t)| + \mathbb{M}_1|z(t)-x(t)|\right)\Bigg] \\
&\quad + \sum_{i=1}^{m}\left(\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\cdot\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}|z(t)-x(t)| + \mathbb{M}_1|z(t)-x(t)|\right) + \epsilon\frac{\Psi_m^{\alpha_m}(t_m,T)}{\rho_m^{\alpha_m}\Gamma(\alpha_m+1)} \\
&\quad + \frac{\epsilon}{|\Lambda|}\Bigg[\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\alpha_i+\gamma_i}(t_i,\xi_i)}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i+1)} + \frac{|\beta|\Psi_m^{\alpha_m}(t_m,T)}{\rho_m^{\alpha_m}\Gamma(\alpha_m+1)} + |\beta|\sum_{i=1}^{m}\left(\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+1\right) \\
&\quad + \sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\left(\frac{\Psi_{j-1}^{\alpha_{j-1}}(t_{j-1},t_j)}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1}+1)}+1\right)\Bigg] + \epsilon\sum_{i=1}^{m}\left(\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+1\right) \\
&= \left(\frac{2\mathbb{L}_1\Omega_1}{1-\mathbb{L}_2} + \mathbb{M}_1\Omega_2\right)|z(t)-x(t)| + (\Omega_1+\Omega_2)\,\epsilon.
\end{aligned}$$

This further implies that $|z(t)-x(t)| \leq C_f\epsilon$, where

$$C_f := \frac{\Omega_1+\Omega_2}{1-\left(\frac{2\mathbb{L}_1\Omega_1}{1-\mathbb{L}_2}+\mathbb{M}_1\Omega_2\right)}.$$

Hence, the problem (1.12) is $\mathbb{UH}$ stable. □

Corollary 4.1. *In Theorem* 4.1*, if we set* $\phi(\epsilon) = C_f\epsilon$ *such that* $\phi(0) = 0$*, then* (1.12) *is generalized* $\mathbb{UH}$ *stable.*

4.2. The $\mathbb{UHR}$ *stability*

Before the proof of the next result, we give the following assumption:

($\mathcal{A}_5$) There exist a nondecreasing function $\phi \in C(\mathcal{J},\mathbb{R})$ and constants $\kappa_\phi > 0$, $\epsilon > 0$ such that

$${}^{\rho}\mathcal{I}_a^{\alpha,\psi}\phi(t) \leq \kappa_\phi\phi(t).$$

Theorem 4.2. *Assume that* $f : \mathcal{J}\times\mathbb{R}^3 \to \mathbb{R}$*,* $J_k : \mathbb{R} \to \mathbb{R}$ *are continuous functions. If* ($\mathcal{A}_1$), ($\mathcal{A}_2$), ($\mathcal{A}_5$) *and* (3.4) *are fulfilled, then* (1.12) *is* $\mathbb{UHR}$ *stable with respect to* (υ,ϕ)*, where* ϕ *is a nondecreasing function and* $\upsilon \geq 0$*.*

Proof. Let z be any solution of (4.3) and let x be a unique solution of (1.12). Then, for $t \in \mathcal{J}_k$, we have

$$\begin{aligned}
|z(t)-x(t)| &\leq {}^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}|F_z(t)-F_x(t)| + \frac{1}{|\Lambda|}\Bigg[\sum_{i=0}^{m}|\delta_i|{}^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}|F_z(\xi_i)-F_x(\xi_i)| \\
&\quad + |\beta|{}^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}|F_z(T)-F_x(T)| + |\beta|\sum_{i=1}^{m}\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|F_z(t_i)-F_x(t_i)| + |J_i(z(t_i))-J_i(x(t_i))|\right)
\end{aligned}$$

$$+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\left({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}|F_z(t_j)-F_x(t_j)|+|J_j(z(t_j))-J_j(x(t_j))|\right)\Bigg]$$

$$+\sum_{i=1}^{m}\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|F_z(t_i)-F_x(t_i)|+|J_i(z(t_i))-J_i(x(t_i))|\right)+{}^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}|w(t)|$$

$$+\frac{1}{|\Lambda|}\Bigg[\sum_{i=0}^{m}|\delta_i|^{\rho_i}\mathcal{I}_{t_i}^{\alpha_i+\gamma_i,\psi_i}|w(\xi_i)|+|\beta|^{\rho_m}\mathcal{I}_{t_m}^{\alpha_m,\psi_m}|w(T)|+|\beta|\sum_{i=1}^{m}\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|w(t_i)|+|w_i|\right)$$

$$+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\left({}^{\rho_{j-1}}\mathcal{I}_{t_{j-1}}^{\alpha_{j-1},\psi_{j-1}}|w(t_j)|+|w_i|\right)\Bigg]+\sum_{i=1}^{m}\left({}^{\rho_{i-1}}\mathcal{I}_{t_{i-1}}^{\alpha_{i-1},\psi_{i-1}}|w(t_i)|+|w_i|\right).$$

By using (c_1) in Remark 4.4 with $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_5)$ and the fact of $0 < e^{\frac{\rho_l-1}{\rho_l}(\psi_l(u)-\psi_l(s))} \leq 1$ for $0 \leq s \leq u \leq T$, $l = 0, 1, \ldots, m$, we estimate that

$$\begin{aligned}
|z(t)-x(t)| &\leq \Bigg(\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\Bigg[\left(1+\frac{|\beta|}{|\Lambda|}\right)\sum_{i=1}^{m+1}\frac{\Psi_{i-1}^{\alpha_{i-1}}(t_{i-1},t_i)}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}\\
&\quad+\frac{1}{|\Lambda|}\Bigg(\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\alpha_i+\gamma_i}(t_i,\xi_i)}{\rho_i^{\alpha_i+\gamma_i}\Gamma(\alpha_i+\gamma_i+1)}+\sum_{i=0}^{m}\frac{|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\sum_{j=1}^{i}\frac{\Psi_{j-1}^{\alpha_{j-1}}(t_{j-1},t_j)}{\rho_{j-1}^{\alpha_{j-1}}\Gamma(\alpha_{j-1}+1)}\Bigg)\Bigg]\\
&\quad+\mathbb{M}_1\left[m\left(1+\frac{|\beta|}{|\Lambda|}\right)+\frac{1}{|\Lambda|}\sum_{i=0}^{m}\frac{i|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\right]\Bigg)|z(t)-x(t)|\\
&\quad\Bigg(\left[1+\frac{1}{|\Lambda|}\left(|\beta|+\sum_{i=0}^{m}|\delta_i|\right)+m\left(1+\frac{|\beta|}{|\Lambda|}\right)+\frac{1}{|\Lambda|}\sum_{i=0}^{m}\frac{i|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\right]\kappa_\phi\phi(t)\\
&\quad+\left[m\left(1+\frac{|\beta|}{|\Lambda|}\right)+\frac{1}{|\Lambda|}\sum_{i=0}^{m}\frac{i|\delta_i|\Psi_i^{\gamma_i}(t_i,\xi_i)}{\rho_i^{\gamma_i}\Gamma(\gamma_i+1)}\right]\upsilon\Bigg)\epsilon\\
&= \left(\frac{2\mathbb{L}_1\Omega_1}{1-\mathbb{L}_2}+\mathbb{M}_1\Omega_2\right)|z(t)-x(t)|+\left(\left[1+\frac{1}{|\Lambda|}\left(|\beta|+\sum_{i=0}^{m}|\delta_i|\right)+\Omega_2\right]\kappa_\phi\phi(t)+\Omega_2\upsilon\right)\epsilon\\
&\leq \left(\frac{2\mathbb{L}_1\Omega_1}{1-\mathbb{L}_2}+\mathbb{M}_1\Omega_2\right)|z(t)-x(t)|+\left(\left[1+\frac{1}{|\Lambda|}\left(|\beta|+\sum_{i=0}^{m}|\delta_i|\right)+\Omega_2\right]\kappa_\phi+\Omega_2\right)\epsilon\,(\upsilon+\phi(t)).
\end{aligned}$$

This further implies that $|z(t)-x(t)| \leq C_{f,\kappa_\phi}\epsilon(\upsilon+\phi(t))$, where

$$C_{f,\kappa_\phi}=\frac{\left[1+\frac{1}{|\Lambda|}\left(|\beta|+\sum_{i=0}^{m}|\delta_i|\right)+\Omega_2\right]\kappa_\phi+\Omega_2}{1-\left(\frac{2\mathbb{L}_1\Omega_1}{1-\mathbb{L}_2}+\mathbb{M}_1\Omega_2\right)}.$$

Hence, the problem (1.12) is $\mathbb{UHR}$ stable. □

Corollary 4.2. *In Theorem* 4.2*, if we set* $\epsilon = 1$*, then* (1.12) *is generalized* $\mathbb{UHR}$ *stable.*

5. Numerical examples

This section provides three numerical problems, which indicate the exactitude and applicability of the main results.

Example 5.1. *Consider the following an impulsive pantograph fractional boundary value problem:*

$$
\begin{cases}
{}^{C\frac{2k+1}{10}}\mathfrak{D}_{t_k^+}^{\frac{k+1}{3k+2},\exp(t^{\frac{2}{k+3}})}x(t) = f\left(t, x(t), x\left(\frac{\sqrt{3}}{2}t\right), {}^{C\frac{2k+1}{10}}\mathfrak{D}_{t_k^+}^{\frac{k+1}{3k+2},\exp(t^{\frac{2}{k+3}})}x(t)\right), \ t \neq t_k, \\
\Delta x(t_k) = J_k(x(t_k)), \quad k = 1, 2, \\
4x(0) + \frac{1}{2}x\left(\frac{3}{2}\right) = \sum_{i=0}^{2}\left(\frac{i+3}{2i+8}\right) {}^{\frac{2i+1}{10}}\mathcal{I}_{t_i}^{\frac{i+3}{3i+2},\exp(t^{\frac{2}{i+3}})} x\left(\frac{2i+1}{4}\right).
\end{cases}
\tag{5.1}
$$

By giving $\alpha_k = (k+1)/(3k+2)$, $\psi_k(t) = \exp(t^{\frac{2}{k+3}})$, $t_k = k/2$, $\rho_k = (2k+1)/10$, $k = 0, 1, 2$, $\lambda = \sqrt{3}/2$, $m = 2$, $T = 3/2$, $\eta = 4$, $\beta = 1/2$, $\delta_i = (i+3)/(2i+8)$, $\gamma_i = (i+3)/(3i+2)$, $\xi_i = (2i+1)/4$, $i = 0, 1, 2$. *From the given all data, we can find that* $\Lambda \approx 3.962094671 \neq 0$, $\Omega_1 \approx 11.27074721$ *and* $\Omega_2 \approx 2.532952962$. *For the theoretical confirmation, we will consider the various functions as below:*

(i) *To demonstrate the application of Theorem 3.1, let us take the following nonlinear functions:*

$$
f(t, x, y, z) = \frac{e^{5+2\cos t}}{t+3} + \frac{1}{(3+\sin^2 \pi t)^2 + 1}\left(\frac{|x|}{3+|x|} + \frac{|y|}{3+|y|}\right) + \frac{4t}{(3t+5)^2+5} \cdot \frac{|z|}{2+|z|}, \tag{5.2}
$$

$$
J_k(x(t_k)) = \frac{1}{(2k+3)^2}\sin x(t_k) + 2t_k, \quad k = 1, 2. \tag{5.3}
$$

By $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$, *for any* $x_i, y_i, z_i \in \mathbb{R}$, $i = 1, 2$ *and* $t \in \mathcal{J}$, *we have* $|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leq (1/30)(|x_1 - x_2| + |y_1 - y_2|) + (1/10)|z_1 - z_2|$ *and* $|J_k(x) - J_k(y)| \leq (1/25)|x(t_k) - y(t_k)|$, *for* $k = 1, 2$. *The conditions* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *are satisfied with* $\mathbb{L}_1 = 1/30$, $\mathbb{L}_2 = 1/10$ *and* $\mathbb{M}_1 = 1/25$. *Hence,*

$$
\frac{2\mathbb{L}_1\Omega_1}{1 - \mathbb{L}_2} + \mathbb{M}_1\Omega_2 \approx 0.9361882824 < 1.
$$

Then, all the conditions of Theorem 3.1 *are satisfied, which implies that the numerical problem* (5.1), *where the functions* f *and* J_k *are given by* (5.2) *and* (5.3), *has a unique solution on* $[0, 3/2]$.

Furthermore, we also compute the constant

$$
C_f = \frac{\Omega_1 + \Omega_2}{1 - \left(\frac{2\mathbb{L}_1\Omega_1}{1-\mathbb{L}_2} + \mathbb{M}_1\Omega_2\right)} \approx 51.80636573 > 0.
$$

Hence, by Theorem 4.1, *the numerical problem* (5.1) *is* $\mathbb{UH}$ *stable on* $[0, 3/2]$. *In addition, if we set* $\phi(\epsilon) = C_f\epsilon$ *with* $\phi(0) = 0$, *then, by Corollary* 4.1, *the numerical problem* (5.1) *is generalized* $\mathbb{UH}$ *stable on* $[0, 3/2]$. *By setting* $\phi(t) = e^{\frac{\rho_k - 1}{\rho_k}\psi_k(t)}(\psi_k(t) - \psi_k(t_k))^{\frac{5}{2}}$ *with* $\upsilon = 1$, *we have*

$$
{}^{\frac{2k+1}{10}}\mathcal{I}_{t_k}^{\frac{k+1}{3k+2}, e^{t^{\frac{2}{k+3}}}} \phi(t) \leq \frac{\Gamma(\frac{7}{2}) e^{\frac{2k-9}{2k+1} e^{t^{\frac{2}{k+3}}}} \left(e^{t^{\frac{2}{k+3}}} - e^{t_k^{\frac{2}{k+3}}}\right)^{\frac{17k+12}{6k+4}}}{\left(\frac{2k+1}{10}\right)^{\alpha_k} \Gamma\left(\frac{23k+16}{6k+4}\right)} \phi(t).
$$

By using $(\mathcal{A}_5)$, *we get*

$$
\kappa_\phi = \frac{\Gamma(\frac{7}{2}) e^{\frac{2k-9}{2k+1} e^{t^{\frac{2}{k+3}}}} \left(e^{t^{\frac{2}{k+3}}} - e^{t_k^{\frac{2}{k+3}}}\right)^{\frac{17k+12}{6k+4}}}{\left(\frac{2k+1}{10}\right)^{\alpha_k} \Gamma\left(\frac{23k+16}{6k+4}\right)} > 0, \quad \forall t \in [0, 3/2].
$$

We have

$$C_{f,\kappa_\phi} = \frac{\left[1 + \frac{1}{|\Lambda|}\left(|\beta| + \sum_{i=0}^{m}|\delta_i|\right) + \Omega_2\right]\kappa_\phi + \Omega_2}{1 - \left(\frac{2\mathbb{L}_1\Omega_1}{1-\mathbb{L}_2} + \mathbb{M}_1\Omega_2\right)} \approx 39.80812832 > 0.$$

Therefore, by all assumptions in Theorem 4.2*, the numerical problem* (5.1) *is* $\mathbb{UHR}$ *stable on* $[0, 3/2]$*. Additionally, if we set* $\phi(\epsilon) = C_f\epsilon$ *with* $\phi(0) = 0$*, then, by Corollary* 4.1*, the numerical problem* (5.1) *is generalized* $\mathbb{UHR}$ *stable with respect to* (υ, ϕ)*.*

(*ii*) *To demonstrate the application of Theorem 3.2, we consider the following nonlinear functions:*

$$f(t,x,y,z) = \frac{\ln(3t^2+4)}{3t+1} + \frac{3+\tan^2\pi t}{3t^2+2}(\cos(\pi - |x|) + \sin(|y|)) + \frac{3t^3+6}{2t+5}\cdot\frac{|z|}{3+|z|}, \tag{5.4}$$

$$J_k(x(t_k)) = (3k-2)^2\frac{\sin x(t_k)}{\cos x(t_k)+2} + 3t_k + 2, \qquad k = 1, 2. \tag{5.5}$$

By $(\mathcal{A}_3)$ *and* $(\mathcal{A}_4)$*, for any* $x, y, z \in \mathbb{R}$ *and* $t \in \mathcal{J}$*, we have*

$$\begin{aligned} |f(t,x,y,z)| &\le \frac{\ln(3t^2+4)}{3t+1} + \frac{3+\tan^2\pi t}{3t^2+2}(|x|+|y|) + \frac{t^3+1}{2t+5}|z|, \\ |J_k(x)| &\le 4|x| + 5, \quad k = 1, 2. \end{aligned}$$

The $(\mathcal{A}_3)$ *and* $(\mathcal{A}_4)$ *are satisfied with* $g_1 = (\ln(3t^2+4))/(3t+1)$*,* $g_2 = (3+\tan^2\pi t)/(3t^2+2)$*,* $g_3 = (t^3+1)/(2t+5)$*,* $\mathbb{N}_1 = 4$ *and* $\mathbb{N}_2 = 5$*. Hence, all the conditions of Theorem* 3.2 *are satisfied, which implies that the numerical problem* (5.1) *has at least one solution on* $[0, 3/2]$ *,where* f *and* J_k *are given by* (5.4) *and* (5.5)*.*

(*iii*) *We consider the linear impulsive fractional boundary value problem:*

$$\begin{cases} {}^{C}\frac{2k+1}{10}\mathfrak{D}_{t_k^+}^{\frac{k+1}{3k+2},\exp(t^{\frac{2}{k+3}})}x(t) = 0, \quad t \in \left[0, \frac{3}{2}\right] \setminus \left\{\frac{1}{2}, 1\right\}, \\ \Delta x(t_k) = \frac{3}{2}k - 2, \quad k = 1, 2, \\ 4x(0) + \frac{1}{2}x\left(\frac{3}{2}\right) = \sum_{i=0}^{2}\left(\frac{i+3}{2i+8}\right)^{\frac{2k+1}{10}}\mathcal{I}_{t_i}^{\frac{i+3}{3i+2},\exp(t^{\frac{2}{i+3}})}x\left(\frac{2i+1}{4}\right). \end{cases} \tag{5.6}$$

Here, $f(t,x,y,z) = 0$ *and* $J_k(x(t_k)) = (3k/2) - 2$*,* $k = 1, 2$*. Clearly, all conditions of Theorem 3.1 are satisfied. Then, the numerical problem* (5.6) *has a unique solution on* $[0, 3/2]$*. By setting* $F_x(t) = 0$*,* $J_0(x(t_0)) = -2$ *and* $J_1(x(t_1)) = -1/2$ *in* (2.3)*, it is easy to compute that*

$$x(t) = \begin{cases} 0.2523917481e^{-9e^{t^{2/3}}+9}, \quad t \in [0, \frac{1}{2}], \\ -0.4999830704e^{-2.333333333e^{\sqrt{t}}+4.732268288}, \quad t \in (\frac{1}{2}, 1], \\ 0.9000957055e^{-1e^{t^{2/5}}+2.718281828}, \quad t \in (1, \frac{3}{2}]. \end{cases} \tag{5.7}$$

Thanks of (5.7)*, we present the numerical solution of* (5.6) *by using MATLAB program (see Figure* 1*).*

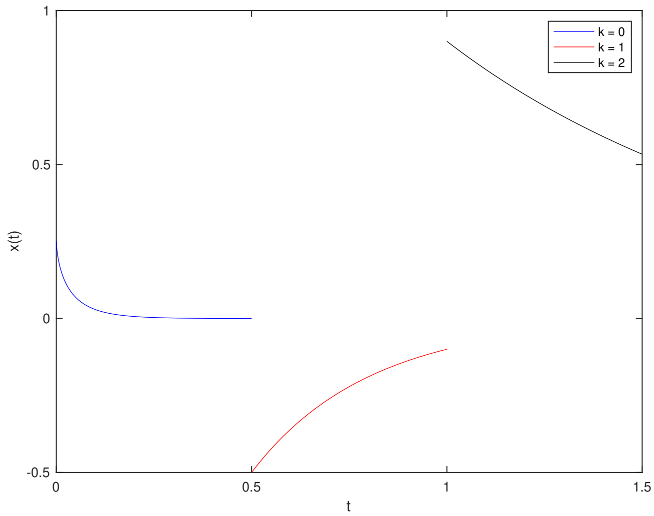

Figure 1. The numerical solution of (5.6).

6. Conclusions

We discussed the important role of qualitative theory, which is a favorable trend to study the existence and stability analysis of solutions for the impulsive boundary value problems with general boundary conditions involving the Caputo proportional fractional derivative type of a function with respect to another function (1.12). Firstly, the uniqueness result for the problem (1.12) was investigated by applying Banach's contraction principle. Afterward, the existence result was established by applying fixed point theory of Schaefer's type. Furthermore, by the application of qualitative theory and nonlinear functional analysis techniques, we examined results concerning different kinds of $\mathbb{UH}$ stability concepts. The concerned results have been guaranteed by numerical examples to demonstrate the application of our main results. This paper has flourished the literature of qualitative theory on nonlinear impulsive fractional initial/boundary value problems concerning a certain function in future works.

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Conflict of interest

The authors declare no conflicts of interest.

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