Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 7781–7816.
DOI: 10.3934/math.2022437
Received: 06 November 2021
Revised: 03 February 2022
Accepted: 14 February 2022
Published: 17 February 2022

Research article

To investigate a class of multi-singular pointwise defined fractional q–integro-differential equation with applications

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Abstract: In the research work, we discuss a multi-singular pointwise defined fractional q–integro-differential equation under some boundary conditions via the Riemann-Liouville q–integral and Caputo fractional q–derivatives. New existence results rely on the α-admissible map and fixed point theorem for α-ψ-contraction map. At the end, we present an example with application and some algorithms to illustrate the primary effects.

Keywords: singularity; pointwise defined equations; integral boundary conditions; Caputo q–derivation
Mathematics Subject Classification: 34A08, 34B16, 39A13

1. Introduction

Jackson [25] introduced quantum calculus. Then, it was later developed by Al-Salam who started fitting the concept of q–fractional calculus [7]. Agarwal continued studying certain q–fractional integrals and derivatives [3]. Furthermore, some researchers have also studied q–difference equations (for more details, see [1, 2, 5, 6, 8, 9, 15, 23, 24, 26, 27, 33, 39, 40]). On the one hand, fractional differential equations have gained a considerable importance due to their applications in various fields of sciences, such as physics, mechanics, chemistry, and engineering (see [17–21]). In [22], El-Sayed discussed a class of nonlinear functional differential equations of arbitrary orders, and Lakshmikantham [30] initiated the basic theory for fractional functional differential equations.

In 1996, Delbosco et al. investigated $\mathbb{D}^{\beta}u(t) = \hbar(t, u)$ with initial condition: $u(a) = \eta$, where $a > 0$, $\eta \in \mathbb{R}$ and $\beta \in J := (0, 1)$ [16]. In 2005, Bai *et al.* presented the boundary problem:

$$\mathbb{D}_0^{\beta}u(t) = h(t, u(t)),$$

under conditions: $u(0) = u(1) = 0$, where $t \in J$, $0 < \beta \leq 2$, and $\mathbb{D}_0^{\beta}$ is the Riemann-Liouville standard derivative [11]. In 2008, Qiu *et al.* studied the equation with conditions: $u(0) = u'(1) = u''(1) = 0$, where $t \in J$, $2 < \beta < 3$, $\mathbb{D}_{0^+}^{\beta}$ is the Caputo derivative and $h : \bar{J} \times [0, \infty) \to [0, \infty)$, here $\bar{J} := [0, 1]$, is such that $\lim_{t\to 0^+} h(t, .) = \infty$ [34]. In 2010, Agarwal et al. considered the singular fractional Dirichlet problem:

$$\mathbb{D}^{\beta}u(t) + h\left(t, u(t), \mathbb{D}^{\gamma}u(t)\right) = 0,$$

with the boundary value condition: $u(0) = u(1) = 0$, where $\beta \in (1, 2]$, $\gamma > 0$, $\beta - \gamma \geq 1$, $h \in \mathrm{Car}(\bar{J} \times (0, \infty) \times \mathbb{R})$, h is positive and singular at $t = 0$, and $\mathbb{D}$ is the usual Riemann-Liouville derivative [4]. In 2012, Cabada et al. investigated the existence of positive solution for the following nonlinear fractional differential equation:

$$\begin{cases} \mathbb{D}^{\beta}u(t) = h(t, u(t)) \\ u(0) = u''(1) = 0, u(1) = \displaystyle\int_0^1 u(\xi)\,\mathrm{d}\xi, \end{cases}$$

where $0 < t < 1$, $2 < \beta < 3$ and $h : \bar{J} \times [0, \infty) \to [0, \infty)$ is a continuous function [13]. In 2014, Li reviewed the problem:

$${}^{C}\mathbb{D}^{\beta}u(t) + h\left(t, u(t), \mathbb{D}^{\gamma}u(t)\right) = 0,$$

for each $t \in J$, under conditions: $u(0) = u'(0) = 0$ and $u'(1) = {}^{C}\mathbb{D}^{\beta}u(1)$, where $\beta \in (2, 3)$, $\gamma \in J$, $h : (0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous function that may be singular at $t = 0$, ${}^{C}\mathbb{D}^{\beta}$ is the standard Caputo derivative [31]. In 2016, the fractional integro-differential equation

$$\mathbb{D}^{\gamma}u(t) = h\left(t, u(t), u'(t), \mathbb{D}^{\alpha}u(t), \mathbb{I}^{\beta}u(t)\right),$$

under conditions $u'(0) = u(\eta)$, $u(1) = \int_0^{\nu} u(\xi)\,\mathrm{d}\xi$ and $u^{(i)}(0) = 0$ for $i = 2, \ldots, [\gamma] - 1$ was investigated, where $t \in J$, $\gamma \in [2, 3)$, $u \in \bar{\mathcal{B}} = C^1(\bar{J})$, $\alpha, \eta, \nu \in J$, $\beta > 1$ and $h : \bar{J} \times \mathbb{R}^4 \to \mathbb{R}$ is a function such that $h(t, ., ., ., .)$ is singular at some point $t \in \bar{J}$ [44]. In 2017, Shabibi et al. studied the singular fractional integro-differential equation:

$${}^{C}\mathbb{D}^{\beta}u(t) + h\left(t, u(t), u'(t), {}^{c}\mathbb{D}^{\gamma}u(t), \mu(u(t))\right) = 0,$$

where $\mu(u(t)) = \int_0^t f(\xi)u(\xi)\,\mathrm{d}\xi$, under boundary conditions: $u(0) = u'(0)$ and $u(1) = {}^{C}\mathbb{D}^{\gamma}u(t)$, where $t \in J$, $u \in \bar{\mathcal{B}}$, $\beta > 2$, $0 < \gamma, a < 1$, $f \in \bar{\mathcal{L}} = L_1(\bar{J})$, $\|f\|_1 = m$, $h(t, u_1, u_2, u_3, u_4)$ is singular at some points $t \in \bar{J}$ and ${}^{C}\mathbb{D}^{\beta}$ is the Caputo fractional derivative [45]. In 2020, Samei considered the singular system of q–differential equations:

$$\begin{cases} \mathbb{D}_q^{\alpha_1}u(t) = g_1(t, u(t), v(t)), \\ \mathbb{D}_q^{\alpha_2}v(t) = g_2(t, u(t), v(t)), \end{cases}$$

with conditions: $u(0) = v(0) = 0$, $u^{(i)}(0) = v^{(i)}(0) = 0$, for $i = 2, \ldots, n-1$ and

$$u(1) = \left[\mathbb{I}_q^{\gamma_1}(w_1(t)u(t))\right]_{t=1}, \qquad v(1) = \left[\mathbb{I}_q^{\gamma_2}(w_2(t)v(t))\right]_{t=1},$$

where $\mathbb{D}_q^{\alpha_j}$ is the q–derivative of fractional order α_j, $\alpha_j \in (n, n+1]$ with $n \geq 3$, $\mathbb{I}_q^{\gamma_j}$ is the q–integral of fractional order γ_j, $\gamma_j \geq 1$, $g_j \in C(E)$, g_j are singular at $t = 0$ and satisfy the local Carathéodory condition on $E = (0, 1] \times (0, \infty) \times (0, \infty)$, and $w_j \in \bar{\mathcal{L}}$ are non-negative such that

$$\left[I_q^{\gamma_j}(w_j(t))\right]_{t=1} \in \left[0, \frac{1}{2}\right),$$

for $j = 1, 2$ [37]. Also, Liang et al. [32] investigated a nonlinear problem of regular and singular fractional q–differential equation:

$${}^c\mathbb{D}_q^{\alpha} u(t) = h\left(t, u(t), u'(t), {}^c\mathbb{D}_q^{\beta} u(t)\right),$$

with conditions: $u(0) = c_1 u(1)$, $u'(0) = c_2 {}^c\mathbb{D}_q^{\beta} u(1)$ and $u^{(k)}(0) = 0$ for all $2 \leq k \leq n-1$, here $n-1 < \alpha < n$ with $n \geq 3$, $\beta, q, c_1 \in J$, $0 < c_2 < \Gamma_q(2-\beta)$, function h is a L^{κ}-Carathéodory and $h(t, u_1, u_2, u_3)$ may be singular. Similarly, some related results have been obtained in [28, 36, 38]. Dassios *et al.* used a generalized system of differential equations of fractional order:

$$T_\lambda \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} = -H_d \lambda(t) + K_E \left(\omega^{\mathrm{ref}} - \omega_{Col}(t)\right),$$

to incorporate memory into an electricity market model by constructing the fractional-order dynamical model, studying its solutions, and providing the closed formulas of solutions, where $\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}$, $\lambda(t)$ are the marginal electricity price and electricity price, respectively, ω^{ref} represents the reference frequency, $\omega_{Col}(t)$ represents the frequency of the Col, that is, $\omega^{\mathrm{ref}} - \omega_{Col}(t)$ is the deviation frequency of the CoI with respect to the reference frequency, T_λ is the time constant, H_d is the deviation with respect to a perfect tracking integrator, and for a low-pass filter, it is $H_d = 1$, and K_E can be used as feedback gain [14].

Using the ideas from these works, we investigate the existence of solutions for the following nonlinear pointwise defined fractional q–integro-differential equation:

$$\mathbb{D}_q^{\alpha} u(t) = w\Big(t, u(t), \mathbb{D}_q^{\beta} u(t), \int_0^t f(\xi) u(\xi)\, \mathrm{d}\xi, \varphi(u(t))\Big), \tag{1.1}$$

for $q \in J$, under boundary conditions: $\int_0^b u(r)\,\mathrm{d}r = 0$, $u'(1) = u(a)$ and $u^{(j)}(0) = 0$ for $j \geq 2$, here $\alpha \geq 2$, $a, b, \beta \in J$, $\varphi : \bar{\mathcal{B}} \to \bar{\mathcal{B}}$ is a map such that

$$\|\varphi(u_1) - \varphi(u_2)\| \leq c_1 \|u_1 - u_2\| + c_2 \|u_1' - u_2'\|,$$

for some non-negative real numbers c_1 and c_2 belonging to $[0, \infty)$ and all $u_1, u_2 \in \bar{\mathcal{B}}$, where $\mathbb{D}_q^{\alpha}$ and $\mathbb{D}_q^{\beta}$ are the Caputo fractional q–derivatives of order α and β, respectively, which are defined in (2.11), and $w \in \bar{\mathcal{L}}$ is singular at some points $t \in \bar{J}$.

In fact, the non-constant real-valued function u on the interval $I = [a, b]$ is said to be singular on I, if it is continuous, and there exists a set $S \subseteq I$ of measure 0 such that for all t outside of S, $u'(t)$ exists, and it is zero, that is, the derivative of u vanish almost everywhere. We say that, $\mathbb{D}_q^\alpha u(t) + g(t) = 0$ is a pointwise defined equation on $\bar{J}$ if there exists set $S \subset \bar{J}$ such that the measure of S^c is zero, and the equation holds on S [44].

In Section 2, we recall some essential definitions of Caputo fractional q–derivative. Section 3 contains our main results of this work, while an example is presented to support the validity of our obtained results. An application with some needed algorithms for the problems are given in Section 4. In Section 5, conclusion is presented.

2. Basic definitions for the problem

Throughout the paper, we apply the notations of time scales calculus [12]. The Caputo fractional q–derivative is considered here on

$$\mathbb{T}_{\mathrm{s}_0} = \{0\} \cup \left\{\mathrm{s} \,:\, \mathrm{s} = \mathrm{s}_0 q^{\aleph}\right\},$$

for all $\aleph \in \mathbb{N}$, $\mathrm{s}_0 \in \mathbb{R}$ and $q \in J$. If there is no confusion concerning s_0, we denote $\mathbb{T}_{\mathrm{s}_0}$ by $\mathbb{T}$. Let $p \in \mathbb{R}$. Let us define $[p]_q = (1 - q^p)(1 - q)^{-1}$ [25]. The q–factorial function $(\mathrm{v} - \mathrm{w})_q^{(\aleph)}$ with $\aleph \in \mathbb{N}_0$ is defined by

$$(\mathrm{v} - \mathrm{w})_q^{(\aleph)} = \prod_{\mathrm{k}=0}^{\aleph-1} (\mathrm{v} - \mathrm{w}q^{\mathrm{k}}), \qquad (\forall \mathrm{v}, \mathrm{w} \in \mathbb{R}), \tag{2.1}$$

and $(\mathrm{v} - \mathrm{w})_q^{(0)} = 1$, where $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$ [2]. Also, for $\sigma \in \mathbb{R}$, we have:

$$(v - w)_q^{(\sigma)} = v^\sigma \prod_{k=0}^{\infty} \frac{v - wq^k}{v - wq^{\sigma+k}}, \qquad (\forall v, w \in \mathbb{R}). \tag{2.2}$$

In [10], the authors proved that $(v - w)_q^{(\sigma+v)} = (v - w)_q^{(\sigma)} (v - q^\sigma w)_q^{(v)}$ and

$$(av - aw)_q^{(\sigma)} = a^\sigma (v - w)_q^{(\sigma)},$$

for each $\mathrm{v}, \mathrm{w} \in \mathbb{R}$. If $w = 0$, then it is clear that $v^{(\sigma)} = v^\sigma$. The q–Gamma function is given by

$$\Gamma_q(v) = (1 - q)^{1-v} (1 - q)_q^{(v-1)},$$

where $v \in \mathbb{R} \backslash \{\cdots, -2, -1, 0\}$ [25]. In fact, by using (2.2), we have

$$\Gamma_q(v) = (1 - q)^{1-v} \prod_{k=0}^{\infty} \frac{1 - q^{k+1}}{1 - q^{v+k-1}}, \qquad (\forall v \in \mathbb{R}). \tag{2.3}$$

Note that, $\Gamma_q(v + 1) = [v]_q \Gamma_q(v)$ [10, Lemma 1]. For a function $u : \mathbb{T} \to \mathbb{R}$, the q–derivative of u, is

$$\mathbb{D}_q[u](t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_q u(t) = \frac{u(t) - u(qt)}{(1 - q)t}, \tag{2.4}$$

for all $t \in \mathbb{T} \backslash \{0\}$, and $\mathbb{D}_q[u](0) = \lim_{t \to 0} \mathbb{D}_q[u](t)$ [2]. Also, the higher order q–derivative of the function u is defined by $\mathbb{D}_q^n[u](t) = \mathbb{D}_q \left[\mathbb{D}_q^{n-1}[u]\right](t)$, for all $n \geq 1$, where $\mathbb{D}_q^0[u](t) = u(t)$ [2]. In fact,

$$\mathbb{D}_q^n[u](t) = \frac{1}{t^n (1 - q)^n} \sum_{k=0}^{n} \frac{(1 - q^{-n})_q^{(k)}}{(1 - q)_q^{(k)}} q^k u(tq^k), \tag{2.5}$$

for $t \in \mathbb{T} \setminus \{0\}$ [9].

Remark 2.1. *By using Eq* (2.1)*, we can change Eq* (2.5) *into the following:*

$$\mathbb{D}_q^n[u](t) = \frac{1}{t^n(1-q)^n} \sum_{k=0}^{n} \frac{\prod_{i=0}^{k-1}\left(1-q^{i-n}\right)}{\prod_{i=0}^{k-1}\left(1-q^{i+1}\right)} q^k u(tq^k). \tag{2.6}$$

The q–integral of the function u is defined by

$$\mathbb{I}_q[u](t) = \int_0^t u(\xi)\,\mathrm{d}_q\xi = t(1-q)\sum_{k=0}^{\infty} q^k u(tq^k), \tag{2.7}$$

for $0 \le t \le b$, provided that the series is absolutely convergent [2]. If a is in $[0, b]$, then

$$\int_a^b u(\xi)\,\mathrm{d}_q\xi = \mathbb{I}_q[u](b) - \mathbb{I}_q[u](a) = (1-q)\sum_{k=0}^{\infty} q^k \left[bu(bq^k) - au(aq^k)\right], \tag{2.8}$$

whenever the series converges. The operator $\mathbb{I}_q^n$ is given by $\mathbb{I}_q^0[u](t) = u(t)$ and

$$\mathbb{I}_q^n[u](t) = \mathbb{I}_q\left[\mathbb{I}_q^{n-1}[u]\right](t),$$

for $n \ge 1$ and $u \in C([0, b])$ [2]. It has been proven that

$$\mathbb{D}_q\left[\mathbb{I}_q[u]\right](t) = u(t), \qquad \mathbb{I}_q\left[\mathbb{D}_q[u]\right](t) = u(t) - u(0),$$

whenever the function u is continuous at $t = 0$ [2]. The fractional Riemann-Liouville type q–integral of the function u is defined by

$$\mathbb{I}_q^\sigma[u](t) = \frac{1}{\Gamma_q(\sigma)} \int_0^t (t-\xi)_q^{(\sigma-1)} u(\xi)\,\mathrm{d}_q\xi, \qquad \mathbb{I}_q^0[u](t) = u(t), \tag{2.9}$$

for $t \in \bar{J}$ and $\sigma > 0$ [9, 23].

Remark 2.2. *By using Eqs* (2.2)*,* (2.3) *and* (2.7)*, we obtain:*

$$\begin{aligned}
&\frac{1}{\Gamma_q(\sigma)} \int_0^t (t-\xi)_q^{(\sigma-1)} u(\xi)\,\mathrm{d}_q\xi \\
&\quad = \frac{1}{\Gamma_q(\sigma)} \int_0^t t^{\sigma-1} \prod_{i=0}^{\infty} \frac{t-\xi q^i}{t-\xi q^{\sigma+i-1}} u(\xi)\,\mathrm{d}_q\xi \\
&\quad = t^\sigma (1-q)^\sigma \prod_{i=0}^{\infty} \frac{1-q^{\sigma+i-1}}{1-q^{i+1}} \sum_{k=0}^{\infty} q^k \prod_{i=0}^{\infty} \frac{1-q^{k+i}}{1-q^{\sigma+k+i-1}} u(tq^k).
\end{aligned}$$

Therefore, we have:

$$\mathbb{I}_q^\sigma[u](t) = t^\sigma(1-q)^\sigma \lim_{n\to\infty} \sum_{k=0}^{n} q^k \prod_{i=0}^{n} \frac{\left(1-q^{\sigma+i-1}\right)\left(1-q^{k+i}\right)}{\left(1-q^{i+1}\right)\left(1-q^{\sigma+k+i-1}\right)} u(tq^k), \tag{2.10}$$

The Caputo fractional q–derivative of the function u is defined by

$$\begin{aligned} {}^{C}\mathbb{D}_q^{\sigma}[u](t) &= \mathbb{I}_q^{[\sigma]-\sigma}\left[\mathbb{D}_q^{[\sigma]}[u]\right](t) \\ &= \frac{1}{\Gamma_q([\sigma]-\sigma)}\int_0^t (t-\xi)_q^{([\sigma]-\sigma-1)}\,\mathbb{D}_q^{[\sigma]}[u](\xi)\,\mathrm{d}_q\xi \end{aligned} \tag{2.11}$$

for $t \in \bar{J}$ and $\sigma > 0$ [23,35]. It has been proven that

$$\mathbb{I}_q^{\nu}\left[\mathbb{I}_q^{\sigma}[u]\right](t) = \mathbb{I}_q^{\sigma+\nu}[u](t), \qquad {}^{C}\mathbb{D}_q^{\sigma}\left[\mathbb{I}_q^{\sigma}[u]\right](t) = u(t),$$

where $\sigma, \nu \geq 0$ [23]. Also,

$$\mathbb{I}_q^{\sigma}\left[\mathbb{D}_q^{n}[u]\right](t) = \mathbb{D}_q^{n}\left[\mathbb{I}_q^{\sigma}[u]\right](t) - \sum_{k=0}^{n-1}\frac{t^{\sigma+k-n}}{\Gamma_q(\sigma+k-n+1)}\mathbb{D}_q^{k}[u](0),$$

where $\sigma > 0$ and $n \geq 1$ [23].

Remark 2.3. *From Eq* (2.3)*, Remark 2.1, and Eq* (2.10) *in Remark 2.2, we obtain:*

$$\begin{aligned} &\frac{1}{\Gamma_q([\sigma]-\sigma)}\int_0^t (t-\xi)_q^{([\sigma]-\sigma-1)}\,\mathbb{D}_q^{[\sigma]}[u](\xi)\,\mathrm{d}_q\xi \\ &\quad = \frac{1}{\Gamma_q([\sigma]-\sigma)}\int_0^t t^{[\sigma]-\sigma-1}\Big[\prod_{i=0}^{\infty}\frac{t-sq^i}{t-sq^{[\sigma]-\sigma-1+i}}\Big] \\ &\quad \times\left(\frac{1}{t^{[\sigma]}(1-q)^{[\sigma]}}\sum_{k=0}^{[\sigma]}\Big[\prod_{i=0}^{k-1}\frac{\left(1-q^{i-[\sigma]}\right)}{(1-q^{i+1})}\Big]q^k\,u(xq^k)\right)\mathrm{d}_q s \\ &\quad = \frac{1}{t^{\sigma}(1-q)^{\sigma-[\sigma]}}\sum_{k=0}^{\infty}\bigg(\Big[\prod_{i=0}^{\infty}\frac{(1-q^{[\sigma]-\sigma+i-1})(1-q^{k+i})}{(1-q^{i+1})(1-q^{[\sigma]-\sigma-1+k+i})}\Big] \\ &\quad \times\left(\sum_{m=0}^{[\sigma]}\Big[\prod_{i=0}^{m-1}\frac{\left(1-q^{i-[\sigma]}\right)}{(1-q^{i+1})}\Big]q^m u\left(tq^{k+m}\right)\right)\bigg). \end{aligned}$$

Thus, we have:

$$\begin{aligned} {}^{C}\mathbb{D}_q^{\sigma}[u](t) &= \frac{1}{t^{\sigma}(1-q)^{\sigma-[\sigma]}}\lim_{n\to\infty}\sum_{k=0}^{n}\bigg(\Big[\prod_{i=0}^{n}\frac{(1-q^{[\sigma]-\sigma+i-1})(1-q^{k+i})}{(1-q^{i+1})(1-q^{[\sigma]-\sigma-1+k+i})}\Big] \\ &\quad \times\left(\sum_{m=0}^{[\sigma]}\Big[\prod_{i=0}^{m-1}\frac{\left(1-q^{i-[\sigma]}\right)}{(1-q^{i+1})}\Big]q^m u\left(tq^{k+m}\right)\right)\bigg). \end{aligned} \tag{2.12}$$

The authors in [41] presented all algorithms and MATLAB code's lines to simplify q–factorial functions $(v-w)_q^{(n)}$, $(v-w)_q^{(\sigma)}$, $\Gamma_q(v)$, $\mathbb{I}_q[u](t)$, and some necessary equations.

Lemma 2.4. *[27, 29] For $\sigma > 0$, the general solution of the fractional q–differential equation ${}^{C}\mathbb{D}^{\sigma}u(t) = 0$ is given by $u(t) = \sum_{i=0}^{n-1} e_i t^i$, where $e_i \in \mathbb{R}$ for $i = 0, 1, 2, \ldots, n-1$ and $n = [\sigma] + 1$ here $[\sigma]$ denotes the integer part of the real number σ.*

We use the three norms: $\|u\| = \sup_{t\in\bar{J}} |u(t)|$,

$$\|(u, u')\|_* = \max\left\{\|u\|, \|u'\|\right\},$$

and $\|u\|_1 = \int_J |u(\xi)|\, d\xi$ in $\bar{\mathcal{A}} = C(\bar{J})$, $\bar{\mathcal{B}} = C^1(\bar{J})$, and $\bar{\mathcal{L}} = L_1(\bar{J})$, respectively. Let Ψ be the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, for all $t > 0$. Let $T : \mathcal{X} \to \mathcal{X}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to (0, \infty)$. T is called an α-admissible mapping if $\alpha(u_1, u_2) \geq 1$ implies that $\alpha(T(u_1), T(u_2)) \geq 1$ for each u_1, u_2 in $\mathcal{X}$.

Definition 2.5. *[42] Let $(\mathcal{X}, \rho)$ be a metric space, where $\psi \in \Psi$ and $\alpha : \mathcal{X}^2 \to [0, \infty)$ is a map. A self-map T defined on $\mathcal{X}$ is called an α-ψ-contraction whenever*

$$\alpha(u_1, u_2)\rho\left(T(u_1), T(u_2)\right) \leq \psi\left(\rho(u_1, u_2)\right),$$

for each $u_1, u_2 \in \mathcal{X}$.

Lemma 2.6. *[42] Let $(\mathcal{X}, \rho)$ be a complete metric space and $T : \mathcal{X} \to \mathcal{X}$ be a continuous, α−admissible and α–ψ–contraction, then T has a fixed point whenever there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, T(u_0)) \geq 1$.*

Lemma 2.7. *[43, 46] If $x \in \bar{\mathcal{A}} \cap \bar{\mathcal{L}}$ with $\mathbb{D}_q^{\alpha} x \in \mathcal{A} \cap \mathcal{L}$, then*

$$\mathbb{I}_q^{\alpha}\mathbb{D}_q^{\alpha}u(t) = u(t) + \sum_{i=1}^{n} c_i t^{\alpha-i},$$

where $[\alpha] \leq n < [\alpha] + 1$, and c_i is some real number.

3. Main results

Let us first prove the following essential lemma:

Lemma 3.1. *Suppose that $\alpha \geq 2$, $q \in J$ and $g \in \bar{\mathcal{L}}$. The solution of the boundary value problem: $\mathbb{D}_q^{\alpha}u(t) = g(t)$ with boundary conditions is expressed as:*

$$\begin{cases} u^{(j)}(0) = 0 & ; j = 2, 3, 4, \ldots, \\ u'(1) = u(a) & ; \forall a \in J, \\ \int_0^b u(\xi)\, d\xi = 0 & ; \forall b \in J, \end{cases}$$

is

$$u(t) = \int_0^1 G_q(t, \xi) g(\xi)\, d_q\xi,$$

on a time scale $\mathbb{T}_{t_0}$ where $G_q(t,s)$ is expressed as:

$$\begin{cases} -A_0(t-s)_q^{(\alpha-1)} + A_1(t)(1-s)_q^{(\alpha-2)} + A_2(t)(a-s)_q^{(\alpha-1)} + A_3(t)(b-s)_q^{(\alpha)} & s \leq \min\{a,b\}; \\ -A_0(t-s)_q^{(\alpha-1)} + A_1(t)(1-s)_q^{(\alpha-2)} + A_2(t)(a-s)_q^{(\alpha-1)} & b \leq s \leq a; \\ -A_0(t-s)_q^{(\alpha-1)} + A_1(t)(1-s)_q^{(\alpha-2)} + A_3(t)(b-s)_q^{(\alpha)} & a \leq s \leq b; \\ -A_0(t-s)_q^{(\alpha-1)} + A_1(t)(1-s)_q^{(\alpha-2)} & s \geq \max\{a,b\}; \end{cases} \tag{3.1}$$

whenever $0 \leq s \leq t \leq 1$*,*

$$\begin{cases} A_1(t)(1-s)_q^{(\alpha-2)} + A_2(t)(a-s)_q^{(\alpha-1)} + A_3(b-s)_q^{(\alpha)} & s \leq \min\{a,b\}; \\ A_1(t)(1-s)_q^{(\alpha-2)} + A_2(t)(a-s)_q^{(\alpha-1)} & b \leq s \leq a; \\ A_1(t)(1-s)_q^{(\alpha-2)} + A_3(t)(b-s)_q^{(\alpha)} & a \leq s \leq b; \\ A_1(t)(1-s)_q^{(\alpha-2)} & s \geq \max\{a,b\}; \end{cases} \tag{3.2}$$

whenever $0 \leq t \leq s \leq 1$*. Also*

$$\begin{cases} A_0 & = \frac{1}{\Gamma_q(\alpha)}, \\ A_1(t) & = \frac{b(1-a+t)-\mu(a,b)}{\mu(a,b)\Gamma_q(\alpha-1)}, \\ A_2(t) & = \frac{\mu(a,b)+b(a+t-1)}{\mu(a,b)\Gamma_q(\alpha)}, \\ A_3(t) & = \frac{\mu(a,b)(1-a)+t}{\mu(a,b)\Gamma_q(\alpha+1)}, \end{cases} \tag{3.3}$$

and

$$\mu(a,b) = b(1-a) + \frac{b^2}{2} > 0. \tag{3.4}$$

Proof. Consider the problem: $\mathbb{D}_q^\alpha u(t) = g(t)$. Using Lemma 2.7, it is deduced that $u(t) = -\mathbb{I}_q^\alpha g(t) + c_0 + c_1 t$, where c_0, c_1 are some real numbers, and $\mathbb{I}_q^\alpha$ is Riemann-Liouville type q–integral of order α. Hence, $u'(t) = -\mathbb{I}_q^{\alpha-1} g(t) + c_1$ where $\mathbb{I}_q^{\alpha-1}$ is a fractional Riemann-Liouville type q–integral of order $\alpha - 1$. By applying condition $u'(1) = u(a)$, we get:

$$-\mathbb{I}_q^{\alpha-1} g(1) + c_1 = -\mathbb{I}_q^\alpha g(a) + c_0 + c_1 a,$$

and so $c_0 = -\mathbb{I}_q^{\alpha-1} g(1) + \mathbb{I}_q^\alpha g(a) + (1-a)c_1$. one can easily check that

$$\int_0^b u(r)\,\mathrm{d}r = -\mathbb{I}_q^{\alpha+1} g(b) - b\mathbb{I}_q^{\alpha-1} g(1) + \mu\mathbb{I}_q^\alpha g(a) + bc_1(1-a) + \frac{1}{2}c_1 b^2.$$

Since $\int_0^b u(r)\,\mathrm{d}r = 0$, we get:

$$c_1 = \frac{1}{\mu(a,b)}\mathbb{I}_q^{\alpha+1}g(b) + \frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha-1}g(1) - \frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha+1}g(a).$$

Thus,

$$\begin{aligned} c_0 = -\mathbb{I}_q^{\alpha-1}g(1) + \mathbb{I}_q^{\alpha}g(a) + \frac{1-a}{\mu(a,b)}\mathbb{I}_q^{\alpha+1}g(b) \\ + \frac{b(1-a)}{\mu(a,b)}\mathbb{I}_q^{\alpha-1}g(1) - \frac{b(1-a)}{\mu(a,b)}\mathbb{I}_q^{\alpha}g(a) \end{aligned}$$

and so

$$\begin{aligned} u(t) = {} & -\mathbb{I}_q^{\alpha}g(t) - \mathbb{I}_q^{\alpha-1}g(1) + \mathbb{I}_q^{\alpha}g(a) + \frac{1-a}{\mu(a,b)}\mathbb{I}_q^{\alpha+1}g(b) \\ & + \frac{b(1-a)}{\mu(a,b)}\mathbb{I}_q^{\alpha-1}g(1) - \frac{b(1-a)}{\mu(a,b)}\mathbb{I}_q^{\alpha}g(a) \\ & + \frac{t}{\mu(a,b)}\mathbb{I}_q^{\alpha+1}g(b) + \frac{bt}{\mu(a,b)}\mathbb{I}_q^{\alpha-1}g(1) - \frac{bt}{\mu(a,b)}\mathbb{I}_q^{\alpha}g(a). \end{aligned}$$

Hence,

$$u(t) = -\mathbb{I}_q^{\alpha}g(t) + A_1(t)\mathbb{I}_q^{\alpha-1}g(1) + A_2(t)\mathbb{I}_q^{\alpha}g(a) + A_3(t)\mathbb{I}_q^{\alpha+1}g(b).$$

Now, some easy evaluations show us that $u(t) = \int_0^1 G_q(t,s)g(s)\,\mathrm{d}_q s$. □

Remark 3.2. *Note that, the mappings $G_q(t,s)$ and $\frac{\partial G_q(t,s)}{\partial t}$ are continuous with respect to t. Let w be a map on $\bar{J} \times \bar{\mathcal{B}}^2$ such that w is singular at some points of $\bar{J}$. Let us define the function $\Theta_u : \bar{\mathcal{B}} \to \bar{\mathcal{B}}$ by*

$$\begin{aligned} \Theta_u(t) = {} & -\mathbb{I}_q^{\alpha}w\left(t, u(t), \mathbb{D}_q^{\beta}u(t), \int_0^t f(\xi)u(\xi)\,\mathrm{d}\xi, \varphi(u(t))\right) \\ & + A_1(t)\mathbb{I}_q^{\alpha-1}w\left(1, u(1), \mathbb{D}_q^{\beta}u(1), \int_0^1 f(\xi)u(\xi)\,\mathrm{d}\xi, \varphi(u(1))\right) \\ & + A_2(t)\mathbb{I}_q^{\alpha}w\left(a, u(a), \mathbb{D}_q^{\beta}u(a), \int_0^a f(\xi)u(\xi)\,\mathrm{d}\xi, \varphi(u(a))\right) \\ & + A_3(t)\mathbb{I}_q^{\alpha+1}w\left(b, u(b), \mathbb{D}_q^{\beta}u(b), \int_0^b f(\xi)u(\xi)\,\mathrm{d}\xi, \varphi(u(b))\right), \end{aligned}$$

for all $t \in \bar{J}$, where $\mathbb{I}_q^{\alpha}$ is the fractional Riemann-Liouville q–integral of order α which is defined in (2.9)*, and $\mathbb{D}_q^{\beta}$ is the Caputo fractional q–derivative of order β which is defined in* (2.11)*. Then, by taking the first order derivative related to t, we have:*

$$\begin{aligned}\Theta_u'(t) &= \int_0^1 \frac{\partial G_q(t,\xi)}{\partial t} w\left(s, u(s), \mathbb{D}_q^{\beta} u(s), \int_0^s f(\xi)u(\xi)\,\mathrm{d}\xi, \varphi(u(s))\right) \mathrm{d}_q s \\ &= -\mathbb{I}_q^{\alpha-1} w\left(t, u(t), \mathbb{D}_q^{\beta} u(t), \int_0^t f(\xi)u(\xi)\,\mathrm{d}\xi, \varphi(u(t))\right) \\ &\quad + \frac{b}{\mu(a,b)} \mathbb{I}_q^{\alpha-1} w\left(1, u(1), \mathbb{D}_q^{\beta} u(1), \int_0^1 f(\xi)u(\xi)\,\mathrm{d}\xi, \varphi(u(1))\right) \\ &\quad + \frac{b}{\mu(a,b)} \mathbb{I}_q^{\alpha} w\left(a, u(a), \mathbb{D}_q^{\beta} u(a), \int_0^a f(\xi)u(\xi)\,\mathrm{d}\xi, \varphi(u(a))\right) \\ &\quad + \frac{1}{\mu(a,b)} \mathbb{I}_q^{\alpha+1} w\left(b, u(b), \mathbb{D}_q^{\beta} u(b), \int_0^b f(\xi)u(\xi)\,\mathrm{d}\xi, \varphi(u(b))\right).\end{aligned}$$

Obviously, the singular pointwise defined Eq (1.1) *has a solution iff the map* Θ_u *has a fixed point.*

Now, we give our main result as follows:

Theorem 3.3. *Assume that* $\alpha \geq 2$*,* $[\alpha] = n - 1$*,* $a, b, q \in J$*,* $f \in \bar{\mathcal{L}}$ *with* $\|f\|_1 = m$*,* $\varphi : \bar{\mathcal{B}} \to \mathbb{R}$ *is such that*

$$|\varphi(u(t)) - \varphi(v(t))| \leq c_1|u(t) - v(t)| + c_2|u'(t) - v'(t)|,$$

for some $c_1, c_2 \in [0, \infty)$*. Let* $\Omega : \bar{J} \times \bar{\mathcal{B}}^5 \to \mathbb{R}$ *be a mapping which is singular on some points* $\bar{J}$ *and*

$$|w(t, u_1, \ldots, u_5) - w(t, v_1, \ldots, v_5)| \leq \sum_{i=1}^{k_0} \mu_i(t)\Omega_i(u_1 - v_1, \ldots, u_5 - v_5),$$

for all $u_1, u_2, v_1, v_2 \in \bar{\mathcal{B}}$ *and almost all* $t \in \bar{J}$*, where* k_0 *is a natural number,* $\mu_i : \bar{J} \to \mathbb{R}^+$*,* $\hat{\mu}_i \in \bar{\mathcal{L}}$*,*

$$\hat{\mu}_i(s) = (1 - s)_q^{\alpha-2}\mu_i(s),$$

$\Omega_i : \bar{\mathcal{B}}^5 \to \mathbb{R}^+$ *is a nondecreasing mapping with respect to all components with*

$$\frac{\Omega_i(\nu, \nu, \nu, \nu, \nu)}{\nu^{\gamma_i}} \to p_i,$$

as $\nu \to 0^+$ *for some* $\gamma_i > 0$*,* $p_i \in \mathbb{R}^+$ *with* $1 \leq i \leq k_0$*. Suppose that*

$$|w(t, u_1, \ldots, u_5)| \leq h(t)T(u_1, \ldots, u_5),$$

for all $(u_1, \ldots, u_5) \in \bar{\mathcal{B}}^5$ *and almost all* $t \in \bar{J}$*, where* $h : \bar{J} \to \mathbb{R}^+$*,* $\hat{h} \in \bar{\mathcal{L}}$*,* $T : \bar{\mathcal{B}}^5 \to \mathbb{R}^+$ *is a nondecreasing mapping respect all their components such that*

$$\lim_{\nu \to 0^+} \frac{T(\nu, \nu, \nu, \nu, \nu)}{\nu} \in [0, \tau),$$

where $\tau = \left(\ell \|\hat{h}\|_1 M_{\alpha,a,b}\right)^{-1}$,

$$\ell = \max\left\{1, \frac{1}{\Gamma_q(2-\beta)}, m, c_1 + c_2\right\},$$

$\mu(a,b)$ *define by Eq* (3.4) *in Lemma 3.1 and*

$$\begin{aligned} M_{\alpha,a,b} = \max\Bigg\{ & \frac{1}{\Gamma_q(\alpha)} + \frac{b(2-a) - \mu(a,b)}{\mu(a,,b)\Gamma_q(\alpha-1)} + \frac{\mu(a,b) + ab}{\mu(a,b)\Gamma_q(\alpha)} \\ & + \frac{\mu(a,b)(1-a) + 1}{\mu(a,b)\Gamma_q(\alpha+1)}, \frac{1}{\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha-1)} \\ & + \frac{b}{\mu(a,b)\Gamma_q(\alpha)} + \frac{1}{\mu(a,b)\Gamma_q(\alpha+1)}\Bigg\}. \end{aligned}$$

If

$$M_{\alpha,a,b} \sum_{i=1}^{k_0} p_i \ell^{\gamma_i} \|\hat{\mu}_i\|_{\bar{J}} < 1,$$

then the pointwise defined Eq (1.1) *under boundary conditions:* $u^{(j)}(0) = 0$ *for* $j \geq 2$, $\int_0^b u(r)\,\mathrm{d}r = 0$ *and* $u'(1) = u(a)$ *has a solution.*

Proof. Let $u, v \in \bar{\mathcal{B}}$. Then, we get:

$$\begin{aligned} |\Theta_u(t) - \Theta_v(t)| \leq \Bigg| & - \mathbb{I}_q^{\alpha} w\left(t, u(t), u'(t), \mathbb{D}_q^{\beta} u(t), \int_0^t f(r)u(r)\,\mathrm{d}r, \varphi(u(t))\right) \\ & + A_1(t)\mathbb{I}_q^{\alpha-1} w\left(1, u(1), u'(1), \mathbb{D}_q^{\beta} u(1), \int_0^1 f(r)u(r)\mathrm{d}r, \varphi(u(1))\right) \\ & + A_2(t)\mathbb{I}_q^{\alpha} w\left(a, u(a), u'(a), \mathbb{D}_q^{\beta} u(a), \int_0^a f(r)u(r)\mathrm{d}r, \varphi(u(a))\right) \\ & + A_3(t)\mathbb{I}_q^{\alpha+1} w\left(b, u(b), u'(b), \mathbb{D}_q^{\beta} u(b), \int_0^1 f(r)u(r)\,\mathrm{d}r, \varphi(u(b))\right) \\ & + \mathbb{I}_q^{\alpha} w\left(t, v(t), v'(t), \mathbb{D}_q^{\beta} v(t), \int_0^t f(r)v(r)\,\mathrm{d}r, \varphi(v(t))\right) \\ & - A_1(t)\mathbb{I}_q^{\alpha-1} w\left(1, v(1), v'(1), \mathbb{D}_q^{\beta} v(1), \int_0^1 f(r)v(r)\,\mathrm{d}r, \varphi(v(1))\right) \\ & - A_2(t)\mathbb{I}_q^{\alpha} w\left(a, v(a), v'(a), \mathbb{D}_q^{\beta} v(a), \int_0^a f(r)v(r)\,\mathrm{d}r, \varphi(v(a))\right) \\ & - A_3(t)\mathbb{I}_q^{\alpha+1} w\left(b, v(b), v'(b), \mathbb{D}_q^{\beta} v(b), \int_0^1 f(r)v(r)\,\mathrm{d}r, \varphi(v(b))\right)\Bigg| \end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{I}_q^{\alpha}\bigg|w\bigg(t, u(t), u'(t), \mathbb{D}_q^{\beta}u(t), \int_0^t f(r)u(r)\,\mathrm{d}r, \varphi(u(t))\bigg)\\
&\quad - w\bigg(t, v(t), v'(t), \mathbb{D}_q^{\beta}v(t), \int_0^t f(r)v(r)\,\mathrm{d}r, \varphi(v(t))\bigg)\bigg|\\
&\quad + A_1(t)\bigg[\mathbb{I}_q^{\alpha-1}\bigg|w\bigg(1, u(1), u'(1), \mathbb{D}_q^{\beta}u(1), \int_0^1 f(r)u(r)\,\mathrm{d}r, \varphi(u(1))\bigg)\\
&\quad - w\bigg(1, v(1), v'(1), \mathbb{D}_q^{\beta}v(1), \int_0^1 f(r)v(r)\,\mathrm{d}r, \varphi(v(1))\bigg)\bigg|\bigg]\\
&\quad + A_2(t)\bigg[\mathbb{I}_q^{\alpha}\bigg|w\bigg(a, u(a), u'(a), \mathbb{D}_q^{\beta}u(a), \int_0^a f(r)u(r)\,\mathrm{d}r, \varphi(u(a))\bigg)\\
&\quad - w\bigg(a, v(a), v'(a), \mathbb{D}_q^{\beta}v(a), \int_0^a f(r)v(r)\,\mathrm{d}r, \varphi(v(a))\bigg)\bigg|\bigg]\\
&\quad + A_3(t)\bigg[\mathbb{I}_q^{\alpha+1}\bigg|w\bigg(b, u(b), u'(b), \mathbb{D}_q^{\beta}u(b), \int_0^1 f(r)u(r)\,\mathrm{d}r, \varphi(u(b))\bigg)\\
&\quad - w\bigg(b, v(b), v'(b), \mathbb{D}_q^{\beta}v(b), \int_0^1 f(r)v(r)\,\mathrm{d}r, \varphi(v(b))\bigg)\bigg|\bigg]\\
&\leq \mathbb{I}_q^{\alpha}\bigg(\sum_{i=1}^{k_0}\mu_i(t)\bigg[\Omega_i\bigg(u(t)-v(t), u'(t)-v'(t), \mathbb{D}_q^{\beta}u(t)-\mathbb{D}_q^{\beta}v(t),\\
&\qquad \int_0^t f(r)u(r)\,\mathrm{d}r - \int_0^t f(r)v(r)\,\mathrm{d}r, \varphi(u(t))-\varphi(v(t))\bigg)\bigg]\bigg)\\
&\quad + A_1(t)\mathbb{I}_q^{\alpha-1}\bigg(\sum_{i=1}^{k_0}\mu_i(1)\\
&\quad \times\bigg[\Omega_i\bigg(u(1)-v(1), u'(1)-v'(1), \mathbb{D}_q^{\beta}u(1)-\mathbb{D}_q^{\beta}v(1),\\
&\qquad \int_0^1 f(r)u(r)\,\mathrm{d}r - \int_0^1 f(r)v(r)\,\mathrm{d}r, \varphi(u(1))-\varphi(v(1))\bigg)\bigg]\bigg)\\
&\quad + A_2(t)\mathbb{I}_q^{\alpha}\bigg(\sum_{i=1}^{k_0}\mu_i(a)\\
&\quad \times\bigg[\Omega_i\bigg(u(a)-v(a), u'(a)-v'(a), \mathbb{D}_q^{\beta}u(a)-\mathbb{D}_q^{\beta}v(a),\\
&\qquad \int_0^a f(r)u(r)\,\mathrm{d}r - \int_0^q f(r)v(r)\,\mathrm{d}r, \varphi(u(a))-\varphi(v(a))\bigg)\bigg]\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ A_3(t)\mathbb{I}_q^{\alpha+1}\Big(\sum_{i=1}^{k_0}\mu_i(b) \\
&\times\Big[\Omega_i\Big(u(b)-v(b), u'(b)-v'(b), \mathbb{D}_q^{\beta}u(b)-\mathbb{D}_q^{\beta}v(b), \\
&\qquad \int_0^b f(r)u(r)\,\mathrm{d}r - \int_0^b f(r)v(r)\,\mathrm{d}r, \varphi(u(b))-\varphi(v(b))\Big)\Big]\Big) \\
\le{}& \sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha}\Big(\mu_i(t)\Big[\Omega_i\Big(\,|u(t)-v(t)|\,, |u'(t)-v'(t)|\,, \left|\mathbb{D}_q^{\beta}u(t)-\mathbb{D}_q^{\beta}v(t)\right|, \\
&\qquad \left|\int_0^t f(r)u(r)\,\mathrm{d}r - \int_0^t f(r)v(r)\,\mathrm{d}r\right|, |\varphi(u(t))-\varphi(v(t))|\,\Big)\Big]\Big) \\
&+ A_1(t)\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha-1}\Big(\mu_i(1) \\
&\times\Big[\Omega_i\Big(|u(1)-v(1)|, |u'(1)-v'(1)|, \left|\mathbb{D}_q^{\beta}u(1)-\mathbb{D}_q^{\beta}v(1)\right|, \\
&\qquad \left|\int_0^1 f(r)u(r)\,\mathrm{d}r - \int_0^1 f(r)v(r)\,\mathrm{d}r\right|, |\varphi(u(1))-\varphi(v(1))|\Big)\Big]\Big) \\
&+ A_2(t)\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha}\Big(\mu_i(a) \\
&\times\Big[\Omega_i\Big(|u(a)-v(a)|, |u'(a)-v'(a)|, \left|\mathbb{D}_q^{\beta}u(a)-\mathbb{D}_q^{\beta}v(a)\right|, \\
&\qquad \left|\int_0^a f(r)u(r)\,\mathrm{d}r - \int_0^q f(r)v(r)\,\mathrm{d}r\right|, |\varphi(u(a))-\varphi(v(a))|\Big)\Big]\Big) \\
&+ A_3(t)\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha+1}\Big(\mu_i(b) \\
&\times\Big[\Omega_i\Big(|u(b)-v(b)|, |u'(b)-v'(b)|, \left|\mathbb{D}_q^{\beta}u(b)-\mathbb{D}_q^{\beta}v(b)\right|, \\
&\qquad \left|\int_0^b f(r)u(r)\,\mathrm{d}r - \int_0^b f(r)v(r)\,\mathrm{d}r\right|, |\varphi(u(b))-\varphi(v(b))|\Big)\Big]\Big).
\end{aligned}
$$

Since $\mathbb{D}_q^{\beta}u(t) = \mathbb{I}_q^{1-\beta}u'(t)$ for $\beta \in J$, we have

$$
|\mathbb{D}_q^{\beta}u(t)| \le \mathbb{I}_q^{1-\beta}|u'(t)| \le \|u'\|\mathbb{I}_q^{1-\beta}(1) = \frac{\|u'\|}{\Gamma_q(2-\beta)},
$$

and so

$$\left|\mathbb{D}_q^{\beta}u(t)-\mathbb{D}_q^{\beta}v(t)\right|=\left|\mathbb{D}_q^{\beta}(u(t)-v(t))\right|\leq\frac{\|u'-v'\|}{\Gamma_q(2-\beta)}.$$

Thus, by considering $\xi=\|u-v\|_*$, we have:

$$\begin{aligned}|\Theta_u(t)-\Theta_v(t)|\leq&\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha}\Omega_i\Big(\|u-v\|,\|u'-v'\|,\frac{\|u'-v'\|}{\Gamma_q(2-\beta)},\\
&m\|u-v\|,c_1\|u-v\|+c_2\|u'-v'\|\Big)\\
&+A_1(t)\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha-1}\Big(\mu_i(1)\Big[\Omega_i\Big(\|u-v\|,\|u'-v'\|,\\
&\frac{\|u'-v'\|}{\Gamma_q(2-\beta)},m\|u-v\|,c_1\|u-v\|+c_2\|u'-v'\|\Big)\Big]\Big)\\
&+A_2(t)\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha}\Big(\mu_i(a)\Big[\Omega_i\Big(\|u-v\|,\|u'-v'\|,\\
&\frac{\|u'-v'\|}{\Gamma_q(2-\beta)},m\|u-v\|,c_1\|u-v\|+c_2\|u'-v'\|\Big)\Big]\Big)\\
&+A_3(t)\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha+1}\Big(\mu_i(b)\Big[\Omega_i\Big(\|u-v\|,\|u'-v'\|,\\
&\frac{\|u'-v'\|}{\Gamma_q(2-\beta)},m\|u-v\|,c_1\|u-v\|+c_2\|u'-v'\|\Big)\Big]\Big)\\
\leq&\sum_{i=1}^{k_0}\Omega_i\Big(\xi,\xi,\frac{\xi}{\Gamma_q(2-\beta)},m\xi,c_1\xi+c_2\xi\Big)\mathbb{I}_q^{\alpha}\mu_i(t)\\
&+A_1(t)\sum_{i=1}^{k_0}\Omega_i\Big(\xi,\xi,\frac{\xi}{\Gamma_q(2-\beta)},m\xi,c_1\xi+c_2\xi\Big)\mathbb{I}_q^{\alpha-1}\mu_i(1)\\
&+A_2(t)\sum_{i=1}^{k_0}\Omega_i\Big(\xi,\xi,\frac{\xi}{\Gamma_q(2-\beta)},m\xi,c_1\xi+c_2\xi\Big)\mathbb{I}_q^{\alpha}\mu_i(a)\\
&+A_3(t)\sum_{i=1}^{k_0}\Omega_i\Big(\xi,\xi,\frac{\xi}{\Gamma_q(2-\beta)},m\xi,c_1\xi+c_2\xi\Big)\mathbb{I}_q^{\alpha+1}\mu_i(b)\\
\leq&\sum_{i=1}^{k_0}\Omega_i(\ell\xi,\ell\xi,\ell\xi,\ell\xi,\ell\xi)\mathbb{I}_q^{\alpha}\mu_i(1)\end{aligned}$$

$$
\begin{aligned}
&+ A_1(t) \sum_{i=1}^{k_0} \Omega_i(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi) \mathbb{I}_q^{\alpha-1} \mu_i(1) \\
&+ A_2(t) \sum_{i=1}^{k_0} \Omega_i(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi) \mathbb{I}_q^{\alpha} \mu_i(1) \\
&+ A_3(t) \sum_{i=1}^{k_0} \Omega_i(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi) \mathbb{I}_q^{\alpha+1} \mu_i(1) \\
= {}& A_0 \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 \Omega_i \left(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi\right) \\
&+ A_1(t) \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 \Omega_i \left(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi\right) \\
&+ A_2(t) \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 \Omega_i \left(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi\right) \\
&+ A_3(t) \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 \Omega_i \left(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi\right) \\
= {}& \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 \Omega_i \left(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi\right) \\
&\times \left[A_0 + A_1(t) + A_2(t) + A_3(t)\right].
\end{aligned}
$$

This implies that

$$
\|\Theta_u - \Theta_v\| \leq \left[A_0 + A_1(t) + A_2(t) + A_3(t)\right] \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 \Omega_i \left(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi\right).
$$

Assume that $u, v \in \bar{\mathcal{B}}$. Then, we get:

$$
\begin{aligned}
|\Theta'_u - \Theta'_v| \leq \Bigg| &- \mathbb{I}_q^{\alpha-1} w\Big(t, u(t), u'(t), \mathbb{D}_q^{\beta} u(t), \int_0^t f(r) u(r)\, \mathrm{d}r, \varphi(u(t))\Big) \\
&+ \frac{b}{\mu(a,b)} \mathbb{I}_q^{\alpha-1} w\Big(1, u(1), u'(1), \mathbb{D}_q^{\beta} u(1), \int_0^1 f(r) u(r)\, \mathrm{d}r, \varphi(u(1))\Big) \\
&+ \frac{b}{\mu(a,b)} \mathbb{I}_q^{\alpha} w\Big(a, u(a), u'(a), \mathbb{D}_q^{\beta} u(a), \int_0^a f(r) u(r)\, \mathrm{d}r, \varphi(u(a))\Big) \\
&+ \frac{1}{\mu(a,b)} \mathbb{I}_q^{\alpha+1} w\Big(b, u(b), u'(b), \mathbb{D}_q^{\beta} u(b), \int_0^b f(r) u(r)\, \mathrm{d}r, \varphi(u(b))\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathbb{I}_q^{\alpha-1} w\Big(t, v(t), v'(t), \mathbb{D}_q^{\beta} v(t), \int_0^t f(r)v(r)\,\mathrm{d}r, \varphi(v(t))\Big) \\
&- \frac{b}{\mu(a,b)} \mathbb{I}_q^{\alpha-1} w\Big(1, v(1), v'(1), \mathbb{D}_q^{\beta} v(1), \int_0^1 f(r)v(r)\,\mathrm{d}r, \varphi(v(1))\Big) \\
&- \frac{b}{\mu(a,b)} \mathbb{I}_q^{\alpha} w\Big(a, v(a), v'(a), \mathbb{D}_q^{\beta} v(a), \int_0^a f(r)v(r)\,\mathrm{d}r, \varphi(v(a))\Big) \\
&- \frac{1}{\mu(a,b)} \mathbb{I}_q^{\alpha+1} w\Big(b, v(b), v'(b), \mathbb{D}_q^{\beta} v(b), \int_0^b f(r)v(r)\,\mathrm{d}r, \varphi(u(b))\Big)\Big| \\
\leq\ & \mathbb{I}_q^{\alpha-1}\Big| w\Big(t, u(t), u'(t), \mathbb{D}_q^{\beta} u(t), \int_0^t f(r)u(r)\,\mathrm{d}r, \varphi(u(t))\Big) \\
&- w\Big(t, v(t), v'(t), \mathbb{D}_q^{\beta} v(t), \int_0^t f(r)v(r)\,\mathrm{d}r, \varphi(v(t))\Big)\Big| \\
&+ \frac{b}{\mu(a,b)} \mathbb{I}_q^{\alpha-1}\Big| w\Big(1, u(1), u'(1), \mathbb{D}_q^{\beta} u(1), \int_0^1 f(r)u(r)\,\mathrm{d}r, \varphi(u(1))\Big) \\
&- \mathbb{I}_q^{\alpha-1} w\Big(1, v(1), v'(1), \mathbb{D}_q^{\beta} v(1), \int_0^1 f(r)v(r)\,\mathrm{d}r, \varphi(v(1))\Big)\Big| \\
&+ \frac{b}{\mu(a,b)} \mathbb{I}_q^{\alpha}\Big| w\Big(a, u(a), u'(a), \mathbb{D}_q^{\beta} u(a), \int_0^a f(r)u(r)\,\mathrm{d}r, \varphi(u(a))\Big) \\
&- \mathbb{I}_q^{\alpha} w\Big(a, v(a), v'(a), \mathbb{D}_q^{\beta} v(a), \int_0^a f(r)v(r)\,\mathrm{d}r, \varphi(v(a))\Big)\Big| \\
&+ \frac{1}{\mu(a,b)} \mathbb{I}_q 3^{\alpha+1}\Big| w\Big(b, u(b), u'(b), \mathbb{D}_q^{\beta} u(b), \int_0^b f(r)u(r)\,\mathrm{d}r, \varphi(u(b))\Big) \\
&- w\Big(b, v(b), v'(b), \mathbb{D}_q^{\beta} v(b), \int_0^b f(r)v(r)\,\mathrm{d}r, \varphi(u(b))\Big)\Big| \\
\leq\ & \mathbb{I}_q^{\alpha-1} \sum_{i=1}^{k_0} \mu_i(t)\Big[\Omega_i\Big(u(t)-v(t), u'(t)-v'(t), \mathbb{D}_q^{\beta} u(t) - \mathbb{D}_q^{\beta} v(t), \\
&\int_0^t f(r)u(r)\,\mathrm{d}r - \int_0^t f(r)v(r)\,\mathrm{d}r, \varphi(u(t)) - \varphi(v(t))\Big)\Big] \\
&+ \frac{b}{\mu(a,b)} \mathbb{I}_q^{\alpha-1} \sum_{i=1}^{k_0} \mu_i(1)
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big[\Omega_i\Big(u(1)-v(1), u'(1)-v'(1), \mathbb{D}_q^{\beta}u(1)-\mathbb{D}_q^{\beta}v(1),\\
&\int_0^1 f(r)u(r)\,\mathrm{d}r-\int_0^1 f(r)v(r)\,\mathrm{d}r, \varphi(u(1))-\varphi(v(1))\Big)\Big]\\
&+\frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha}\sum_{i=1}^{k_0}\mu_i(a)\\
&\times\Big[\Omega_i\Big(u(a)-v(a), u'(a)-v'(a), \mathbb{D}_q^{\beta}u(a)-\mathbb{D}_q^{\beta}v(a),\\
&\int_0^a f(r)u(r)\,\mathrm{d}r-\int_0^a f(r)v(r)\,\mathrm{d}r, \varphi(u(a))-\varphi(v(a))\Big)\Big]\\
&+\frac{1}{\mu(a,b)}\mathbb{I}_q^{\alpha+1}\sum_{i=1}^{k_0}\mu_i(b)\\
&\times\Big[\Omega_i\Big(u(b)-v(b), u'(b)-v'(b), \mathbb{D}_q^{\beta}u(b)-\mathbb{D}_q^{\beta}v(b),\\
&\int_0^b f(r)u(r)\,\mathrm{d}r-\int_0^b f(r)v(r)\,\mathrm{d}r, \varphi(u(b))-\varphi(v(b))\Big)\Big]\\
\leq&\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha-1}\mu_i(t)\Big[\Omega_i\Big(|u(t)-v(t)|, |u'(t)-v'(t)|, |\mathbb{D}_q^{\beta}(u(t)-v(t))|,\\
&\Big|\int_0^t f(r)(u(r)-v(r))\,\mathrm{d}r\Big|, |\varphi(u(t))-\varphi(v(t))|\Big)\Big]\\
&+\frac{b}{\mu(a,b)}\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha-1}\mu_i(1)\\
&\times\Big[\Omega_i\Big(|u(1)-v(1)|, |u'(1)-v'(1)|, |\mathbb{D}_q^{\beta}(u(1)-v(1))|,\\
&\Big|\int_0^1 f(r)(u(r)-v(r))\,\mathrm{d}r\Big|, |\varphi(u(1))-\varphi(v(1))|\Big)\Big]\\
&+\frac{b}{\mu(a,b)}\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha}\mu_i(1)\\
&\times\Big[\Omega_i\Big(|u(1)-v(1)|, |u'(1)-v'(1)|, |\mathbb{D}_q^{\beta}(u(1)-v(1))|,
\end{aligned}
$$

$$\Big|\int_0^1 f(r)(u(r)-v(r))\,\mathrm{d}r\Big|, |\varphi(u(1))-\varphi(v(1))|\Big)\Big]$$

$$+\frac{1}{\mu(a,b)}\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha+1}\mu_i(1)$$

$$\times\Big[\Omega_i\Big(|u(1)-v(1)|, |u'(1)-v'(1)|, |\mathbb{D}_q^{\beta}(u(1)-v(1))|,$$

$$\Big|\int_0^1 f(r)(u(r)-v(r))\,\mathrm{d}r\Big|, |\varphi(u(1))-\varphi(v(1))|\Big)\Big]$$

$$\leq\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha-1}\Big[\Omega_i\Big(\|u(t)-v(t)\|, \|u'(t)-v'(t)\|, \frac{\|u'(t)-v'(t)\|}{\Gamma_q(2-\beta)},$$

$$m\|u(t)-v(t)\|, c_1\|u(t)-v(t)\|+c_2\|u'(t)-v'(t)\|\Big)\Big]$$

$$+\frac{b}{\mu(a,b)}\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha-1}\Big[\Omega_i\Big(\|u(1)-v(1)\|, \|u'(1)-v'(1)\|, \frac{\|u'(1)-v'(1)\|}{\Gamma_q(2-\beta)},$$

$$m\|u(1)-v(1)\|, c_1\|u(1)-v(1)\|+c_2\|u'(1)-v'(1)\|\Big)\Big]$$

$$+\frac{b}{\mu(a,b)}\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha}\Big[\Omega_i\Big(\|u(a)-v(a)\|, \|u'(a)-v'(a)\|, \frac{\|u'(a)-v'(a)\|}{\Gamma_q(2-\beta)},$$

$$m\|u(a)-v(a)\|, c_1\|u(a)-v(a)\|+c_2\|u'(a)-v'(a)\|\Big)\Big]$$

$$+\frac{1}{\mu(a,b)}\sum_{i=1}^{k_0}\mathbb{I}_q^{\alpha+1}\Big[\Omega_i\Big(\|u(b)-v(b)\|, \|u'(b)-v'(b)\|, \frac{\|u'(b)-v'(b)\|}{\Gamma_q(2-\beta)},$$

$$m\|u(b)-v(b)\|, c_1\|u(b)-v(b)\|+c_2\|u'(b)-v'(b)\|\Big)\Big]$$

$$\leq\sum_{i=1}^{k_0}\Omega_i\Big(\xi,\xi,\frac{\xi}{\Gamma_q(2-\beta)}, m\xi, c_1\xi+c_2\xi\Big)\mathbb{I}_q^{\alpha-1}\mu_i(t)$$

$$+\frac{b}{\mu(a,b)}\sum_{i=1}^{k_0}\Omega_i\Big(\xi,\xi,\frac{\xi}{\Gamma_q(2-\beta)}, m\xi, c_1\xi+c_2\xi\Big)\mathbb{I}_q^{\alpha-1}\mu_i(1)$$

$$+\frac{b}{\mu(a,b)}\sum_{i=1}^{k_0}\Omega_i\Big(\xi,\xi,\frac{\xi}{\Gamma_q(2-\beta)}, m\xi, c_1\xi+c_2\xi\Big)\mathbb{I}_q^{\alpha}\mu_i(a)$$

$$
\begin{aligned}
&+ \frac{1}{\mu(a,b)} \sum_{i=1}^{k_0} \Omega_i\Big(\xi, \xi, \frac{\xi}{\Gamma_q(2-\beta)}, m\xi, c_1\xi + c_2\xi\Big)\mathbb{I}_q^{\alpha+1}\mu_i(b) \\
\leq &\sum_{i=1}^{k_0} \Omega_i\Big(\ell\xi\ell\xi, \ell\xi, \ell\xi, \ell\xi\Big)\mathbb{I}_q^{\alpha-1}\mu_i(1) \\
&+ \frac{b}{\mu(a,b)} \sum_{i=1}^{k_0} \Omega_i\Big(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi\Big)\mathbb{I}_q^{\alpha-1}\mu_i(1) \\
&+ \frac{b}{\mu(a,b)} \sum_{i=1}^{k_0} \Omega_i\Big(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi\Big)\mathbb{I}_q^{\alpha}\mu_i(1) \\
&+ \frac{1}{\mu(a,b)} \sum_{i=1}^{k_0} \Omega_i\Big(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi\Big)\mathbb{I}_q^{\alpha+1}\mu_i(1) \\
= &\frac{1}{\Gamma_q(\alpha-1)} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 \Omega_i(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi) \\
&+ \frac{b}{\mu(a,b)\Gamma_q(\alpha-1)} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 \Omega_i(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi) \\
&+ \frac{b}{\mu(a,b)\Gamma_q(\alpha)} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 \Omega_i(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi) \\
&+ \frac{1}{\mu(a,b)\Gamma_q(\alpha+1)} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 \Omega_i(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi) \\
= &\sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 \Omega_i(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi)\Big[\frac{1}{\Gamma_q(\alpha-1)} \\
&+ \frac{b}{\mu(a,b)\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha)} + \frac{1}{\mu(a,b)\Gamma_q(\alpha+1)}\Big].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|\Theta'_u - \Theta'_v\| \leq &\sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 \Omega_i(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi)\Big[\frac{1}{\Gamma_q(\alpha-1)} \\
&+ \frac{b}{\mu(a,b)\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha)} + \frac{1}{\mu(a,b)\Gamma_q(\alpha+1)}\Big]
\end{aligned}
$$

and so

$$\|\Theta_u - \Theta_v\|_* \leq \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 \Omega_i(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi) \max\Big\{\frac{1}{\Gamma_q(\alpha)} + \frac{b(2-a) - \mu(a,b)}{\mu(a,b)\Gamma_q(\alpha-1)} + \frac{\mu(a,b) + ab}{\mu(a,b)\Gamma_q(\alpha)} + \frac{\mu(a,b)(1-a) + 1}{\mu(a,b)\Gamma_q(\alpha+1)}, \frac{1}{\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha)} + \frac{1}{\mu(a,b)\Gamma_q(\alpha+1)}\Big\}.$$

If

$$M_{\alpha,a,b} = \max\Big\{\frac{1}{\Gamma_q(\alpha)} + \frac{b(2-a) - \mu(a,b)}{\mu(a,b)\Gamma_q(\alpha-1)} + \frac{\mu(a,b) + ab}{\mu(a,b)\Gamma_q(\alpha)} + \frac{\mu(a,b)(1-a) + 1}{\mu(a,b)\Gamma_q(\alpha+1)}, \frac{1}{\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha)} + \frac{1}{\mu(a,b)\Gamma_q(\alpha+1)}\Big\},$$

then

$$\|\Theta_u - \Theta_v\|_* \leq M_{\alpha,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 \Omega_i(\ell\xi, \ell\xi, \ell\xi, \ell\xi, \ell\xi). \tag{3.5}$$

Let $0 < \varepsilon \leq 1$ be given. Since

$$\lim_{\nu\to 0^+} \frac{\Omega_i(\nu,\nu,\nu,\nu,\nu)}{\nu^{\gamma_i}} = p_i,$$

for $1 \leq i \leq k_0$, $\exists\, \delta_i = \delta_i(\varepsilon)$ such that $\nu \in (0, \delta_i]$ implies

$$\left|\frac{\Omega_i(\nu,\nu,\nu,\nu,\nu)}{\nu^{\gamma_i}} - p_i\right| < \varepsilon,$$

and so $\Omega_i(\nu,\nu,\nu,\nu,\nu)/\nu^{\gamma_i} < \varepsilon + p_i$. This consequents

$$0 \leq \Omega_i(\nu,\nu,\nu,\nu,\nu) < (\varepsilon + p_i)\nu^{\gamma_i}.$$

We take $\delta = \min\{\delta_1, \ldots, \delta_{k_0}, \varepsilon\}$. In this case, $\nu \in (0, \delta]$ implies

$$0 \leq \Omega_i(\nu,\nu,\nu,\nu,\nu) < (\varepsilon + p_i)\nu^{\gamma_i} \tag{3.6}$$

for all $1 \leq i \leq k_0$. By using (3.6), we obtain:

$$\Omega_i(\ell\xi, \ldots, \ell\xi) \leq (\varepsilon + p_i)(\ell\xi)^{\gamma_i} \leq (\varepsilon + p_i)\ell^{\gamma_i}\varepsilon^{\gamma_i}. \tag{3.7}$$

At present, by applying (3.5) and (3.7), we obtain:

$$\|\Theta_u - \Theta_v\|_* \le M_{\alpha,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 (\varepsilon + p_i)\ell^{\gamma_i}\varepsilon^{\gamma_i}.$$

Now, we consider: $\gamma = \min\{\gamma_1, \cdots, \gamma_{k_0}\}$. Hence,

$$\|\Theta_u - \Theta_v\|_* \le \varepsilon^{\gamma} M_{\alpha,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 (\varepsilon + p_i)\ell^{\gamma_i}.$$

Therefore, this implies that Θ is continuous. Since

$$M_{\alpha,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 p_i \ell^{\gamma_i} < 1,$$

there is $\varepsilon_1 > 0$ such that

$$M_{\alpha,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 (p_i + \varepsilon_1)\ell^{\gamma_i} < 1.$$

Let

$$\lambda = \lim_{\nu \to 0^+} \frac{T(\nu, \nu, \nu, \nu, \nu)}{\nu} \in [0, \tau).$$

Then, we have:

$$\lambda = \lim_{\nu \to 0^+} T(\ell\nu, \dots, \ell\nu)/(\ell\nu),$$

and so for each $\varepsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\nu \in (0, \delta(\varepsilon)]$ implies

$$0 \le \frac{T(\ell\nu, \dots, \ell\nu)}{\ell\nu} - \lambda < \varepsilon.$$

Hence, $0 \le T(\ell\nu, \dots, \ell\nu) < (\lambda + \varepsilon)\ell\nu$ and

$$0 \le T(\ell\delta(\varepsilon), \dots, \ell\delta(\varepsilon)) < (\lambda + \varepsilon)\ell\delta(\varepsilon).$$

Since $\lambda \in [0, \tau)$, choose $\varepsilon_0 > 0$ such that $\lambda + \varepsilon_0 < \tau$. Assume that

$$\eta_0 = \min\left\{\delta(\varepsilon_0), \delta(\varepsilon_1)\right\}.$$

Then, $\eta \le \eta_0$ implies $0 \le T(\ell\eta, \dots, \ell\eta) < (\lambda + \varepsilon_0)\ell\eta$. Since

$$\lim_{\nu \to 0^+} \frac{\Omega_i(\nu, \nu, \nu, \nu, \nu)}{\nu^{\gamma_i}} = p_i,$$

there exists $\eta_1 > 0$ such that $\nu \in (0, \eta_1]$ implies

$$\Omega_i(\ell\nu, \dots, \ell\nu) < (p_i + \varepsilon_0)(\ell\nu)^{\gamma_i} \tag{3.8}$$

for $i = 1, \dots, k_0$. Let $\eta = \min\{\eta_0, \frac{\eta_1}{2}, \frac{1}{2}\}$ and

$$E = \left\{u \in \bar{\mathcal{B}} : \|u\|_* \le \eta\right\}.$$

Define $\alpha : \bar{\mathcal{B}}^2 \to \mathbb{R}$ by

$$\alpha(u,v) = \begin{cases} 1 & u = v, \\ 0 & u \neq v. \end{cases}$$

Assume that $u, v \in \bar{\mathcal{B}}$ be given. If $\alpha(u,v) \geq 1$, then for every $t \in \bar{J}$, we have:

$$\begin{aligned}
|\Theta_u(t)| &\leq \int_0^t |G_q(t,s)| w\Big(s, u(s), u'(s), \mathbb{D}_q^\beta u(s), \int_0^s f(r)u(r)\,\mathrm{d}r, \varphi(u(s))\Big)\,\mathrm{d}_q s \\
&\leq \mathbb{I}_q^\alpha \Big| w\Big(t, u(t), u'(t), \mathbb{D}_q^\beta u(t), \int_0^t f(r)u(r)\,\mathrm{d}r, \varphi(u(t))\Big)\Big| \\
&\quad + A_1(t)\mathbb{I}_q^{\alpha-1} \Big| w\Big(1, u(1), u'(1), \mathbb{D}_q^\beta u(1), \int_0^1 f(r)u(r)\,\mathrm{d}r, \varphi(u(1))\Big)\Big| \\
&\quad + A_2(t)\mathbb{I}_q^{\alpha} \Big| w\Big(a, u(a), u'(a), \mathbb{D}_q^\beta u(a), \int_0^a f(r)u(r)\,\mathrm{d}r, \varphi(u(a))\Big)\Big| \\
&\quad + A_3(t)\mathbb{I}_q^{\alpha+1} \Big| w\Big(b, u(b), u'(b), \mathbb{D}_q^\beta u(b), \int_0^b f(r)u(r)\,\mathrm{d}r, \varphi(u(b))\Big)\Big| \\
&\leq \mathbb{I}_q^\alpha \Big(h(t) T\Big(u(t), u'(t), \mathbb{D}_q^\beta u(t), \int_0^t f(r)u(r)\,\mathrm{d}r, \varphi(u(t))\Big)\Big) \\
&\quad + A_1(t)\mathbb{I}_q^{\alpha-1} \Big(h(1) T\Big(u(1), u'(1), \mathbb{D}_q^\beta u(1), \int_0^1 f(r)u(r)\,\mathrm{d}r, \varphi(u(1))\Big)\Big) \\
&\quad + A_2(t)\mathbb{I}_q^{\alpha} \Big(h(a) T\Big(u(a), u'(a), \mathbb{D}_q^\beta u(a), \int_0^a f(r)u(r)\,\mathrm{d}r, \varphi(u(a))\Big)\Big) \\
&\quad + A_3(t)\mathbb{I}_q^{\alpha+1} \Big(h(b) T\Big(u(b), u'(b), \mathbb{D}_q^\beta u(b), \int_0^b f(r)u(r)\,\mathrm{d}r, \varphi(u(b))\Big)\Big) \\
&\leq \mathbb{I}_q^\alpha \Big(h(t) T\Big(|u(t)|, |u'(t)|, |\mathbb{D}_q^\beta u(t)|, \int_0^t |f(r)||u(r)|\,\mathrm{d}r, |\varphi(u(t))|\Big)\Big) \\
&\quad + A_1(t)\mathbb{I}_q^{\alpha-1} \Big(h(1) T\Big(|u(1)|, |u'(1)|, |\mathbb{D}_q^\beta u(1)|, \\
&\quad \int_0^1 |f(r)||u(r)|\,\mathrm{d}r, |\varphi(u(1))|\Big)\Big) \\
&\quad + A_2(t)\mathbb{I}_q^{\alpha} \Big(h(a) T\Big(|u(a)|, |u'(a)|, |\mathbb{D}_q^\beta u(a)|, \\
&\quad \int_0^a |f(r)||u(r)|\,\mathrm{d}r, |\varphi(u(a))|\Big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ A_3(t)\mathbb{I}_q^{\alpha+1}\Big(h(b)T\Big(|u(b)|, |u'(b)|, |\mathbb{D}_q^{\beta}u(b)|,\\
&\int_0^b |f(r)||u(r)|\,\mathrm{d}r, |\varphi(u(b))|\Big)\Big)\\
\leq\ & \mathbb{I}_q^{\alpha}\Big(h(t)T\Big(\|u(t)\|, \|u'(t)\|, \frac{\|u'(t)\|}{\Gamma_q(2-\beta)},\\
&m\|u(t)\|, c_1\|u(t)\| + c_2\|u'(t)\|\Big)\Big)\\
&+ A_1(t)\mathbb{I}_q^{\alpha-1}\Big(h(1)T\Big(\|u(t)\|, \|u'(t)\|,\\
&\frac{\|u'(t)\|}{\Gamma_q(2-\beta)}, m\|u(t)\|, c_1\|u(t)\| + c_2\|u'(t)\|\Big)\Big)\\
&+ A_2(t)\mathbb{I}_q^{\alpha}\Big(h(a)T\Big(\|u(t)\|, \|u'(t)\|,\\
&\frac{\|u'(t)\|}{\Gamma_q(2-\beta)}, m\|u(t)\|, c_1\|u(t)\| + c_2\|u'(t)\|\Big)\Big)\\
&+ A_3(t)\mathbb{I}_q^{\alpha+1}\Big(h(b)T\Big(\|u(t)\|, \|u'(t)\|,\\
&\frac{\|u'(t)\|}{\Gamma_q(2-\beta)}, m\|u(t)\|, c_1\|u(t)\| + c_2\|u'(t)\|\Big)\Big)\\
\leq\ & T\Big(\|u(t)\|, \|u'(t)\|, \frac{\|u'(t)\|}{\Gamma_q(2-\beta)},\\
&m\|u(t)\|, c_1\|u(t)\| + c_2\|u'(t)\|\Big)\mathbb{I}_q^{\alpha}h(t)\\
&+ A_1(t)T\Big(\|u(t)\|, \|u'(t)\|, \frac{\|u'(t)\|}{\Gamma_q(2-\beta)},\\
&m\|u(t)\|, c_1\|u(t)\| + c_2\|u'(t)\|\Big)\mathbb{I}_q^{\alpha-1}h(1)\\
&+ A_2(t)T\Big(\|u(t)\|, \|u'(t)\|, \frac{\|u'(t)\|}{\Gamma_q(2-\beta)},\\
&m\|u(t)\|, c_1\|u(t)\| + c_2\|u'(t)\|\Big)\mathbb{I}_q^{\alpha}h(a)\\
&+ A_3(t)T\Big(\|u(t)\|, \|u'(t)\|, \frac{\|u'(t)\|}{\Gamma_q(2-\beta)},
\end{aligned}
$$

$$
\begin{aligned}
&m\|u(t)\|, c_1\|u(t)\| + c_2\|u'(t)\|\Big)\mathbb{I}_q^{\alpha+1}h(b)\\
&\leq T(\ell\|u(t)\|_*, \ell\|u(t)\|_*, \ell\|u(t)\|_*, \ell\|u(t)\|_*, \ell\|u(t)\|_*)\|\hat{h}\|_1\\
&\quad \times [A_0 + A_1(t) + A_2(t) + A_3(t)]\\
&\leq T(\ell r, \ell r, \ell r, \ell r, \ell r)\|\hat{h}\|_1[A_0 + A_1(t) + A_2(t) + A_3(t)]\\
&\leq \ell r(\lambda+\varepsilon)\|\hat{h}\|_1[A_0 + A_1(t) + A_2(t) + A_3(t)]\\
&= \eta\Big(\ell(\lambda+\varepsilon)\|\hat{h}\|_1[A_0 + A_1(t) + A_2(t) + A_3(t)]\Big).
\end{aligned}
$$

Therefore,

$$
\|\Theta_u\| \leq \eta\Big(\ell(\lambda+\varepsilon)\|\hat{h}\|_1[A_0 + A_1(t) + A_2(t) + A_3(t)]\Big) \leq \eta.
$$

Also,

$$
\begin{aligned}
|\Theta_u'(t)| \leq &\bigg| - \mathbb{I}_q^{\alpha-1} w\Big(t, u(t), u'(t), \mathbb{D}_q^{\beta}u(t), \int_0^t f(r)u(r)\,\mathrm{d}r, \varphi(u(t))\Big)\\
&+ \frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha-1} w\Big(1, u(1), u'(1), \mathbb{D}_q^{\beta}u(1), \int_0^1 f(r)u(r)\,\mathrm{d}r, \varphi(u(1))\Big)\\
&+ \frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha} w\Big(a, u(a), u'(a), \mathbb{D}_q^{\beta}u(a), \int_0^a f(r)u(r)\,\mathrm{d}r, \varphi(u(a))\Big)\\
&+ \frac{1}{\mu(a,b)}\mathbb{I}_q^{\alpha+1} w\Big(b, u(b), u'(b), \mathbb{D}_q^{\beta}u(b), \int_0^b f(r)u(r)\,\mathrm{d}r, \varphi(u(b))\Big)\bigg|\\
\leq\ & \mathbb{I}_q^{\alpha-1}\bigg| w\Big(t, u(t), u'(t), \mathbb{D}_q^{\beta}u(t), \int_0^t f(r)u(r)\,\mathrm{d}r, \varphi(u(t))\Big)\bigg|\\
&+ \frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha-1}\bigg| w\Big(1, u(1), u'(1), \mathbb{D}_q^{\beta}u(1), \int_0^1 f(r)u(r)\,\mathrm{d}r, \varphi(u(1))\Big)\bigg|\\
&+ \frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha}\bigg| w\Big(a, u(a), u'(a), \mathbb{D}_q^{\beta}u(a), \int_0^a f(r)u(r)\,\mathrm{d}r, \varphi(u(a))\Big)\bigg|\\
&+ \frac{1}{\mu(a,b)}\mathbb{I}_q^{\alpha+1}\bigg| w\Big(b, u(b), u'(b), \mathbb{D}_q^{\beta}u(b), \int_0^b f(r)u(r)\,\mathrm{d}r, \varphi(u(b))\Big)\bigg|\\
\leq\ & \mathbb{I}_q^{\alpha-1}\Big(h(t)T\Big(u(t), u'(t), \mathbb{D}_q^{\beta}u(t), \int_0^t f(r)u(r)\,\mathrm{d}r, \varphi(u(t))\Big)\Big)\\
&+ \frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha-1}\Big(h(1)T\Big(u(1), u'(1), \mathbb{D}_q^{\beta}u(1), \int_0^1 f(r)u(r)\,\mathrm{d}r, \varphi(u(1))\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha}\Big(h(a)T\Big(u(a),u'(a),\mathbb{D}_q^{\beta}u(a),\int_0^a f(r)u(r)\,\mathrm{d}r,\varphi(u(a))\Big)\Big)\\
&+\frac{1}{\mu(a,b)}\mathbb{I}_q^{\alpha+1}\Big(h(b)T\Big(u(b),u'(b),\mathbb{D}_q^{\beta}u(b),\int_0^b f(r)u(r)\,\mathrm{d}r,\varphi(u(b))\Big)\Big)\\
\le{}& \mathbb{I}_q^{\alpha-1}\Big(h(t)T\Big(u(t),u'(t),\mathbb{D}_q^{\beta}u(t),\int_0^t f(r)u(r)\,\mathrm{d}r,\varphi(u(t))\Big)\Big)\\
&+\frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha-1}\Big(h(1)T\Big(u(1),u'(1),\mathbb{D}_q^{\beta}u(1),\int_0^1 f(r)u(r)\,\mathrm{d}r,\varphi(u(1))\Big)\Big)\\
&+\frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha}\Big(h(a)T\Big(u(a),u'(a),\mathbb{D}_q^{\beta}u(a),\int_0^a f(r)u(r)\,\mathrm{d}r,\varphi(u(a))\Big)\Big)\\
&+\frac{1}{\mu(a,b)}\mathbb{I}_q^{\alpha+1}\Big(h(b)T\Big(u(b),u'(b),\mathbb{D}_q^{\beta}u(b),\int_0^b f(r)u(r)\,\mathrm{d}r,\varphi(u(b))\Big)\Big)\\
\le{}& \mathbb{I}_q^{\alpha-1}\Big(h(t)T\Big(\|u(t)\|,\|u'(t)\|,\frac{\|u'(t)\|}{\Gamma_q(2-\beta)},\\
&m\|u(t)\|,c_1\|u(t)\|+c_2\|u'(t)\|\Big)\Big)\\
&+\frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha-1}\Big(h(1)T\Big(\|u(t)\|,\|u'(t)\|,\frac{\|u'(t)\|}{\Gamma_q(2-\beta)},\\
&m\|u(t)\|,c_1\|u(t)\|+c_2\|u'(t)\|\Big)\Big)\\
&+\frac{b}{\mu(a,b)}\mathbb{I}_q^{\alpha}\Big(h(a)T\Big(\|u(t)\|,\|u'(t)\|,\frac{\|u'(t)\|}{\Gamma_q(2-\beta)},\\
&m\|u(t)\|,c_1\|u(t)\|+c_2\|u'(t)\|\Big)\Big)\\
&+\frac{1}{\mu(a,b)}\mathbb{I}_q^{\alpha+1}\Big(h(b)T\Big(\|u(t)\|,\|u'(t)\|,\frac{\|u'(t)\|}{\Gamma_q(2-\beta)},\\
&m\|u(t)\|,c_1\|u(t)\|+c_2\|u'(t)\|\Big)\Big)\\
\le{}& T\Big(\|u(t)\|,\|u'(t)\|,\frac{\|u'(t)\|}{\Gamma_q(2-\beta)},m\|u(t)\|,c_1\|u(t)\|+c_2\|u'(t)\|\Big)\mathbb{I}_q^{\alpha-1}(h(t))\\
&+\frac{b}{\mu(a,b)}T\Big(\|u(t)\|,\|u'(t)\|,\frac{\|u'(t)\|}{\Gamma_q(2-\beta)},m\|u(t)\|,c_1\|u(t)\|+c_2\|u'(t)\|\Big)\mathbb{I}_q^{\alpha-1}(h(1))
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{b}{\mu(a,b)} T\Big(\|u(t)\|, \|u'(t)\|, \frac{\|u'(t)\|}{\Gamma_q(2-\beta)}, m\|u(t)\|, c_1\|u(t)\| + c_2\|u'(t)\|\Big) \mathbb{I}_q^{\alpha}(h(a)) \\
&+ \frac{1}{\mu(a,b)} T\Big(\|u(t)\|, \|u'(t)\|, \frac{\|u'(t)\|}{\Gamma_q(2-\beta)}, m\|u(t)\|, c_1\|u(t)\| + c_2\|u'(t)\|\Big) \mathbb{I}_q^{\alpha+1}(h(b)) \\
\le\ & T(\ell\|u\|_*, \cdots, \ell\|u\|_*)\|\hat{h}\|_1 \\
&\times \Big[\frac{1}{\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha)} + \frac{1}{\mu(a,b)\Gamma_q(\alpha+1)}\Big] \\
\le\ & T(\ell r, \cdots, \ell r)\|\hat{h}\|_1 \\
&\times \Big[\frac{1}{\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha)} + \frac{1}{\mu(a,b)\Gamma_q(\alpha+1)}\Big] \\
\le\ & (\ell r)(\lambda + \varepsilon_0)\|\hat{h}\|_1 \\
&\times \Big[\frac{1}{\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha)} + \frac{1}{\mu(a,b)\Gamma_q(\alpha+1)}\Big].
\end{aligned}
$$

Indeed,

$$
\begin{aligned}
|\Theta_u'(t)| \le\ & (\ell r)(\lambda + \varepsilon_0)\|\hat{h}\|_1 \\
&\times \Big[\frac{1}{\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha)} \\
&+ \frac{1}{\mu(a,b)\Gamma_q(\alpha+1)}\Big] \\
\le\ & r.
\end{aligned}
$$

Hence, $\|\Theta_u\|_* \le \eta$ and so $\Theta_u \in E$. Using a similar proof, we can show that $\Theta_v \in E$. This implies $\alpha(\Theta_u, \Theta_v) \ge 1$ and so Θ_u is α-admissible. It is obvious that, $E \neq \emptyset$. Choose $u_0 \in E$. Hence, $\Theta_{u_0} \in E$, and so $\alpha(u_0, \Theta_{u_0}) \ge 1$. Let $u, v \in E$. Then,

$$
\xi \le \|u\|_* + \|v\|_* \le 2\eta \le \eta_1,
$$

where $\xi = \|u - v\|_*$. Also using (3.5), we have

$$
\|\Theta_u - \Theta_v\|_* \le M_{\alpha,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 M_i(\ell\xi, \dots, \ell\xi).
$$

Now, by using (3.8), we conclude that

$$\begin{aligned}\|\Theta_u - \Theta_v\|_* &\le M_{\alpha,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 (p_i + \varepsilon_1)(\ell\xi)^{\gamma_i} \\ &\le M_{\alpha,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 (p_i + \varepsilon_1)\ell^{\gamma_i}\xi^{\gamma_i} \\ &\le M_{\alpha,a,b}\Big[\sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 (p_i + \varepsilon_1)\ell^{\gamma_i}\Big]\xi^{\gamma},\end{aligned}$$

where $\gamma = \min\{\gamma_1, \dots, \gamma_{k_0}\}$. We take:

$$\eta = M_{\alpha,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 p_i \ell^{\gamma_i}.$$

Note that, $\eta \in [0, 1)$. Define the map $\psi : [0, \infty) \to \mathbb{R}^+$ by

$$\psi(t) = \begin{cases} \eta t^{\gamma} & t \in [0, 1), \\ \eta t & t \in [1, \infty). \end{cases}$$

Then, ψ is nondecreasing and

$$\sum_{i=1}^{\infty} \psi^i(t) = \eta t^{\gamma} + \eta^2 t^{2\gamma} + \cdots \le \sum_{i=1}^{\infty} \eta^i t^{\gamma} = \frac{\tau}{1-\eta} t^{\gamma} < \infty,$$

for $0 \le t < 1$. Also, we obtain

$$\sum_{i=1}^{\infty} \psi^i(t) = \frac{\eta}{1-\eta} t < \infty,$$

for $t \in [1, \infty)$. Thus, $\sum_{i=1}^{\infty} \psi^i(t)$ is a convergent series for all $t \ge 0$ and so $\psi \in \Psi$. Also, we have

$$\alpha(u, v)\|\Theta_u - \Theta_v\|_* \le \phi(\xi).$$

If $u \notin E$ or $v \notin E$, then the last inequality holds obviously. This shows that

$$\alpha(u, v) d(\Theta_u, \Theta_v) \le \phi(d(u, v)),$$

for all $u, v \in \bar{\mathcal{B}}$. Now, Lemma 2.6 implies that Θ has a fixed point that is the solution for problem (1.1). □

4. An illustrative example with application

The following illustrative example is given to support the validity of our main results. A computational method is provided here to test the proposed problem (1.1). Linear motion is commonly basic among all other motions. From the 1st law of Newton's motion, objects that are not experiencing any net force will continue to move in a straight line with a constant velocity until they are subjected to a net force.

Example 4.1. *We consider a constrained motion of a particle along a straight line restrained by two linear springs with equal spring constant (stiffness coefficient) under external force and fractional damping along the t-axis (Figure 1).*

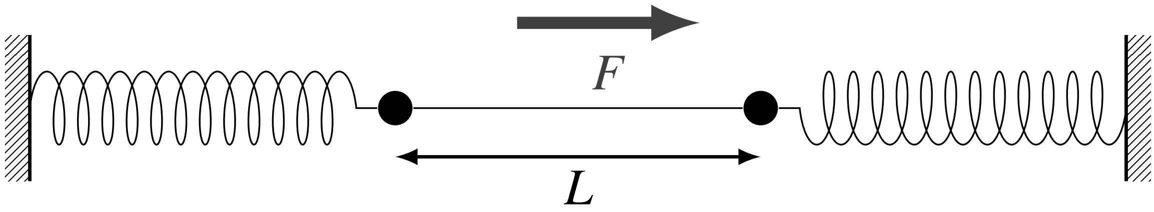

Figure 1. A particle along a straight line restrained by two linear springs with equal spring constant.

We consider the pointwise defined equation:

$$100\theta(t)^c\mathbb{D}_q^{2.5}u(t) + p(t)u(t) = -p(t)\Big(|u'(t)| + \Big|\mathbb{D}_q^{\frac{1}{2}}u(t)\Big| + \Big|\int_0^t \frac{u(r)}{\sqrt{r}}\,\mathrm{d}r\Big| + |\sin(u(t))|\Big), \tag{4.1}$$

where

$$p(t) = \frac{1}{8}\left(2 - 2L - \eta^2 L - \eta^2 L\cos t\right),$$

η is constant and L is the unstretched length of the spring. We change Eq (4.1) *into a form of the problem* (1.1) *as follows:*

$$\mathbb{D}_q^{\frac{5}{2}}u(t) = \frac{1}{100\,\theta(t)}\Big(|u(t)| + |u'(t)| + \Big|\mathbb{D}_q^{\frac{1}{2}}u(t)\Big| + \Big|\int_0^t \frac{u(r)}{\sqrt{r}}\,\mathrm{d}r\Big| + |\sin(u(t))|\Big) \tag{4.2}$$

with boundary conditions:

$$\int_0^{\frac{1}{3}} u(r)\,\mathrm{d}r = 0, \quad u'(1) = u(\frac{1}{4}), \quad u''(0) = 0.$$

Also

$$\theta(t) = \begin{cases} 0 & t \in \bar{J} \cap \mathbb{Q}, \\ 1 - t & t \in \bar{J} \cap \mathbb{Q}^c. \end{cases}$$

Take $\alpha = \frac{5}{2} \geq 2$, $\beta = \frac{1}{2} \in J$, $a = \frac{1}{4} \in J$, $b = \frac{1}{3} \in J$, $k_0 = 1$, $\gamma_1 = 1$, $\mu_1(t) = h(t) = \frac{1}{\theta(t)}$, $c_1 = \frac{1}{3}$, $c_2 = \frac{2}{3}$, $f(\xi) = \frac{u(\xi)}{\sqrt{\xi}}$, $\varphi(x) = \sin(x)$ and

$$T(u_1, \ldots, u_5) = \Omega_1(u_1, \ldots, u_5) = \frac{1}{500}\Big(|u_1| + \cdots + |u_5|\Big).$$

Then, we get:

$$|\varphi(u)-\varphi(v)| = |\sin(u)-\sin(v)| \leq |u-v| = c_1|u-v| + c_2|u'-v'|,$$

$$|w(t,u_1,\ldots,u_5)-w(t,v_1,\ldots,v_5)| \leq \mu_1(t)\Big[|u_1-v_1|+\cdots+|u_5-v_5|\Big],$$

$$p_1 = \lim_{\nu\to 0^+}\frac{\Omega_1(\nu,\nu,\nu,\nu,\nu)}{\nu^{\gamma_1}} = \lim_{\nu\to 0^+}\frac{5|\nu|}{500\nu} = 0.01,$$

$\mu_1, h \in L^1$, *$m = \|h\|_1 = 2$,*

$$\|\hat{h}\|_{\bar{J}} = \|\hat{\mu}_1\|_{\bar{J}} = \int_0^1 \frac{1}{\theta(s)}(1-s)^{\alpha-2}\,\mathrm{d}s = \int_0^1 \frac{(1-s)^{\frac{1}{2}}}{1-s}\,\mathrm{d}s = 2,$$

$$|w(t,u_1,\ldots,u_5)| \leq h(t)T(u_1,\ldots,u_5),$$

T, Ω_1 are non-negative and non-decreasing with respect to $u_1,\ldots,u_5$,

$$\mu(a,b) = b(1-a) + \frac{b^2}{2} = \frac{11}{36},$$

$$\ell = \max\left\{1, \frac{1}{\Gamma_q(2-\beta)}, m, c_1+c_2\right\} = \max\left\{1, \frac{1}{\Gamma_q(\frac{3}{2})}, 2, 1\right\} = 2,$$

$$\begin{aligned} M_{\alpha,a,b} = \max\Bigg\{&\frac{1}{\Gamma_q(\alpha)} + \frac{b(2-a)-\mu(a,b)}{\mu(a,b)\Gamma_q(\alpha-1)} + \frac{\mu(a,b)+ab}{\mu(a,b)\Gamma_q(\alpha)} \\ &+ \frac{\mu(a,b)(1-a)+1}{\mu(a,b)\Gamma_q(\alpha+1)}, \frac{1}{\Gamma_q(\alpha-1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha-1)} \\ &+ \frac{b}{\mu(a,b)\Gamma_q(\alpha)} + \frac{1}{\mu(a,b)\Gamma_q(\alpha+1)}\Bigg\} \\ = \max\Bigg\{&\frac{25}{11\Gamma_q(\frac{5}{2})} + \frac{10}{11\Gamma_q(\frac{3}{2})} + \frac{177}{44\Gamma_q(\frac{7}{2})}, \\ &\frac{23}{11\Gamma_q(\frac{3}{2})} + \frac{12}{11\Gamma_q(\frac{5}{2})} + \frac{36}{11\Gamma_q(\frac{7}{2})}\Bigg\}. \end{aligned}$$

We put:

$$\begin{aligned} \Lambda_1 &= \frac{25}{11\Gamma_q(\frac{5}{2})} + \frac{10}{11\Gamma_q(\frac{3}{2})} + \frac{177}{44\Gamma_q(\frac{7}{2})}, \\ \Lambda_2 &= \frac{23}{11\Gamma_q(\frac{3}{2})} + \frac{12}{11\Gamma_q(\frac{5}{2})} + \frac{36}{11\Gamma_q(\frac{7}{2})}. \end{aligned} \tag{4.3}$$

Table 1 shows the values of Λ_1 *and* Λ_2 *for* $q = \left\{\frac{1}{8}, \frac{1}{2}, \frac{4}{5}, \frac{8}{9}\right\}$*. We can see that*

$$M_{\alpha,a,b} = 33.170478, 21.551855, 16.363257, 15.234356,$$

for $q = \frac{1}{8}$, $\frac{1}{2}$, $\frac{4}{5}$ *and* $\frac{8}{9}$*, respectively. Thus, by using the numerical results, we obtain:*

$$\tau = \left(\ell \|\hat{h}\|_1 M_{\alpha,a,b}\right)^{-1} \geq \frac{1}{4 \times 33.1704} = 0.0075,$$

whenever $q = \frac{1}{8}$,

$$\tau = \left(\ell \|\hat{h}\|_1 M_{\alpha,a,b}\right)^{-1} \geq \frac{1}{4 \times 21.5518} = 0.0116,$$

whenever $q = \frac{1}{2}$,

$$\tau = \left(\ell \|\hat{h}\|_1 M_{\alpha,a,b}\right)^{-1} \geq \frac{1}{4 \times 16.3632} = 0.0153,$$

whenever $q = \frac{4}{5}$ *and*

$$\tau = \left(\ell \|\hat{h}\|_1 M_{\alpha,a,b}\right)^{-1} \geq \frac{1}{4 \times 15.2343} = 0.0164,$$

whenever $q = \frac{8}{9}$*. Also, we can check that*

$$\lim_{\nu \to 0^+} \frac{T(\nu, \nu, \nu, \nu, \nu)}{\nu} = 0.01 \in [0, \tau),$$

and for all $q \in J$

$$M_{\alpha,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_{\bar{J}} p_i \ell^{\gamma_i} = M_{\alpha,a,b} \times 2 \times 0.01 \times 2^1 = 0.04 M_{\alpha,a,b} < 1.$$

Table 2 shows numerical results for different values of $q \in J$*. Figure 2 shows the curve of these results. Now, according to the obtained results, Theorem 3.3 implies that problem* (4.2) *has a solution.*

Table 1. The results of Λ_1, Λ_2 in Eq (4.3) in Example 4.1 for $q \in \left\{\frac{1}{8}, \frac{1}{2}, \frac{4}{5}, \frac{8}{9}\right\}$.

	$q = \frac{1}{8}$		$q = \frac{1}{2}$		$q = \frac{4}{5}$		$q = \frac{8}{9}$	
n	Λ_1	Λ_2	Λ_1	Λ_2	Λ_1	Λ_2	Λ_1	Λ_2
1	6.4269	33.0986	4.1726	17.6569	1.6844	4.9657	0.9465	2.2669
2	6.4386	33.1615	4.5536	19.5549	2.1098	6.6125	1.1971	3.0219
3	6.4401	33.1694	4.7492	20.5409	2.4808	8.1416	1.4377	3.8100
4	$\underline{6.4403}$	33.1703	4.8483	21.0433	2.7983	9.5087	1.6670	4.6128
5	6.4403	$\underline{33.1705}$	4.8982	21.2968	3.0660	10.6983	1.8838	5.4129
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
16	6.4403	33.1705	4.9482	21.5517	4.1529	15.8174	3.4095	11.8900
17	6.4403	33.1705	4.9482	21.5518	4.1750	15.9256	3.4840	12.2356
18	6.4403	33.1705	$\underline{4.9483}$	21.5518	4.1928	16.0126	3.5509	12.5482
19	6.4403	33.1705	4.9483	21.5518	4.2070	16.0823	3.6110	12.8303
20	6.4403	33.1705	4.9483	$\underline{21.5519}$	4.2184	16.1383	3.6649	13.0844
21	6.4403	33.1705	4.9483	21.5519	4.2275	16.1831	3.7132	13.3130
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	6.4403	33.1705	4.9483	21.5519	4.2640	16.3630	4.0987	15.1686
51	6.4403	33.1705	4.9483	21.5519	4.2640	16.3630	4.1002	15.1759
52	6.4403	33.1705	4.9483	21.5519	4.2640	16.3631	4.1015	15.1824
53	6.4403	33.1705	4.9483	21.5519	4.2640	16.3631	4.1027	15.1882
54	6.4403	33.1705	4.9483	21.5519	$\underline{4.2641}$	16.3631	4.1037	15.1933
55	6.4403	33.1705	4.9483	21.5519	4.2641	$\underline{16.3632}$	4.1047	15.1979
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
91	6.4403	33.1705	4.9483	21.5519	4.2641	16.3633	4.1120	15.2339
92	6.4403	33.1705	4.9483	21.5519	4.2641	16.3633	$\underline{4.1121}$	15.2339
93	6.4403	33.1705	4.9483	21.5519	4.2641	16.3633	4.1121	15.2340
94	6.4403	33.1705	4.9483	21.5519	4.2641	16.3633	4.1121	15.2340
95	6.4403	33.1705	4.9483	21.5519	4.2641	16.3633	4.1121	15.2341
96	6.4403	33.1705	4.9483	21.5519	4.2641	16.3633	4.1121	15.2341
97	6.4403	33.1705	4.9483	21.5519	4.2641	16.3633	4.1121	15.2341
98	6.4403	33.1705	4.9483	21.5519	4.2641	16.3633	4.1121	15.2341
99	6.4403	33.1705	4.9483	21.5519	4.2641	16.3633	4.1121	15.2342
100	6.4403	33.1705	4.9483	21.5519	4.2641	16.3633	4.1121	15.2342
101	6.4403	33.1705	4.9483	21.5519	4.2641	16.3633	4.1121	15.2342
102	6.4403	33.1705	4.9483	21.5519	4.2641	16.3633	4.1121	15.2342
103	6.4403	33.1705	4.9483	21.5519	4.2641	16.3633	4.1121	$\underline{15.2343}$
104	6.4403	33.1705	4.9483	21.5519	4.2641	16.3633	4.1121	15.2343

Table 2. The results of $M_{\alpha,a,b}$ and $(*) = M_{\alpha,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_{\bar{J}} p_i \ell^{\gamma_i}$ in Example 4.1 for $q \in \left\{\frac{1}{8}, \frac{1}{2}, \frac{4}{5}, \frac{8}{9}\right\}$.

	$q=\frac{1}{8}$		$q=\frac{1}{2}$		$q=\frac{4}{5}$		$q=\frac{8}{9}$	
n	$M_{\alpha,a,b}$	$(*)$	$M_{\alpha,a,b}$	$(*)$	$M_{\alpha,a,b}$	$(*)$	$M_{\alpha,a,b}$	$(*)$
1	33.0986	1.3239	17.6569	0.7063	4.9657	0.1986	2.2669	0.0907
2	33.1615	1.3265	19.5549	0.7822	6.6125	0.2645	3.0219	0.1209
3	33.1694	1.3268	20.5409	0.8216	8.1416	0.3257	3.8100	0.1524
4	33.1703	1.3268	21.0433	0.8417	9.5087	0.3803	4.6128	0.1845
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
12	33.1705	1.3268	21.5499	0.8620	15.0506	0.6020	10.1272	0.4051
13	33.1705	1.3268	21.5509	0.8620	15.3077	0.6123	10.6296	0.4252
14	33.1705	1.3268	21.5514	0.8621	15.5153	0.6206	11.0894	0.4436
15	33.1705	1.3268	21.5516	0.8621	15.6827	0.6273	11.5088	0.4604
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
73	33.1705	1.3268	21.5519	0.8621	16.3633	0.6545	15.2300	0.6092
74	33.1705	1.3268	21.5519	0.8621	16.3633	0.6545	15.2305	0.6092
75	33.1705	1.3268	21.5519	0.8621	16.3633	0.6545	15.2309	0.6092
76	33.1705	1.3268	21.5519	0.8621	16.3633	0.6545	15.2313	0.6093
77	33.1705	1.3268	21.5519	0.8621	16.3633	0.6545	15.2316	0.6093
78	33.1705	1.3268	21.5519	0.8621	16.3633	0.6545	15.2319	0.6093
79	33.1705	1.3268	21.5519	0.8621	16.3633	0.6545	15.2322	0.6093
80	33.1705	1.3268	21.5519	0.8621	16.3633	0.6545	15.2325	0.6093

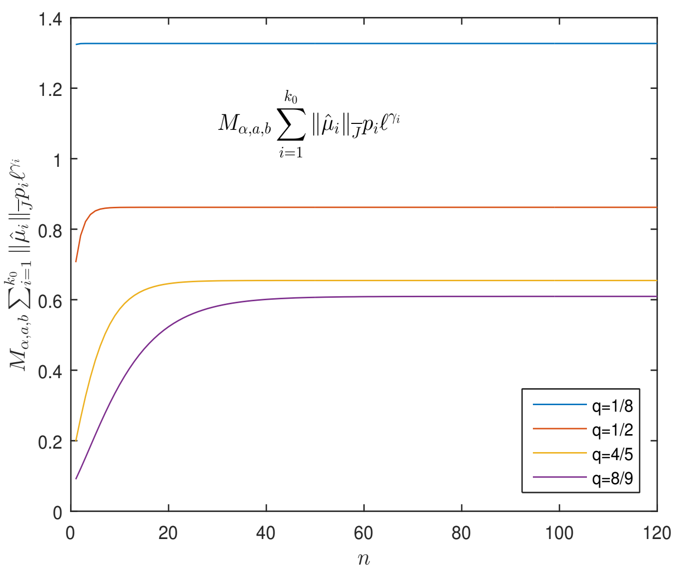

Figure 2. Numerical results of $M_{\alpha,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_{\bar{J}} p_i \ell^{\gamma_i}$ where $q = \frac{1}{8}, \frac{1}{2}, \frac{4}{5}$ and $\frac{8}{9}$ in Example 4.1.

5. Conclusions

The multi-singular pointwise defined fractional q–integro-differential equation has been successfully investigated in this work. The investigation of this particular equation provides us with a powerful tool in modeling most scientific phenomena without the need to remove most parameters which have an essential role in the physical interpretation of the studied phenomena. Multi-singular pointwise defined fractional q–integro-differential equation (1.1) has been studied on a time scale under some boundary conditions. An application that describes the motion of a particle in the plane has been provided in this work to support our results' validity and applicability in the fields of physics and engineering.

Acknowledgments

The first author was supported by Bu-Ali Sina University.

Conflict of interest

The authors declare that they have no competing interests.

References

1. T. Abdeljawad, J. Alzabut, D. Baleanu, A generalized q–fractional gronwall inequality and its applications to nonlinear delay q–fractional difference systems, *J. Inequal. Appl.*, **2016** (2016), 240. https://doi.org/10.1186/s13660-016-1181-2
2. C. R. Adams, The general theory of a class of linear partial q–difference equations, *T. Am. Math. Soc.*, **26** (1924), 283–312. https://doi.org/10.2307/1989141
3. R. P. Agarwal, Certain fractional q–integrals and q–derivatives, *Mathematical Proceedings of the Cambridge Philosophical Society*, **66** (1965), 365–370. https://doi.org/10.1017/S0305004100045060
4. R. P. Agarwal, D. O'Regan, S. Staněk, Positive solutions for Dirichlet problem of singular nonlinear fractional differential equations, *J. Math. Anal. Appl.*, **371** (2010), 57–68. https://doi.org/10.1016/j.jmaa.2010.04.034
5. B. Ahmad, S. Etemad, M. Ettefagh, S. Rezapour, On the existence of solutions for fractional q–difference inclusions with q–antiperiodic boundary conditions, *Bull. Math. Soc. Sci. Math. Roumanie*, **59** (2016), 119–134.
6. B. Ahmad, S. K. Ntouyas, I. Purnaras, Existence results for nonlocal boundary value problems of nonlinear fractional q–difference equations, *Adv. Differ. Equ.*, **2012** (2012), 140. https://doi.org/10.1186/1687-1847-2012-140
7. W. A. Al-Salam, q–analogues of Cauchy's formulas, *Proc. Am. Math. Soc.*, **17** (1966), 616–621. https://doi.org/10.2307/2035378

8. J. Alzabut, B. Mohammadaliee, M. E. Samei, Solutions of two fractional q–integro-differential equations under sum boundary value conditions on a time scale, *Adv. Differ. Equ.*, **2020** (2020), 304. https://doi.org/10.1186/s13662-020-02766-y

9. M. Annaby, Z. Mansour, *q–fractional calculus and equations*, Springer Heidelberg, Cambridge, 2012. https://doi.org/10.1007/978-3-642-30898-7

10. F. Atici, P. W. Eloe, Fractional q–calculus on a time scale, *J. Nonlinear Math. Phys.*, **14** (2007), 341–352. https://doi.org/10.2991/jnmp.2007.14.3.4

11. Z. Bai, H. Lu, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **311** (2005), 495–505. https://doi.org/10.1016/j.jmaa.2005.02.052

12. M. Bohner, A. Peterson, *Dynamic equations on time scales: An introduction with applications*, Birkhäuser, Boston, 2001.

13. A. Cabada, G. Wang, Positive solution of nonlinear fractional differential equations with integral boundary value conditions, *J. Math. Anal. Appl.*, **389** (2012), 403–411. https://doi.org/10.1016/j.jmaa.2011.11.065

14. I. Dassios, T. Kerci, D. Baleanu, F. Milano, Fractional-order dynamical model for electricity markets, *Math. Methods Appl. Sci.*, 2021. https://doi.org/10.1002/mma.7892

15. K. Deimling, *Nonlinear functional analysis*, Springer, Berlin, Germany, 1985.

16. D. Delbosco, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **204** (1996), 609–625. https://doi.org/10.1006/jmaa.1996.0456

17. J. Alzabut, A. G. M. Selvam, R. Dhineshbabu, M. K. A. Kaabar, The existence, uniqueness, and stability analysis of the discrete fractional three-point boundary value problem for the elastic beam equation, *Symmetry*, **13** (2021), 789. https://doi.org/10.3390/sym13050789

18. M. Abu-Shady, M. K. A. Kaabar, A generalized definition of the fractional derivative with applications, *Math. Probl. Eng.*, **2021** (2021), 1–9. https://doi.org/10.1155/2021/9444803

19. S. J. Achar, C. Baishya, M. K. A. Kaabar, Dynamics of the worm transmission in wireless sensor network in the framework of fractional derivatives, *Math. Methods Appl. Sci.*, 2021, 1–17. https://doi.org/10.1002/mma.8039

20. M. K. A. Kaabar, F. Martínez, J. F. Gómez-Aguilar, B. Ghanbari, M. Kaplan, H. Günerhan, New approximate analytical solutions for the nonlinear fractional Schrödinger equation with second-order spatio-temporal dispersion via double Laplace transform method, *Math. Methods Appl. Sci.*, **44** (2021), 11138–11156. https://doi.org/10.1002/mma.7476

21. S. Rashid, M. K. A. Kaabar, A. Althobaiti, M. S. Alqurashi, Constructing analytical estimates of the fuzzy fractional-order Boussinesq model and their application in oceanography, *J. Ocean Eng. Sci.*, 2022. https://doi.org/10.1016/j.joes.2022.01.003

22. A. M. A. El-Sayed, Nonlinear functional differential equations of arbitrary orders, *Nonlinear Anal.*, **33** (1998), 181–186. https://doi.org/10.1016/S0362-546X(97)00525-7

23. R. Ferreira, Nontrivials solutions for fractional q–difference boundary value problems, *Electron. J. Qual. Theo.*, **70** (2010), 1–10.

24. C. Goodrich, A. C. Peterson, *Discrete fractional calculus*, Switzerland: Springer International Publishing, 2015. https://doi.org/10.1007/978-3-319-25562-0

25. F. Jackson, q–difference equations, *Am. J. Math.*, **32** (1910), 305–314. https://doi.org/10.2307/2370183

26. H. Jafari, V. Daftardar-Gejji, Positive solutions of nonlinear fractional boundary value problems using adomian decomposition method, *Appl. Math. Comput.*, **180** (2006), 700–706. https://doi.org/10.1016/j.amc.2006.01.007

27. V. Kac, P. Cheung, *Quantum calculus*, New York: Springer, 2002. https://doi.org/10.1007/978-1-4613-0071-7

28. V. Kalvandi, M. E. Samei, New stability results for a sum-type fractional q–integro-differential equation, *J. Adv. Math. Stud.*, **12** (2019), 201–209.

29. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Elsevier, 2006.

30. V. Lakshmikantham, Theory of fractional functional differential equations, *Nonlinear Anal.*, **69** (2008), 3337–3343. https://doi.org/10.1016/j.na.2007.09.025

31. R. Li, Existence of solutions for nonlinear singular fractional differential equations with fractional derivative condition, *Adv. Differ. Equ.*, **2014** (2014), 292. https://doi.org/10.1186/1687-1847-2014-292

32. S. Liang, M. E. Samei, New approach to solutions of a class of singular fractional q–differential problem via quantum calculus, *Adv. Differ. Equ.*, **2020** (2020), 14. https://doi.org/10.1186/s13662-019-2489-2

33. S. Liang, J. Zhang, Existence and uniqueness of positive solutions to m-point boundary value problemfor nonlinear fractional differential equation, *J. Appl. Math. Comput.*, **38** (2012), 225–241. https://doi.org/10.1007/s12190-011-0475-2

34. T. Qiu, Z. Bai, Existence of positive solutions for singular fractional differential equations, *Electron. J. Differ. Eq.*, **2008** (2008), 1–9.

35. P. M. Rajković, S. D. Marinković, M. S. Stanković, Fractional integrals and derivatives in q–calculus, *Appl. Anal. Discr. Math.*, **1** (2007), 311–323.

36. S. Rezapour, M. E. Samei, On the existence of solutions for a multi-singular pointwise defined fractional q–integro-differential equation, *Bound. Value Probl.*, **2020** (2020), 38. https://doi.org/10.1186/s13661-020-01342-3

37. S. M. Esmael, Employing kuratowski measure of non-compactness for positive solutions of system of singular fractional q–differential equations with numerical effects, *Filomat*, **34** (2020), 2971–2989. https://doi.org/10.2298/FIL2009971S

38. M. E. Samei, A. Ahmadi, S. N. Hajiseyedazizi, S. K. Mishra, B. Ram, The existence of non-negative solutions for a nonlinear fractional q–differential problem via a different numerical approach, *J. Inequal. Appl.*, **2021** (2021), 75. https://doi.org/10.1186/s13660-021-02612-z

39. M. E. Samei, V. Hedayati, S. Rezapour, Existence results for a fraction hybrid differential inclusion with Caputo-Hadamard type fractional derivative, *Adv. Differ. Equ.*, **2019** (2019), 163. https://doi.org/10.1186/s13662-019-2090-8

40. M. E. Samei, S. Rezapour, On a system of fractional q–differential inclusions via sum of two multi-term functions on a time scale, *Bound. Value Probl.*, **2020** (2020), 135. https://doi.org/10.1186/s13661-020-01433-1
41. M. E. Samei, H. Zanganeh, S. M. Aydogan, Investigation of a class of the singular fractional integro-differential quantum equations with multi-step methods, *J. Math. Ext.*, **15** (2021), 1–54. https://doi.org/10.30495/JME.SI.2021.2070
42. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α–ψ–contractive type mappings, *Nonlinear Anal.*, **75** (2012), 2154–2165. https://doi.org/10.1016/j.na.2011.10.014
43. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach Science Publishers, 1993.
44. M. Shabibi, M. Postolache, S. Rezapour, S. M. Vaezpour, Investigation of a multisingular pointwise defined fractional integro-differential equation, *J. Math. Anal.*, **7** (2016), 61–77.
45. M. Shabibi, S. Rezapour, S. M. Vaezpour, A singular fractional integro-differential equation, *UPB Sci. Bull., Ser. A*, **79** (2017), 109–118.
46. N. E. Tatar, An impulsive nonlinear singular version of the gronwall-bihari inequality, *J. Inequal. Appl.*, **2006** (2006), 1–12. https://doi.org/10.1155/JIA/2006/84561

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