Research article

To investigate a class of multi-singular pointwise defined fractional \(q\)-integro-differential equation with applications

Mohammad Esmael Samei\(^1\), Lotfollah Karimi\(^2\) and Mohammed K. A. Kaabar\(^3,4,\star\)

\(^1\) Department of Mathematics, Faculty of Basic Science, Bu-Ali Sina University, Hamedan, Iran
\(^2\) Department of Mathematics, Hamedan University of Technology, Hamedan, Iran
\(^3\) Jabalia Camp, United Nations Relief and Works Agency (UNRWA), Palestinian Refugee Camp, Gaza Strip Jabalya, Palestine
\(^4\) Institute of Mathematical Sciences, Faculty of Science, University of Malaya, Kuala Lumpur 50603, Malaysia

\* Correspondence: Email: mohammed.kaabar@wsu.edu; Tel: +971547746073.

Abstract: In the research work, we discuss a multi-singular pointwise defined fractional \(q\)-integro-differential equation under some boundary conditions via the Riemann-Liouville \(q\)-integral and Caputo fractional \(q\)-derivatives. New existence results rely on the \(\alpha\)-admissible map and fixed point theorem for \(\alpha\)-\(\psi\)-contraction map. At the end, we present an example with application and some algorithms to illustrate the primary effects.

Keywords: singularity; pointwise defined equations; integral boundary conditions; Caputo \(q\)-derivation

Mathematics Subject Classification: 34A08, 34B16, 39A13

1. Introduction

Jackson [25] introduced quantum calculus. Then, it was later developed by Al-Salam who started fitting the concept of \(q\)-fractional calculus [7]. Agarwal continued studying certain \(q\)-fractional integrals and derivatives [3]. Furthermore, some researchers have also studied \(q\)-difference equations (for more details, see [1, 2, 5, 6, 8, 9, 15, 23, 24, 26, 27, 33, 39, 40]). On the one hand, fractional differential equations have gained a considerable importance due to their applications in various fields of sciences, such as physics, mechanics, chemistry, and engineering (see [17–21]). In [22], El-Sayed discussed a class of nonlinear functional differential equations of arbitrary orders, and Lakshmikantham [30] initiated the basic theory for fractional functional differential equations.
In 1996, Delbosco et al. investigated $D^\beta u(t) = h(t, u)$ with initial condition: $u(a) = \eta$, where $a > 0$, $\eta \in \mathbb{R}$ and $\beta \in J := (0, 1)$. In 2005, Bai et al. presented the boundary problem:

$$D^\beta_0 u(t) = h(t, u(t)),$$

under conditions: $u(0) = u(1) = 0$, where $t \in J$, $0 < \beta \leq 2$, and $D^\beta_0$ is the Riemann-Liouville standard derivative [11]. In 2008, Qiu et al. studied the equation with conditions: $u(0) = u'(1) = u''(1) = 0$, where $t \in J$, $2 < \beta < 3$, $D^\beta_0$ is the Caputo derivative and $h : \tilde{J} \times [0, \infty) \to [0, \infty)$, here $\tilde{J} := [0, 1]$, is such that $\lim_{t \to 0^+} h(t, \cdot) = \infty$ [34]. In 2010, Agarwal et al. considered the singular fractional Dirichlet problem:

$$D^\beta u(t) + h(t, u(t), D^\gamma u(t)) = 0,$$

with the boundary value condition: $u(0) = u(1) = 0$, where $\beta \in (1, 2]$, $\gamma > 0$, $\beta - \gamma \geq 1$, $h \in \text{Car}(\tilde{J} \times (0, \infty) \times \mathbb{R})$, $h$ is positive and singular at $t = 0$, and $D^\beta$ is the usual Riemann-Liouville derivative [4]. In 2012, Cabada et al. investigated the existence of positive solution for the following nonlinear fractional differential equation:

$$
\begin{cases}
D^\beta u(t) = h(t, u(t)) \\
u(0) = u''(1) = 0, u(1) = \int_0^1 u(\xi) d\xi,
\end{cases}
$$

where $0 < t < 1$, $2 < \beta < 3$ and $h : \tilde{J} \times [0, \infty) \to [0, \infty)$ is a continuous function [13]. In 2014, Li reviewed the problem:

$$C^D^\beta u(t) + h(t, u(t), D^\gamma u(t)) = 0,$$

for each $t \in J$, under conditions: $u(0) = u'(0) = 0$ and $u'(1) = C^D^\beta u(1)$, where $\beta \in (2, 3)$, $\gamma \in J$, $h : (0, 1) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function that may be singular at $t = 0$, $C^D^\beta$ is the standard Caputo derivative [31]. In 2016, the fractional integro-differential equation

$$D^\gamma u(t) = h(t, u(t), u'(t), D^\beta u(t), D^\beta_0 u(t)),$$

under conditions $u'(0) = u(\eta)$, $u(1) = \int_0^\gamma u(\xi) d\xi$ and $u^{(i)}(0) = 0$ for $i = 2, \ldots, \lfloor\gamma\rfloor - 1$ was investigated, where $t \in J$, $\gamma \in [2, 3)$, $u \in \mathcal{B} = C^1(\tilde{J})$, $\alpha, \eta, \nu \in J$, $\beta > 1$ and $h : J \times \mathbb{R}^4 \to \mathbb{R}$ is a function such that $h(t, \cdot, \ldots, \cdot)$ is singular at some point $t \in \tilde{J}$ [44]. In 2017, Shabibi et al. studied the singular fractional integro-differential equation:

$$C^D^\beta u(t) + h(t, u(t), u'(t), C^D^\gamma u(t), \mu(u(t))) = 0,$$

where $\mu(u(t)) = \int_0^1 f(\xi)u(\xi) d\xi$, under boundary conditions: $u(0) = u'(0)$ and $u(1) = C^D^\gamma u(t)$, where $t \in J$, $u \in \mathcal{B}$, $\beta > 2$, $0 < \gamma < 1$, $f \in \mathcal{L} = L_1(\tilde{J})$, $\|f\|_1 = m$, $h(t, u_1, u_2, u_3, u_4)$ is singular at some points $t \in \tilde{J}$ and $C^D^\beta$ is the Caputo fractional derivative [45]. In 2020, Samei considered the singular system of $q$–differential equations:

$$
\begin{cases}
D_q^\alpha u(t) = g_1(t, u(t), v(t)), \\
D_q^\gamma v(t) = g_2(t, u(t), v(t)),
\end{cases}
$$
with conditions: \( u(0) = v(0) = 0, u^{(i)}(0) = v^{(i)}(0) = 0 \), for \( i = 2, \ldots, n - 1 \) and

\[
u(1) = \left[ I_q^\gamma(w_1(t))u(t) \right]_{t=1}, \quad v(1) = \left[ I_q^\gamma(w_2(t))v(t) \right]_{t=1},
\]

where \( D_q^\alpha \) is the \( q \)-derivative of fractional order \( \alpha \), \( \alpha \in (n, n + 1] \) with \( n \geq 3 \), \( I_q^\gamma \) is the \( q \)-integral of fractional order \( \gamma \), \( \gamma \geq 1 \), \( g \in C(E) \), \( g \) are singular at \( t = 0 \) and satisfy the local Carathéodory condition on \( E = (0,1] \times (0, \infty) \times (0, \infty) \), and \( w_j \in \mathcal{L} \) are non-negative such that

\[
\left[ I_q^\gamma(w_j(t)) \right]_{t=1} \in \left[ 0, \frac{1}{2} \right],
\]

for \( j = 1, 2 \) [37]. Also, Liang et al. [32] investigated a nonlinear problem of regular and singular fractional \( q \)-differential equation:

\[
^cD_q^\alpha u(t) = h(t, u(t), u'(t), ^cD_q^\beta u(t)),
\]

with conditions: \( u(0) = c_1u(1), u'(0) = c_2^cD_q^\alpha u(1) \) and \( u^{(k)}(0) = 0 \) for all \( 2 \leq k \leq n - 1 \), here \( n - 1 < \alpha < n \) with \( n \geq 3 \), \( \beta, q, c_1 \in J, 0 < c_2 < \Gamma_q(2 - \beta) \), function \( h \) is a \( L^2 \)-Carathéodory and \( h(t, u_1, u_2, u_3) \) may be singular. Similarly, some related results have been obtained in [28, 36, 38]. Dassios et al. used a generalized system of differential equations of fractional order:

\[
T_\lambda \frac{d\lambda(t)}{dr} = -H_d\lambda(t) + K_E \left( \omega_{\text{ref}} - \omega_{\text{Col}}(t) \right),
\]

to incorporate memory into an electricity market model by constructing the fractional-order dynamical model, studying its solutions, and providing the closed formulas of solutions, where \( \frac{d\lambda(t)}{dr} \), \( \lambda(t) \) are the marginal electricity price and electricity price, respectively, \( \omega_{\text{ref}} \) represents the reference frequency, \( \omega_{\text{Col}}(t) \) represents the frequency of the Col, that is, \( \omega_{\text{ref}} - \omega_{\text{Col}}(t) \) is the deviation frequency of the Col with respect to the reference frequency, \( T_\lambda \) is the time constant, \( H_d \) is the deviation with respect to a perfect tracking integrator, and for a low-pass filter, it is \( H_d = 1 \), and \( K_E \) can be used as feedback gain [14].

Using the ideas from these works, we investigate the existence of solutions for the following nonlinear pointwise defined fractional \( q \)-integro-differential equation:

\[
D_q^\alpha u(t) = w(t, u(t), D_q^\beta u(t), \int_0^\tau f(\xi)u(\xi) \, d\xi, \varphi(u(t))),
\]

for \( q \in J \), under boundary conditions: \( \int_0^b u(r) \, dr = 0, u'(1) = u(a) \) and \( u^{(j)}(0) = 0 \) for \( j \geq 2 \), here \( \alpha \geq 2 \), \( a, b, \beta \in J \), \( \varphi : \overline{B} \to \overline{B} \) is a map such that

\[
||\varphi(u_1) - \varphi(u_2)|| \leq c_1 ||u_1 - u_2|| + c_2 ||u'_1 - u'_2||,
\]

for some non-negative real numbers \( c_1 \) and \( c_2 \) belonging to \([0, \infty)\) and all \( u_1, u_2 \in \overline{B} \), where \( D_q^\alpha \) and \( D_q^\beta \) are the Caputo fractional \( q \)-derivatives of order \( \alpha \) and \( \beta \), respectively, which are defined in (2.11), and \( w \) in \( \mathcal{L} \) is singular at some points \( t \in J \).

In fact, the non-constant real-valued function \( u \) on the interval \( I = [a, b] \) is said to be singular on \( I \), if it is continuous, and there exists a set \( S \subseteq I \) of measure 0 such that for all \( t \) outside of \( S \), \( u'(t) \) exists,
and it is zero, that is, the derivative of \( u \) vanish almost everywhere. We say that, \( D_q^au(t) + g(t) = 0 \) is a pointwise defined equation on \( J \) if there exists set \( S \subset J \) such that the measure of \( S^c \) is zero, and the equation holds on \( S \) [44].

In Section 2, we recall some essential definitions of Caputo fractional \( q \)-derivative. Section 3 contains our main results of this work, while an example is presented to support the validity of our obtained results. An application with some needed algorithms for the problems are given in Section 4. In Section 5, conclusion is presented.

2. Basic definitions for the problem

Throughout the paper, we apply the notations of time scales calculus [12]. The Caputo fractional \( q \)-derivative is considered here on

\[
T_{s_0} = \{ 0 \} \cup \{ s : s = s_0^n, n \in \mathbb{N}_0 \},
\]

for all \( q \in \mathbb{R} \) and \( q \in J \). If there is no confusion concerning \( s_0 \), we denote \( T_{s_0} \) by \( T \). Let \( p \in \mathbb{R} \). Let us define \([p]_q = (1 - q^p)(1 - q)^{-1} [25]\). The \( q \)-factorial function \((v - w)_q^N\) with \( N \in \mathbb{N}_0 \) is defined by

\[
(v - w)_q^N = \prod_{k=0}^{N-1} (v - w_q^k), \quad (\forall v, w \in \mathbb{R}),
\]

and \((v - w)_q^0 = 1\), where \( \mathbb{N}_0 := \{ 0, 1, 2, 3, \ldots \} \) [2]. Also, for \( \sigma \in \mathbb{R} \), we have:

\[
(v - w)_q^{(\sigma)} = v^\sigma \prod_{k=0}^{\infty} \frac{v - w_q^k}{v - w_q^{\sigma+k}}, \quad (\forall v, w \in \mathbb{R}).
\]

In [10], the authors proved that \((v - w)_q^{(\sigma+v)} = (v - w)_q^{(\sigma)}(v - q^\sigma w)_q^{(v)}\) and

\[
(a v - a w)_q^{(\sigma)} = a^\sigma (v - w)_q^{(\sigma)},
\]

for each \( v, w \in \mathbb{R} \). If \( w = 0 \), then it is clear that \( v^{(\sigma)} = v^\sigma \). The \( q \)-Gamma function is given by

\[
\Gamma_q(v) = (1 - q)^{1-v}(1 - q)_q^{(v-1)},
\]

where \( v \in \mathbb{R}\backslash \cdots, -2, -1, 0 \) [25]. In fact, by using (2.2), we have

\[
\Gamma_q(v) = (1 - q)^{1-v} \prod_{k=0}^{\infty} \frac{1 - q^{k+1}}{1 - q^{\sigma+k-1}}, \quad (\forall v \in \mathbb{R}).
\]

Note that, \( \Gamma_q(v + 1) = [v]_q \Gamma_q(v) \) [10, Lemma 1]. For a function \( u : T \rightarrow \mathbb{R} \), the \( q \)-derivative of \( u \), is

\[
D_q^a u(t) = \left( \frac{d}{dt} \right)_q u(t) = \frac{u(t) - u(qt)}{(1 - q)t},
\]

for all \( t \in T \backslash \{ 0 \} \), and \( D_q^0 u(0) = \lim_{t \rightarrow 0} D_q u(t) \) [2]. Also, the higher order \( q \)-derivative of the function \( u \) is defined by \( D_q^{n+1} u(t) = D_q \left[ D_q^{n-1} u(t) \right] \) for all \( n \geq 1 \), where \( D_q^0 u(t) = u(t) \) [2]. In fact,

\[
D_q^n u(t) = \frac{1}{\Gamma^n(1 - q)^n} \sum_{k=0}^{\infty} \frac{(1 - q^{-n})_q^{(k)}}{(1 - q_q^k)^k} q^k u(t q^k),
\]

for \( t \in T \backslash \{ 0 \} \) [9].
Remark 2.1. By using Eq (2.1), we can change Eq (2.5) into the following:

\[
\frac{\mathcal{D}_q^n[u]}{t^n(1-q)^n} \sum_{k=0}^{n} \prod_{i=k+1}^{n} \left(1 - q^{i-k}\right) q^k u(tq^k). \tag{2.6}
\]

The \( q \)-integral of the function \( u \) is defined by

\[
\mathcal{I}_q[u](t) = \int_0^t u(\xi) \, d_{q} \xi = t(1-q) \sum_{k=0}^{\infty} q^k u(tq^k), \tag{2.7}
\]

for \( 0 \leq t \leq b \), provided that the series is absolutely convergent \([2]\). If \( a \) is in \([0, b)\), then

\[
\int_a^b u(\xi) \, d_{q} \xi = \mathcal{I}_q[u](b) - \mathcal{I}_q[u](a) = (1-q) \sum_{k=0}^{\infty} q^k \left[ bu(bq^k) - au(aq^k) \right], \tag{2.8}
\]

whenever the series converges. The operator \( \mathcal{I}_q^n \) is given by \( \mathcal{I}_q^n[u](t) = u(t) \) and

\[
\mathcal{I}_q^n[u](t) = \mathcal{I}_q \left[ \mathcal{I}_q^{n-1}[u] \right](t),
\]

for \( n \geq 1 \) and \( u \in C([0, b]) \) \([2]\). It has been proven that

\[
\mathcal{D}_q \left[ \mathcal{I}_q[u] \right](t) = u(t), \quad \mathcal{I}_q \left[ \mathcal{D}_q[u] \right](t) = u(t) - u(0),
\]

whenever the function \( u \) is continuous at \( t = 0 \) \([2]\). The fractional Riemann-Liouville type \( q \)-integral of the function \( u \) is defined by

\[
\mathcal{I}_q^\sigma[u](t) = \frac{1}{\Gamma_q(\sigma)} \int_0^t (t-\xi)^{(\sigma-1)} q^k u(\xi) \, d_{q} \xi, \quad \mathcal{I}_q^0[u](t) = u(t), \tag{2.9}
\]

for \( t \in \bar{J} \) and \( \sigma > 0 \) \([9, 23]\).

Remark 2.2. By using Eqs (2.2), (2.3) and (2.7), we obtain:

\[
\frac{1}{\Gamma_q(\sigma)} \int_0^t (t-\xi)^{(\sigma-1)} q^k u(\xi) \, d_{q} \xi
\]

\[
= \frac{1}{\Gamma_q(\sigma)} \int_0^t q^{\sigma-1} \sum_{i=0}^{\infty} \frac{t-\xi q^i}{t-\xi q^{\sigma+i-1}} u(\xi) \, d_{q} \xi
\]

\[
= t^\sigma (1-q)^\sigma \sum_{i=0}^{\infty} \frac{1-q^{\sigma+i-1}}{1-q^{i+1}} \sum_{k=0}^{\infty} q^k \prod_{i=0}^{\infty} \frac{1-q^{k+i}}{1-q^{\sigma+i+k+1}} u(tq^k).
\]

Therefore, we have:

\[
\mathcal{I}_q^\sigma[u](t) = t^\sigma (1-q)^\sigma \lim_{n \to \infty} \sum_{k=0}^{n} q^k \prod_{i=0}^{n} \frac{1-q^{\sigma+i-1}}{1-q^{i+1}} \frac{(1-q^{k+i})(1-q^{k+i})}{(1-q^{\sigma+i+k+1})(1-q^{\sigma+i+k+1})} u(tq^k), \tag{2.10}
\]
The Caputo fractional $q$–derivative of the function $u$ is defined by

$$
\mathcal{C}D_q^{\sigma}[u](t) = \mathcal{I}_q^{[\sigma]-\sigma} \left[ D_q^{[\sigma]}[u] \right](t)
$$

$$
= \frac{1}{\Gamma_q([\sigma] - \sigma)} \int_0^t (t - \xi)_{q}^{([\sigma]-\sigma-1)} D_q^{[\sigma]}[u](\xi) \, dq\xi
$$

(2.11)

for $t \in \bar{J}$ and $\sigma > 0$ [23, 35]. It has been proven that

$$
\mathcal{I}_q^{[\sigma]} \left[ \mathcal{I}_q^{[\sigma]}[u] \right](t) = \mathcal{I}_q^{[\sigma+\nu]}[u](t),
$$

$$
\mathcal{C}D_q^{\sigma} \left[ \mathcal{I}_q^{[\sigma]}[u] \right](t) = u(t),
$$

where $\sigma, \nu \geq 0$ [23]. Also,

$$
\mathcal{I}_q^{[\sigma]} \left[ D_q^{[\sigma]}[u] \right](t) = D_q^{\sigma} \left[ \mathcal{I}_q^{[\sigma]}[u] \right](t) - \sum_{k=0}^{n-1} \frac{t^{\sigma+k-n}}{\Gamma_q(\sigma + k - n + 1)} D_q^{[\sigma]}[u](0),
$$

where $\sigma > 0$ and $n \geq 1$ [23].

Remark 2.3. From Eq (2.3), Remark 2.1, and Eq (2.10) in Remark 2.2, we obtain:

$$
\frac{1}{\Gamma_q([\sigma] - \sigma)} \int_0^t (t - \xi)_{q}^{([\sigma]-\sigma-1)} D_q^{[\sigma]}[u](\xi) \, dq\xi
$$

$$
= \frac{1}{\Gamma_q([\sigma] - \sigma)} \int_0^t \left[ \prod_{i=0}^{\infty} \frac{t - sq^i}{q^i [1 - sq^i]} \right] u(xq^k) \, dq s
$$

$$
= \frac{1}{t^{\sigma}(1 - q)^{[\sigma]-\sigma}} \sum_{k=0}^{\infty} \left( \prod_{i=0}^{\infty} \frac{1 - q^{i+1} - q^{i+1}(1 - q^{i+1})}{1 - q^{i+1}(1 - q^{i+1})} \right)
$$

$$
\times \left( \prod_{n=0}^{\infty} \left( \prod_{i=0}^{m-1} \frac{1 - q^{i+1}}{1 - q^{i+1}} \right) q^m u \left( t q^{k+m} \right) \right)
$$

Thus, we have:

$$
\mathcal{C}D_q^{\sigma}[u](t) = \frac{1}{t^{\sigma}(1 - q)^{[\sigma]-\sigma}} \lim_{n \to \infty} \sum_{k=0}^{n} \left( \prod_{i=0}^{n} \frac{1 - q^{i+1} - q^{i+1}(1 - q^{i+1})}{1 - q^{i+1}(1 - q^{i+1})} \right)
$$

$$
\times \left( \prod_{m=0}^{\infty} \left( \prod_{i=0}^{m-1} \frac{1 - q^{i+1}}{1 - q^{i+1}} \right) q^m u \left( t q^{k+m} \right) \right)
$$

(2.12)

The authors in [41] presented all algorithms and MATLAB code’s lines to simplify $q$–factorial functions $(v - w)_q^n$, $(v - w)_q\sigma^n$, $\Gamma_q(v)$, $\mathcal{I}_q[u](t)$, and some necessary equations.
Lemma 2.4. [27, 29] For $\sigma > 0$, the general solution of the fractional $q$–differential equation $^cD^\sigma u(t) = 0$ is given by $u(t) = \sum_{i=0}^{n-1} e_i t^i$, where $e_i \in \mathbb{R}$ for $i = 0, 1, 2, \ldots, n-1$ and $n = [\sigma] + 1$ here $[\sigma]$ denotes the integer part of the real number $\sigma$.

We use the three norms: $\|u\| = \sup_{t \in J} |u(t)|$,

$$
\|(u, u')\|_{\psi} = \max \{\|u\|, \|u'\|\},
$$
and $\|u\|_1 = \int_{J} |u(\xi)| d\xi$ in $\tilde{A} = C(J)$, $\bar{B} = C^1(J)$, and $\tilde{L} = L_1(J)$, respectively. Let $\Psi$ be the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, for all $t > 0$. Let $T : X \to X$ and $\alpha : X \times X \to (0, \infty)$. $T$ is called an $\alpha$-admissible mapping if $\alpha(u_1, u_2) \geq 1$ implies that $\alpha(T(u_1), T(u_2)) \geq 1$ for each $u_1, u_2 \in X$.

Definition 2.5. [42] Let $(X, \rho)$ be a metric space, where $\psi \in \Psi$ and $\alpha : X^2 \to [0, \infty)$ is a map. A self-map $T$ defined on $X$ is called an $\alpha$-$\psi$-contraction whenever

$$
\alpha(u_1, u_2)\rho(T(u_1), T(u_2)) \leq \psi(\rho(u_1, u_2)),
$$
for each $u_1, u_2 \in X$.

Lemma 2.6. [42] Let $(X, \rho)$ be a complete metric space and $T : X \to X$ be a continuous, $\alpha$–admissible and $\alpha$–$\psi$–contraction, then $T$ has a fixed point whenever there exists $u_0 \in X$ such that $\alpha(u_0, T(u_0)) \geq 1$.

Lemma 2.7. [43, 46] If $x \in \tilde{A} \cap \tilde{L}$ with $D_q^n x \in \mathcal{A} \cap \mathcal{L}$, then

$$
\Gamma_q^a D_q^\alpha u(t) = u(t) + \sum_{i=1}^{n} c_i t^{\alpha-i},
$$
where $[\alpha] \leq n < [\alpha] + 1$, and $c_i$ is some real number.

3. Main results

Let us first prove the following essential lemma:

Lemma 3.1. Suppose that $\alpha \geq 2$, $q \in J$ and $g \in \tilde{L}$. The solution of the boundary value problem: $D_q^\alpha u(t) = g(t)$ with boundary conditions is expressed as:

$$
\begin{align*}
\begin{cases}
|u(0)| = 0 &; j = 2, 3, 4, \ldots, \\
u' = u(\alpha) &; \forall a \in J,
\end{cases}
\end{align*}
$$

is

$$
u(t) = \int_{0}^{1} G_q(t, \xi) g(\xi) d\xi,
$$

AIMS Mathematics

on a time scale $\mathbb{T}_{h_0}$ where $G_q(t, s)$ is expressed as:

$$
\begin{cases}
-A_0(t - s)_{q}^{(a-1)} + A_1(t)(1 - s)_{q}^{(a-2)} + A_2(t)(a - s)_{q}^{(a-1)} + A_3(t)(b - s)_{q}^{(a)} & s \leq \min\{a, b\}; \\
-A_0(t - s)_{q}^{(a-1)} + A_1(t)(1 - s)_{q}^{(a-2)} + A_2(t)(a - s)_{q}^{(a-1)} & b \leq s \leq a; \\
-A_0(t - s)_{q}^{(a-1)} + A_1(t)(1 - s)_{q}^{(a-2)} + A_3(t)(b - s)_{q}^{(a)} & a \leq s \leq b; \\
-A_0(t - s)_{q}^{(a-1)} + A_1(t)(1 - s)_{q}^{(a-2)} & s \geq \max\{a, b\};
\end{cases}
$$

whenever $0 \leq s \leq t \leq 1$.

$$
\begin{cases}
A_1(t)(1 - s)_{q}^{(a-2)} + A_2(t)(a - s)_{q}^{(a-1)} + A_3(b - s)_{q}^{(a)} & s \leq \min\{a, b\}; \\
A_1(t)(1 - s)_{q}^{(a-2)} + A_2(t)(a - s)_{q}^{(a-1)} & b \leq s \leq a; \\
A_1(t)(1 - s)_{q}^{(a-2)} + A_3(t)(b - s)_{q}^{(a)} & a \leq s \leq b; \\
A_1(t)(1 - s)_{q}^{(a-2)} & s \geq \max\{a, b\};
\end{cases}
$$

whenever $0 \leq t \leq s \leq 1$. Also

$$
\begin{aligned}
A_0 &= \frac{1}{\Gamma_q(a)}, \\
A_1(t) &= \frac{b(1-a+t)-\mu(a,b)}{\mu(a,b)\Gamma_q(a-1)}, \\
A_2(t) &= \frac{\mu(a,b)+b(a+1-1)}{\mu(a,b)\Gamma_q(a)}, \\
A_3(t) &= \frac{b(b(1-a)+1)}{\mu(a,b)\Gamma_q(a+1)},
\end{aligned}
$$

and

$$
\mu(a, b) = b(1 - a) + \frac{b^2}{2} > 0.
$$

Proof. Consider the problem: $D_q^a u(t) = g(t)$. Using Lemma 2.7, it is deduced that $u(t) = -\Gamma_q^a g(t) + c_0 + c_1 t$, where $c_0$, $c_1$ are some real numbers, and $\Gamma_q^a$ is Riemann-Liouville type $q$–integral of order $\alpha$. Hence, $u'(t) = -\Gamma_q^{a-1} g(t) + c_1$ where $\Gamma_q^{a-1}$ is a fractional Riemann-Liouville type $q$–integral of order $\alpha - 1$. By applying condition $u'(1) = u(a)$, we get:

$$-\Gamma_q^{a-1} g(1) + c_1 = -\Gamma_q^a g(a) + c_0 + c_1 a,$$

and so $c_0 = -\Gamma_q^{a-1} g(1) + \Gamma_q^a g(a) + (1 - a)c_1$. One can easily check that

$$\int_0^b u(r) \, dr = -\Gamma_q^{a+1} g(b) - b\Gamma_q^{a-1} g(1) + \mu \Gamma_q^a g(a) + bc_1(1 - a) + \frac{1}{2} c_1 b^2.$$
Since $\int_0^b u(r) \, dr = 0$, we get:

$$c_1 = \frac{1}{\mu(a, b)} \Gamma^{\alpha+1}_q g(b) + \frac{b}{\mu(a, b)} \Gamma^{\alpha-1}_q g(1) - \frac{b}{\mu(a, b)} \Gamma^{\alpha+1}_q g(a).$$

Thus,

$$c_0 = -\Gamma^{\alpha-1}_q g(1) + \Gamma^{\alpha}_q g(a) + \frac{1 - a}{\mu(a, b)} \Gamma^{\alpha+1}_q g(b) + \frac{b(1 - a)}{\mu(a, b)} \Gamma^{\alpha-1}_q g(1) - \frac{b(1 - a)}{\mu(a, b)} \Gamma^{\alpha}_q g(a)$$

and so

$$u(t) = -\Gamma^{\alpha}_q g(t) - \Gamma^{\alpha-1}_q g(1) + \Gamma^{\alpha}_q g(a) + \frac{1 - a}{\mu(a, b)} \Gamma^{\alpha+1}_q g(b) + \frac{b(1 - a)}{\mu(a, b)} \Gamma^{\alpha-1}_q g(1) - \frac{b(1 - a)}{\mu(a, b)} \Gamma^{\alpha}_q g(a)$$

$$+ \frac{t}{\mu(a, b)} \Gamma^{\alpha+1}_q g(b) + \frac{b t}{\mu(a, b)} \Gamma^{\alpha-1}_q g(1) - \frac{b t}{\mu(a, b)} \Gamma^{\alpha}_q g(a).$$

Hence,

$$u(t) = -\Gamma^{\alpha}_q g(t) + A_1(t) \Gamma^{\alpha-1}_q g(1) + A_2(t) \Gamma^{\alpha}_q g(a) + A_3(t) \Gamma^{\alpha+1}_q g(b).$$

Now, some easy evaluations show us that $u(t) = \int_0^1 G_q(t, s) g(s) \, dq \, ds$. \qed

**Remark 3.2.** Note that, the mappings $G_q(t, s)$ and $\frac{\partial G_q(t,s)}{\partial t}$ are continuous with respect to $t$. Let $w$ be a map on $\tilde{J} \times \tilde{B}^2$ such that $w$ is singular at some points of $\tilde{J}$. Let us define the function $\Theta_u : \tilde{B} \to \tilde{B}$ by

$$\Theta_u(t) = -\Gamma^{\alpha}_q w \left( t, u(t), \nabla^\beta_q u(t), \int_0^t f(\xi)u(\xi) \, d\xi, \varphi(u(t)) \right)$$

$$+ A_1(t) \Gamma^{\alpha-1}_q w \left( 1, u(1), \nabla^\beta_q u(1), \int_0^1 f(\xi)u(\xi) \, d\xi, \varphi(u(1)) \right)$$

$$+ A_2(t) \Gamma^{\alpha}_q w \left( a, u(a), \nabla^\beta_q u(a), \int_0^a f(\xi)u(\xi) \, d\xi, \varphi(u(a)) \right)$$

$$+ A_3(t) \Gamma^{\alpha+1}_q w \left( b, u(b), \nabla^\beta_q u(b), \int_0^b f(\xi)u(\xi) \, d\xi, \varphi(u(b)) \right),$$

for all $t \in \tilde{J}$, where $\Gamma^{\alpha}_q$ is the fractional Riemann-Liouville $q$–integral of order $\alpha$ which is defined in (2.9), and $\nabla^\beta_q$ is the Caputo fractional $q$–derivative of order $\beta$ which is defined in (2.11). Then, by taking the first order derivative related to $t$, we have:
\[ \Theta_a(t) = \int_0^1 \frac{\partial G_q(t, \xi)}{\partial t} w(s, u(s), D_q^a u(s), \int_0^s f(\xi) u(\xi) \, d\xi, \varphi(u(s))) \, ds \]
\[ = -E_q^{a-1} w(t, u(t), D_q^a u(t), \int_0^t f(\xi) u(\xi) \, d\xi, \varphi(u(t))) \]
\[ + \frac{b}{\mu(a, b)} E_q^{a-1} w(1, u(1), D_q^a u(1), \int_0^1 f(\xi) u(\xi) \, d\xi, \varphi(u(1))) \]
\[ + \frac{b}{\mu(a, b)} E_q^a w(a, u(a), D_q^a u(a), \int_0^a f(\xi) u(\xi) \, d\xi, \varphi(u(a))) \]
\[ + \frac{1}{\mu(a, b)} E_q^{a+1} w(b, u(b), D_q^a u(b), \int_0^b f(\xi) u(\xi) \, d\xi, \varphi(u(b))). \]

Obviously, the singular pointwise defined Eq (1.1) has a solution iff the map \( \Theta_a \) has a fixed point.

Now, we give our main result as follows:

**Theorem 3.3.** Assume that \( \alpha \geq 2, [\alpha] = n - 1, a, b, q \in J, f \in \mathcal{L} \) with \( \|f\|_1 = m, \varphi : \bar{B} \rightarrow \mathbb{R} \) is such that
\[ |\varphi(u(t)) - \varphi(v(t))| \leq c_1|u(t) - v(t)| + c_2|u'(t) - v'(t)|, \]
for some \( c_1, c_2 \in [0, \infty) \). Let \( \Omega : \bar{J} \times \bar{B}^5 \rightarrow \mathbb{R} \) be a mapping which is singular on some points \( \bar{J} \) and
\[ |w(t, u_1, \ldots, u_5) - w(t, v_1, \ldots, v_5)| \leq \sum_{i=1}^{k_0} \mu_i(t) \Omega_i(u_1 - v_1, \ldots, u_5 - v_5), \]
for all \( u_1, u_2, v_1, v_2 \in \bar{B} \) and almost all \( t \in \bar{J} \), where \( k_0 \) is a natural number, \( \mu_i : \bar{J} \rightarrow \mathbb{R}^+, \hat{\mu}_i \in \mathcal{L}, \]
\[ \hat{\mu}_i(s) = (1 - s)_{q}^{\alpha-2} \mu_i(s), \]
\( \Omega_i : \bar{B}^5 \rightarrow \mathbb{R}^+ \) is a nondecreasing mapping with respect to all components with
\[ \frac{\Omega_i(v, v, v, v, v)}{\gamma_i} \rightarrow p_i, \]
as \( v \rightarrow 0^+ \) for some \( \gamma_i > 0, p_i \in \mathbb{R}^+ \) with \( 1 \leq i \leq k_0 \). Suppose that
\[ |w(t, u_1, \ldots, u_5)| \leq h(t) T(u_1, \ldots, u_5), \]
for all \( (u_1, \ldots, u_5) \in \bar{B}^5 \) and almost all \( t \in \bar{J} \), where \( h : \bar{J} \rightarrow \mathbb{R}^+, \hat{h} \in \mathcal{L}, T : \bar{B}^5 \rightarrow \mathbb{R}^+ \) is a nondecreasing mapping respect all their components such that
\[ \lim_{v \rightarrow 0^+} \frac{T(v, v, v, v, v)}{v} \in [0, \tau), \]

AIMS Mathematics

where \( \tau = \left( \ell \| \hat{h} \|_{M_{a,a,b}} \right)^{-1} \),

\[
\ell = \max \left\{ 1, \frac{1}{\Gamma_q(2 - \beta)}, m, c_1 + c_2 \right\},
\]

\( \mu(a,b) \) define by Eq (3.4) in Lemma 3.1 and

\[
M_{a,a,b} = \max \left\{ \frac{1}{\Gamma_q(\alpha)} + \frac{b(2 - a) - \mu(a,b)}{\mu(a,b)\Gamma_q(\alpha - 1)} + \frac{\mu(a,b) + ab}{\mu(a,b)\Gamma_q(\alpha)} \right. \\
+ \frac{\mu(a,b)(1 - a) + 1}{\mu(a,b)\Gamma_q(\alpha + 1)} \left\{ \frac{1}{\Gamma_q(\alpha - 1)} + \frac{b}{\mu(a,b)\Gamma_q(\alpha + 1)} \right\}
\]

If

\[
M_{a,a,b} \sum_{k=1}^{k_0} p_i \ell^p \| \tilde{\mu}_i \| < 1,
\]

then the pointwise defined Eq (1.1) under boundary conditions: \( u^{(j)}(0) = 0 \) for \( j \geq 2 \), \( \int_0^b u(r) \, dr = 0 \) and \( u'(1) = u(a) \) has a solution.

**Proof.** Let \( u, v \in \tilde{B} \). Then, we get:

\[
|\Theta_u(t) - \Theta_v(t)| \leq \left| - \int_q^w \left( t, u(t), u'(t), D_q^\beta u(t), \int_0^t f(r)u(r) \, dr, \varphi(u(t)) \right) \right.
\]

\[
+ A_1(t) \int_q^\alpha w \left( 1, u(1), u'(1), D_q^\beta u(1), \int_0^1 f(r)u(r)dr, \varphi(u(1)) \right)
\]

\[
+ A_2(t) \int_q^\alpha w \left( a, u(a), u'(a), D_q^\beta u(a), \int_0^a f(r)u(r)dr, \varphi(u(a)) \right)
\]

\[
+ A_3(t) \int_q^\alpha w \left( b, u(b), u'(b), D_q^\beta u(b), \int_0^b f(r)u(r)dr, \varphi(u(b)) \right)
\]

\[
+ \int_q^w \left( t, v(t), v'(t), D_q^\beta v(t), \int_0^t f(r)v(r) \, dr, \varphi(v(t)) \right)
\]

\[
- A_1(t) \int_q^\alpha w \left( 1, v(1), v'(1), D_q^\beta v(1), \int_0^1 f(r)v(r) \, dr, \varphi(v(1)) \right)
\]

\[
- A_2(t) \int_q^\alpha w \left( a, v(a), v'(a), D_q^\beta v(a), \int_0^a f(r)v(r) \, dr, \varphi(v(a)) \right)
\]

\[
- A_3(t) \int_q^\alpha w \left( b, v(b), v'(b), D_q^\beta v(b), \int_0^b f(r)v(r) \, dr, \varphi(v(b)) \right)
\]
\[
\leq \|w\left(t, u(t), u'(t), D_q^\alpha u(t), \int_0^t f(r)u(r) \, dr, \varphi(u(t))\right) \\
- w\left(t, v(t), v'(t), D_q^\alpha v(t), \int_0^t f(r)v(r) \, dr, \varphi(v(t))\right)\] \\
+ A_1(t)\left[\|w\left(1, u(1), u'(1), D_q^\alpha u(1), \int_0^1 f(r)u(r) \, dr, \varphi(u(1))\right) \\
- w\left(1, v(1), v'(1), D_q^\alpha v(1), \int_0^1 f(r)v(r) \, dr, \varphi(v(1))\right)\right]\] \\
+ A_2(t)\left[\|w\left(a, u(a), u'(a), D_q^\alpha u(a), \int_0^a f(r)u(r) \, dr, \varphi(u(a))\right) \\
- w\left(a, v(a), v'(a), D_q^\alpha v(a), \int_0^a f(r)v(r) \, dr, \varphi(v(a))\right)\right]\] \\
+ A_3(t)\left[\|w\left(b, u(b), u'(b), D_q^\alpha u(b), \int_0^1 f(r)u(r) \, dr, \varphi(u(b))\right) \\
- w\left(b, v(b), v'(b), D_q^\alpha v(b), \int_0^1 f(r)v(r) \, dr, \varphi(v(b))\right)\right]\] \\
\leq \|w\left(1,0,0,0,0,0\right)\| + \sum_{i=1}^{k_0} \mu_i(t) \left[\Omega\left(u(t) - v(t), u'(t) - v'(t), D_q^\alpha u(t) - D_q^\alpha v(t), \\
\int_0^t f(r)u(r) \, dr - \int_0^t f(r)v(r) \, dr, \varphi(u(t)) - \varphi(v(t))\right)\right]\] \\
+ A_1(t)\|w\left(1,0,0,0,0,0\right)\| \times \left[\Omega\left(u(1) - v(1), u'(1) - v'(1), D_q^\alpha u(1) - D_q^\alpha v(1), \\
\int_0^1 f(r)u(r) \, dr - \int_0^1 f(r)v(r) \, dr, \varphi(u(1)) - \varphi(v(1))\right)\right]\] \\
+ A_2(t)\|w\left(1,0,0,0,0,0\right)\| \times \left[\Omega\left(u(a) - v(a), u'(a) - v'(a), D_q^\alpha u(a) - D_q^\alpha v(a), \\
\int_0^a f(r)u(r) \, dr - \int_0^a f(r)v(r) \, dr, \varphi(u(a)) - \varphi(v(a))\right)\right]\]
\[\begin{align*}
+ A_3(t) & \Gamma_q^{a+1} \left( \sum_{i=1}^{k_l} \mu_i(b) \right) \\
\times & \left[ \Omega \left( \left| u(b) - v(b), u'(b) - v'(b), D^\beta_q u(b) - D^\beta_q v(b), \right. \right. \\
& \left. \left. \int_0^b f(r)u(r) \, dr - \int_0^b f(r)v(r) \, dr, \varphi(u(b)) - \varphi(v(b)) \right) \right] \right] \\
\leq & \sum_{i=1}^{k_l} \Gamma_q^a \left( \mu_i(t) \right) \left[ \Omega \left( \left| u(t) - v(t), u'(t) - v'(t), \right. \right. \\
& \left. \left. \left| D^\beta_q u(t) - D^\beta_q v(t) \right|, \right. \right. \\
& \left. \left. \left| \int_0^t f(r)u(r) \, dr - \int_0^t f(r)v(r) \, dr \right|, \varphi(u(t)) - \varphi(v(t)) \right) \right] \right] \\
+ A_1(t) & \sum_{i=1}^{k_l} \Gamma_q^{a-1} \left( \mu_i(1) \right) \\
\times & \left[ \Omega \left( \left| u(1) - v(1), u'(1) - v'(1), \right. \right. \\
& \left. \left. \left| D^\beta_q u(1) - D^\beta_q v(1) \right|, \right. \right. \\
& \left. \left. \left| \int_0^1 f(r)u(r) \, dr - \int_0^1 f(r)v(r) \, dr \right|, \varphi(u(1)) - \varphi(v(1)) \right) \right] \right] \\
+ A_2(t) & \sum_{i=1}^{k_l} \Gamma_q^a \left( \mu_i(a) \right) \\
\times & \left[ \Omega \left( \left| u(a) - v(a), u'(a) - v'(a), \right. \right. \\
& \left. \left. \left| D^\beta_q u(a) - D^\beta_q v(a) \right|, \right. \right. \\
& \left. \left. \left| \int_0^a f(r)u(r) \, dr - \int_0^a f(r)v(r) \, dr \right|, \varphi(u(a)) - \varphi(v(a)) \right) \right] \right] \\
+ A_3(t) & \sum_{i=1}^{k_l} \Gamma_q^{a+1} \left( \mu_i(b) \right) \\
\times & \left[ \Omega \left( \left| u(b) - v(b), u'(b) - v'(b), \right. \right. \\
& \left. \left. \left| D^\beta_q u(b) - D^\beta_q v(b) \right|, \right. \right. \\
& \left. \left. \left| \int_0^b f(r)u(r) \, dr - \int_0^b f(r)v(r) \, dr \right|, \varphi(u(b)) - \varphi(v(b)) \right) \right] \right].
\end{align*}\]

Since \(D^\beta_q u(t) = \Gamma_q^{1-\beta} u'(t)\) for \(\beta \in J\), we have
\[|D^\beta_q u(t)| \leq \Gamma_q^{1-\beta}|u'(t)| \leq ||u'||^\beta \Gamma_q(1) = \frac{||u'||}{\Gamma_q(2 - \beta)},\]
and so
\[ |\mathcal{D}_q^\alpha u(t) - \mathcal{D}_q^\alpha v(t)| = |\mathcal{D}_q^\alpha (u(t) - v(t))| \leq \frac{||u' - v'||}{\Gamma_q(2 - \beta)}.
\]

Thus, by considering \( \xi = ||u - v|| \), we have:

\[
|\Theta_u(t) - \Theta_v(t)| \leq \sum_{i=1}^{k_0} \Gamma_q^{\alpha(i)} \left( \|u - v\|, \|u' - v'\|, \frac{||u' - v'||}{\Gamma_q(2 - \beta)} \right)
\]

\[
+ A_1(t) \sum_{i=1}^{k_0} \Gamma_q^{\alpha-1} \left( \mu_1(1) \left[ \Omega_q \left( \|u - v\|, \|u' - v'\| \right) \right] \right)
\]

\[
+ A_2(t) \sum_{i=1}^{k_0} \Gamma_q^\alpha \left( \mu_2(a) \left[ \Omega_q \left( \|u - v\|, \|u' - v'\| \right) \right] \right)
\]

\[
+ A_3(t) \sum_{i=1}^{k_0} \Gamma_q^{\alpha+1} \left( \mu_3(b) \left[ \Omega_q \left( \|u - v\|, \|u' - v'\| \right) \right] \right)
\]

\[
\leq \sum_{i=1}^{k_0} \Omega_q \left( \xi, \xi, \frac{\xi}{\Gamma_q(2 - \beta)}, m\xi, c_1\xi + c_2\xi \right) \Gamma_q^\alpha \mu_1(t)
\]

\[
+ A_1(t) \sum_{i=1}^{k_0} \Omega_q \left( \xi, \xi, \frac{\xi}{\Gamma_q(2 - \beta)}, m\xi, c_1\xi + c_2\xi \right) \Gamma_q^{\alpha-1} \mu_1(1)
\]

\[
+ A_2(t) \sum_{i=1}^{k_0} \Omega_q \left( \xi, \xi, \frac{\xi}{\Gamma_q(2 - \beta)}, m\xi, c_1\xi + c_2\xi \right) \Gamma_q^\alpha \mu_2(a)
\]

\[
+ A_3(t) \sum_{i=1}^{k_0} \Omega_q \left( \xi, \xi, \frac{\xi}{\Gamma_q(2 - \beta)}, m\xi, c_1\xi + c_2\xi \right) \Gamma_q^{\alpha+1} \mu_3(b)
\]

\[
\leq \sum_{i=1}^{k_0} \Omega_q (\xi, \xi, \xi, \xi, \xi) \Gamma_q^\alpha \mu_1(1)
\]
\[ + A_1(t) \sum_{i=1}^{k_0} \Omega_i(\ell \xi, \ell \xi, \ell \xi, \ell \xi) \mu_i(1) \]
\[ + A_2(t) \sum_{i=1}^{k_0} \Omega_i(\ell \xi, \ell \xi, \ell \xi, \ell \xi) \mu_i(1) \]
\[ + A_3(t) \sum_{i=1}^{k_0} \Omega_i(\ell \xi, \ell \xi, \ell \xi, \ell \xi) \mu_i(1) \]
\[ = A_0 \sum_{i=1}^{k_0} ||\hat{\mu}_i|| \Omega_i(\ell \xi, \ell \xi, \ell \xi, \ell \xi) \]
\[ + A_1(t) \sum_{i=1}^{k_0} ||\hat{\mu}_i|| \Omega_i(\ell \xi, \ell \xi, \ell \xi, \ell \xi) \]
\[ + A_2(t) \sum_{i=1}^{k_0} ||\hat{\mu}_i|| \Omega_i(\ell \xi, \ell \xi, \ell \xi, \ell \xi) \]
\[ + A_3(t) \sum_{i=1}^{k_0} ||\hat{\mu}_i|| \Omega_i(\ell \xi, \ell \xi, \ell \xi, \ell \xi) \]
\[ = \sum_{i=1}^{k_0} ||\hat{\mu}_i|| \Omega_i(\ell \xi, \ell \xi, \ell \xi, \ell \xi) \times [A_0 + A_1(t) + A_2(t) + A_3(t)]. \]

This implies that
\[ ||\Theta_u - \Theta_v|| \leq [A_0 + A_1(t) + A_2(t) + A_3(t)] \sum_{i=1}^{k_0} ||\hat{\mu}_i|| \Omega_i(\ell \xi, \ell \xi, \ell \xi, \ell \xi). \]

Assume that \( u, v \in \bar{B} \). Then, we get:
\[ |\Theta_u' - \Theta_v'| \leq \left| - \Gamma_{q}^{w-1} \left( t, u(t), u'(t), D_q^w u(t), \int_{0}^{1} f(r) u(r) \, dr, \varphi(u(t)) \right) \right| \]
\[ + \frac{b}{\mu(a,b)} \Gamma_{q}^{w-1} \left( 1, u(1), u'(1), D_q^w u(1), \int_{0}^{1} f(r) u(r) \, dr, \varphi(u(1)) \right) \]
\[ + \frac{b}{\mu(a,b)} \Gamma_{q}^{w} \left( a, u(a), u'(a), D_q^w u(a), \int_{0}^{a} f(r) u(r) \, dr, \varphi(u(a)) \right) \]
\[ + \frac{1}{\mu(a,b)} \Gamma_{q}^{w+1} \left( b, u(b), u'(b), D_q^w u(b), \int_{0}^{b} f(r) u(r) \, dr, \varphi(u(b)) \right) \]
\[ + \mathcal{I}_q^{\alpha-1} w \left( t, v(t), v'(t), D^\beta_q v(t), \int_0^t f(r)v(r) \, dr, \varphi(v(t)) \right) \]

\[- \frac{b}{\mu(a,b)} \mathcal{I}_q^{\alpha-1} w \left( 1, v(1), v'(1), D^\beta_q v(1), \int_0^1 f(r)v(r) \, dr, \varphi(v(1)) \right) \]

\[- \frac{b}{\mu(a,b)} \mathcal{I}_q^\alpha w \left( a, v(a), v'(a), D^\beta_q v(a), \int_0^a f(r)v(r) \, dr, \varphi(v(a)) \right) \]

\[- \frac{1}{\mu(a,b)} \mathcal{I}_q^{\alpha+1} w \left( b, v(b), v'(b), D^\beta_q v(b), \int_0^b f(r)v(r) \, dr, \varphi(u(b)) \right) \]

\[ \leq \mathcal{I}_q^{\alpha-1} w \left( t, u(t), u'(t), D^\beta_q u(t), \int_0^t f(r)u(r) \, dr, \varphi(u(t)) \right) \]

\[- w \left( t, v(t), v'(t), D^\beta_q v(t), \int_0^t f(r)v(r) \, dr, \varphi(v(t)) \right) \]

\[ + \frac{b}{\mu(a,b)} \mathcal{I}_q^{\alpha-1} \left| w \left( 1, u(1), u'(1), D^\beta_q u (1), \int_0^1 f(r)u(r) \, dr, \varphi(u(1)) \right) \right| \]

\[- \mathcal{I}_q^{\alpha-1} w \left( 1, v(1), v'(1), D^\beta_q v(1), \int_0^1 f(r)v(r) \, dr, \varphi(v(1)) \right) \]

\[ + \frac{b}{\mu(a,b)} \mathcal{I}_q^\alpha w \left( a, u(a), u'(a), D^\beta_q u(a), \int_0^a f(r)u(r) \, dr, \varphi(u(a)) \right) \]

\[- \mathcal{I}_q^\alpha w \left( a, v(a), v'(a), D^\beta_q v(a), \int_0^a f(r)v(r) \, dr, \varphi(v(a)) \right) \]

\[ + \frac{1}{\mu(a,b)} \mathcal{I}_q^{\alpha+1} \left| w \left( b, u(b), u'(b), D^\beta_q u(b), \int_0^b f(r)u(r) \, dr, \varphi(u(b)) \right) \right| \]

\[- w \left( b, v(b), v'(b), D^\beta_q v(b), \int_0^b f(r)v(r) \, dr, \varphi(u(b)) \right) \]

\[ \leq \mathcal{I}_q^{\alpha-1} \sum_{i=1}^{k_0} \mu_i(1) \left[ \Omega_q \left( u(t) - v(t), u'(t) - v'(t), D^\beta_q u(t) - D^\beta_q v(t), \int_0^t f(r)u(r) \, dr - \int_0^t f(r)v(r) \, dr, \varphi(u(t)) - \varphi(v(t)) \right) \right] \]

\[ + \frac{b}{\mu(a,b)} \mathcal{I}_q^{\alpha-1} \sum_{i=1}^{k_0} \mu_i(1) \]
\[
\times \left[ \Omega \left( u(1) - v(1), u'(1) - v'(1), D_q^\beta u(1) - D_q^\beta v(1), \right. \right.
\]
\[
\int_0^1 f(r)u(r) \, dr - \int_0^1 f(r)v(r) \, dr, \varphi(u(1)) - \varphi(v(1)) \left. \right] \right]
\]
\[
+ \frac{b}{\mu(a, b)} \sum_{i=1}^{k_0} \mu_i(a)
\times \left[ \Omega \left( u(a) - v(a), u'(a) - v'(a), D_q^\beta u(a) - D_q^\beta v(a), \right. \right.
\]
\[
\int_0^a f(r)u(r) \, dr - \int_0^a f(r)v(r) \, dr, \varphi(u(a)) - \varphi(v(a)) \left. \right] \right]
\]
\[
+ \frac{1}{\mu(a, b)} \sum_{i=1}^{k_0} \mu_i(b)
\times \left[ \Omega \left( u(b) - v(b), u'(b) - v'(b), D_q^\beta u(b) - D_q^\beta v(b), \right. \right.
\]
\[
\int_0^b f(r)u(r) \, dr - \int_0^b f(r)v(r) \, dr, \varphi(u(b)) - \varphi(v(b)) \left. \right] \right]
\]
\[
\leq \sum_{i=1}^{k_0} E_q^{a-1} \mu_i(t) \left[ \Omega \left( |u(t) - v(t)|, |u'(t) - v'(t)|, |D_q^\beta(u(t) - v(t))|, \right. \right.
\]
\[
\left| \int_0^t f(r)(u(r) - v(r)) \, dr \right|, |\varphi(u(t)) - \varphi(v(t))| \right]\right]
\]
\[
+ \frac{b}{\mu(a, b)} \sum_{i=1}^{k_0} E_q^{a-1} \mu_i(1)
\times \left[ \Omega \left( |u(1) - v(1)|, |u'(1) - v'(1)|, |D_q^\beta(u(1) - v(1))|, \right. \right.
\]
\[
\left| \int_0^1 f(r)(u(r) - v(r)) \, dr \right|, |\varphi(u(1)) - \varphi(v(1))| \right]\right]
\]
\[
+ \frac{b}{\mu(a, b)} \sum_{i=1}^{k_0} E_q^a \mu_i(1)
\times \left[ \Omega \left( |u(1) - v(1)|, |u'(1) - v'(1)|, |D_q^\beta(u(1) - v(1))|, \right. \right.
\]
\[
\left| \int_0^1 f(r)(u(r) - v(r)) \, dr \right|, |\varphi(u(1)) - \varphi(v(1))| \right]\right]
\[
\left| \int_0^1 f(r)(u(r) - v(r)) \, dr \right|, |\varphi(u(1)) - \varphi(v(1))| \right|
\]
\[
+ \frac{1}{\mu(a, b)} \sum_{i=1}^{k_0} I^{a+1}_q \mu_i(1)
\]
\[
\times \left[ \Omega \left( |u(1) - v(1)|, |u'(1) - v'(1)|, |D_q^{a}(u(1) - v(1))|, \right. \right.
\]
\[
\left. \left| \int_0^1 f(r)(u(r) - v(r)) \, dr \right|, |\varphi(u(1)) - \varphi(v(1))| \right|
\]
\[
\leq \sum_{i=1}^{k_0} E^{-1}_q \left[ \Omega \left( |u(t) - v(t)|, |u'(t) - v'(t)|, \frac{|u'(t) - v'(t)|}{\Gamma_q(2-\beta)}, \right. \right.
\]
\[
\left. m|u(t) - v(t)|, c_1|u(t) - v(t)| + c_2|u'(t) - v'(t)| \right] \right]
\]
\[
+ \frac{b}{\mu(a, b)} \sum_{i=1}^{k_0} E^{-1}_q \left[ \Omega \left( |u(1) - v(1)|, |u'(1) - v'(1)|, \frac{|u'(1) - v'(1)|}{\Gamma_q(2-\beta)}, \right. \right.
\]
\[
\left. m|u(1) - v(1)|, c_1|u(1) - v(1)| + c_2|u'(1) - v'(1)| \right] \right]
\]
\[
+ \frac{b}{\mu(a, b)} \sum_{i=1}^{k_0} E_q \left[ \Omega \left( |u(a) - v(a)|, |u'(a) - v'(a)|, \frac{|u'(a) - v'(a)|}{\Gamma_q(2-\beta)}, \right. \right.
\]
\[
\left. m|u(a) - v(a)|, c_1|u(a) - v(a)| + c_2|u'(a) - v'(a)| \right] \right]
\]
\[
+ \frac{1}{\mu(a, b)} \sum_{i=1}^{k_0} E^{a+1}_q \left[ \Omega \left( |u(b) - v(b)|, |u'(b) - v'(b)|, \frac{|u'(b) - v'(b)|}{\Gamma_q(2-\beta)}, \right. \right.
\]
\[
\left. m|u(b) - v(b)|, c_1|u(b) - v(b)| + c_2|u'(b) - v'(b)| \right] \right]
\]
\[
\leq \sum_{i=1}^{k_0} \Omega \left( \xi, \xi, \frac{\xi}{\Gamma_q(2-\beta)}, m\xi, c_1\xi + c_2\xi \right] E^{a-1}_q \mu_i(t)
\]
\[
+ \frac{b}{\mu(a, b)} \sum_{i=1}^{k_0} \Omega \left( \xi, \xi, \frac{\xi}{\Gamma_q(2-\beta)}, m\xi, c_1\xi + c_2\xi \right] E^{a-1}_q \mu_i(1)
\]
\[
+ \frac{b}{\mu(a, b)} \sum_{i=1}^{k_0} \Omega \left( \xi, \xi, \frac{\xi}{\Gamma_q(2-\beta)}, m\xi, c_1\xi + c_2\xi \right] E_q \mu_i(a)
\]
\[+ \frac{1}{\mu(a, b)} \sum_{i=1}^{k_0} \Omega_i \left( \frac{\xi, \xi, \xi}{\Gamma_q(2 - \beta)}, \xi, c_1 \xi + c_2 \xi \right) \Gamma_q^{a+1} \mu_i(b)\]

\[\leq \sum_{i=1}^{k_0} \Omega_i \left( \xi, \xi, \xi, \xi, \xi, \xi \right) \Gamma_q^{a-1} \mu_i(1)\]

\[+ \frac{b}{\mu(a, b)} \sum_{i=1}^{k_0} \Omega_i \left( \xi, \xi, \xi, \xi, \xi, \xi \right) \Gamma_q^{a-1} \mu_i(1)\]

\[+ \frac{b}{\mu(a, b)} \sum_{i=1}^{k_0} \Omega_i \left( \xi, \xi, \xi, \xi, \xi, \xi \right) \Gamma_q^{a} \mu_i(1)\]

\[+ \frac{1}{\mu(a, b)} \sum_{i=1}^{k_0} \Omega_i \left( \xi, \xi, \xi, \xi, \xi, \xi \right) \Gamma_q^{a+1} \mu_i(1)\]

\[= \frac{1}{\Gamma_q(\alpha - 1)} \sum_{i=1}^{k_0} ||\hat{\mu}_i|| \Omega_i (\xi, \xi, \xi, \xi, \xi)\]

\[+ \frac{b}{\mu(a, b)\Gamma_q(\alpha - 1)} \sum_{i=1}^{k_0} ||\hat{\mu}_i|| \Omega_i (\xi, \xi, \xi, \xi, \xi)\]

\[+ \frac{b}{\mu(a, b)\Gamma_q(\alpha)} \sum_{i=1}^{k_0} ||\hat{\mu}_i|| \Omega_i (\xi, \xi, \xi, \xi, \xi)\]

\[+ \frac{1}{\mu(a, b)\Gamma_q(\alpha + 1)} \sum_{i=1}^{k_0} ||\hat{\mu}_i|| \Omega_i (\xi, \xi, \xi, \xi, \xi)\]

\[= \sum_{i=1}^{k_0} ||\hat{\mu}_i|| \Omega_i (\xi, \xi, \xi, \xi, \xi) \left[ \frac{1}{\Gamma_q(\alpha - 1)} + \frac{b}{\mu(a, b)\Gamma_q(\alpha - 1)} + \frac{b}{\mu(a, b)\Gamma_q(\alpha)} + \frac{1}{\mu(a, b)\Gamma_q(\alpha + 1)} \right].\]

Hence,

\[||\Theta' - \Theta'|| \leq \sum_{i=1}^{k_0} ||\hat{\mu}_i|| \Omega_i (\xi, \xi, \xi, \xi, \xi) \left[ \frac{1}{\Gamma_q(\alpha - 1)} + \frac{b}{\mu(a, b)\Gamma_q(\alpha - 1)} + \frac{b}{\mu(a, b)\Gamma_q(\alpha)} + \frac{1}{\mu(a, b)\Gamma_q(\alpha + 1)} \right].\]
and so
\[ \| \Theta_u - \Theta_v \|_* \leq \sum_{i=1}^{k_0} \| \hat{\mu}_i \|_1 \Omega_i(\ell \xi, \ell \xi, \ell \xi) \max \left\{ \frac{1}{\Gamma_q(\alpha)} + \frac{b(2 - a) - \mu(a, b)}{\mu(a, b)\Gamma_q(\alpha - 1)} + \frac{\mu(a, b) + ab}{\mu(a, b)\Gamma_q(\alpha)} + \frac{\mu(a, b)(1 - a) + 1}{\mu(a, b)\Gamma_q(\alpha + 1)}, \right. \]
\[ \left. + \frac{1}{\Gamma_q(\alpha - 1)} + \frac{b}{\mu(a, b)\Gamma_q(\alpha - 1)} + \frac{b}{\mu(a, b)\Gamma_q(\alpha)} + \frac{1}{\mu(a, b)\Gamma_q(\alpha + 1)} \right\}. \]

If
\[ M_{a,b} = \max \left\{ \frac{1}{\Gamma_q(\alpha)} + \frac{b(2 - a) - \mu(a, b)}{\mu(a, b)\Gamma_q(\alpha - 1)} + \frac{\mu(a, b) + ab}{\mu(a, b)\Gamma_q(\alpha)} + \frac{\mu(a, b)(1 - a) + 1}{\mu(a, b)\Gamma_q(\alpha + 1)}, \right. \]
\[ \left. + \frac{1}{\Gamma_q(\alpha - 1)} + \frac{b}{\mu(a, b)\Gamma_q(\alpha - 1)} + \frac{b}{\mu(a, b)\Gamma_q(\alpha)} + \frac{1}{\mu(a, b)\Gamma_q(\alpha + 1)} \right\}, \]

then
\[ \| \Theta_u - \Theta_v \|_* \leq M_{a,b} \sum_{i=1}^{k_0} \| \hat{\mu}_i \|_1 \Omega_i(\ell \xi, \ell \xi, \ell \xi, \ell \xi). \]  

(3.5)

Let \( 0 < \varepsilon \leq 1 \) be given. Since
\[ \lim_{\nu \to 0^+} \frac{\Omega_i(\nu, \nu, \nu, \nu)}{\nu^{\gamma_i}} = p_i, \]
for \( 1 \leq i \leq k_0 \), \( \exists \delta_i = \delta_i(\varepsilon) \) such that \( \nu \in (0, \delta_i] \) implies
\[ \left| \frac{\Omega_i(\nu, \nu, \nu, \nu)}{\nu^{\gamma_i}} - p_i \right| < \varepsilon, \]
and so \( \Omega_i(\nu, \nu, \nu, \nu)/\nu^{\gamma_i} < \varepsilon + p_i \). This consequents
\[ 0 \leq \Omega_i(\nu, \nu, \nu, \nu) < (\varepsilon + p_i)\nu^{\gamma_i}. \]

We take \( \delta = \min[\delta_1, \ldots, \delta_{k_0}, \varepsilon] \). In this case, \( \nu \in (0, \delta] \) implies
\[ 0 \leq \Omega_i(\nu, \nu, \nu, \nu) < (\varepsilon + p_i)\nu^{\gamma_i} \]  

(3.6)

for all \( 1 \leq i \leq k_0 \). By using (3.6), we obtain:
\[ \Omega_i(\ell \xi, \ldots, \ell \xi) \leq (\varepsilon + p_i)(\ell \xi)^{\gamma_i} \leq (\varepsilon + p_i)\ell^{\gamma_i} \varepsilon^{\gamma_i}. \]  

(3.7)
At present, by applying (3.5) and (3.7), we obtain:

\[ \|\Theta_u - \Theta_v\|_* \leq M_{a,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 (\varepsilon + p_i) \ell^\gamma. \]

Now, we consider: \( \gamma = \min\{\gamma_1, \ldots, \gamma_{k_0}\} \). Hence,

\[ \|\Theta_u - \Theta_v\|_* \leq \varepsilon^\gamma M_{a,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 (\varepsilon + p_i) \ell^\gamma. \]

Therefore, this implies that \( \Theta \) is continuous. Since

\[ M_{a,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 p_i \ell^\gamma < 1, \]

there is \( \varepsilon_1 > 0 \) such that

\[ M_{a,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_1 (p_i + \varepsilon_1) \ell^\gamma < 1. \]

Let

\[ \lambda = \lim_{\nu \to 0^+} \frac{T(\nu, \nu, \nu, \nu, \nu)}{\nu} \in [0, \tau). \]

Then, we have:

\[ \lambda = \lim_{\nu \to 0^+} T(\ell\nu, \ldots, \ell\nu)/(\ell\nu), \]

and so for each \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) > 0 \) such that \( \nu \in (0, \delta(\varepsilon)) \) implies

\[ 0 \leq \frac{T(\ell\nu, \ldots, \ell\nu)}{\ell\nu} - \lambda < \varepsilon. \]

Hence, \( 0 \leq T(\ell\nu, \ldots, \ell\nu) < (\lambda + \varepsilon)\ell\nu \) and

\[ 0 \leq T(\ell\delta(\varepsilon), \ldots, \ell\delta(\varepsilon)) < (\lambda + \varepsilon)\ell\delta(\varepsilon). \]

Since \( \lambda \in [0, \tau) \), choose \( \varepsilon_0 > 0 \) such that \( \lambda + \varepsilon_0 < \tau \). Assume that

\[ \eta_0 = \min\{\delta(\varepsilon_0), \delta(\varepsilon_1)\}. \]

Then, \( \eta \leq \eta_0 \) implies \( 0 \leq T(\ell\eta, \ldots, \ell\eta) < (\lambda + \varepsilon_0)\ell\eta. \) Since

\[ \lim_{\nu \to 0^+} \frac{\Omega(\nu, \nu, \nu, \nu, \nu)}{\nu^\gamma} = p_i, \]

there exists \( \eta_1 > 0 \) such that \( \nu \in (0, \eta_1] \) implies

\[ \Omega_\nu(\ell\nu, \ldots, \ell\nu) < (p_i + \varepsilon_0)(\ell\nu)^\gamma \] (3.8)

for \( i = 1, \ldots, k_0 \). Let \( \eta = \min\{\eta_0, \frac{\eta_1}{2}, \frac{1}{2}\} \) and

\[ E = \{u \in \bar{B} : \|u\|_* \leq \eta\}. \]
Define $\alpha : \mathcal{B}^2 \to \mathbb{R}$ by

$$\alpha(u, v) = \begin{cases} 1 & u = v, \\ 0 & u \neq v. \end{cases}$$

Assume that $u, v \in \mathcal{B}$ be given. If $\alpha(u, v) \geq 1$, then for every $t \in \bar{J}$, we have:

$$|\Theta_u(t)| \leq \int_0^t |G_q(t, s)|w\left(s, u(s), u'(s), D_q^\alpha u(s), \int_0^s f(r)u(r) \, dr, \varphi(u(s))\right) \, ds$$

$$\leq \Gamma_q^a \left|w(t, u(t), u'(t), D_q^\alpha u(t), \int_0^t f(r)u(r) \, dr, \varphi(u(t)))\right|$$

$$+ A_1(t)\Gamma_q^{a-1} \left|w(1, u(1), u'(1), D_q^\alpha u(1), \int_0^1 f(r)u(r) \, dr, \varphi(u(1)))\right|$$

$$+ A_2(t)\Gamma_q^a \left|w(a, u(a), u'(a), D_q^\alpha u(a), \int_0^a f(r)u(r) \, dr, \varphi(u(a)))\right|$$

$$+ A_3(t)\Gamma_q^{a-1} \left|w(b, u(b), u'(b), D_q^\alpha u(b), \int_0^b f(r)u(r) \, dr, \varphi(u(b)))\right|$$

$$\leq \Gamma_q^a \left|h(t)T\left(u(t), u'(t), D_q^\alpha u(t), \int_0^t f(r)u(r) \, dr, \varphi(u(t))\right)\right|$$

$$+ A_1(t)\Gamma_q^{a-1} \left|h(1)T\left(u(1), u'(1), D_q^\alpha u(1), \int_0^1 f(r)u(r) \, dr, \varphi(u(1))\right)\right|$$

$$+ A_2(t)\Gamma_q^a \left|h(a)T\left(u(a), u'(a), D_q^\alpha u(a), \int_0^a f(r)u(r) \, dr, \varphi(u(a))\right)\right|$$

$$+ A_3(t)\Gamma_q^{a-1} \left|h(b)T\left(u(b), u'(b), D_q^\alpha u(b), \int_0^b f(r)u(r) \, dr, \varphi(u(b))\right)\right|$$

$$\leq \Gamma_q^a \left|h(t)T\left|u(t)|, |u'(t)|, |D_q^\alpha u(t)|, \int_0^t |f(r)||u(r)| \, dr, |\varphi(u(t))|\right|\right|$$

$$+ A_1(t)\Gamma_q^{a-1} \left|h(1)T\left|u(1)|, |u'(1)|, |D_q^\alpha u(1)|, \int_0^1 |f(r)||u(r)| \, dr, |\varphi(u(1))|\right|\right|$$

$$+ A_2(t)\Gamma_q^a \left|h(a)T\left|u(a)|, |u'(a)|, |D_q^\alpha u(a)|, \int_0^a |f(r)||u(r)| \, dr, |\varphi(u(a))|\right|\right|$$

$$+ A_3(t)\Gamma_q^{a-1} \left|h(b)T\left|u(b), u'(b), D_q^\alpha u(b), \int_0^b |f(r)||u(r)| \, dr, |\varphi(u(b))|\right|\right|$$
\[
+ A_3(t)\Gamma_q^{\alpha+1}\left(h(b)T\left(\|u(b)\|, \|u'(b)\|, |D_q^a u(b)|, \right. \right.
\]
\[
\left. \int_0^b |f(r)||u(r)| \, dr, |\varphi(u(b))| \right) \right) \]
\[
\leq \Gamma_q^{\alpha}\left(h(t)T\left(\|u(t)\|, \|u'(t)\|, \frac{\|u'(t)\|}{\Gamma_q(2-\beta)}, m\|u(t)\|, c_1\|u(t)\| + c_2\|u'(t)\| \right) \right)
\]
\[
+ A_1(t)\Gamma_q^{\alpha}\left(h(1)T\left(\|u(\alpha)\|, \|u'(\alpha)\|, \frac{\|u'(\alpha)\|}{\Gamma_q(2-\beta)}, m\|u(\alpha)\|, c_1\|u(\alpha)\| + c_2\|u'(\alpha)\| \right) \right)
\]
\[
+ A_2(t)\Gamma_q^{\alpha}\left(h(b)T\left(\|u(b)\|, \|u'(b)\|, \frac{\|u'(b)\|}{\Gamma_q(2-\beta)}, m\|u(b)\|, c_1\|u(b)\| + c_2\|u'(b)\| \right) \right)
\]
\[
\leq T\left(\|u(t)\|, \|u'(t)\|, \frac{\|u'(t)\|}{\Gamma_q(2-\beta)}, m\|u(t)\|, c_1\|u(t)\| + c_2\|u'(t)\| \right)\Gamma_q^\alpha h(t)
\]
\[
+ A_1(t)T\left(\|u(\alpha)\|, \|u'(\alpha)\|, \frac{\|u'(\alpha)\|}{\Gamma_q(2-\beta)}, m\|u(\alpha)\|, c_1\|u(\alpha)\| + c_2\|u'(\alpha)\| \right)\Gamma_q^{\alpha-1} h(1)
\]
\[
+ A_2(t)T\left(\|u(b)\|, \|u'(b)\|, \frac{\|u'(b)\|}{\Gamma_q(2-\beta)}, m\|u(b)\|, c_1\|u(b)\| + c_2\|u'(b)\| \right)\Gamma_q^\alpha h(a)
\]
\[
+ A_3(t)T\left(\|u(t)\|, \|u'(t)\|, \frac{\|u'(t)\|}{\Gamma_q(2-\beta)}, m\|u(t)\|, c_1\|u(t)\| + c_2\|u'(t)\| \right)
\]
\[
m\|u(t)\|, c_1\|u(t)\| + c_2\|u'(t)\| \leq T(\ell\|u(t)\|, \ell\|u(t)\|, \ell\|u(t)\|, \ell\|u(t)\|)\hat{h}(b)
\]
\[
\leq T(\ell, \ell, \ell, \ell)\hat{h}_1[A_0 + A_1(t) + A_2(t) + A_3(t)]
\]
\[
\leq \ell r(\lambda + \varepsilon)\hat{h}_1[A_0 + A_1(t) + A_2(t) + A_3(t)]
\]
\[
= \eta(\ell(\lambda + \varepsilon))\hat{h}_1[A_0 + A_1(t) + A_2(t) + A_3(t)].
\]

Therefore,
\[
\|\Theta_u\| \leq \eta(\ell(\lambda + \varepsilon))\hat{h}_1[A_0 + A_1(t) + A_2(t) + A_3(t)] \leq \eta.
\]

Also,
\[
|\Theta'_u(t)| \leq \eta^{-1}\left| w\left(t, u(t), u'(t), \mathcal{D}_q^\beta u(t), \int_0^t f(r)u(r) \, dr, \varphi(u(t)) \right) \right|
\]
\[
+ \frac{b}{\mu(a, b)} \eta^{-1}\left| w\left(1, u(1), u'(1), \mathcal{D}_q^\beta u(1), \int_0^1 f(r)u(r) \, dr, \varphi(u(1)) \right) \right|
\]
\[
+ \frac{b}{\mu(a, b)} \eta^{-1}\left| w\left(a, u(a), u'(a), \mathcal{D}_q^\beta u(a), \int_0^a f(r)u(r) \, dr, \varphi(u(a)) \right) \right|
\]
\[
+ \frac{1}{\mu(a, b)} \eta^{-1}\left| w\left(b, u(b), u'(b), \mathcal{D}_q^\beta u(b), \int_0^b f(r)u(r) \, dr, \varphi(u(b)) \right) \right|
\]
\[
\leq \eta^{-1}\left| w\left(t, u(t), u'(t), \mathcal{D}_q^\beta u(t), \int_0^t f(r)u(r) \, dr, \varphi(u(t)) \right) \right|
\]
\[
+ \frac{b}{\mu(a, b)} \eta^{-1}\left| w\left(1, u(1), u'(1), \mathcal{D}_q^\beta u(1), \int_0^1 f(r)u(r) \, dr, \varphi(u(1)) \right) \right|
\]
\[
+ \frac{b}{\mu(a, b)} \eta^{-1}\left| w\left(a, u(a), u'(a), \mathcal{D}_q^\beta u(a), \int_0^a f(r)u(r) \, dr, \varphi(u(a)) \right) \right|
\]
\[
+ \frac{1}{\mu(a, b)} \eta^{-1}\left| w\left(b, u(b), u'(b), \mathcal{D}_q^\beta u(b), \int_0^b f(r)u(r) \, dr, \varphi(u(b)) \right) \right|
\]
\[
\leq \eta^{-1}\left( h(t)T\left(u(t), u'(t), \mathcal{D}_q^\beta u(t), \int_0^t f(r)u(r) \, dr, \varphi(u(t)) \right) \right)
\]
\[
+ \frac{b}{\mu(a, b)} \eta^{-1}\left( h(1)T\left(u(1), u'(1), \mathcal{D}_q^\beta u(1), \int_0^1 f(r)u(r) \, dr, \varphi(u(1)) \right) \right)
\]
\[ + \frac{b}{\mu(a, b)} \Gamma_q \left( h(a) T \left( u(a), u'(a), \nabla_q u(a), \int_0^a f(r) u(r) \, dr, \varphi(u(a)) \right) \right) \]
\[ + \frac{1}{\mu(a, b)} \Gamma_q \left( h(b) T \left( u(b), u'(b), \nabla_q u(b), \int_0^b f(r) u(r) \, dr, \varphi(u(b)) \right) \right) \]
\[ \leq \Gamma_q \left( h(t) T \left( u(t), u'(t), \nabla_q u(t), \int_0^t f(r) u(r) \, dr, \varphi(u(t)) \right) \right) \]
\[ + \frac{b}{\mu(a, b)} \Gamma_q \left( h(1) T \left( u(1), u'(1), \nabla_q u(1), \int_0^1 f(r) u(r) \, dr, \varphi(u(1)) \right) \right) \]
\[ + \frac{b}{\mu(a, b)} \Gamma_q \left( h(a) T \left( u(a), u'(a), \nabla_q u(a), \int_0^a f(r) u(r) \, dr, \varphi(u(a)) \right) \right) \]
\[ + \frac{1}{\mu(a, b)} \Gamma_q \left( h(b) T \left( u(b), u'(b), \nabla_q u(b), \int_0^b f(r) u(r) \, dr, \varphi(u(b)) \right) \right) \]
\[ \leq \Gamma_q \left( h(t) T \left( ||u(t)||, ||u'(t)||, \frac{||u'(t)||}{\Gamma_q(2 - \beta)} \right) \right) \]
\[ m ||u(t)||, c_1 ||u(t)|| + c_2 ||u'(t)||) \]
\[ + \frac{b}{\mu(a, b)} \Gamma_q \left( h(1) T \left( ||u(t)||, ||u'(t)||, \frac{||u'(t)||}{\Gamma_q(2 - \beta)} \right) \right) \]
\[ m ||u(t)||, c_1 ||u(t)|| + c_2 ||u'(t)||) \]
\[ + \frac{b}{\mu(a, b)} \Gamma_q \left( h(a) T \left( ||u(t)||, ||u'(t)||, \frac{||u'(t)||}{\Gamma_q(2 - \beta)} \right) \right) \]
\[ m ||u(t)||, c_1 ||u(t)|| + c_2 ||u'(t)||) \]
\[ + \frac{1}{\mu(a, b)} \Gamma_q \left( h(b) T \left( ||u(t)||, ||u'(t)||, \frac{||u'(t)||}{\Gamma_q(2 - \beta)} \right) \right) \]
\[ m ||u(t)||, c_1 ||u(t)|| + c_2 ||u'(t)||) \]
\[ \leq T \left( ||u(t)||, ||u'(t)||, \frac{||u'(t)||}{\Gamma_q(2 - \beta)}, m ||u(t)||, c_1 ||u(t)|| + c_2 ||u'(t)|| \right) \Gamma_q^{-1} (h(t)) \]
\[ + \frac{b}{\mu(a, b)} T \left( ||u(t)||, ||u'(t)||, \frac{||u'(t)||}{\Gamma_q(2 - \beta)}, m ||u(t)||, c_1 ||u(t)|| + c_2 ||u'(t)|| \right) \Gamma_q^{-1} (h(1)) \]
where ξ
\[\sum\int\frac{b}{\mu(a, b)} T\left(||u(t)||, ||u'(t)||, \frac{||u'(t)||}{\Gamma q(2 - \beta)}, m||u(t)||, c_1||u(t)|| + c_2||u'(t)||\right)\Gamma q^\alpha(h)\] + \frac{1}{\mu(a, b)} T\left(||u(t)||, ||u'(t)||, \frac{||u'(t)||}{\Gamma q(2 - \beta)}, m||u(t)||, c_1||u(t)|| + c_2||u'(t)||\right)\Gamma q^{\alpha+1}(h)\] ≤ T(ℓ||u||, · · ·, ℓ||u||)||h||

Indeed,
\[|\Theta'_u(t)| \leq (\ell r)(\lambda + \varepsilon_0)||h||\]
\[
\times \left[\frac{1}{\Gamma q(\alpha - 1)} + \frac{b}{\mu(a, b)\Gamma q(\alpha - 1)} + \frac{b}{\mu(a, b)\Gamma q(\alpha)} + \frac{1}{\mu(a, b)\Gamma q(\alpha + 1)}\right]
\]

Hence, ||\Theta_u||* \leq \eta and so \Theta_u \in E. Using a similar proof, we can show that \Theta_v \in E. This implies \alpha(\Theta_u, \Theta_v) \geq 1 and so \Theta_v is \alpha-admissible. It is obvious that, \(E \neq \emptyset\). Choose \(u_0 \in E\). Hence, \(\Theta_{u_0} \in E\), and so \(\alpha(u_0, \Theta_{u_0}) \geq 1\). Let \(u, v \in E\). Then,
\[\xi \leq ||u||_* + ||v||_* \leq 2\eta \leq \eta_1,\]
where \(\xi = ||u - v||_*\). Also using (3.5), we have
\[||\Theta_u - \Theta_v||_* \leq M_{a, b, k} \sum_{i=1}^{k_0} ||\hat{u}_i||_1 M_i(\ell_1, \ell_1, \ell_1).\]

Now, by using (3.8), we conclude that
\[ ||\Theta_u - \Theta_v||_* \leq M_{\alpha,a,b} \sum_{i=1}^{k_0} ||\hat{\mu}_i||_1 (p_i + \epsilon_1) (\ell \xi_i) \gamma_i \]
\[ \leq M_{\alpha,a,b} \sum_{i=1}^{k_0} ||\hat{\mu}_i||_1 (p_i + \epsilon_1) \ell \xi_i \gamma_i \]
\[ \leq M_{\alpha,a,b} \left[ \sum_{i=1}^{k_0} ||\hat{\mu}_i||_1 (p_i + \epsilon_1) \ell \right] \xi_i \gamma_i, \]

where \( \gamma = \min\{\gamma_1, \ldots, \gamma_{k_0}\} \). We take:
\[ \eta = M_{\alpha,a,b} \sum_{i=1}^{k_0} ||\hat{\mu}_i||_1 p_i \ell \gamma_i. \]

Note that, \( \eta \in [0, 1) \). Define the map \( \psi : [0, \infty) \to \mathbb{R}^+ \) by
\[ \psi(t) = \begin{cases} \eta t^\gamma & t \in [0, 1), \\ \eta t & t \in [1, \infty). \end{cases} \]

Then, \( \psi \) is nondecreasing and
\[ \sum_{i=1}^{\infty} \psi^i(t) = \eta t^\gamma + \eta^2 t^{2\gamma} + \cdots \leq \sum_{i=1}^{\infty} \eta^i t^\gamma = \frac{\tau}{1 - \eta} t^\gamma < \infty, \]
for \( 0 \leq t < 1 \). Also, we obtain
\[ \sum_{i=1}^{\infty} \psi^i(t) = \frac{\eta}{1 - \eta} t < \infty, \]
for \( t \in [1, \infty) \). Thus, \( \sum_{i=1}^{\infty} \psi^i(t) \) is a convergent series for all \( t \geq 0 \) and so \( \psi \in \Psi \). Also, we have
\[ \alpha(u, v)||\Theta_u - \Theta_v||_* \leq \phi(\xi). \]

If \( u \notin E \) or \( v \notin E \), then the last inequality holds obviously. This shows that
\[ \alpha(u, v) d(\Theta_u, \Theta_v) \leq \phi(d(u, v)), \]
for all \( u, v \in \bar{B} \). Now, Lemma 2.6 implies that \( \Theta \) has a fixed point that is the solution for problem (1.1).

\[ \square \]

4. An illustrative example with application

The following illustrative example is given to support the validity of our main results. A computational method is provided here to test the proposed problem (1.1). Linear motion is commonly basic among all other motions. From the 1st law of Newton’s motion, objects that are not experiencing any net force will continue to move in a straight line with a constant velocity until they are subjected to a net force.
Example 4.1. We consider a constrained motion of a particle along a straight line restrained by two linear springs with equal spring constant (stiffness coefficient) under external force and fractional damping along the $t$-axis (Figure 1).

![Figure 1. A particle along a straight line restrained by two linear springs with equal spring constant.](image)

We consider the pointwise defined equation:

$$
100\theta(t)\frac{D^{2.5}_q u(t)}{D_q} + p(t)u(t) = -p(t)\left(|u'(t)| + \left|\frac{D^{\frac{3}{2}}_q u(t)}{D_q}\right|\right)
+ \left|\int_0^t \frac{u(r)}{\sqrt{r}} \, dr \right| + |\sin(u(t))|, \quad (4.1)
$$

where

$$
p(t) = \frac{1}{8} \left(2 - 2L - \eta^2 L - \eta^2 L \cos t\right),
$$

$\eta$ is constant and $L$ is the unstretched length of the spring. We change Eq (4.1) into a form of the problem (1.1) as follows:

$$
\frac{D^{\frac{3}{2}}_q u(t)}{D_q} = \frac{1}{100 \theta(t)}\left(|u(t)| + |u'(t)| + \left|\frac{D^{\frac{3}{2}}_q u(t)}{D_q}\right|\right)
+ \left|\int_0^t \frac{u(r)}{\sqrt{r}} \, dr \right| + |\sin(u(t))|, \quad (4.2)
$$

with boundary conditions:

$$
\int_0^1 u(r) \, dr = 0, \quad u'(1) = u(\frac{1}{4}), \quad u''(0) = 0.
$$

Also

$$
\theta(t) = \begin{cases} 
0 & t \in J \cap \mathbb{Q}, \\
1 - t & t \in J \cap \mathbb{Q}^c.
\end{cases}
$$

Take $\alpha = \frac{5}{2} \geq 2$, $\beta = \frac{1}{2} \in J$, $a = \frac{1}{3} \in J$, $b = \frac{1}{3} \in J$, $k_0 = 1$, $\gamma_1 = 1$, $\mu_1(t) = h(t) = \frac{1}{\theta(t)}$, $c_1 = \frac{1}{3}$, $c_2 = \frac{2}{3}$, $f(\xi) = \frac{\varphi(\xi)}{\sqrt{\xi}}$, $\varphi(x) = \sin(x)$ and

$$
T(u_1, \ldots, u_5) = \Omega_1(u_1, \ldots, u_5) = \frac{1}{500}(|u_1| + \cdots + |u_5|).
$$
Then, we get:

\[ |\varphi(u) - \varphi(v)| = |\sin(u) - \sin(v)| \leq |u - v| = c_1|u - v| + c_2|u' - v'|, \]

\[ |w(t, u_1, \ldots, u_5) - w(t, v_1, \ldots, v_5)| \leq \mu_1(t) \left[ |u_1 - v_1| + \cdots + |u_5 - v_5| \right]. \]

\[ p_1 = \lim_{\nu \to 0^+} \frac{\Omega_1(v, v, v, v)}{v^5} = \lim_{\nu \to 0^+} \frac{5|\nu|}{500\nu} = 0.01, \]

\[ \mu_1, h \in L^1, m = \|h\|_1 = 2, \]

\[ ||\hat{h}||_f = ||\hat{\mu}_1||_f = \int_0^1 \frac{1}{\tilde{g}(s)}(1 - s)^{\alpha - 2} \, ds = \int_0^1 \frac{(1 - s)^2}{1 - s} \, ds = 2, \]

\[ |w(t, u_1, \ldots, u_5)| \leq h(t)T(u_1, \ldots, u_5), \]

\( T, \Omega_1 \) are non-negative and non-decreasing with respect to \( u_1, \ldots, u_5, \)

\[ \mu(a, b) = b(1 - a) + \frac{b^2}{2} = \frac{11}{36}, \]

\[ \ell = \max \left\{ 1, \frac{1}{\Gamma_q(2 - \beta)}, m, c_1 + c_2 \right\} = \max \left\{ 1, \frac{1}{\Gamma_q(\frac{3}{2})}, 2, 1 \right\} = 2, \]

\[ M_{a,b} = \max \left\{ \frac{1}{\Gamma_q(\alpha)} + \frac{b(2 - a) - \mu(a, b)}{\mu(a, b)\Gamma_q(\alpha - 1)} + \frac{\mu(a, b) + ab}{\mu(a, b)\Gamma_q(\alpha)} \right. \]

\[ + \frac{\mu(a, b)(1 - a) + 1}{\mu(a, b)\Gamma_q(\alpha + 1)} \cdot \Gamma_q(\alpha - 1) + \frac{b}{\mu(a, b)\Gamma_q(\alpha - 1)} \]

\[ + \left. \frac{b}{\mu(a, b)\Gamma_q(\alpha)} + \frac{1}{\mu(a, b)\Gamma_q(\alpha + 1)} \right\} \]

\[ = \max \left\{ \frac{25}{11\Gamma_q(\frac{3}{2})} + \frac{10}{11\Gamma_q(\frac{5}{2})} + \frac{177}{44\Gamma_q(\frac{3}{2})}, \right. \]

\[ \frac{23}{11\Gamma_q(\frac{3}{2})} + \frac{12}{11\Gamma_q(\frac{5}{2})} + \frac{36}{44\Gamma_q(\frac{3}{2})} \right\} \]

We put:

\[ \Lambda_1 = \frac{25}{11\Gamma_q(\frac{3}{2})} + \frac{10}{11\Gamma_q(\frac{5}{2})} + \frac{177}{44\Gamma_q(\frac{3}{2})}, \]

\[ \Lambda_2 = \frac{23}{11\Gamma_q(\frac{3}{2})} + \frac{12}{11\Gamma_q(\frac{5}{2})} + \frac{36}{44\Gamma_q(\frac{3}{2})}. \]
Table 1 shows the values of $\Lambda_1$ and $\Lambda_2$ for $q = \left\{ \frac{1}{8}, \frac{1}{2}, \frac{4}{5}, \frac{8}{9} \right\}$. We can see that

\[ M_{a,a,b} = 33.170478, 21.551855, 16.363257, 15.234356, \]

for $q = \frac{1}{8}, \frac{1}{2}, \frac{4}{5}$ and $\frac{8}{9}$, respectively. Thus, by using the numerical results, we obtain:

\[ \tau = \left( \ell \left\| \hat{h} \right\| M_{a,a,b} \right)^{-1} \geq \frac{1}{4 \times 33.1704} = 0.0075, \]

whenever $q = \frac{1}{8}$,

\[ \tau = \left( \ell \left\| \hat{h} \right\| M_{a,a,b} \right)^{-1} \geq \frac{1}{4 \times 21.5518} = 0.0116, \]

whenever $q = \frac{1}{2}$,

\[ \tau = \left( \ell \left\| \hat{h} \right\| M_{a,a,b} \right)^{-1} \geq \frac{1}{4 \times 16.3632} = 0.0153, \]

whenever $q = \frac{4}{5}$ and

\[ \tau = \left( \ell \left\| \hat{h} \right\| M_{a,a,b} \right)^{-1} \geq \frac{1}{4 \times 15.2343} = 0.0164, \]

whenever $q = \frac{8}{9}$. Also, we can check that

\[ \lim_{\nu \to 0^+} \frac{T(\nu, \nu, \nu, \nu, \nu)}{\nu} = 0.01 \in [0, \tau), \]

and for all $q \in J$

\[ M_{a,a,b} \sum_{i=1}^{k_0} \left\| \hat{\mu}_i \right\| J P_i \ell_{\nu_0} = M_{a,a,b} \times 2 \times 0.01 \times 2^1 = 0.04 M_{a,a,b} < 1. \]

Table 2 shows numerical results for different values of $q \in J$. Figure 2 shows the curve of these results. Now, according to the obtained results, Theorem 3.3 implies that problem (4.2) has a solution.
Table 1. The results of $\Lambda_1, \Lambda_2$ in Eq (4.3) in Example 4.1 for $q \in \{\frac{1}{8}, \frac{1}{5}, \frac{4}{5}, \frac{8}{5}\}$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\Lambda_1$</th>
<th>$\Lambda_2$</th>
<th>$\Lambda_1$</th>
<th>$\Lambda_2$</th>
<th>$\Lambda_1$</th>
<th>$\Lambda_2$</th>
<th>$\Lambda_1$</th>
<th>$\Lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.4269</td>
<td>33.0986</td>
<td>4.1726</td>
<td>17.6569</td>
<td>1.6844</td>
<td>4.9657</td>
<td>0.9465</td>
<td>2.2669</td>
</tr>
<tr>
<td>3</td>
<td>6.4401</td>
<td>33.1694</td>
<td>4.7492</td>
<td>20.5409</td>
<td>2.4808</td>
<td>8.1416</td>
<td>1.4377</td>
<td>3.8100</td>
</tr>
</tbody>
</table>
Table 2. The results of $M_{\alpha,a,b}$ and $(*) = M_{\alpha,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_{J_p\ell^\infty}$ in Example 4.1 for $q \in \{\frac{1}{8}, \frac{1}{2}, \frac{4}{5}, \frac{8}{9}\}$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$M_{\alpha,a,b}$ $(*)$</th>
<th>$M_{\alpha,a,b}$ $(*)$</th>
<th>$M_{\alpha,a,b}$ $(*)$</th>
<th>$M_{\alpha,a,b}$ $(*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>33.0986</td>
<td>1.3239</td>
<td>17.6569</td>
<td>0.7063</td>
</tr>
<tr>
<td>2</td>
<td>33.1615</td>
<td>1.3265</td>
<td>19.5549</td>
<td>0.7822</td>
</tr>
<tr>
<td>3</td>
<td>33.1694</td>
<td>1.3268</td>
<td>20.5409</td>
<td>0.8216</td>
</tr>
<tr>
<td>4</td>
<td>33.1703</td>
<td>1.3268</td>
<td>21.0433</td>
<td>0.8417</td>
</tr>
<tr>
<td>12</td>
<td>33.1705</td>
<td>1.3268</td>
<td>21.5499</td>
<td>0.8620</td>
</tr>
<tr>
<td>13</td>
<td>33.1705</td>
<td>1.3268</td>
<td>21.5509</td>
<td>0.8620</td>
</tr>
<tr>
<td>14</td>
<td>33.1705</td>
<td>1.3268</td>
<td>21.5514</td>
<td>0.8621</td>
</tr>
<tr>
<td>15</td>
<td>33.1705</td>
<td>1.3268</td>
<td>21.5516</td>
<td>0.8621</td>
</tr>
<tr>
<td>73</td>
<td>33.1705</td>
<td>1.3268</td>
<td>21.5519</td>
<td>0.8621</td>
</tr>
<tr>
<td>74</td>
<td>33.1705</td>
<td>1.3268</td>
<td>21.5519</td>
<td>0.8621</td>
</tr>
<tr>
<td>75</td>
<td>33.1705</td>
<td>1.3268</td>
<td>21.5519</td>
<td>0.8621</td>
</tr>
<tr>
<td>76</td>
<td>33.1705</td>
<td>1.3268</td>
<td>21.5519</td>
<td>0.8621</td>
</tr>
<tr>
<td>77</td>
<td>33.1705</td>
<td>1.3268</td>
<td>21.5519</td>
<td>0.8621</td>
</tr>
<tr>
<td>78</td>
<td>33.1705</td>
<td>1.3268</td>
<td>21.5519</td>
<td>0.8621</td>
</tr>
<tr>
<td>79</td>
<td>33.1705</td>
<td>1.3268</td>
<td>21.5519</td>
<td>0.8621</td>
</tr>
<tr>
<td>80</td>
<td>33.1705</td>
<td>1.3268</td>
<td>21.5519</td>
<td>0.8621</td>
</tr>
</tbody>
</table>

Figure 2. Numerical results of $M_{\alpha,a,b} \sum_{i=1}^{k_0} \|\hat{\mu}_i\|_{J_p\ell^\infty}$ where $q = \frac{1}{8}$, $\frac{1}{2}$, $\frac{4}{5}$ and $\frac{8}{9}$ in Example 4.1.
5. Conclusions

The multi-singular pointwise defined fractional $q$–integro-differential equation has been successfully investigated in this work. The investigation of this particular equation provides us with a powerful tool in modeling most scientific phenomena without the need to remove most parameters which have an essential role in the physical interpretation of the studied phenomena. Multi-singular pointwise defined fractional $q$–integro-differential equation (1.1) has been studied on a time scale under some boundary conditions. An application that describes the motion of a particle in the plane has been provided in this work to support our results’ validity and applicability in the fields of physics and engineering.

Acknowledgments

The first author was supported by Bu-Ali Sina University.

Conflict of interest

The authors declare that they have no competing interests.

References


©2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)