AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 7728–7741.
DOI: 10.3934/math.2022434
Received: 18 November 2021
Revised: 12 February 2022
Accepted: 14 February 2022
Published: 17 February 2022

Research article

On coupled Gronwall inequalities involving a ψ-fractional integral operator with its applications

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Abstract: In this paper, we obtain a new generalized coupled Gronwall inequality through the Caputo fractional integral with respect to another function ψ. Based on this result, we prove the existence and uniqueness of solutions for nonlinear delay coupled ψ-Caputo fractional differential system. Moreover, the Ulam-Hyers stability of solutions for ψ-Caputo fractional differential system is discussed. An example is also presented to demonstrate the application of main results.

Keywords: ψ-fractional operators; generalized coupled Gronwall inequality; delay ψ-Caputo fractional differential system; existence and uniqueness; Ulam-Hyers stability
Mathematics Subject Classification: 26A24, 26A33, 26D10, 34A08, 34D20, 35A23, 39B82, 47H10

1. Introduction

The fractional calculus is an important branch of mathematics and it has wide applications to many fields of science and engineering. We know that the fractional calculus is a wonderful technique to understand of memory and hereditary properties of materials and processes. Some contributions to fractional calculus have been carried out, see the monographs [1–3], and the references cited therein.

The theory of generalized fractional calculus was proposed by Kiryakova in [4]. One of the proposed generalizations of the fractional calculus operators is the ψ-fractional operator which has wide applications, some properties of this operator could be found in [5–10]. The Gronwall inequality plays an important role in the study of qualitative and qualitative properties of solution of fractional differential and integral equations [11–18]. In order to work with continuous dependence of differential equations via ψ-Hilfer fractional derivative, the generalized Gronwall inequality by means of the fractional integral with respect to another function ψ was first given and proved by Vanterler et al. in [19]. Indeed, they obtained the theorem given below.

Theorem 1.1 [19]. *Let u, v be two integrable functions and g continuous, with domain $[a, b]$. Let $\psi \in C^1[a, b]$ be an increasing function such that $\psi'(t) \neq 0$, $\forall t \in [a, b]$. Assume that*

(1) u and v are nonnegative;

(2) g is nonnegative and nondecreasing.

If

$$u(t) \leq v(t) + g(t) \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} u(\tau) d\tau,$$

then

$$u(t) \leq v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} v(\tau) d\tau, \quad \forall t \in [a, b].$$

The Hilfer version of the fractional derivative with another function called ψ-Hilfer FDO has been presented by Sousa et al [20]. Recently, the existence and uniqueness of the solution of a nonlinear ψ-Hilfer fractional differential equations with different kinds of initial and boundary conditions and the Ulam-Hyers stabilities of its solutions have been investigated [21–25].

The Ulam stability, which can be considered as a special type of date dependence was initiated by Ulam [26,27]. Since then, there are many development of this field, we refer the reader to [28–32] and the references therein.

The main objective of this paper is to extend Theorem 1.1 to the generalized coupled Gronwall inequality by the implementation of ψ-fractional operator. As applications, we prove the existence and uniqueness of solutions for the following nonlinear delay coupled ψ-Caputo fractional differential system

$$\begin{cases} {}^C_{t_0}D^{\alpha}_{\psi} x(t) = F(t, y(t), y(t-\tau)), \quad t \in [t_0, t_1], \\ {}^C_{t_0}D^{\beta}_{\psi} y(t) = G(t, x(t), x(t-\tau)), \quad t \in [t_0, t_1], \\ x(t) = \phi(t), \quad y(t) = \theta(t), \quad t \in [t_0 - \tau, t_0], \end{cases} \tag{1.1}$$

where $F, G \in C([t_0, t_1] \times \mathbb{R}^2, \mathbb{R})$, $\phi, \theta \in C[t_0 - \tau, t_0]$, and ${}^C_{t_0}D^{\alpha}_{\psi} x(t)$ is the left Caputo fractional derivative of x of order α, $0 < \alpha \leq 1$ with respect to the continuous function ψ with $\psi'(t) > 0$, $t \in [t_0, t_1]$. The meaning of ${}^C_{t_0}D^{\beta}_{\psi} y(t)$ $(0 < \beta \leq 1)$ is the same as ${}^C_{t_0}D^{\alpha}_{\psi} x(t)$. Moreover, we investigate the Ulam-Hyers stability of solutions for (1.1). Our results extend the main results of [33].

This paper is organized as follows: In Section 2, we give some notations, definitions and preliminaries. Section 3 is devoted to proving a new generalized coupled Gronwall inequality. In Section 4, the existence and uniqueness of the solution of system (1.1) are given and proved, and the Ulam-Hyers stability theorem of (1.1) is obtained. In Section 5, an example is given to illustrate our theoretical result. Finally, the paper is concluded in Section 6.

2. Preliminaries

In this section, we provided some basic definitions and lemmas which are used in the sequel.

Definition 2.1 [10,34]. Let $\alpha > 0$, f be an integrable function defined on $[a, b]$ and $\psi \in C^1([a, b])$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a, b]$. The left ψ-Riemann-Liouville fractional integral operator of order α of a function f is defined by

$$({}_{t_0}I^{\alpha}_{\psi} f)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (\psi(t) - \psi(s))^{\alpha-1} f(s) \psi'(s) ds. \tag{2.1}$$

Definition 2.2 [10,34]. Let $n-1<\alpha<n$, $f\in C^n([a,b])$ and $\psi\in C^n([a,b])$ be an increasing function with $\psi'(t)\neq 0$ for all $t\in[a,b]$. The left ψ-Caputo fractional derivative of order α of a function f is defined by

$$({}^{C}_{t_0}D^{\alpha}_{\psi}f)(t)=({}_{t_0}I^{n-\alpha}_{\psi}f^{[n]})(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(\psi(t)-\psi(s))^{n-\alpha-1}f^{[n]}(s)\psi'(s)ds, \tag{2.2}$$

where $n=[\alpha]+1$ and $f^{[n]}(t):=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t)$ on $[a,b]$.

Lemma 2.1 [34]. *Let $\alpha>0$ and $\beta>0$, then*

(i) ${}_{t_0}I^{\alpha}_{\psi}(\psi(s)-\psi(t_0))^{\beta-1}(t)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi(t)-\psi(t_0))^{\beta+\alpha-1}$,

(ii) ${}^{C}_{t_0}D^{\alpha}_{\psi}(\psi(s)-\psi(t_0))^{\beta-1}(t)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(t)-\psi(t_0))^{\beta-\alpha-1}$,

(iii) ${}^{C}_{t_0}D^{\alpha}_{\psi}(\psi(s)-\psi(t_0))^{k}(t)=0,\quad n-1<\alpha<n,\quad k=0,1,...,n-1.$

In the following, we will give the combinations of the fractional integral and the fractional derivatives of a function with respect to another function.

Lemma 2.2 [34]. *Let $f\in C^n([a,b])$ and $n-1<\alpha<n$. Then we have*

(1) ${}^{C}_{t_0}D^{\alpha}_{\psi}{}_{t_0}I^{\alpha}_{\psi}f(t)=f(t)$;

(2) ${}_{t_0}I^{\alpha}_{\psi}{}^{C}_{t_0}D^{\alpha}_{\psi}f(t)=f(t)-\sum_{k=0}^{n-1}\frac{f^{[k]}(t_0^+)}{k!}(\psi(t)-\psi(t_0))^k.$

In particular, given $\alpha\in(0,1)$, one has

$${}_{t_0}I^{\alpha}_{\psi}{}^{C}_{t_0}D^{\alpha}_{\psi}f(t)=f(t)-f(t_0).$$

Let $X=C([t_0-\tau,t_1],\mathbb{R})\cap C^1([t_0,t_1],\mathbb{R})$, then the space X is a Banach space with respect to the norm defined by $||u||=\max_{t\in[t_0-\tau,t_1]}|u(t)|$.

The following definition of Ulam stability of (1.1) is similar to the definition stated in [35].

Definition 2.3. System (1.1) is said to be Ulam-Hyers stable if there exists a real number c such that for all $\epsilon>0$ and for each $(u,v)\in X\times X$ with $(u(t),v(t))=(\phi(t),\theta(t))$ for $t\in[t_0-\tau,t_0]$ satisfying the inequalities

$$|{}^{C}_{t_0}D^{\alpha}_{\psi}u(t)-F(t,v(t),v(t-\tau))|\le\epsilon,\quad t\in[t_0,t_1], \tag{2.3}$$

$$|{}^{C}_{t_0}D^{\beta}_{\psi}v(t)-G(t,u(t),u(t-\tau))|\le\epsilon,\quad t\in[t_0,t_1], \tag{2.4}$$

there exists a solution $(x,y)\in X\times X$ of (1.1) satisfying

$$||u-x||\le c\epsilon,\quad ||v-y||\le c\epsilon. \tag{2.5}$$

3. A generalized coupled Gronwall inequality

Now we state and prove a new generalized coupled Gronwall inequality as follows.

Theorem 3.1. *Assume that* x, y, *and* a_i $(i = 1, 2)$ *are integrable and nonnegative functions, and* b_i $(i = 1, 2)$ *are continuous, nonnegative and nondecreasing functions, with domain* $[t_0, t_1]$. *Let* $\psi \in C^1[t_0, t_1]$ *be an increasing function such that* $\psi'(t) \neq 0$, $\forall t \in [t_0, t_1]$.
If

$$\begin{cases} x(t) \leq a_1(t) + b_1(t) \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} y(s) ds, \\ y(t) \leq a_2(t) + b_2(t) \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\beta-1} x(s) ds, \end{cases} \tag{3.1}$$

then

$$\begin{aligned} x(t) \leq \ & a_1(t) + b_1(t) \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} a_2(s) ds \\ & + \int_{t_0}^{t} \sum_{k=1}^{\infty} \frac{\Gamma^k(\alpha)\Gamma^k(\beta)}{\Gamma(k(\alpha+\beta))} b_1^k(t) b_2^k(t) \psi'(s)(\psi(t) - \psi(s))^{k(\alpha+\beta)-1} \\ & \cdot \left(a_1(s) + b_1(s) \int_{t_0}^{s} \psi'(\tau)(\psi(s) - \psi(\tau))^{\alpha-1} a_2(\tau) d\tau \right) ds, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} y(t) \leq \ & a_2(t) + b_2(t) \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\beta-1} a_1(s) ds \\ & + \int_{t_0}^{t} \sum_{k=1}^{\infty} \frac{\Gamma^k(\alpha)\Gamma^k(\beta)}{\Gamma(k(\alpha+\beta))} b_1^k(t) b_2^k(t) \psi'(s)(\psi(t) - \psi(s))^{k(\alpha+\beta)-1} \\ & \cdot \left(a_2(s) + b_2(s) \int_{t_0}^{s} \psi'(\tau)(\psi(s) - \psi(\tau))^{\beta-1} a_1(\tau) d\tau \right) ds. \end{aligned} \tag{3.3}$$

Proof. Let

$$Ay(t) = b_1(t) \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} y(s) ds,$$

and

$$Bx(t) = b_2(t) \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\beta-1} x(s) ds.$$

Then from system (3.1), one has

$$x(t) \leq a_1(t) + Ay(t), \quad y(t) \leq a_2(t) + Bx(t). \tag{3.4}$$

By (3.4) and the monotonicity of the operators A and B, we obtain

$$\begin{aligned} x(t) & \leq a_1(t) + A(a_2(t) + Bx(t)) = a_1(t) + Aa_2(t) + ABx(t) \\ & \leq a_1(t) + Aa_2(t) + AB[a_1(t) + Aa_2(t) + ABx(t)] \\ & = a_1(t) + ABa_1(t) + Aa_2(t) + ABAa_2(t) + (AB)^2 x(t). \end{aligned}$$

Thus, through iteration, for $n \in \mathbb{N}$, one has

$$x(t) \leq \sum_{k=0}^{n-1} (AB)^k a_1(t) + \sum_{k=0}^{n-1} (AB)^k Aa_2(t) + (AB)^n x(t), \ t \in [t_0, t_1]. \tag{3.5}$$

Similarly, we have

$$y(t) \leq \sum_{k=0}^{n-1}(BA)^k a_2(t) + \sum_{k=0}^{n-1}(BA)^k Ba_1(t) + (BA)^n y(t),\ t \in [t_0, t_1], \tag{3.6}$$

where $(AB)^0 a_1(t) = a_1(t)$ and $(BA)^0 a_2(t) = a_2(t)$.

In the following, we will prove that

$$\begin{aligned} (AB)^n x(t) &\leq \frac{\Gamma^n(\alpha)\Gamma^n(\beta)}{\Gamma(n(\alpha+\beta))} b_1^n(t) b_2^n(t) \int_{t_0}^t \psi'(s)(\psi(t)-\psi(s))^{n(\alpha+\beta)-1} x(s)ds, \\ (BA)^n y(t) &\leq \frac{\Gamma^n(\alpha)\Gamma^n(\beta)}{\Gamma(n(\alpha+\beta))} b_1^n(t) b_2^n(t) \int_{t_0}^t \psi'(s)(\psi(t)-\psi(s))^{n(\alpha+\beta)-1} y(s)ds, \end{aligned} \tag{3.7}$$

where $t \in [t_0, t_1]$, and

$$\lim_{n\to\infty}(AB)^n x(t) = 0, \quad \lim_{n\to\infty}(BA)^n y(t) = 0. \tag{3.8}$$

We know that (3.7) is true for $n = 1$. In fact, one has

$$ABx(t) = A(Bx(t)) = b_1(t) \int_{t_0}^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} b_2(s) \int_{t_0}^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1} x(\tau) ds d\tau$$

$$\leq b_1(t)b_2(t) \int_{t_0}^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \int_{t_0}^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1} x(\tau) ds d\tau$$

$$= b_1(t)b_2(t) \int_{t_0}^t \psi'(\tau)x(\tau)d\tau \int_{\tau}^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}(\psi(s)-\psi(\tau))^{\beta-1} ds.$$

Introducing a change of variables $v = \frac{\psi(s)-\psi(\tau)}{\psi(t)-\psi(\tau)}$ and using the definition of beta function, we obtain

$$\int_{\tau}^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}(\psi(s)-\psi(\tau))^{\beta-1} ds$$

$$= \int_{\tau}^t \psi'(s)(\psi(t)-\psi(\tau))^{\alpha-1} \left[1 - \frac{\psi(s)-\psi(\tau)}{\psi(t)-\psi(\tau)}\right]^{\alpha-1} (\psi(s)-\psi(\tau))^{\beta-1} ds$$

$$= (\psi(t)-\psi(\tau))^{\alpha+\beta-1} \int_0^1 (1-v)^{\alpha-1} v^{\beta-1} dv$$

$$= (\psi(t)-\psi(\tau))^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Thus

$$ABx(t) \leq \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} b_1(t)b_2(t) \int_{t_0}^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1} x(\tau)d\tau.$$

Similarly, one has

$$BAy(t) \leq \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} b_1(t)b_2(t) \int_{t_0}^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1} y(\tau)d\tau.$$

Now, by using mathematical induction, for $n = k$ and $t \in [t_0, t_1]$, we obtain

$$\begin{aligned} (AB)^k x(t) &\le \frac{\Gamma^k(\alpha)\Gamma^k(\beta)}{\Gamma(k(\alpha+\beta))} b_1^k(t) b_2^k(t) \int_{t_0}^t \psi'(s)(\psi(t)-\psi(s))^{k(\alpha+\beta)-1} x(s)ds, \\ (BA)^k y(t) &\le \frac{\Gamma^k(\alpha)\Gamma^k(\beta)}{\Gamma(k(\alpha+\beta))} b_1^n(t) b_2^n(t) \int_{t_0}^t \psi'(s)(\psi(t)-\psi(s))^{k(\alpha+\beta)-1} y(s)ds. \end{aligned} \tag{3.9}$$

For $n = k+1$ and $t \in [t_0, t_1]$, we get by using the nondecreasing of functions $b_1(t)$ and $b_2(t)$, and the induction hypothesis that

$$\begin{aligned} (AB)^{k+1}x(t) &= AB((AB)^k x(t)) \\ &\le \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} b_1(t)b_2(t) \int_{t_0}^t \psi'(s)(\psi(t)-\psi(s))^{\alpha+\beta-1} \\ &\quad \cdot \frac{\Gamma^k(\alpha)\Gamma^k(\beta)}{\Gamma(k(\alpha+\beta))} b_1^k(s) b_2^k(s) \int_{t_0}^s \psi'(\tau)(\psi(s)-\psi(\tau))^{k(\alpha+\beta)-1} x(\tau)d\tau ds \\ &\le \frac{\Gamma^{k+1}(\alpha)\Gamma^{k+1}(\beta)}{\Gamma(\alpha+\beta)\Gamma(k(\alpha+\beta))} b_1^{k+1}(t) b_2^{k+1}(t) \\ &\quad \cdot \int_{t_0}^t \psi'(\tau)x(\tau)d\tau \int_{\tau}^t \psi'(s)(\psi(t)-\psi(s))^{\alpha+\beta-1}(\psi(s)-\psi(\tau))^{k(\alpha+\beta)-1} ds \\ &\le \frac{\Gamma^{k+1}(\alpha)\Gamma^{k+1}(\beta)}{\Gamma(\alpha+\beta)\Gamma(k(\alpha+\beta))} b_1^{k+1}(t) b_2^{k+1}(t) \\ &\quad \cdot \int_{t_0}^t \psi'(\tau)x(\tau)(\psi(t)-\psi(\tau))^{(k+1)(\alpha+\beta)-1} \frac{\Gamma(\alpha+\beta)\Gamma(k(\alpha+\beta))}{\Gamma((k+1)(\alpha+\beta))} d\tau \\ &= \frac{\Gamma^{k+1}(\alpha)\Gamma^{k+1}(\beta)}{\Gamma((k+1)(\alpha+\beta))} b_1^{k+1}(t) b_2^{k+1}(t) \int_{t_0}^t \psi'(\tau)(\psi(t)-\psi(\tau))^{(k+1)(\alpha+\beta)-1} x(\tau)d\tau. \end{aligned} \tag{3.10}$$

Similar to the proof of (3.10), we can obtain

$$(BA)^{k+1}y(t) \le \frac{\Gamma^{k+1}(\alpha)\Gamma^{k+1}(\beta)}{\Gamma((k+1)(\alpha+\beta))} b_1^{k+1}(t) b_2^{k+1}(t) \int_{t_0}^t \psi'(\tau)(\psi(t)-\psi(\tau))^{(k+1)(\alpha+\beta)-1} y(\tau)d\tau. \tag{3.11}$$

That is, (3.7) is proved. Now we prove that (3.8) holds. Since b_1 and b_2 are two continuous functions on $[t_0, t_1]$, there exists a constant $M > 0$ such that $b_1(t) \le M$ and $b_2(t) \le M$ for $t \in [t_0, t_1]$. Thus, we have

$$(AB)^n x(t) \le \frac{(M^2\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n(\alpha+\beta))} \int_{t_0}^t \psi'(\tau)(\psi(t)-\psi(\tau))^{n(\alpha+\beta)-1} x(\tau)d\tau.$$

Consider the series

$$\sum_{n=1}^{\infty} \frac{(M^2\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n(\alpha+\beta))}.$$

Using the ratio test to the series and the asymptotic approximation [36], we obtain

$$\lim_{n\to\infty} \frac{\Gamma(n(\alpha+\beta))}{\Gamma(n(\alpha+\beta)+\alpha+\beta)} = 0.$$

Hence, the series converges and we conclude that

$$x(t) \le a_1(t) + b_1(t)\int_{t_0}^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}a_2(s)ds$$
$$+\int_{t_0}^{t}\sum_{k=1}^{\infty}\frac{\Gamma^k(\alpha)\Gamma^k(\beta)}{\Gamma(k(\alpha+\beta))}b_1^k(t)b_2^k(t)\psi'(s)(\psi(t)-\psi(s))^{k(\alpha+\beta)-1}$$
$$\cdot\left(a_1(s)+b_1(s)\int_{t_0}^{s}\psi'(\tau)(\psi(s)-\psi(\tau))^{\alpha-1}a_2(\tau)d\tau\right)ds.$$

Similarly, we can obtain that (3.3) holds.

Corollary 3.2. *Under the hypotheses of Theorem 3.1, assume that* $a_1(t)$ *and* $a_2(t)$ *are two nondecreasing functions for* $t \in [t_0, t_1]$. *Then*

$$x(t) \le \left(a_1(t) + \frac{b_1(t)a_2(t)}{\alpha}(\psi(t)-\psi(t_0))^{\alpha}\right)E_{\alpha+\beta}(b_1(t)b_2(t)\Gamma(\alpha)\Gamma(\beta)(\psi(t)-\psi(t_0))^{\alpha+\beta}), \tag{3.12}$$

and

$$y(t) \le \left(a_2(t) + \frac{b_2(t)a_1(t)}{\beta}(\psi(t)-\psi(t_0))^{\beta}\right)E_{\alpha+\beta}(b_1(t)b_2(t)\Gamma(\alpha)\Gamma(\beta)(\psi(t)-\psi(t_0))^{\alpha+\beta}). \tag{3.13}$$

Proof. Since a_2 is nondecreasing, one has

$$\int_{t_0}^{s}\psi'(\tau)(\psi(s)-\psi(\tau))^{\alpha-1}a_2(\tau)d\tau \le a_2(s)\int_{t_0}^{s}\psi'(\tau)(\psi(s)-\psi(\tau))^{\alpha-1}d\tau$$
$$= \frac{a_2(s)}{\alpha}(\psi(s)-\psi(t_0))^{\alpha}. \tag{3.14}$$

Thus, from (3.2) and (3.14), we get

$$x(t) \le \left(a_1(t) + \frac{b_1(t)a_2(t)}{\alpha}(\psi(t)-\psi(t_0))^{\alpha}\right)$$
$$\cdot\left[1+\int_{t_0}^{t}\sum_{k=1}^{\infty}\frac{\Gamma^k(\alpha)\Gamma^k(\beta)}{\Gamma(k(\alpha+\beta))}b_1^k(t)b_2^k(t)\psi'(s)(\psi(t)-\psi(s))^{k(\alpha+\beta)-1}ds\right]$$
$$= \left(a_1(t) + \frac{b_1(t)a_2(t)}{\alpha}(\psi(t)-\psi(t_0))^{\alpha}\right)$$
$$\cdot\left[1+\sum_{k=1}^{\infty}\frac{\Gamma^k(\alpha)\Gamma^k(\beta)}{\Gamma(k(\alpha+\beta)+1)}b_1^k(t)b_2^k(t)(\psi(t)-\psi(t_0))^{k(\alpha+\beta)}\right]$$
$$= \left(a_1(t) + \frac{b_1(t)a_2(t)}{\alpha}(\psi(t)-\psi(t_0))^{\alpha}\right)E_{\alpha+\beta}(b_1(t)b_2(t)\Gamma(\alpha)\Gamma(\beta)(\psi(t)-\psi(t_0))^{\alpha+\beta}).$$

Similarly, we obtain (3.13) holds.

4. Main results

By Lemma 2.2, we can easily show that the following lemma holds.

Lemma 4.1. $(x(t), y(t))$ *satisfies (1.1) if and only if* $(x(t), y(t))$ *satisfies the coupled integral system*

$$\begin{cases} x(t) = \phi(t_0) + {}_{t_0}I_\psi^\alpha F(t, y(t), y(t-\tau)), & t \in [t_0, t_1], \\ y(t) = \theta(t_0) + {}_{t_0}I_\psi^\beta G(t, x(t), x(t-\tau)), & t \in [t_0, t_1], \\ x(t) = \phi(t), \quad y(t) = \theta(t), & t \in [t_0 - \tau, t_0]. \end{cases}$$

The product space $X \times X$ is a Banach space with norm $\|(u, v)\| = \|u\| + \|v\|$. Now we give and prove the existence uniqueness theorem.

Theorem 4.2. *Let*

(H1) $F, G \in C([t_0, t_1] \times \mathbb{R}^2, \mathbb{R})$ *and* $\phi, \theta \in C[t_0 - \tau, t_0]$;

(H2) there exist two positive constants L_1 *and* L_2 *such that*

$$|F(t, x_1, x_2) - F(t, y_1, y_2)| \le L_1(|x_1 - y_1| + |x_2 - y_2|),$$

$$|G(t, x_1, x_2) - G(t, y_1, y_2)| \le L_2(|x_1 - y_1| + |x_2 - y_2|);$$

(H3) $M = \max\left\{\dfrac{2L_1(\psi(t_1) - \psi(t_0))^\alpha}{\Gamma(\alpha+1)}, \dfrac{2L_2(\psi(t_1) - \psi(t_0))^\beta}{\Gamma(\beta+1)}\right\} < 1.$

Then the system (1.1) has a unique solution in $X \times X$.

Proof. Define the operator $T(x, y)(t) := (T_1 x(t), T_2 y(t))$ as follows :

$$T_1 x(t) = \begin{cases} \phi(t), & t \in [t_0 - \tau, t_0], \\ \phi(t_0) + {}_{t_0}I_\psi^\alpha F(t, y(t), y(t-\tau)), & t \in [t_0, t_1], \end{cases}$$

$$T_2 y(t) = \begin{cases} \theta(t), & t \in [t_0 - \tau, t_0], \\ \theta(t_0) + {}_{t_0}I_\psi^\beta G(t, x(t), x(t-\tau)), & t \in [t_0, t_1]. \end{cases}$$

For $t \in [t_0 - \tau, t_0]$, we have $|T_1 x(t) - T_1 u(t)| = 0$ and $|T_2 y(t) - T_2 v(t)| = 0$ if $(x, y), (u, v) \in C([t_0 - \tau, t_1], \mathbb{R}) \times C([t_0 - \tau, t_1], \mathbb{R})$. For $t \in [t_0, t_1]$, one has

$$\begin{aligned} |T_1 x(t) - T_1 u(t)| &= |{}_{t_0}I_\psi^\alpha F(t, y(t), y(t-\tau)) - {}_{t_0}I_\psi^\alpha F(t, v(t), v(t-\tau))| \\ &\le {}_{t_0}I_\psi^\alpha (|F(t, y(t), y(t-\tau)) - F(t, v(t), v(t-\tau))|) \\ &\le {}_{t_0}I_\psi^\alpha (L_1|y(t) - v(t)| + L_1|y(t-\tau) - v(t-\tau)|) \\ &\le L_1(\max_{t_0-\tau\le t\le t_1} |y(t) - v(t)| + \max_{t_0-\tau\le t\le t_1} |y(t-\tau) - v(t-\tau)|)_{t_0}I_\psi^\alpha 1 \\ &\le \frac{2L_1(\psi(t) - \psi(t_0))^\alpha}{\Gamma(\alpha+1)}\|y - v\| \\ &\le \frac{2L_1(\psi(t_1) - \psi(t_0))^\alpha}{\Gamma(\alpha+1)}\|y - v\|. \end{aligned} \tag{4.1}$$

Similarly, we can obtain

$$|T_2 y(t) - T_2 v(t)| \le \frac{2L_2(\psi(t_1) - \psi(t_0))^\beta}{\Gamma(\beta+1)}\|x - u\|. \tag{4.2}$$

Thus, by (4.1) and (4.2), we have

$$\begin{aligned}\|T(x,y)-T(u,v)\| &= \|T_1x-T_1u\|+\|T_2y-T_2v\| \\ &\le \max\left\{\frac{2L_1(\psi(t_1)-\psi(t_0))^{\alpha}}{\Gamma(\alpha+1)},\frac{2L_2(\psi(t_1)-\psi(t_0))^{\beta}}{\Gamma(\beta+1)}\right\}(\|x-u\|+\|y-v\|) \\ &= M\|(x-u,y-v)\| = M\|(x,y)-(u,v)\|,\end{aligned}$$

which implies that the operator T is a contraction by (H3). Thus T has a unique fixed point by Banach fixed point theorem.

Theorem 4.3. *Under the hypotheses of Theorem 4.2, system (1.1) is Ulam-Hyers stable.*

Proof. Let $(u(t),v(t)) \in X\times X$ be a solution of the inequalities (2.3) and (2.4), and let $(x(t),y(t))$ be the unique solution of system (1.1) satisfying the conditions

$$x(t)=u(t)=\phi(t),\quad y(t)=v(t)=\theta(t),\quad t\in[t_0-\tau,t_0].$$

Thus we have

$$x(t)=\begin{cases} u(t), & t\in[t_0-\tau,t_0],\\ u(t_0)+{}_{t_0}I_{\psi}^{\alpha}F(t,y(t),y(t-\tau)), & t\in[t_0,t_1],\end{cases}$$

$$y(t)=\begin{cases} v(t), & t\in[t_0-\tau,t_0],\\ v(t_0)+{}_{t_0}I_{\psi}^{\beta}G(t,x(t),x(t-\tau)), & t\in[t_0,t_1].\end{cases}$$

Which is guaranteed by Theorem 4.2. Obviously, $(u(t),v(t))$ satisfies (2.3)-(2.4) if and only if there exist two functions $h_1(t),h_2(t)\in C[t_0,t_1]$ such that $|h_i(t)|\le\epsilon$ $(i=1,2)$ and

$$ {}_{t_0}^{C}D_{\psi}^{\alpha}u(t)-F(t,v(t),v(t-\tau))=h_1(t),\quad t\in[t_0,t_1], \tag{4.3}$$

$$ {}_{t_0}^{C}D_{\psi}^{\beta}v(t)-G(t,u(t),u(t-\tau))=h_2(t),\quad t\in[t_0,t_1]. \tag{4.4}$$

Applying the ψ-fractional integral (2.1) to both sides of (4.3) and using Lemma 2.2 we obtain

$$\begin{aligned}&|u(t)-u(t_0)-{}_{t_0}I_{\psi}^{\alpha}F(t,v(t),v(t-\tau))| = |{}_{t_0}I_{\psi}^{\alpha}h_1(t)|\\ &\le {}_{t_0}I_{\psi}^{\alpha}|h_1(t)| \le {}_{t_0}I_{\psi}^{\alpha}\epsilon \le \frac{(\psi(t)-\psi(t_0))^{\alpha}}{\Gamma(\alpha+1)}\epsilon\\ &\le \frac{(\psi(t_1)-\psi(t_0))^{\alpha}}{\Gamma(\alpha+1)}\epsilon.\end{aligned} \tag{4.5}$$

Similarly, we get

$$|v(t)-v(t_0)-{}_{t_0}I_{\psi}^{\beta}G(t,u(t),u(t-\tau))| \le \frac{(\psi(t_1)-\psi(t_0))^{\beta}}{\Gamma(\beta+1)}\epsilon. \tag{4.6}$$

For $t\in[t_0-\tau,t_0]$, we have $|x(t)-u(t)|=0$ and $|y(t)-v(t)|=0$. For $t\in[t_0,t_0+\tau]$, one has

$$\begin{aligned}|u(t)-x(t)| &= |u(t)-u(t_0)-{}_{t_0}I_{\psi}^{\alpha}F(t,y(t),y(t-\tau))|\\ &\le |u(t)-u(t_0)-{}_{t_0}I_{\psi}^{\alpha}F(t,v(t),v(t-\tau))|\\ &\quad +|{}_{t_0}I_{\psi}^{\alpha}F(t,v(t),v(t-\tau))-{}_{t_0}I_{\psi}^{\alpha}F(t,y(t),y(t-\tau))|\\ &\le |u(t)-u(t_0)-{}_{t_0}I_{\psi}^{\alpha}F(t,v(t),v(t-\tau))|\\ &\quad +{}_{t_0}I_{\psi}^{\alpha}(|F(t,v(t),v(t-\tau))-F(t,y(t),y(t-\tau))|)\\ &\le |u(t)-u(t_0)-{}_{t_0}I_{\psi}^{\alpha}F(t,v(t),v(t-\tau))| + L_1\cdot{}_{t_0}I_{\psi}^{\alpha}(|v(t)-y(t)|).\end{aligned} \tag{4.7}$$

Similarly, for $t \in [t_0, t_0 + \tau]$, we get

$$|v(t) - y(t)| \leq |v(t) - v(t_0) - {}_{t_0}I_{\psi}^{\beta}G(t, u(t), u(t - \tau))| + L_2 \cdot {}_{t_0}I_{\psi}^{\beta}(|u(t) - x(t)|). \tag{4.8}$$

Using (4.5) and (4.7), (4.6) and (4.8), respectively, we obtain

$$|u(t) - x(t)| \leq \frac{(\psi(t_1) - \psi(t_0))^{\alpha}}{\Gamma(\alpha + 1)}\epsilon + \frac{L_1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}|v(s) - y(s)|ds, \tag{4.9}$$

$$|v(t) - y(t)| \leq \frac{(\psi(t_1) - \psi(t_0))^{\beta}}{\Gamma(\beta + 1)}\epsilon + \frac{L_2}{\Gamma(\beta)} \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\beta-1}|u(s) - x(s)|ds. \tag{4.10}$$

By using Corollary 3.2, we get from (4.9) and (4.10) that

$$|u(t) - x(t)| \leq \left(\frac{(\psi(t_1) - \psi(t_0))^{\alpha}\epsilon}{\Gamma(\alpha + 1)} + \frac{L_1}{\Gamma(\alpha + 1)}\frac{(\psi(t_1) - \psi(t_0))^{\beta}\epsilon}{\Gamma(\beta + 1)}\right) E_{\alpha+\beta}\left(L_1 L_2(\psi(t) - \psi(t_0))^{\alpha+\beta}\right).$$

Therefore, for any $t \in [t_0, t_0 + \tau]$, one has

$$\begin{aligned} |u(t) - x(t)| \leq & \left(\frac{(\psi(t_1) - \psi(t_0))^{\alpha}}{\Gamma(\alpha + 1)} + \frac{L_1}{\Gamma(\alpha + 1)}\frac{(\psi(t_1) - \psi(t_0))^{\beta}}{\Gamma(\beta + 1)}\right) \\ & \cdot E_{\alpha+\beta}\left(L_1 L_2(\psi(t_0 + \tau) - \psi(t_0))^{\alpha+\beta}\right)\epsilon. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |v(t) - y(t)| \leq & \left(\frac{(\psi(t_1) - \psi(t_0))^{\beta}}{\Gamma(\beta + 1)} + \frac{L_2}{\Gamma(\beta + 1)}\frac{(\psi(t_1) - \psi(t_0))^{\alpha}}{\Gamma(\alpha + 1)}\right) \\ & \cdot E_{\alpha+\beta}\left(L_1 L_2(\psi(t_0 + \tau) - \psi(t_0))^{\alpha+\beta}\right)\epsilon, \quad \forall t \in [t_0, t_0 + \tau]. \end{aligned}$$

For $t \in [t_0 + \tau, t_1]$, we adopt the similar steps as above, we may have

$$\begin{aligned} |u(t) - x(t)| \leq & \frac{(\psi(t_1) - \psi(t_0))^{\alpha}}{\Gamma(\alpha + 1)}\epsilon + \frac{L_1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}|v(s) - y(s)|ds \\ & + \frac{L_1}{\Gamma(\alpha)} \int_{t_0+\tau}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}|v(s - \tau) - y(s - \tau)|ds, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} |v(t) - y(t)| \leq & \frac{(\psi(t_1) - \psi(t_0))^{\beta}}{\Gamma(\beta + 1)}\epsilon + \frac{L_2}{\Gamma(\beta)} \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\beta-1}|u(s) - x(s)|ds \\ & + \frac{L_2}{\Gamma(\beta)} \int_{t_0+\tau}^{t} \psi'(s)(\psi(t) - \psi(s))^{\beta-1}|u(s - \tau) - x(s - \tau)|ds. \end{aligned} \tag{4.12}$$

Let $z(t) = \max\limits_{r\in[-\tau,0]} |u(t + r) - x(t + r)|$ and $w(t) = \max\limits_{r\in[-\tau,0]} |v(t + r) - y(t + r)|$, then we obtain by (4.11) and (4.12) that

$$\begin{aligned} z(t) \leq & \frac{(\psi(t_1) - \psi(t_0))^{\alpha}}{\Gamma(\alpha + 1)}\epsilon + \frac{L_1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}w(s)ds \\ & + \frac{L_1}{\Gamma(\alpha)} \int_{t_0+\tau}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}w(s)ds \\ \leq & \frac{(\psi(t_1) - \psi(t_0))^{\alpha}}{\Gamma(\alpha + 1)}\epsilon + \frac{2L_1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}w(s)ds, \end{aligned} \tag{4.13}$$

and

$$w(t) \le \frac{(\psi(t_1)-\psi(t_0))^{\beta}}{\Gamma(\beta+1)}\epsilon + \frac{2L_2}{\Gamma(\beta)}\int_{t_0}^{t}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}z(s)ds. \tag{4.14}$$

By utilizing Corollary 3.2, for any $t \in [t_0+\tau, t_1]$, we get by (4.13) and (4.14) that

$$z(t) \le \left(\frac{(\psi(t_1)-\psi(t_0))^{\alpha}}{\Gamma(\alpha+1)} + \frac{2L_1}{\Gamma(\alpha+1)}\frac{(\psi(t_1)-\psi(t_0))^{\beta}}{\Gamma(\beta+1)}\right)E_{\alpha+\beta}\left(2L_1L_2(\psi(t_1)-\psi(t_0))^{\alpha+\beta}\right)\epsilon,$$

and

$$w(t) \le \left(\frac{(\psi(t_1)-\psi(t_0))^{\beta}}{\Gamma(\beta+1)} + \frac{2L_2}{\Gamma(\beta+1)}\frac{(\psi(t_1)-\psi(t_0))^{\alpha}}{\Gamma(\alpha+1)}\right)E_{\alpha+\beta}\left(2L_1L_2(\psi(t_1)-\psi(t_0))^{\alpha+\beta}\right)\epsilon.$$

Since $|u(t)-x(t)| \le z(t)$ and $|v(t)-y(t)| \le w(t)$, for each $t \in [t_0+\tau, t_1]$, we have

$$|u(t)-x(t)| \le \left(\frac{(\psi(t_1)-\psi(t_0))^{\alpha}}{\Gamma(\alpha+1)} + \frac{2L_1}{\Gamma(\alpha+1)}\frac{(\psi(t_1)-\psi(t_0))^{\beta}}{\Gamma(\beta+1)}\right)E_{\alpha+\beta}\left(2L_1L_2(\psi(t_1)-\psi(t_0))^{\alpha+\beta}\right)\epsilon,$$

and

$$|v(t)-y(t)| \le \left(\frac{(\psi(t_1)-\psi(t_0))^{\beta}}{\Gamma(\beta+1)} + \frac{2L_2}{\Gamma(\beta+1)}\frac{(\psi(t_1)-\psi(t_0))^{\alpha}}{\Gamma(\alpha+1)}\right)E_{\alpha+\beta}\left(2L_1L_2(\psi(t_1)-\psi(t_0))^{\alpha+\beta}\right)\epsilon.$$

5. Example

Example 5.1. Consider the following coupled delay ψ-Caputo fractional differential system

$$\begin{cases} {}^{C}_{1}D^{\frac{2}{3}}_{\sqrt[3]{t}}x(t) = \frac{\ln(t)}{4}(\arctan(y(t)) + \sin(y(t-1))), & t \in [1,6], \\ {}^{C}_{1}D^{\frac{3}{4}}_{\sqrt[3]{t}}y(t) = \frac{\sqrt{t}}{5}(\sin(x(t)) + x(t-1)), & t \in [1,6], \\ x(t) = t, \quad y(t) = \sin(\frac{\pi}{2}t), & t \in [0,1]. \end{cases} \tag{5.1}$$

Here

$$F(t,u,v) = \frac{\ln(t)}{4}(\arctan(u(t)) + \sin(v(t))), \quad G(t,u,v) = \frac{\sqrt{t}}{5}(\sin(u(t)) + v(t)).$$

It is easy to know that F is continuous with the Lipschitz constant $L_1 = \frac{\ln 6}{4}$, and G is continuous with the Lipschitz constant $L_2 = \frac{\sqrt{6}}{5}$. Since $\psi(t) = \sqrt[3]{t}$, $\alpha = \frac{2}{3}$ and $\beta = \frac{3}{4}$, we have

$$\frac{2L_1(\psi(t_1)-\psi(t_0))^{\alpha}}{\Gamma(\alpha+1)} = \frac{\ln 6}{2}\frac{1}{\Gamma(\frac{5}{3})}(\sqrt[3]{6} - \sqrt[3]{1})^{\frac{2}{3}} = 0.8674 < 1,$$

and

$$\frac{2L_2(\psi(t_1)-\psi(t_0))^{\beta}}{\Gamma(\beta+1)} = \frac{2\sqrt{6}}{5}\frac{1}{\Gamma(\frac{7}{4})}(\sqrt[3]{6} - \sqrt[3]{1})^{\frac{3}{4}} = 0.9162 < 1.$$

Thus, all the conditions of Theorem 4.3 are satisfied. Hence (5.1) is Ulam-Hyers stable.

6. Conclusions

In this paper, we introduced and proved a new generalized coupled Gronwall inequality. We examined the validity and applicability of our results by considered the existence and uniqueness of solutions of nonlinear delay coupled ψ-Caputo fractional differential system. Moreover, some result to verify sufficient conditions has been provided in this paper to determine the Ulam-Hyers stability of solutions for the considered system. Finally, a example is given to illustrate the effectiveness and feasibility of our criterion.

In the future, we will consider the nonlinear delay coupled ψ-Hilfer fractional differential systems, and we will study the existence and multiplicity of solutions, and the Ulam-Hyers and Ulam-Hyers-Rassias stabilities for there systems.

Acknowledgments

We are really thankful to the reviewers for their careful reading of our manuscript and their many insightful comments and suggestions that have improved the quality of our manuscript. This work is supported by Natural Science Foundation of China (11571136).

Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives Theory and Applications*, Gordon and Breach Science Publishers, 1993.
2. A. A. Kilbas, H. M. Srivastava, J. J Trujillo, *Theory and Applications of Fractional Differential Equations*, Amsterdam: Elsevier Science, 2006.
3. M. Feckan, J. Wang, M. Pospsil, *Fractional-Order Equations and Inclusions*, Berlin/Boston: de Gruyter, 2017.
4. V. Kiryakova, *Generalized Fractional Calculus and Applications*, New York: J. Wiley & Sons, Inc., 1994.
5. V. Kiryakova, A brief story about the operators of generalized fractional calculus, *Fract. Calc. Appl. Anal.,* **11** (2008), 203–220.
6. J. V.da C. Sousa, E. C. de Oliveira, On the ψ-Hilfer fractional derivative, *Commun. Nonlinear Sci.,* **60** (2018), 72–91.
7. A. Seemab, M. Ur Rehman, J. Alzabut, A. Hamdi, On the existence of positive solutions for generalized fractional boundary value problems, *Bound. Value Probl.,* **2019** (2019), 186. https://doi.org/10.1186/s13661-019-01300-8
8. R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci.,* **44** (2017), 460–481. https://doi.org/10.1016/j.cnsns.2016.09.006

9. R. Almeida, A. B. Malinowska, M. T. T. Monteiro, Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications, *Math. Methods Appl. Sci.,* **41** (2018), 336–352. https://doi.org/10.1002/mma.4617

10. F. Jarad, T. Abdeljawad, Generalized fractional derivatives and Laplace transform, *Discrete Contin. Dyn. Syst. Ser. S*, **13** (2019), 709–722. https://doi.org/10.3934/dcdss.2020039

11. J. V.da C. Sousa, E. C. de Oliveira, A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator, *Diff. Equations Appl.,* **11** (2019), 87–106.

12. H. Ye, J. Gao, Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math, Anal. Appl.,* **328** (2007), 1075–1081. https://doi.org/10.1016/j.jmaa.2006.05.061

13. Y. Adjabi, F. Jarad, T. Abdeljawad, On generalized fractional operators and a Gronwall type inequality with applications, *Filomat,* **31** (2017), 5457–5473. https://doi.org/10.2298/FIL1717457A

14. J. Alzabut, T. Abdeljawad, A generalized discrete fractional Gronwall inequality and its application on the uniqueness of solutions for nonlinear delay fractional difference system, *Appl. Anal. Discrete Math.,* **12** (2018), 36–48. https://doi.org/10.2298/AADM1801036A

15. R. I. Butt, T. Abdeljawad, M. A. Alqudah, M. Ur Rehman, Ulam stability of Caputo q-fractional delay difference equation: q-fractional Gronwall inequality approach, *J. Inequal. Appl.,* **2019** (2019), 305. https://doi.org/10.1186/s13660-019-2257-6

16. S. S. Dragomir, *Some Gronwall type inequalities and applications*, Hauppauge: Nova Science Pub Inc, 2003.

17. J. Alzabut, T. Abdeljawad, F. Jarad, W. Sudsutad, A Gronwall inequality via the generalized proportional fractional derivative with applications, *J. Inequal. Appl.,* **2019** (2019), 101. https://doi.org/10.1186/s13660-019-2052-4

18. J. Alzabut, Y. Adjabi, W. Sudsutad, Mutti-Ur. Rehman, New generalizations for Gronwall type inequalities involving a ψ-fractional operator and their applications, *AIMS Math.,* **6** (2021), 5053–5077. https://doi.org/10.3934/math.2021299

19. J. Vanterler, D. C. Sousa, E. C. D. Oliveira, A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator, arXiv:1709.03634v1.

20. J. Vanterler, D. C. Sousa, E. C. D. Oliveira, On the ψ-Hilfer fractional derivative, *Commun. Nonlinear Sci.,* **60** (2018), 72–91.

21. A. D. Mali, K. D. Kucche, J. Vanterier, On coupled system of nonlinear ψ-Hilfer hybrid fractional differential equations, *Int. J. Nonlin. Sci. Num.,*. https://doi.org/10.1515/ijnsns-2021-0012.

22. K. D. Kucche, A. D. Mali, On the nonlinear (k, ψ)-Hilfer fractional differential equations, *Chaos Soliton. Fract.,* **152** (2021), 111335.

23. K. D. Kucche, J. P. Kharade, Analysis of impulsive φ-Hilfer fractional differential equations, *Mediterr. J. Math.,* **17** (2020), 163. https://doi.org/10.1007/s00009-020-01575-7

24. A. D. Mali, K. D. Kucche, Nonlocal boundary value problem for generalized Hilfer implicit fractional differential equations, *Math. Meth, Appl. Sci.,* **2020** (2020), 1–24. https://doi.org/10.1002/mma.6521

25. K. D. Kucche, A. D. Mali, J. V. da C. Sousa, On the nonlinear Ψ-Hilfer fractional differential equations, *Comput. Appl. Math.,* **8** (2019), 73.
26. S. M. Ulam, *Problems in Modern Mathematics*, Chapter 6, John Wiley and Sons, New York, 1940.
27. S. M. Ulam, *A collection of Mathematical Problems*, Interscience, New York, 1960.
28. Th. M. Rassias, On a modified Hyers-Ulam sequence, *J. Math. Anal. Appl.,* **158** (2003), 106–113. https://doi.org/10.1016/0022-247X(91)90270-A
29. M. Benchohra, J. E. Lazreg, On stability of nonlinear implicit fractional differential equations, *Matematiche* (Catania), **70** (2015), 49–61.
30. J. Wang, L. Lv, Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative, *Elect. J. Qual. Theory Differ. Equ.,* **63** (2011), 1–10. https://doi.org/10.14232/ejqtde.2011.1.63
31. Q. Dai, S. Liu, Stability of the mixed Caputo fractional integro-differential equation by means of weighted space method, *AIMS Math.,* **7** (2022), 2498–2511. https://doi.org/10.3934/math.2022140
32. H. Khan, W. Chen, A. Khan, T. S. Khan, Q. M. Al-Madlal, Hyers-Ulam stability and existence criteria for coupled fractional differential equations involving p-Laplacian operator, *Adv. Differ. Equ.,* **2018** (2018), 455. https://doi.org/10.1186/s13662-018-1899-x
33. R. Ameen, F. Jarad, T. Abdeljawad, Ulam stability for delay fractional differential equations with a generalized Caputo derivative, *Filomat,* **32** (2018), 5265–5274. https://doi.org/10.2298/FIL1815265A
34. R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci.,* **44** (2017), 460–481. https://doi.org/10.1016/j.cnsns.2016.09.006
35. I. A. Rus, Ulam stabilities of ordinay differential equations in a Banach space, *Carp. J. Math.,* **26** (2010), 103–107.
36. R. Wong, *Approximations of Integrals*, SIAM. Philadelphia, 2001.