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Research article

Conjugacy classes of left ideals of Sweedler's four-dimensional algebra H_4

Fengxia Gao[1] **and Jialei Chen**[2,*]

[1] School of Science, Henan Institute of Technology, Henan, Xinxiang, Henan 453003, China
[2] College of Mathematics, Faculty of Science, Beijing University of Technology, Beijing 100124, China

* **Correspondence:** Email: chenjialei@bjut.edu.cn.

Abstract: Let A be a finite-dimensional algebra with identity over the field $\mathbb{F}$, $U(A)$ be the group of units of A and $L(A)$ be the set of left ideals of A. It is well known that there is an equivalence relation $\sim$ on $L(A)$ by defining $L_1 \sim L_2 \in L(A)$ if and only if there exists some $u \in U(A)$ such that $L_1 = L_2u$. $C(A) = \{[L] | L \in L(A)\}$ is the set of equivalence classes determined by the relation $\sim$, which is a semigroup with respect to the natural operation $[L_1][L_2] = [L_1L_2]$ for any $L_1, L_2 \in L(A)$. The purpose of this paper is to describe the structures of semigroup of conjugacy classes of left ideals for the Sweedler's four-dimensional Hopf algebra H_4.

Keywords: left ideal; idempotent; semigroup; nilpotent left ideal; conjugacy class
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1. Introduction

Throughout the paper, let A be a finite-dimensional associative algebra with identity over an algebraically closed field $\mathbb{F}$ of characteristic 0 and $U(A)$ be the group of units of A. By $L(A)$ we denote the set of left ideals in A. According to [1], there exists an equivalent relation on $L(A)$ by defining $L_1 \sim L_2 \in L(A)$ if and only if there exists some $u \in U(A)$ such that $L_1 = L_2u$. The corresponding set of equivalence classes is denoted by $C(A)$, which coincides with the set of conjugacy classes, and is also a semigroup [2] with respect to the natural operation $[L_1][L_2] = [L_1L_2]$ for any $L_1, L_2 \in L(A)$.

A study of $C(A)$ is in part motivated by a general program of searching for semigroup invariants of associative algebras [3]. In addition, the semigroup $C(A)$ is related to the subspace semigroup of an associative algebra, studied in [4, 5], which is an analogue of the semigroup of closed subsets in an algebraic monoid. In the context of ring theory, various related actions of $U(A)$ have been considered on a ring A, see [6], and also [7, 8], leading to certain finiteness conditions for A. Especially, the class of algebras with $C(A)$ finite includes the algebras of finite representation type, see Theorem 6 in [1].

Let H_4 be the Sweedler's four-dimensional Hopf algebra. From the classification result of finite-dimensional Hopf algebras, it is known that H_4 is a non-commutative and non-cocommutative Hopf algebra of minimal dimension, which is closely related to non-semisimple eight-dimensional Hopf algebras. It is also an algebra of finite representation type from the view of representation theory of algebras. Furthermore, H_4 is a very important example in the development of the theory of Hopf algebras [9, 10]. Indeed, the Sweedler's Algebra H_4 is a Taft's Hopf algebra (see for instance the book [11] for more details) that are examples of non-commutative and non-cocommutative Hopf algebras [12–14], too. Therefore, it is necessary to further consider the congucacy class of left ideals of H_4.

The paper is organized as follows. First of all, we recall the necessary prerequisites on idempotents used in this paper. Secondly, for H_4 we consider all conjugacy classes of nilpotent left ideals with the help of Jacobson radical of H_4, and all conjugacy classes of principle left ideals generated by idempotents. In addition, we also give the products of the principal left ideals generated by idempotents and the nilpotent left ideals. In Section 3 we discuss conjugacy classes of those left ideals which are neither nilpotent nor generated by idempotents. The last section is our main result. In this section we describe the semigroup of conjugacy classes of left ideals of H_4 by means of generators and relations.

2. Prerequisites

In the study of indecomposable modules over A, an important role is played by idempotent elements of A. An element $e \in A$ is called an idempotent if $e^2 = e$. The idempotents $e_1, e_2 \in A$ are called orthogonal if $e_1e_2 = e_2e_1 = 0$. The idempotent e is said to be primitive if e cannot be written as a sum $e = e_1 + e_2$, where e_1 and e_2 are nonzero orthogonal idempotents of A. According to [15] a set $E = \{e_1, e_2, \cdots, e_n\} \subset A$ of primitive orthogonal-pair idempotents is called a complete set of primitive orthogonal idempotents of A if $e_1 + e_2 + \cdots + e_n = \mathbf{1}$. As for the complete set of primitive orthogonal idempotents, the following conclusion holds.

Lemma 2.1. ([16], Theorem 3.4.1) Let $\{e_1, \cdots, \cdots, e_n\}$ and $\{f_1, \cdots, \cdots, f_m\}$ be two complete sets of primitive orthogonal idempotents of A. Then $n = m$ and there is an invertible element a in the algebra A such that, up to a suitable reindexing, $f_i = ae_ia^{-1}$ for all i.

Lemma 2.2. ([3], Proposition 2.2) Let e, f be idempotents in A. The following conditions are equivalent:

(1) e and f are conjugate in A;

(2) $[Ae] = [Af]$ in $C(A)$.

Let us recall that an ideal I or a left ideal of A is nilpotent if there exists a positive integer n such that $I^n = 0$ and the minimal positive integer is called the nilpotent index of I. It is well known that for a finite-dimensional algebra A, the Jacobson radical $J(A)$ is the unique maximal nilpotent (left) ideal. The following lemma can be obtained from Lemma 3.1 and Proposition 3.3 in [3].

Lemma 2.3. *Let $L_i \in L(A)$ for $i = 1, 2$. Then*

(1) there exist idempotents $e_i \in L_i$ such that $L_i = Ae_i \oplus L_i(1 - e_i)$ and $L_i(1 - e_i) \subseteq J(A)$;

(2) moreover, $[L_1] = [L_2]$ in $C(A)$ if and only if $[Ae_1] = [Ae_2]$ and $[L_1(1 - e_1)] = [L_2(1 - e_2)]$.

By the lemmas above, for a finite-dimensional algebra A, to investigate the conjugacy classes of left ideals of A, it needs to describe the conjugacy classes of nilpotent ideals and principal left ideals generated by idempotents.

3. Conjugacy classes of nilpotent left ideals and principle left ideals generated by idempotents

In this section we mainly describe conjugacy classes of nilpotent left ideals and principle left ideals generated by idempotents of H_4. For the purpose we need to determine the unit group $U(H_4)$ and the Jacobson radical $J(H_4)$.

Recall that Sweedler's four-dimensional Hopf algebra H_4 is an associative algebra with unity $\mathbf{1}$ over $\mathbb{F}$ generated by g, x with the following relations

$$g^2 = \mathbf{1}, x^2 = 0, gx + xg = 0.$$

Note that there is an $\mathbb{F}$-basis $\mathbf{1}$, g, x, gx in H_4. Let $\varepsilon_1 = \dfrac{\mathbf{1}+g}{2}$ and $\varepsilon_2 = \dfrac{\mathbf{1}-g}{2}$. Then ε_1 and ε_2 are orthogonal primitive idempotents such that $\mathbf{1} = \varepsilon_{\mathbf{1}} + \varepsilon_2$ is the decomposition of the unity. Moreover, $J(H_4)$ is a linear space of dimension 2 with a basis x, gx, which contains all nilpotent ideals and as left ideal $J(H_4)$ is only conjugate to itself.

Since H_4 is finite-dimensional it follows from [16] that an element $a = l_1\cdot\mathbf{1}+l_g\cdot g+l_x\cdot x+l_{gx}\cdot gx \in H_4$ is invertible or a unit if and only if a is a left unit, i.e., there exists an element $b = m_1 \cdot 1 + m_g \cdot g + m_x \cdot x + m_{gx} \cdot gx \in H_4$ such that $ab = \mathbf{1}$. At the same time, we have

$$\begin{aligned} ab = (l_1m_1 + m_gl_g)\cdot\mathbf{1} + (l_1m_g + l_gm_1)\cdot g + (l_1m_x + l_xm_1 + l_gm_{gx} - l_{gx}m_g)\cdot x \\ +(l_1m_{gx} + l_{gx}m_1 + l_gm_x - l_xm_g)\cdot gx. \end{aligned}$$

Hence, $ab = 1$ implies that the following system of equations with four unknowns m_1, m_g, m_x, m_{gx}

$$\begin{cases} l_1m_1 + l_gm_g = 1, \\ l_gm_1 + l_1m_g = 0, \\ l_1m_x + l_gm_{gx} = l_{gx}m_g - l_xm_1, \\ l_gm_x + l_1m_{gx} = l_xm_g - l_{gx}m_1 \end{cases}$$

has a unique solution. Of course, the system of equations with two unknowns m_1 and m_g

$$\begin{cases} l_1m_1 + l_gm_g = 1, \\ l_gm_1 + l_1m_g = 0 \end{cases}$$

has a unique solution, which is equivalent to the determinant of coefficient matrix $\begin{pmatrix} l_1 & l_g \\ l_g & l_1 \end{pmatrix}$ is not zero, i.e., ${l_1}^2 - {l_g}^2 \neq 0$ or ${l_1}^2 \neq {l_g}^2$. Furthermore, ${l_1}^2 \neq {l_g}^2$ also implies that the system of equations with unknowns m_x, m_{gx}

$$\begin{cases} l_1m_x + l_gm_{gx} = l_{gx}m_g - l_xm_1, \\ l_gm_x + l_1m_{gx} = l_xm_g - l_{gx}m_1 \end{cases}$$

has a unique solution.

In summary, ${l_1}^2 \neq {l_g}^2$ is a sufficient and necessary condition for a to be invertible. The proof of the following lemma is complete.

Lemma 3.1. $U(H_4) = \{\ l_1 1 + l_g g + l_x x + l_{gx} gx \ \mid\ {l_1}^2 \neq {l_g}^2\}$.

Since any nilpotent left ideal is contained in $J(H_4)$ and $J(H_4)$ is of dimension two, it follows from Lemma 2.3 that the classification of nilpotent left ideals is reduced to the study of all one-dimensional nilpotent left ideals. Next we initiate all nilpotent left ideals of dimension one. By J_1, J_2 we denote the one-dimensional linear subspaces of $J(H_4)$ spanned by $x + gx$ and $x - gx$, respectively.

Throughout the remainder of this paper, we fix notations $J_1 = \mathbb{F}(x + gx)$, $J_2 = \mathbb{F}(x - gx)$ and $J = J(H_4)$.

Lemma 3.2. *Let I be any nilpotent left ideal of H_4 of dimension one. Then $I = J_1$ or $I = J_2$.*

Proof. Since I is included in $J(H_4)$, there exists a nonzero element $z = ax + bgx \in I$ which is an $\mathbb{F}$-basis of I, where $a, b \in \mathbb{F}$. We claim that $a \neq 0, b \neq 0$. Otherwise, we suppose $a = 0$ or $b = 0$.

In the case of $a = 0$, without loss of generality we can assume $z = gx$. Since I is a left ideal of H_4, it follows that $gz = g(gx) = x \in I$. It is obvious that x is not a scalar multiple of $z = gx$. This contradicts with the fact that I is a left ideal of dimension one.

In the case of $b = 0$, without loss of generality we can assume $z = x$. It follows that $gz = gx \in I$. This also contradicts with the fact that I is a left ideal of dimension one.

In summary, we have $a \neq 0$ and $b \neq 0$, which implies that $z = a \cdot x + b \cdot gx = b \cdot (\frac{a}{b} \cdot x + gx)$. So we can choose $z = ax + gx$. Next we prove that $a = \pm 1$.

Since I is a left ideal of dimension one, it follows from $0 \neq gz = g(ax + gx) = x + agx \in I$ that $x + agx \in I$ is a scalar multiple of $ax + gx$, which implies that there exists a number $q \in \mathbb{F}$ such that $x + agx = q(ax + gx)$. More specifically, we have $(qa - 1)x + (a - q)gx = 0$, which leads to $qa - 1 = 0, a - q = 0$ and $a = \pm 1$. If $a = 1$, then we have $I = J_1$. Otherwise, we have $I = J_2$. The proof of Lemma 3.2 is complete. □

Theorem 3.3. *For any $u \in U(H_4)$, $J_1 = J_1 u$ and $J_2 = J_2 u$.*

Proof. First, from Lemma 3.1 for any $u \in U(H_4)$ we have $u = a\mathbf{1} + bg + cx + dgx \in U(H_4)$ with $a^2 \neq b^2$. So we get that

$$(x + gx)u = (x + gx)(a1 + bg + cx + dgx) = a(x + gx) - b(x + gx) = (a - b)(x + gx) \in I_1,$$

$$(x - gx)u = (x - gx)(a1 + bg + cx + dgx) = a(x - gx) - b(x - gx) = (a + b)(x - gx) \in I_2,$$

which imply that $J_1 u = J_1, J_2 = J_2 u$ for any $u \in U(H_4)$.□

Remark 1 The conclusion of Theorem 3.3 tells us that the $U(H_4)$-orbit of J_i contains only one element for $i = 1, 2$.

Corollary 3.4. *Let $N(H_4)$ be the set which consists of the conjugacy classes of nilpotent left ideals of H_4. Then $N(H_4) = \{0, [J_1], [J_2], [J]\}$ is a commutative semigroup with the following multiplication*

$$[J_1]^2 = [J_2]^2 = [J]^2 = [J_1][J] = [J_2][J] = [J_1][J_2] = 0.$$

Next, we focus on conjugacy classes of the principal left ideals generated by idempotents. According to Lemma 2.2 it is enough to describe the complete set $\left\{\varepsilon_1 = \dfrac{\mathbf{1} + g}{2}, \quad \varepsilon_2 = \dfrac{\mathbf{1} - g}{2}\right\}$ of the primitive orthogonal idempotent elements.

Lemma 3.5. *For any $u \in U(H_4)$, $u\varepsilon_1 \neq \varepsilon_2 u$.*

Proof. For $u \in H_4$, there exist $a, b, c, d \in \mathbb{F}$ with $a^2 \neq b^2$ such that $u = a \cdot \mathbf{1} + b \cdot g + c \cdot x + d \cdot gx$ by Lemma 3.1. Next we discuss the relation between $u\varepsilon_1$ and $\varepsilon_2 u$.

Direct calculations show that $u\varepsilon_1 = (a\cdot\mathbf{1}+b\cdot g+c\cdot x+d\cdot gx)\frac{1+g}{2} = \frac{1}{2}[(a+b)\cdot\mathbf{1}+(a+b)\cdot g+(c-d)\cdot x+(d-c)\cdot gx]$ and $\varepsilon_2 u = (a\cdot\mathbf{1}+b\cdot g+c\cdot x+d\cdot gx)\frac{1+g}{2} = \frac{1}{2}[(a-b)\cdot\mathbf{1}+(b-a)\cdot g+(c-d)\cdot x+(d-c)\cdot gx]$. If $u\varepsilon_1 = \varepsilon_2 u$, then we have

$$a + b = a - b, \ a + b = b - a,$$

which leads to $a = b = 0$. This contradicts with $a^2 \neq b^2$. The proof of Lemma 3.5 is complete.

Lemma 3.5 shows that ε_1 and ε_2 are not conjugate in H_4, which yields that $H_4\varepsilon_1$ and $H_4\varepsilon_2$ are not conjugate as left ideals. As a consequence, we have the following conclusion on the conjugacy classes of the principal left ideals generated by idempotents.

Lemma 3.6. *The conjugacy classes of the principal left ideals generated by idempotents is a set with three elements* $[H_4\varepsilon_1], [H_4\varepsilon_2]$ *and the conjugate class* $[H_4]$ *determined by the trivial ideal* H_4.

By definition it follows that $H_4\varepsilon_1$ is linearly spanned by $\varepsilon_1 = g\varepsilon_1$ and $x\varepsilon_1 = xg\varepsilon_1$. It is obvious that $1 + g, x - gx$ are linearly independent and they form a basis of $H_4\varepsilon_1$. Similarly, $H_4\varepsilon_2$ has a basis $1 - g, x + gx$. Let $I_1 = H_4\varepsilon_1, I_2 = H_4\varepsilon_2$. It is easy to see that $\{1 + g, x - gx, 1 - g, x + gx\}$ is also a basis of H_4, denoted by I.

Lemma 3.7. *The products of the principal left ideals generated by idempotents are as follows:*

$$I_1{}^2 = I_1, \ I_2{}^2 = I_2, I_1 I_2 = J_1, \ I_2 I_1 = J_2, I_1 H_4 = I_1 + J_1, \ I_2 H_4 = I_2 + J_2.$$

Proof. It is a direct conclusion of Table 1.

Table 1. The multiplication with respect to Basis I.

	$1+g$	$x-gx$	$1-g$	$x+gx$
$1+g$	$2(1+g)$	0	0	$2(x+gx)$
$x-gx$	$2(x-gx)$	0	0	0
$1-g$	0	$2(x-gx)$	$2(1-g)$	0

As a bonus of Lemma 3.7, another two left ideals of H_4: $I_1 + J_1$ and $I_2 + J_2$ are obtained, which are neither principle left ideals generated by idempotents nor nilpotent.

We end this section with the products between the principal left ideals $H_4\varepsilon_j$ generated by idempotents and the nilpotent ideals J_1, J_2 and $J(H_4)$.

Lemma 3.8. *As for the products between the principal left ideals and nilpotent left ideals we have*

	I_2	I_1	H_4
J_1	J_1	0	J_1
J_2	0	J_2	J_2
J	J_1	J_2	J

and

	J_1	J_2	J
I_1	J_1	0	J_1
I_2	0	J_2	J_2
H_4	J_1	J_2	J

.

4. Conjugate classes of other left ideals

Next we consider any left ideal which is neither a principle left ideal generated by an idempotent nor a nilpotent left ideal. Let K be such an ideal. Then according to Lemma 2.2 K must be of the form: $H_4\varepsilon_i + J_j$ with $1 \le i, j \le 2$. So to determine K we only need to discuss $H_4\varepsilon_i + J_j = I_i + J_j$ for $1 \le i, j \le 2$. Since $J_2 \subseteq I_1$, $J_1 \subseteq I_2$, $I_1 \cap J_1 = 0$, $I_2 \cap J_2 = 0$ and $J = J_1 + J_2$, it follows that

$$I_1 + J_1 = I_1 \oplus J_1, \quad I_1 + J_2 = I_1,$$

$$I_2 + J_1 = I_2, \quad I_2 + J_2 = I_2 \oplus J_2,$$

$$I_1 + J = I_1 + J_1, \quad I_2 + J = I_2 + J_2.$$

Therefore, we have the following

Lemma 4.1. *Let K be any left ideal which is neither a principle left ideal generated by an idempotent nor nilpotent. Then we have $[K] = [I_1 \oplus J_1]$ or $[K] = [I_2 \oplus J_2]$.*

For convenience later, we fix the notations $K_1 = I_1 \oplus J_1$ and $K_2 = I_2 \oplus J_2$. From Lemma 3.7 we also know that $K_1 = I_1 H_4, K_2 = I_2 H_4$

Lemma 4.2. *Keep the notations above, we have*

$$K_1{}^2 = K_1, \quad K_1 K_2 = J_1,$$

$$K_2 K_1 = J_2, \quad K_2{}^2 = K_2.$$

5. Results

In this section we give the main result of this paper.

Theorem 5.1. *As a semigroup $C(H_4)$ is isomorphic to the semigroup $S = \{o, e, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta, \gamma_1, \gamma_2\}$, which can be generated by $e, \alpha_1, \alpha_2, \beta, o$ subject to the following relations:*

(1) $ee = e, e\alpha_1 = \alpha_1, e\alpha_2 = \alpha_2, e\beta = \beta;$

(2) $oo = eo = oe = o\alpha_1 = \alpha_1 o = o\alpha_2 = \alpha_2 o = o\beta = \beta o;$

(3) $\alpha_1{}^2 = \alpha_1,\ \alpha_2{}^2 = \alpha_2,\ \beta^2 = o;$

(4) $\alpha_1\alpha_2 = \alpha_1\beta = \beta\alpha_2, \alpha_2\alpha_1 = \alpha_2\beta = \beta\alpha_1.$

Proof. By definition of the isomorphism of semigroup, in order to prove the theorem it is sufficient to find a bijection between $C(H_4)$ and S keeping multiplication. First, we define a map $\varphi : C(H_4) \longrightarrow S$ such that

$$\varphi([H_4]) = e, \varphi([J]) = \beta, \varphi([0]) = o;$$

$$\varphi([H_4\varepsilon_i]) = \alpha_i, \varphi([J_i]) = \beta_i, \varphi([H_4\varepsilon_i + J_i]) = \gamma_i \quad \text{for } i = 1, 2.$$

It is easy to see that φ is bijection. By Corollary 3.4, Lemma 3.7, Lemma 3.8 and Lemma 4.2 it follows that φ keeps the multiplication.

Corollary 5.2. *(1)* *e is a left unit: for any $a \in S$, $ea = a$;*

(2) *o is the zero element: for any $a \in S$, $oa = ao = o$;*

(3) *$s = \{o, \beta_1, \beta_2, \beta\}$ is a commutative ideal of S: for any $a, b \in s$, $ab = ba = o$;*

(4) *$B = \{\alpha_1, \alpha_2, \gamma_1, \gamma_2\}$ consists of all idempotents of S.*

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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