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Research article

Existence and essential stability of Nash equilibria for biform games with Shapley allocation functions

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Abstract: We define the Shapley allocation function (SAF) based on the characteristic function on a set of strategy profiles composed of infinite strategies to establish an n-person biform game model. It is the extension of biform games with finite strategies and scalar strategies. We prove the existence of Nash equilibria for this biform game with SAF, provided that the characteristic function satisfies the linear and semicontinuous conditions. We investigate the essential stability of Nash equilibria for biform games when characteristic functions are perturbed. We identify a residual dense subclass of the biform games whose Nash equilibria are all essential and deduce the existence of essential components of the Nash equilibrium set by proving the connectivity of its minimal essential set.

Keywords: biform games; Nash equilibrium point; essential connected component; generic stability; Shapley allocation function
Mathematics Subject Classification: 46T20, 49J53, 91A10, 91A12, 91A40

1. Introduction

The purpose of this paper is to use infinite strategies to establish a biform game model and study the existence and essential stability of its Nash equilibria. Grossman and Hart [1] started from a noncooperative second-stage game and defined an associated cooperative game to solve the questions of property rights and the nature of the firm. Hart and Moore [2] used this idea in a multiasset, multiindividual economy to study how changes in ownership affect the incentives of nonowners of assets (employees) as well as the incentives of owner-managers. Following this idea, Brandenburger and Stuart [3] proposed a general framework called the biform game to introduce a competitive environment with both strategy selection and benefit allocation (or cost allocation), it is a hybrid noncooperative-cooperative game model with two stages, the first stage involves players choosing strategies in a noncooperative state, the second stage of the cooperative game involves an analysis of how to allocate and how much utility is allocated, it is related to the confidence indices of players. Note that the strategy sets of players in this biform game are finite (please refer to footnote 23 of Brandenburger and Stuart [3]). Based on this biform game with finite strategy sets, Liu et al. [4] established a biform game model with mixed strategy sets, using the Shapley allocation function as the solution of the cooperation stage, so that the utility allocations of the players are independent of the confidence indices.

Shapley value [5] is one of the main solutions in the cooperative stage of biform games. Feess and Thun [6] used Shapley value to calculate the revenue of each firm, which is related to the surplus of a supply chain and the return on the investment. Li et al. [7] applied Shapley value to calculate the players' profits in the E-commerce game and Software firm game. Fiala [8] proposed a profit allocation mechanism in supply chains by using the method of the biform games, the profit-sharing of the cooperation part is based on the Shapley value concept. Nan et al. [9] proposed the Nash equilibrium solution based on the Shapley value to make up for the situation that the core is empty or the core allocation is not unique. For the applications of the core in biform games, please refer to [11–13]. In recent years, the significance of biform games has become increasingly apparent.

At present, almost all studies on biform games are based on finite pure strategies, while there are few studies on infinite strategies (mixed strategies as a special form). In the model of Hart and Moore [2], the scalar strategies of players belong to an interval, which is an infinite set. Example 1 of Liu et al. [4] shows that the players' strategies are the investment proportions. It can be seen that if the infinite strategies of a biform game involve quantitative size, such as the number and proportion of investment, then infinite strategies need to be introduced into the biform game. In this paper, we use the infinite strategies and Shapley value to establish a biform game model, and then we prove the existence of the Nash equilibrium for this model. Researchers usually use Nash equilibrium as the solution of biform games, such as Hart and Moore [2], Brandenburger and Stuart [3], and Feess and Thun [6].

Most games have multiple equilibria, Govindan and Wilson [14] pointed out that Nash's definition [15] is not a complete rational game theory. The refinement of game theory is a unified variant of Kohlberg and Mertens [16] on the definition of hyperstable component of equilibria of a game, they proposed the KM set by summarizing many refinement methods, the main idea of which was to seek a stable balanced set with strategy perturbation. Govindan and Wilson [14] explained that hyper-stability requires two principles: "Hyper" requires that a refinement should be immune to treating a mixed strategy as an additional pure strategy, which excludes presentation effects by ensuring that equivalent equilibria are selected in equivalent games. "Stability" requires that every nearby game has a nearby equilibrium, a nearby game is one with players' payoffs in a neighborhood of those of the given game, represented by them as a point in Euclidean space with the l_∞ norm. They proved that an equilibrium component is uniformly hyperstable if it is essential, and a connected uniformly hyperstable set is an essential component. Wu and Jiang [17] introduced the notion of essential equilibrium points and essential games for the n-person noncooperative finite games and proved that every game can be closely approximated arbitrarily by an essential game. Relying on the equivalence of Nash equilibrium and Ky Fan's point, Yu and Xiang [18] deduced that every n-person game has at least one essential connected component of Nash equilibrium set with perturbation of the payoff functions. Later, the work of essential equilibria was extended to infinite-action games (see [20,21]), the normal-form games with discontinuous payoffs (see [22–24]), the multi-leader-multi-follower games (see [25]), the α-core of games with nonordered preferences (see [26]), and the cooperative equilibria for population games (see [27]). To refine the Nash equilibria of biform games from the perspective of hyperstability [14], this paper studies the minimal essential sets and essential components of the set of the Nash equilibria when characteristic functions are perturbed.

Our main contributions can be summarized as follows: First, we use the infinite strategy and Shapley value to build the Shapley allocation function (SAF) in the second stage of cooperative games for the biform game proposed by Brandenburger and Stuart [3]. Based on the SAF, we establish an n-person biform game model, which is the extension of biform games with scalar strategies [2], mixed strategies [4], and finite strategies. Second, we characterize the existence of Nash equilibria of this biform game by providing linear and semicontinuous conditions for characteristic functions. Third, we find that the sufficiently small perturbations of the characteristic functions can lead to the sufficiently small changes in the SAFs, however, these perturbations have no definite link to changes in core allocations (note that the core here is not a normal core of cooperative games, it is only defined on a strategy profile (Brandenburger and Stuart [3])). Fourth, in the biform games with SAFs, we investigate the essential stability of Nash equilibria when characteristic functions are perturbed. We introduce the notions of essential Nash equilibria, minimal essential sets, and essential components of the set of the Nash equilibria. We prove a residual dense subset of the biform games whose Nash equilibria are essential and deduce the existence of essential components of the Nash equilibrium set for the biform games. Fifth, we give a link between the stabilities of the biform games and noncooperative games. That is, if a biform game with SAF is additive, then it becomes an n-person normal form noncooperative game. In this case, the biform game and the n-person normal form noncooperative game have no difference in their Nash equilibrium set and essential stability.

The rest of the paper is organized as follows. Section 2 establishes a biform game model based on infinite strategy and the SAF. Section 3 proves the existence of Nash equilibria for the biform game model. Section 4 shows the variations of the SAFs and core allocations with the characteristic function. Section 5 studies the essential stability of Nash equilibria for the biform games. Section 6 is the conclusion.

2. The model

Let $N=\{1,\cdots,n\}$ be the set of players, 2^N denote the set of subsets (i.e., coalitions) of N, for each $S\in 2^N$, $|S|$ denote the number of players in S. For each $i\in N$, X_i is the (pure) strategy set of player i, which is a nonempty compact convex subset of a normed linear space E_i. $X=\prod_{i\in N} X_i$ is the strategy profile set of all players, denote $X=(X_i,X_{-i})$. For each $i\in N$, denote $X_{-i}=\prod_{j\in N\backslash\{i\}} X_j$, if $x=(x_1,\cdots,x_n)\in X=(X_i,X_{-i})$, denote $x=(x_i,x_{-i})$. Let $B_i=\{S_{i1},\cdots,S_{ik_i}\}$ be the set of coalitions containing player i (i.e., $i\in S_{ij}$), where $S_{ij}\in 2^N$, $j=1,\cdots,k_i$ $(k_i=2^{n-1})$.

An n-person biform game [3] is a collection $(A_1,\cdots,A_n;V;\alpha_1,\cdots,\alpha_n)$, where V is a map from A to the set of maps from 2^N to the reals, with $V(c)(\varnothing)=0$ for every $c=(c_1,\cdots,c_n)\in A=\prod_{i\in N} A_i$. For each $i\in N$, where the number α_i $(0\le\alpha_i\le 1)$ is player i's confidence index, A_i is a finite pure strategy set of player i. In the second stage of this biform game, Brandenburger and Stuart [3] introduced core as the solution of the cooperative games. We will use the Shapley value as the solution of the cooperative games and consider X_i as the strategy set of each player i.

In the first stage of a biform game, each player can try to use his strategies to select the best game for himself, where by "game" is meant the subsequent (second-stage) game of value. Equally, they can also try to change the game with a strategy, if we define one of the second-stage games as the status quo. In the second stage of the biform game, for a strategy profile $x \in X$ generated by all players selecting one of their own strategies, the Shapley allocation function (SAF) for player $i \in N$ is defined as

$$\Phi_i(x) = \sum_{S_{ij} \in B_i} \frac{(|S_{ij}|-1)!(n-|S_{ij}|)!}{n!}[V_{S_{ij}}(x) - V_{S_{ij}\setminus\{i\}}(x)],$$

where $V_{S_{ij}}(x) - V_{S_{ij}\setminus\{i\}}(x)$ is the marginal contribution of the player i to coalition S_{ij} on x (for each $S \in 2^N$, $V_S(x)$ is defined in Definition 1), $\frac{(|S_{ij}|-1)!(n-|S_{ij}|)!}{n!}$ is the probability of player i entering the coalition S_{ij}. Thus, $\Phi_i(x)$ is the average (expected) marginal contribution of player i to all coalitions in B_i on x, which reflects the fairness of utility allocation to player i. The $\Phi(x) = (\Phi_1(x), \cdots, \Phi_n(x))$ from X to R^n is a vector valued function.

Definition 1. *An n-person biform game with SAFs is defined as a collection*

$$(X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n),$$

where

(1) for each $i \in N$, X_i is the (pure) strategy set of player i, which is a nonempty compact convex subset of a normed linear space E_i;

(2) for each $x = (x_1, \cdots, x_n) \in X$, it defines a cooperative game with characteristic function $V(x)$: $2^N \to \mathbb{R}$. That is, for each $S \in 2^N$, $V_S(x)$ is the value created by coalition S, given that the players chose the strategies $x_1, \cdots, x_n$. As usual, $V_\varnothing(x) = 0$ for any $x \in X$; and

(3) for each $i \in N$ and each $x \in X$,

$$\Phi_i(x) = \sum_{S_{ij} \in B_i} p_{ij}[V_{S_{ij}}(x) - V_{S_{ij}\setminus\{i\}}(x)],$$

where $p_{ij} = \frac{(|S_{ij}|-1)!(n-|S_{ij}|)!}{n!}$.

Definition 2. *For each $x \in X$, let $V(x): 2^N \to \mathbb{R}$ be the characteristic function of a biform game $(X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)$. Then, for any S_1, $S_2 \in 2^N$, $S_1 \cap S_2 = \varnothing$,*

(1) $(X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)$ is said to be superadditive on $x \in X$ if $V_{S_1 \cup S_2}(x) \geq V_{S_1}(x) + V_{S_2}(x)$, it is said to be superadditive on X if $V_{S_1 \cup S_2}(x) \geq V_{S_1}(x) + V_{S_2}(x)$ for any $x \in X$;

(2) $(X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)$ is said to be subadditive on $x \in X$ if $V_{S_1 \cup S_2}(x) \leq V_{S_1}(x) + V_{S_2}(x)$, it is said to be subadditive on X if $V_{S_1 \cup S_2}(x) \leq V_{S_1}(x) + V_{S_2}(x)$ for any $x \in X$;

(3) $(X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)$ is said to be additive on $x \in X$ if $V_{S_1 \cup S_2}(x) = V_{S_1}(x) + V_{S_2}(x)$, it is said to be additive on X if $V_{S_1 \cup S_2}(x) = V_{S_1}(x) + V_{S_2}(x)$ for any $x \in X$.

Remark 1. *The n-person biform game $(X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)$ becomes an n-person normal form noncooperative game, if it is additive on X.*

Hart and Moore [2] take the action (a scalar) $x_i \in [0, \bar{x}_i]$ of agent i to be a pure investment in human capital. Clearly, interval $[0, \bar{x}_i]$ is an infinite pure strategy set of agent i. According to their Assumptions 2 to 6, the share of agent i on strategy profile $x = (x_1, \cdots, x_I)$ is

$$\psi_i(\alpha \mid x) = \sum_{S \mid i \in S} p(S)[v(S, \alpha(S) \mid x) - v(S \setminus \{i\}, \alpha(S \setminus \{i\}) \mid x)],$$

where $p(S) = \frac{(s-1)!(I-s)!}{I!}$, $s = |S|$, α is a control structure, $v(S, \alpha(S)|x)$ is the value of a coalition S. For each agent $i = 1, \cdots, I$, let $[0, \bar{x}_i] = \bar{X}_i$, denote the profit-sharing game of Hart and Moore as

$$(\bar{X}_1, \cdots, \bar{X}_I; v; \psi_1, \cdots, \psi_I).$$

For each $i \in N$, A_i is the finite pure strategy set of player i, let $\tilde{X}_i$ be the mixed strategy set of player i corresponding to A_i. A biform game model with these mixed strategy sets established by Liu et al. [4] is

$$(\tilde{X}_1, \cdots, \tilde{X}_n; V; \Phi_1, \cdots, \Phi_n).$$

Remark 2. *(1)* $(X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)$ *is an extension of* $(\bar{X}_1, \cdots, \bar{X}_I; v; \psi_1, \cdots, \psi_I)$.

(2) $(\tilde{X}_1, \cdots, \tilde{X}_n; V; \Phi_1, \cdots, \Phi_n)$ *is a special form of* $(X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)$.

In Definition 1, X_i is an extension of the mixed strategy set $\tilde{X}_i$ based on the pure strategy set A_i. In the following, we extend the V defined by Brandenburger and Stuart [3].

Assumption 1. *Referring to the extension of Nash* [15] *to the payoff functions, for each nonempty coalition* $S \in 2^N$ *and each fixed* $x_{-i} \in X_{-i}$, *we assume that the function* $V_S(x_i, x_{-i})$ *in Definition 1 is linear on* X_i. *Further, for each nonempty coalition* $S \in 2^N$ *and each* $i \in S$, *we assume that* $V_S(x) - V_{S\setminus\{i\}}(x)$ *is upper semicontinuous on* X *and* $V_S(x_i, x_{-i}) - V_{S\setminus\{i\}}(x_i, x_{-i})$ *is lower semi-continuous on* X_{-i}.

Obviously, if $V_S(x)$ is continuous on X for each nonempty coalition $S \in 2^N$, then V satisfies the two semicontinuous conditions of Assumption 1.

3. The existence of Nash equilibrium

Following Hart and Moore [2] and Brandenburger and Stuart [3], we consider Nash equilibrium as the solution of the biform game $(X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)$.

A profile of strategies $x^* = (x_1^*, \cdots, x_n^*) \in X$ is said to be a Nash equilibrium of the biform game $(X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)$, if it is a Nash equilibrium of the induced noncooperative game generated in the second stage of cooperative games. That is, a strategy profile $x^* \in X$ is a Nash equilibrium of an n-person biform game $(X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)$ if and only if for every $i \in N$,

$$\Phi_i(x_i^*, x_{-i}^*) = \max_{z_i \in X_i} \Phi_i(z_i, x_{-i}^*).$$

The following Lemma comes from Lemma 2.1 of Tan et al. [28].

Lemma 1. *Let* H^1 *and* H^2 *be two Hausdorff topological spaces and* H^1 *be compact. Let* f *be a real-valued function defined on* $H^1 \times H^2$ *such that*
(1) f is upper semicontinuous on $H^1 \times H^2$*;*
(2) for each fixed $x \in H^1$*, the function* $y \mapsto f(x,y)$ *is lower semicontinuous.*
Then the function $\varphi: H^2 \to \mathbb{R}$ *defined by* $\varphi(y) = \max_{z \in H^1} f(z,y)$ *is continuous on* H^2*.*

Theorem 1. *Suppose that an n-person biform game* $(X_1,\cdots,X_n;V;\Phi_1,\cdots,\Phi_n)$ *satisfies Assumption 1. Then* $(X_1,\cdots,X_n;V;\Phi_1,\cdots,\Phi_n)$ *has at least one Nash equilibrium.*

Proof. By Assumption 1, we can easily deduce that:

(1) For each $i \in N$, $\Phi_i(x)$ is upper semicontinuous on X;

(2) For each $i \in N$ and fixed $x_i \in X_i$, the function $x_{-i} \mapsto \Phi_i(x_i, x_{-i})$ is lower simi- continuous on X_{-i};

(3) For each $i \in N$, the function $x_i \mapsto \Phi_i(x_i, x_{-i})$ is linear on X_i.

For a fixed positive integer k, we define the set-valued mapping W_k from X to X as

$$W_k(x) = \prod_{i=1}^{n} \{y_i \in X_i \mid \Phi_i(y_i, x_{-i}) > \max_{z_i \in X_i} \Phi_i(z_i, x_{-i}) - \frac{1}{k}\}.$$

Then for each $x \in X$, $W_k(x)$ is nonempty and is convex by (3). For each $y \in X$, by Lemma 1 and (2), the function $x \mapsto \Phi_i(y_i, x_{-i}) - \max_{z_i \in X_i} \Phi_i(z_i, x_{-i})$ is lower semicontinuous and hence the set

$$W_k^{-1}(y) = \bigcap_{i=1}^{n} \{x \in X \mid \Phi_i(y_i, x_{-i}) > \max_{z_i \in X_i} \Phi_i(z_i, x_{-i}) - \frac{1}{k}\}$$

is open in X. Then Theorem 1 of Browder [29] shows that there exists $x^k \in X$ such that $x^k \in W_k(x^k)$, that is, $\Phi_i(x_i^k, x_{-i}^k) > \max_{z_i \in X_i} \Phi_i(z_i, x_{-i}^k) - \frac{1}{k}$ for each $i \in N$.

Since X is a compact, there is a subnet $\{y^m\}$ of $\{x^k\}$ and $x^* \in X$ such that $y^m \to x^*$. Let $y^m \to x^{k(m)}$ where $k(m) \to \infty$. By Lemma 1 and (1), for every $i \in N$,

$$\Phi_i(x_i^*, x_{-i}^*) \geq \limsup_m \Phi_i(x_i^{k(m)}, x_{-i}^{k(m)}) \geq \lim_m \max_{z_i \in X_i} \Phi_i(z_i, x_{-i}^{k(m)}) = \max_{z_i \in X_i} \Phi_i(z_i, x_{-i}^*).$$

Therefore, x^* is a Nash equilibrium of $(X_1,\cdots,X_n;V;\Phi_1,\cdots,\Phi_n)$. This completes the proof.

4. Variations in SAFs and core allocations relative to the characteristic function

Let M be the set of all V satisfying Assumption 1, and let Ω be the set of all $\Phi = (\Phi_1,\cdots,\Phi_n)$, where $\Phi_1,\cdots,\Phi_n$ satisfy the (1), (2), and (3) in the proof of Theorem 1. For any $V,\ V' \in M$, define the distance between V and V' by

$$\sigma(V,V') = \max_{S \in 2^N} \max_{x \in X} \left| V_S(x) - V'_S(x) \right|.$$

For any $\Phi, \Phi' \in \Omega$, define the distance between Φ and Φ' by

$$\rho(\Phi,\Phi') = \max_{x\in X} \sum_{i=1}^{n} \left|\Phi_i(x) - \Phi_i'(x)\right|.$$

It is easy to prove that (M,σ) and (Ω,ρ) are both complete metric spaces.

For each $V \in M$, denote by $h_V(x) = (h_1(x),\cdots,h_n(x)) \in \mathbb{R}^n$ a core allocation on $x \in X$ relative to V, let $C_V(x)$ be the set of core allocations on $x \in X$ relative to V. Then $C_V : X \to 2^{R^n}$ is a core allocation mapping on X. Denote by $C_M(X)$ the set of core allocation mappings on X relative to M.

Example 1. *Let* $X_1 = X_2 = [0,1], X = X_1 \times X_2, \alpha,\beta > 0$. *Consider the characteristic functions of the superadditive 2-person biform games as:*

$$V_{\{1\}}(x) = V_{\{2\}}(x) = \alpha x_1 + \beta x_2, V_{\{1,2\}}(x) = 4(\alpha x_1 + \beta x_2), \forall (x_1,x_2) \in X;$$

$$V_{\{1\}}^n(x) = V_{\{2\}}^n(x) = \alpha x_1 + \beta x_2, V_{\{1,2\}}^n(x) = 4(\alpha x_1 + \beta x_2) + \frac{2}{n}, \forall (x_1,x_2) \in X.$$

It is easy to obtain that

$$\Phi_1(x) = \Phi_2(x) = 2(\alpha x_1 + \beta x_2), \Phi_1^n(x) = \Phi_2^n(x) = 2(\alpha x_1 + \beta x_2) + \frac{1}{n}, \forall x = (x_1,x_2) \in X,$$

$$C_V(x) = \{h_V(x) = (h_1(x), h_2(x)) | \alpha x_1 + \beta x_2 \le h_1(x), h_2(x) \le 3(\alpha x_1 + \beta x); \\ h_1(x) + h_2(x) = 4(\alpha x_1 + \beta x_2)\},$$

$$C_{V^n}(x) = \{h_{V^n}(x) = (h_1^n(x), h_2^n(x)) | \alpha x_1 + \beta x_2 \le h_1(x), h_2(x) \le 3(\alpha x_1 + \beta x_2) + \frac{2}{n}; \\ h_1^n(x) + h_2^n(x) = 4(\alpha x_1 + \beta x_2) + \frac{2}{n}\}.$$

Take core allocations

$$h_V(x) = (\alpha x_1 + \beta x_2, 3(\alpha x_1 + \beta x_2)) \in C_V(x),$$

$$h_{V^n}(x) = (3(\alpha x_1 + \beta x_2) + \frac{1}{n}, \alpha x_1 + \beta x_2 + \frac{1}{n}) \in C_{V^n}(x).$$

Let $n \to \infty$, then $\sigma(V,V^n) = \frac{2}{n} \to 0$. We have $\rho(\Phi,\Phi^n) = \frac{2}{n} \to 0$, but $h_{V^n}(x)$ can not converge to $h_V(x)$ whenever $(x_1,x_2) \ne (0,0)$. Thus, the sufficiently small change of V can lead to the sufficiently small changes of the SAFs, but cannot guarantee the sufficiently small changes of any two core allocation sets.

Note that if V and V^n are the characteristic functions of the additive biform games, then on each $x \in X$, the core allocation is unique and equal to SAF vector. In this case, the sufficiently small perturbation of V can obtain sufficiently small changes of SAF vector and core allocation. If V and V^n are the characteristic functions of the subadditive biform games, then the core is an empty set on some $x \in X$, while the sufficiently small perturbation of V can still make the sufficiently small changes of SAFs.

Lemma 2. *For* $V, V' \in M, \Phi, \Phi' \in \Omega$, *let* $\sigma(V,V') = \max_{S\in 2^N} \max_{x\in X} |V_S(x) - V'_S(x)| = \delta(\delta > 0)$, *if* $\delta \to 0$, *then* $\rho(\Phi,\Phi') = \max_{x\in X} \sum_{i=1}^{n} |\Phi_i(x) - \Phi'_i(x)| \to 0$.

Proof. For each $i \in N$ and each $x \in X$,

$$|\Phi_i(x) - \Phi'_i(x)| = \left| \sum_{S_{ij}\in B_i} p_{ij}[V_{S_{ij}}(x) - V_{S_{ij}\setminus\{i\}}(x)] - \sum_{S_{ij}\in B_i} p_{ij}[V'_{S_{ij}}(x) - V'_{S_{ij}\setminus\{i\}}(x)] \right|$$

$$= \left| \sum_{S_{ij}\in B_i} p_{ij}[(V_{S_{ij}}(x) - V'_{S_{ij}}(x)) - (V_{S_{ij}\setminus\{i\}}(x) - V'_{S_{ij}\setminus\{i\}}(x))] \right|$$

$$\le \sum_{S_{ij}\in B_i} p_{ij} 2\delta = 2\delta.$$

By $\delta \to 0$, we have $|\Phi_i(x) - \Phi'_i(x)| \to 0$ for each $x \in X$. Thus, $\rho(\Phi,\Phi') \le 2n\delta \to 0$ by $\delta \to 0$. This completes the proof.

Obviously, each V in M determines a unique Φ in Ω. However, a Φ in Ω may correspond to multiple V in M. For instance, in Example 1, if

$$V_{\{1\}}(x) = V_{\{2\}}(x) = 2(\alpha x_1 + \beta x_2), V_{\{1,2\}}(x) = 4(\alpha x_1 + \beta x_2), \forall (x_1, x_2) \in X.$$

We can still get that

$$\Phi_1(x) = \Phi_2(x) = 2(\alpha x_1 + \beta x_2), \forall (x_1, x_2) \in X.$$

5. Essential stability

By Lemma 2, $V^m \to V\ (m \to \infty)$ implies that $\Phi^m \to \Phi\ (m \to \infty)$. In the following, we abbreviate the biform game $(X_1, \cdots, X_n; V; \Phi_1, \cdots, \Phi_n)$ as V. We study the essential stability of Nash equilibria for the biform games when characteristic functions are perturbed.

Denote by $F(V)$ the set of Nash equilibria of the biform game V, then $F(V)$ is nonempty by Theorem 1. Thus, we obtain a Nash equilibrium correspondence $F: M \to 2^X \setminus \{\varnothing\}$, it is a set-valued mapping.

Let Z be a Hausdorff topological space and Y be a metric space. A set-valued mapping F : $Z \to 2^Y \setminus \{\varnothing\}$ is upper semicontinuous, if for each $z \in Z$ and any open set O in Y with $O \supset F(z)$, there exists a open set $U(z)$ of z in Z such that $O \supset F(z')$ for any $z' \in U(z)$; the set-valued mapping $F: Z \to 2^Y \setminus \{\varnothing\}$ is lower semicontinuous, if for each $z \in Z$ and any open set O in Y with $O \cap F(z) \ne \varnothing$, there exists a open set $U(z)$ of z in Z such that $O \cap F(z') \ne \varnothing$ for any $z' \in U(z)$.

For any $x, x' \in X$, define the distance between x and x' by

$$d(x, x') = \sum_{i=1}^{n} \|x_i - x'_i\|_i,$$

where $\|x_i - x'_i\|_i$ is the distance between x_i and x'_i in the nonempty compact convex normed linear space E_i.

5.1. Generic stability

Following the representations of Wu and Jiang [17] and Yu [19], we introduce the notion of essential Nash equilibria for biform games.

Definition 3. *A Nash equilibrium* $x \in F(V)$ *is called an essential equilibrium of the biform game* V *if for any* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that for any* $V' \in M$ *with* $\sigma(V,V') < \delta$, *there exists at least one Nash equilibrium* $x' \in F(V')$ *with* $d(x,x') < \varepsilon$; *a biform game* V *is weakly essential if there exists a Nash equilibrium* $x \in F(V)$ *which is essential; a biform game* V *is essential if all its Nash equilibria are essential.*

Definition 4. *A subset* Q *of* M *is dense if* $clQ = M$. *A subset* Q *of* M *is residual if* $Q = \bigcap_{i=1}^{\infty} Q_i'$, *where each* Q' *is an open dense subset of* M.

Lemma 3. *The Nash equilibrium correspondence* $F : M \to 2^X \setminus \{\varnothing\}$ *is upper semicontinuous with compact values.*

Proof. Let $\{V^m\}$ be a sequence in M, $V^m \to V$ $(m \to \infty)$ (corresponding to a sequence $\{\Phi^m\}$ in Ω, $\Phi^m \to \Phi$ $(m \to \infty)$), and let $x^m \in F(V^m)$, $x^m \to x$ $(m \to \infty)$, we will prove that $x \in F(V)$.

Since $x^m \in F(V^m)$, it follows that

$$\Phi_i^m(x_i^m, x_{-i}^m) = \max_{z_i \in X_i} \Phi_i^m(z_i, x_{-i}^m) \geq \Phi_i^m(z_i^0, x_{-i}^m), \forall z_i^0 \in X_i, \forall i \in N .$$

For any $\varepsilon > 0$ and each $i \in N$, by Φ_i is linear on X_i, there exists $z_i^0 \in X_i$ such that

$$\Phi_i(z_i^0, x_{-i}) > \max_{z_i \in X_i} \Phi_i(z_i, x_{-i}) - \varepsilon .$$

Since $V^m \to V(m \to \infty)$ and Φ_i is lower semicontinuous on X_{-i}, for the same $\varepsilon > 0$ and $z_i^0 \in X_i$ above, we obtain that there exists m_0 such that

$$\sigma(V^m, V) < \frac{\varepsilon}{2n} \text{ and } \Phi_i(z_i^0, x_{-i}^m) > \Phi_i(z_i^0, x_{-i}) - \varepsilon .$$

For the same m_0 and $\varepsilon > 0$ above, by Lemma 2, we have $\rho(\Phi^m, \Phi) < \varepsilon$. Since

$$\begin{aligned}\Phi_i(x_i, x_{-i}) - \Phi_i^m(x_i^m, x_{-i}^m) &= \Phi_i(x_i, x_{-i}) - \Phi_i(x_i^m, x_{-i}^m) + \Phi_i(x_i^m, x_{-i}^m) - \Phi_i^m(x_i^m, x_{-i}^m) \\ &\geq \Phi_i(x_i, x_{-i}) - \Phi_i(x_i^m, x_{-i}^m) - \rho(\Phi^m, \Phi)\end{aligned}$$

and Φ_i is upper semicontinuous on X, we have

$$\begin{aligned}\Phi_i(x_i, x_{-i}) - \limsup_m \Phi_i^m(x_i^m, x_{-i}^m) &\geq \liminf_m [\Phi_i(x_i, x_{-i}) - \Phi_i(x_i^m, x_{-i}^m)] \\ &= \Phi_i(x_i, x_{-i}) - \limsup_m \Phi_i(x_i^m, x_{-i}^m) \geq 0.\end{aligned}$$

Thus,

$$\Phi_i(x_i, x_{-i}) \geq \limsup_m \Phi_i^m(x_i^m, x_{-i}^m) \geq \Phi_i^m(x_i^m, x_{-i}^m) \geq \Phi_i^m(z_i^0, x_{-i}^m) \geq \Phi_i(z_i^0, x_{-i}^m)$$

$$-\rho(\Phi^m, \Phi) \geq \Phi_i(z_i^0, x_{-i}^m) - \varepsilon \geq \Phi_i(z_i^0, x_{-i}) - 2\varepsilon \geq \max_{z_i \in X_i} \Phi_i(z_i, x_{-i}) - 3\varepsilon,$$

thus, $\Phi_i(x_i, x_{-i}) = \max_{z_i \in X_i} \Phi_i(z_i, x_{-i})$ for all $i \in N$. Therefore, $x \in F(V)$, the graph of F is closed. Hence, for each $V \in M$, $F(V)$ must be compact in X since X is compact. By Proposition 3 of Aubin [30], p. 72, it can be sufficiently shown that F is upper semicontinuous on Ω. This completes the proof.

The following Lemma 4 is obtained from theorem 2 of Fort [31].

Lemma 4. *Let Y and H are two metric spaces and H is complete. If the set-valued mapping $\varphi: H \to 2^Y$ is upper semicontinuous with nonempty compact values, then φ is lower simicontinuous on a dense residual subset Q of H.*

Theorem 2. *A biform game V is essential if and only if the Nash equilibrium correspondence $F: M \to 2^X \setminus \{\varnothing\}$ is lower semicontinuous at V.*

Proof. By the definition of the lower semicontinuous of set-valued mapping and the definition of the essential game V, this theorem holds. This completes the proof.

Theorem 3. *There exists a dense residual subset Q of M such that every biform game $V \in Q$ is essential.*

Proof. By Lemma 3, F is upper semicontinuous with compact values. It, along with Lemma 4, yields that F is lower semicontinuous on a dense residual subset Q of M. Then from Theorem 2, it is concluded that every game $V \in Q$ is essential. This completes the proof.

Theorem 4. *If a biform game V is such that $F(V) = \{x\}$ is a singleton set, then V is essential.*

Proof. Let U be any open set of X such that $F(V) \cap U \neq \varnothing$, then $x \in U$ makes $F(V) \subset U$. By Lemma 3, F is upper semicontinuous at V, thus, there exists $\delta > 0$ such that for any $V' \in M$ with $\sigma(V, V') < \delta$, $U \supset F(V')$, that is, $F(V') \cap U \neq \varnothing$ for any $V' \in M$ with $\sigma(V, V') < \delta$. Thus, F is lower semicontinuous at V. By Theorem 2, V is essential. This completes the proof.

5.2. Essential components

Note that components are connected closed subsets of $F(V)$ and are also connected compact. $F(V)$ can be decomposed into union of pairwise disjoint finite or infinite connected components C_α, i.e.,

$$F(V) = \bigcup_{\alpha \in \Lambda} C_\alpha(V),$$

where Λ is an index set, for any $\alpha \in \Lambda$, $C_\alpha(V)$ is a nonempty, connected and compact set, and $C_\alpha(V) \cap C_\beta(V) = \varnothing$ for any $\alpha, \beta \in \Lambda$ $(\alpha \neq \beta)$ (please refer to p. 352 of Engelking [32]).

Following Yu and Xiang [18] and Zhou et al. [20], we introduce the notions of essential sets and essential components of the Nash equilibrium set for the biform game V.

Definition 5. *Let* $V \in M$. *A nonempty closed subset* $e(V)$ *of* $F(V)$ *is essential if for any open set* $O \supset e(V)$, *there exists* $\delta > 0$ *such that* $O \bigcap F(V') \neq \varnothing$ *for any* $V' \in M$ *with* $\sigma(V, V') < \delta$. *An essential set* $m(V)$ *of* $F(V)$ *is said to be a minimal essential set if it is a minimal element ordered by set inclusion.*

Definition 6. *Let* $V \in M$. *A component of* $x \in F(V)$ *is the union of all connected subsets of* $F(V)$ *containing* x . *A component of* $F(V)$ *is an essential component if it is essential.*

Remark 3. *For two closed subsets* $e_1(V)$, $e_2(V)$ *of* $F(V)$, *if* $e_1(V) \subset e_2(V)$ *and* $e_1(V)$ *is essential, then* $e_2(V)$ *is also essential.*

Theorem 5. *For each* $V \in M$, *there exists a minimal essential set of* $F(V)$.

Proof. For each $V \in M$, by F is upper semicontinuous at V , it follows that $F(V)$ is the essential set of itself. Denote by P the set of all essential sets of $F(V)$ ordered by set inclusion, then P is nonempty and every decreasing chain of elements in P has a lower bound (by the compactness, the intersection is in P). By Zorn's lemma, P has a minimal element and this minimal element is a minimal essential set of $F(V)$. This completes the proof.

Theorem 6. *For each* $V \in M$, *every minimal essential set of* $F(V)$ *is connected.*

Proof. For each $V \in M$, let $m(V)$ be a minimal essential set of $F(V)$. Suppose that $m(V)$ is not connected. Following the proof method of Kinoshita [33], Yu and Xiang [18], and Yang et al. [27], we show the family $\{(e_l(V), U_l, V^l)_{l=1,2}, \delta > 0\}$ such that

(1) each $e_l(V)$ is nonempty compact subset of $m(V)$ and $m(V) = e_1(V) \bigcup e_2(V)$;

(2) each U_l is open and $e_l(V) \subset U_l$, $clU_1 \bigcap clU_2 = \varnothing$;

(3) for each $l = 1, 2$, $V^l \in M$, $\sigma(V^l, V) < \delta$ and $F(V^l) \bigcap U_l = \varnothing$;

(4) $F(V') \bigcap (U_1 \bigcup U_2) \neq \varnothing$ for any $V' \in M$ with $\sigma(V', V) < \delta$.

For each nonempty coalition $S \in 2^N$, define a function $V_S'(x): 2^N \to \mathbb{R}$ as follows:

$$V_S'(x) = \mu(x) V_S^1(x) + \eta(x) V_S^1(x), \forall x \in X ,$$

where

$$\mu(x) = \frac{d(x, clU_2)}{d(x, clU_1) + d(x, clU_2)}, \eta(x) = 1 - \mu(x) .$$

It is easy to verify that $V' \in M$ and $\sigma(V', V) < \delta$. This implies that $F(V') \bigcap (U_1 \bigcup U_2) \neq \varnothing$. Thus, there exist an $x^* \in U_1$ such that $\Phi'(x^*) = \Phi^1(x^*)$ if $F(V') \bigcap U_1 \neq \varnothing$, then $x^* \in F(V^1) \bigcap U_1$, which is a contradiction. Similarly, we can also get a contradiction if $F(V') \bigcap U_2 \neq \varnothing$. This completes the proof.

Theorem 7. *For each* $V \in M$, *there exists at least one essential component of* $F(V)$.

Proof. By Theorem 5, Theorem 6, and Remark 3, it immediately follows that this theorem holds. This completes the proof.

Theorem 8. *If* $V \in M$ *is such that* $F(V)$ *is a totally disconnected set, then the biform game*

V is weakly essential.

Proof. Since $F(V)$ is a totally disconnected set, the $F(V)=\bigcup_{\alpha\in\Lambda} C_\alpha(V)$ with $C_\alpha(V)$ is a singleton set for every $\alpha\in\Lambda$. By Theorem 7, there exists an essential component $C_{\alpha_0}(V)=\{x_0\}$ of $F(V)$ for an $\alpha_0\in\Lambda$. It is easy to see that x_0 is essential. Thus, V is weakly essential. This completes the proof.

Remark 4. *For any $V\in M$, if the biform game V is additive on X, then, Remark 1 shows that V becomes an n-person normal form noncooperative game. Therefore, the biform game V and this n-person normal form noncooperative game have no difference in their Nash equilibrium set and essential stability.*

6. Conclusions

In this paper, we build a biform form game model using infinite strategies (of which mixed and scalar strategies are special forms) and show how it relates to the biform game model with finite and scalar strategies and the noncooperative game model. Few articles have investigated the essential stability of Nash equilibria for biform games. In this paper, the research based on the SAF is used to fill this gap, and the existence and generic stability of Nash equilibria for biform games are obtained. Further, we give a connected property of the Nash equilibrium set. In addition, we present a link between the stabilities of biform games and noncooperative games.

As a further study, we will consider the extension of infinite strategies to differentiable functions with respect to time, in order to introduce the method of biform analysis to nonlinear differential games with finite and infinite time domains (e.g., [34,35]), or to practical applications of linear quadratic differential games (e.g., [36]) and stochastic differential games (e.g., [37,38]). The stabilities of these problems under biform analysis will also be considered.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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