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Research article

Existence of solutions to a class of damped random impulsive differential equations under Dirichlet boundary value conditions

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Abstract: In this paper, we study sufficient conditions for the existence of solutions to a class of damped random impulsive differential equations under Dirichlet boundary value conditions. By using variational method we first obtain the corresponding energy functional. Then the existence of critical points are obtained by using Mountain pass lemma and Minimax principle. Finally we assert the critical point of enery functional is the mild solution of damped random impulsive differential equations.

Keywords: damped random impulsive differential equations; mild solution; mountain pass lemma; minimax principle; theoery of critical point; energy functional
Mathematics Subject Classification: 34B37, 37H10, 37J51

1. Introduction

Impulsive differential equations can be used to describe a class of discontinuous dynamic systems. The impulse is a discrete jump that occurs in many evolutionary processes. Since impulses widely exist in finance, population dynamics, optimal control model [7,8,38,39], mechanics problems [1,17,23] and chaos theory, it is of practical significance to study differential equations with pulses. However in some practical situations, such as mechanics, pulses occur randomly and duration of impulses is negligible in comparison with the entire phenomenon, that is, random impulse points are random variables. Owing to the characteristic of random impulse, we can tell the solution of the random impulse differential equation is a random process, which is different from the corresponding fixed impulse differential equation, whose solutions is a piecewise continuous function. Quite a few scholars have studied the fixed impulse differential equation [3, 4, 9, 19, 20, 26, 29, 33, 34], while the random pulse differential equation has not been involved by many people [14, 21, 30–32, 38, 39]. So we're going to expand our work from random impulsive differential equations(RIDE).

Many scholars pay attention to the existence, uniqueness and stability of solutions. Most of results of the existence of solutions obtained using various fixed point theory [4, 10, 15, 16, 21, 27]. For example, by using Schaeffer′s theorem, Li and Nieto have proved new existence theorems for a nonlinear periodic boundary value problem of first-order differential equations with impulses in the literature [15]:

$$\begin{cases} u'(t) + \lambda u(t) = f(t, u(t)), t \in J, t \neq t_k, k = 1, \ldots p. \\ u(t_k^+) = u(t_k^-) + I_k(u(t_k)), k = 1, \ldots p. \\ u(0) = u(T) = 0 \end{cases}$$

where $\lambda \in R$ and $\lambda \neq 0$, $J = [0, T], T > 0, 0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T, I_k \in C(R, R), k = 1, \ldots, p$, and $f : J \times R \to R$ is continuous at every point $(t, u) \in J_0 \times R, J_0 = J - \{t_1, \ldots, t_p\}, f(t_k^+, u)$ and $f(t_k^-, u)$ exist, $f(t_k^-, u) = f(t_k^+, u)$. And Niu et al. [21] have investigated the existence and Hyers-Ulam stability of solution for second order random impulsive differential equations by fixed point:

$$\begin{cases} x''(t) = f(t, x(t)), t \in J, t \neq \xi_k, \\ x(\xi_k) = b_k(\tau_k)x(\xi_k^-), k = 1, 2, \ldots, \\ x(0) = x_0, x'(0) = x_1. \end{cases}$$

where τ_k is random variable. Upper and lower solution method also has be used to study impulsive differential equations [13, 28, 33]. In the literature [13], Lee and Liu have established criteria of the existence of extremal solutions by using the method of upper and lower solutions and the monotone iterative:

$$\begin{cases} u''(t) + f(t, u(t)) = 0, t \in (0, 1), t \neq t_1 \\ \triangle u|_{t=t_1} = I(u(t_1)) \\ \triangle u'|_{t=t_1} = N(u(t_1), u'(t_1)) \\ u(0) = a, u(1) = b. \end{cases}$$

where $a, b \in R, \triangle u|_{t=t_1} = u(t_1^+) - u(t_1), \triangle u'|_{t=t_1} = u'(t_1^+) - u'(t_1^-)$ and $f : D \subset (0, 1) \times R \to R, I : R \to R, N : R \times R \to R$ are continuous. And Li et al [36] have studied the boundary value problem of second order random impulsive differential equation using upper and lower solution method:

$$\begin{cases} -x''(t) = f(t, x(t), x'(t)), t \in J' \\ x(\xi_k^+) = b_k(\tau_k)x(\xi_k^-), k = 1, 2, \ldots, \\ \alpha_0 x(0) - \alpha_1 x'(0) = x_0, \\ \beta_0 x(1) + \beta_1 x'(1) = x_0^*. \end{cases}$$

where $f : J \times R \times R \to R$ is a continuous mapping. $x(t)$ is a stochastic process taking values in the Euclidean space$(R, \|\cdot\|)$. And τ_k is random variable defined from Ω to $E_k := (0, d_k)$, with $0 < d_k < 1$ for every $k \in N^+$.

As we all known, solving a RIDE consists not only in obtaining its solution, which is a stochastic process, but also its main probabilistic properties [39]. Besides, some solutions are explicitly obtained by using the method of statistical analysis via the first probability density function. In the paper [38], Juan C. Cortés et al had studied a randomized version of the following exponential growing/decaying model, which is controlled/pumped by an infinite sequence of instantaneous impulses modeled by the Dirac delta function, $\delta(\cdot)$, at the time instants $T_i > 0$,$i = 1, 2, 3 \ldots$,

$$\begin{cases} \frac{dx(t)}{dt} = \alpha x(t) - \gamma \sum_{i=1}^{\infty} \delta(t - T_i)x(t), t > 0, \\ x(0) = x_0, \end{cases}$$

where x_0 denotes the initial condition and $\alpha, \gamma \in \mathbf{R}$.

And the aim of Juan C. Cortés et al in [39] is to advance in the realm of RDEs whose right-hand side is discontinuous without restricting the probability distributions, they had tackled the study of non-homogeneous linear RDEs of exponential growth/decay controlled by an infinite sequence of square pulses of time duration, τ,

$$\dot{x}(t) = \alpha x(t) + \beta - \gamma x(t) \sum_{n=1}^{\infty} (H(t - (nT - \tau)) - H(t - nT)), x(0) = x_0.$$

where H(t) is the Heaviside function:

$$H(t) = \begin{cases} 1, & \text{if t} < 0 \ , \\ 0, & \text{if t} \geq 0 \ . \end{cases}$$

In recent years, variational method has been used by many scholars to study the solutions of differential equations. In fact, it's very difficult to get a strong solution to a differential equation, and the general approach is to turn the differential equation into an integral equation, and then to get the corresponding energy functional. In this way we can use variational method and critical point theory to study differential equations. Many scholars have done a lot of works in differential equations by means of variational methods and critical point theory such as [6, 11, 12, 18, 22, 24, 25], for the case of differential equations with fixed impulses see [3, 19, 20, 29, 34, 35]. Inspired by [19], we obtain the variational structure of damped ordinary differential equations:

$$-u''(t) + g(t)u'(t) + \lambda u(t) = f(t, u(t)).$$

We usually considers the position u and the velocity $\dot{u}$. In the motion of spacecraft one has to consider instantaneous impulses depending on the position that results in jump discontinuties in velocity, but with no change in position [1, 2, 17, 23]. Therefore, it is reasonable to supplement such an impulse condition: $u'(\xi_j^+) - u'(\xi_j^-) = b_j(\tau_j)u(\xi_j)$, where τ_j is a random variable.

Hence, we consider the existence of solutions to the damped random pulse Dirichlet boundary value problem:

$$\begin{cases} -u''(t) + g(t)u'(t) + \lambda u(t) = f(t, u(t)), & t \in [0, T] \backslash \{\xi_1, \xi_2, \cdots\}, \\ \Delta \dot{u}(\xi_j) = \dot{u}(\xi_j^+) - \dot{u}(\xi_j^-) = b_j(\tau_j)u(\xi_j), & \xi_j \in (0, T), \ j = 1, 2, \cdots , \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where $f : [0, T] \times R \to R$ is continuous; $\tau_j : \Omega \to F_j$, where $F_j := (0, d_j)$ is a random variable, $0 < d_j < +\infty$, and τ_i, τ_j are mutually independent when $i \neq j$, $i, j = 1, 2, \cdots$; $b_j : F_j \to R$, $\forall j = 1, 2, \cdots$. Set $\xi_{j+1} = \xi_j + \tau_j$. $\{\xi_j\}$ is a sequence of strictly increasing random variable and also a random process defined on Ω, i.e. $0 < \xi_1 < \xi_2 < \cdots < \xi_k < \cdots < \xi_\infty = \sup_{k \in N^+} \{\xi_k\} \leq T$. Define $\dot{u}(\xi_j^+) = \lim_{t \to \xi_j^+} \dot{u}(t), \dot{u}(\xi_j^-) = \lim_{t \to \xi_j^-} \dot{u}(t)$ in sense of sample orbit. The above definitions are reasonable because that $\{\xi_k\}$ can be regard as a sequence of fixed points under the realization of each sample orbit. We suppose $\{N(t) : t \geq 0\}$ be a simple counting process generated by $\{\xi_k\}$, that is, $\{N(t) \geq n\} = \{\xi_n \leq t\}$, and denote ψ_t the σ-algebra generated by $\{N(t), t \geq 0\}$.

2. Preliminaries

Let (Ω, ψ, P) be a probability space. Let $L^q([0,T] \times \Omega, R)$ be the collection of all strongly measurable, qth-integrable, ψ_t-measurable R-valued random process: $u : [0,T]\times\Omega \to R$ with the norm $\|u\|_q = \Big(\int_0^T E|x|^q\Big)^{\frac{1}{q}}$, where $Eu = \int_\Omega udP$. Next, define the Banach space $S = S\Big([0,T], L^2(\Omega, R)\Big) := \Big\{u(t) : u(t) = u(t,\omega)$ is random process, $u(t,\cdot) \in L^2(\Omega, R), u(\cdot,\omega)$ is continuous and diffrentiable on $[0,T]\backslash\{\xi_1, \xi_2, \cdots\}$ and continuous on $[0,T]$, $\dot{u}(\xi_j^+), \dot{u}(\xi_j^-)$ exist, $j = 1, 2, \cdots; u(0) = u(T) = 0\Big\}$, with the norm $\|u\|_S = \Big(\int_0^T E|u|^2\Big)^{\frac{1}{2}} + \Big(\int_0^T E|\dot{u}|^2\Big)^{\frac{1}{2}}$. For convenience, we denote $L^q([0,T]\times\Omega, R) := L^q([0,T]\times\Omega)$.

Lemma 2.1 (The norm inequality)**.** *Denote* $\|u\|_{1,2} = \Big(\int_0^T E|\dot{u}|^2\Big)^{\frac{1}{2}}$, $\|u\|_\infty = \sup\limits_{t\in[0,T]}\Big(E|u|\Big)$. *Then the following inequalities hold:*

$$\exists\, C_1 > 0 \text{ satisfies } \|u\|_S \le C_1\|u\|_{1,2}; \tag{2.1}$$

$$\|u\|_S \ge \|u\|_{1,2}; \tag{2.2}$$

$$\exists\, C_2 > 0 \text{ satisfies } \|u\|_\infty \le C_2\|u\|_s. \tag{2.3}$$

Proof. For (2.1), by Poincáre inequality, i.e. $\exists\, C_0 > 0$, satisfying $\Big(\int_0^T |u|^2\Big)^{\frac{1}{2}} \le C_0\Big(\int_0^T |\dot{u}|^2\Big)^{\frac{1}{2}}$,

$$\Rightarrow E\Big(\int_0^T |u|^2\Big) \le C_0^2 E\Big(\int_0^T |\dot{u}|^2\Big),$$

$$\Rightarrow \Big(\int_0^T E|u|^2\Big)^{\frac{1}{2}} \le C_0\Big(\int_0^T E|\dot{u}|^2\Big)^{\frac{1}{2}}.$$

Let $C_1 = C_0 + 1 > 0$. From the definition of $\|\cdot\|_S$:

$$\begin{aligned}\|u\|_S &= \Big(\int_0^T E|u|^2\Big)^{\frac{1}{2}} + \Big(\int_0^T E|\dot{u}|^2\Big)^{\frac{1}{2}}\\ &\le (C_0+1)\Big(\int_0^T E|\dot{u}|^2\Big)^{\frac{1}{2}}\\ &= C_1\|u\|_{1,2}.\end{aligned}$$

For (2.2), it is easy to get the result by the definition of $\|\cdot\|_S$.

For (2.3), under the meaning of one given sample orbit, since $|u| \in C[0,T]$, then $E|u| \in C[0,T]$. There exists a $\theta \in [0,T]$, s.t. $\frac{1}{T}\int_0^T |u(s)|ds = |u(\theta)|$. Hence for arbitrary $t \in [0,T]$,

$$|u(t)| \le \Big|u(\theta) + \int_\theta^t E\dot{u}(s)ds\Big| \le |u(\theta)| + \int_0^T |\dot{u}(s)|ds = \frac{1}{T}\int_0^T |u|ds + \int_0^T |\dot{u}|ds.$$

$$\Rightarrow E|u(t)| \le \frac{1}{T}\int_0^T E|u|ds + \int_0^T E|\dot{u}|ds.$$

Then by Cauchy-Schwarz inequality,

$$E|u(t)| \leq \frac{1}{\sqrt{T}}\Big(\int_0^T E|u|^2 ds\Big)^{\frac{1}{2}} + \sqrt{T}\Big(\int_0^T E|\dot{u}|^2 ds\Big)^{\frac{1}{2}}. \tag{2.4}$$

On one hand,

$$(2.4) = \sqrt{T}\Big(\frac{1}{T}\Big(\int_0^T E|u|^2 ds\Big)^{\frac{1}{2}} + \Big(\int_0^T E|\dot{u}|^2 ds\Big)^{\frac{1}{2}}\Big) \leq \sqrt{T}\|u\|_S,\ T \geq 1;$$

and on the other hand,

$$(2.4) = \frac{1}{\sqrt{T}}\Big(\Big(\int_0^T E|u|^2 ds\Big)^{\frac{1}{2}} + T\Big(\int_0^T E|\dot{u}|^2 ds\Big)^{\frac{1}{2}}\Big) \leq \frac{1}{\sqrt{T}}\|u\|_S,\ T \leq 1.$$

So, $E|u(t)| \leq (\sqrt{T} + \frac{1}{\sqrt{T}})\|u\|_S := C_2\|u\|_S$.

$$\Rightarrow \sup_{t\in[0,T]} E|u(t)| \leq C_2\|u\|_S,\ \text{i.e. } \|u\|_\infty \leq C_2\|u\|_S.$$

□

Lemma 2.2 (Embedding theorem [37]). *$[0,T]$ is a bounded interval, for each $q > 0$, $W_0^{1,2}([0,T]) \hookrightarrow L^p([0,T])$, then there exists an constant $C = C(q) > 0$, such that $\Big(\int_0^T |u|^q\Big)^{\frac{1}{q}} \leq C\Big(\int_0^T |\dot{u}|^2\Big)^{\frac{1}{2}}$, for all $u \in W_0^{1,2}([0,T])$ hold.*

Remark 2.3. Under the sense of Lebesgue-integration, $W_0^{1,2}([0,T])$ is different from the S space only in zero measure set. Then, there exists $C = C(q) > 0$, s.t. $\Big(\int_0^T |u|^q\Big)^{\frac{1}{q}} \leq C\Big(\int_0^T |\dot{u}|^2\Big)^{\frac{1}{2}}$, $\forall u \in S$.

Remark 2.4. An embedding operator defined by $\mathbb{A} : W_0^{1,2}([0,T]) \to L^q([0,T]), \forall q > 0$, is a continuous compact operator. Then by Remark 2.3 and completeness of S space, we know $\mathbb{A}|_S : S \to L^q([0,T]), \forall q > 0$ is a continuous compact operator.

Lemma 2.5. *([37]) Let φ is a function in Banach space E, $u \in E$. If φ has linear bounded Gâteaux differential in a neighborhood and its Gâteaux derivatives $D\varphi(u)$ is continuous at u, then φ is Fréchet differentiable at u, and $D\varphi(u) = \varphi'(u)$.*

Remark 2.6 (Gâteaux derivatives and differential [37]). For arbitrary $u, v \in E$, we call φ is differentiable in u if $\lim_{x\to 0} \frac{\varphi(u+xv)-\varphi(u)}{x}$ exists and denote the value by $D\varphi(u, v)$. If there is a linear bounded function $B \in E^*$, satisfying $D\varphi(u, v) =< B, v >$, we denote by $B = D\varphi(u)$ the Gâteaux derivatives of φ at u.

Lemma 2.7. *([37]) If function $f(t,u)$ satisfies Carathéodory condition on $[0,T] \times R$, i.e.*

1. *For almost every $t \in [0,T]$, $f(t,u)$ is continuous in u;*
2. *For each given $u \in R$, $f(t,u)$ is measurable in t.*

And for each $(t,u) \in [0,T] \times R$, we set $|f(t,u)| \leq a + b|u|^r$, where $a, b > 0, r > 0$. Then operator $\mathbb{A} : u(t) \to f(t, u(t))$ is a bounded continuous operator mapping from $L^{r+1}([0,T])$ to $L^{\frac{r+1}{r}}([0,T])$.

Theorem 2.8 (Minimax principle [18]). *E is a Banach space and φ is Fréchet differential in E, $\varphi \in C^1(E, R)$. If φ has a lower bound in E and satisfies P.-S. condition. Then φ exists a critical point, $c = \inf_{x \in E} \varphi(x)$ is a critical value of φ.*

Remark 2.9 (P.-S. condition [18]). Suppose $\varphi \in C^1(E, R)$. If $\{\varphi(u_k)\}$ is bounded and $\varphi'(u_k) \to 0$ in E^* as $k \to \infty$, which can be deduced that each $\{u_k\}$ is sequentially compact set in E. Then we call φ satisfies P.-S. condition.

Theorem 2.10 (Mountain pass lemma [18]). *E is a Banach space, $\varphi \in C^1(E, R)$, if φ satisfies*

1. *$\varphi(0) = 0$, $\exists\, \rho > 0$, s.t. $\varphi_{\partial B_\rho(0)} \geq \alpha > 0$;*
2. *$\exists\, e \in E \backslash \overline{B_\rho(0)}$, s.t. $\varphi(e) \leq 0$;*
3. *the P.-S. condition is fulfilled,*

then, φ exists a critical point u satisfying $\varphi'(u) = 0$ and $\varphi(u) > \max\{\varphi(0), \varphi(e)\}$.

Remark 2.11. In the equation (1.1), we set $g(t)$ be a Riemannian integrable function, then $G(t), e^{G(t)}$ is continuous on $[0, T]$, where $G(t) = -\int_0^t g(s)ds$. By the boundedness of continuous functions in a closed interval, it is easy to see, there exist constants μ_1, μ_2 which are only associated with g, satisfying $0 < \mu_1 \leq e^{G(t)} \leq \mu_2$.

Now, we present some important conclusions that will be used in the next section.

Result 1: Define the function $\forall u \in S$

$$\begin{aligned}\varphi(u) = E\Big[&\frac{1}{2}\int_0^T e^G|\dot{u}|^2 dt + \frac{\lambda}{2}\int_0^T e^G|u|^2 dt \\ &+ \sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k}\Big(\frac{e^{G(\xi_j)}b_j(\tau_j)u^2(\xi_j)}{2}\Big)I_A(\{\xi_j\}_{j=1}^k)\Big) - \int_0^T F(t,u)dt\Big],\end{aligned} \tag{2.5}$$

where $G(t) = -\int_0^t g(s)ds, F(t, u) = \int_0^u f(t, s)e^{G(t)}ds$.

$$I_A(x) = \begin{cases} 1, & \text{if x} \in A\ , \\ 0, & \text{if x} \notin A\ , \end{cases}$$

here A represent the set consisting of all sample orbits, and $\{\xi_j\}_{j=1}^k$ is a sample orbit.

We can prove that $\varphi(u) \in C^1(S, R)$ and for every $u, v \in S$,

$$\begin{aligned}(\varphi'(u), v) = E\Big[&\int_0^T e^G \dot{u}\dot{v} dt + \lambda\int_0^T e^G uv dt \\ &+ \sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k}\Big(e^{G(\xi_j)}b_j(\tau_j)u(\xi_j)v(\xi_j)\Big)I_A(\{\xi_j\}_{j=1}^k)\Big) - \int_0^T e^G f(t,u)v dt\Big].\end{aligned} \tag{2.6}$$

Detailed proof of these will be given in Section 3.

Result 2: The mild solution u of the random impulsive differential equation (1.1) is a critical point of $\varphi(u)$. That is to say $(\varphi'(u), v) = 0, \forall v \in S$. Conversely, if $u \in S$, u is a critical point of $\varphi(u)$, then u is a mild solution of the equation (1.1).

Proof. Suppose $0 < t_1 < t_2 < \cdots < t_k < T$, where $t_1, t_2, \cdots, t_k$ is a sample orbit. $\{t_i\}_{i=1}^k \in A$. Let $u \in S$ is a mild solution of (1.1). If $u \in C^2(J') \cap S$, then u is the solution of (1.1), here $J' := [0, T] \backslash \{t_1, t_2, \cdots, t_k\}$. u satisfies

$$-u''(t) + g(t)u'(t) + \lambda u(t) = f(t, u(t)).$$

We multiply both sides of above equation by $e^{G(t)}$ and $v \in S$, where $G(t) = -\int_0^t g(s)ds$. Then we get

$$-(e^G u')'v + \lambda e^G uv = e^G f(t, u)v.$$

After integration on $[0, T]$, we have

$$-\int_0^T (e^G u')'vdt + \lambda \int_0^T e^G uvdt = \int_0^T e^G f(t, u)vdt,\ u \in C^2(J') \cap S, \tag{2.7}$$

where

$$\begin{aligned}
&\int_0^T (e^G u')'vdt \\
&= \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} (e^G u')'vdt + \int_0^{t_1} (e^G u')'vdt + \int_{t_k}^T (e^G u')'vdt \\
&= \sum_{i=1}^{k-1} \left[e^G u'v\Big|_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} e^G u'v'dt \right] + e^G u'v\Big|_0^{t_1} \\
&\quad - \int_0^{t_1} e^G u'v'dt + e^G u'v\Big|_{t_k}^T - \int_{t_k}^T e^G u'v'dt \\
&= -\sum_{j=1}^k e^{G(t_j)}(\Delta u'(t_j))v(t_j) - \int_0^T e^G u'v'dt.
\end{aligned} \tag{2.8}$$

Then putting (2.8) into (2.7) and according to the impulsive condition in (1.1), we obtain that

$$\begin{aligned}
&(\int_0^T e^G u'v') + (\lambda \int_0^T e^G uv) + \sum_{j=1}^k e^{G(t_j)} b_j(\tau_j)u(t_j)v(t_j) \\
&\qquad - \int_0^T e^G f(t, u)vdt = 0,\ u \in S,\ \forall \{t_j\}_{j=1}^k \in A.
\end{aligned} \tag{2.9}$$

From (2.6) we know $(\varphi'(u), v) = 0$, i.e. u is a critical point of $\varphi(u)$.

Conversely, suppose $u \in S$ is critical point of φ, i.e. $(\varphi'(u), v) = 0$. $\forall v \in S$,

$$\begin{aligned}
&\int_0^T e^G u'v'dt + \lambda \int_0^T e^G uvdt \\
&\qquad + \sum_{j=1}^k e^{G(t_j)} b_j(\tau_j)u(t_j)v(t_j) - \int_0^T e^G f(t, u)vdt = 0,\ v \in S.
\end{aligned} \tag{2.10}$$

Since $v \in S$, we know $v(t_j^+) = v(t_j^-)$, $j = 1, 2, 3, \cdots$. and $v(0) = v(T) = 0$. When $u \in S \cap C^2(J')$, we will prove u is the solution of (1.1):

$$\sum_{j=1}^{k-1} \int_{t_j}^{t_{j+1}} e^G u'v'dt + \int_0^{t_1} e^G u'v'dt + \int_{t_k}^{T} e^G u'v'dt + \lambda \int_0^T e^G uvdt$$
$$+ \sum_{j=1}^{k} e^{G(t_j)} b_j(\tau_j) u(t_j) v(t_j) - \int_0^T e^G f(t,u) vdt = 0$$

For the convenience, let $t_0 = 0, t_{k+1} = T$ and $v(t_0) = v(t_{k+1}) = 0$,

$$\sum_{j=0}^{k} [e^G u'v\Big|_{t_j}^{t_{j+1}} - \int_{t_j}^{t_{j+1}} (e^G u')'vdt]$$
$$+ \sum_{j=1}^{k} e^{G(t_j)} b_j(\tau_j) u(t_j) v(t_j) + \int_0^T e^G v[\lambda u - f(t,u)]dt = 0$$

$$\Rightarrow -\sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (e^G u')'vdt + \int_0^T e^G v[\lambda u - f(t,u)]dt$$
$$+ \sum_{j=1}^{k} e^{G(t_j)} v(t_j)[b_j(\tau_j) u(t_j) - (\Delta u'(t_j))] = 0$$

$$\Rightarrow \int_0^T e^G v[-u'' + g(t)u' + \lambda u - f(t,u)]dt$$
$$+ \sum_{j=1}^{k} e^{G(t_j)} v(t_j)[b_j(\tau_j) u(t_j) - (\Delta u'(t_j))] = 0 \tag{2.11}$$

Set $a_j := e^{G(t_j)} v(t_j)[b_j(\tau_j) u(t_j) - \Delta u'(t_j)]$, $j = 1, 2, \cdots, k$. And:

$$\delta_j(t) = \begin{cases} 1, & \text{if t} = t_j \ , \\ 0, & \text{if t} \neq t_j \ , \end{cases}$$

Then (2.11) can be written as:

$$\int_0^T \Big\{ e^{G(t)} v(t)[-u''(t) + g(t)u'(t) + \lambda u(t) - f(t,u)] + \sum_{j=1}^{k} a_j \delta_j(t) \Big\} dt = 0 \tag{2.12}$$

then we know $u \in S$ is the mild solution of the equation

$$-u''(t) + g(t)u'(t) + \lambda u(t) = f(t, u(t)), \ t \in J'$$

and (2.12) imply that the random impulsive condition: $\Delta u'(t_j) = b_j(\tau_j) u(t_j)$, $j = 1, 2, \cdots, k$ hold. Thus, u is a mild solution of (1.1), $u \in S$. □

3. Results

Theorem 3.1. *When $f(t,u), b_j(\tau_j)$ satisfy the following assumptions respectively:*

(H1) $\forall (t,u) \in [0,T] \times R$, $|f(t,u)| \leq a + b|u|^r$ holds, where $a, b > 0, r > 0$.

(H2) Let $B = E\Big(\sum_{k=1}^{\infty} \Big(\sum_{j=1}^{k} |b_j(\tau_j)|\Big) I_A(\{\xi_j\}_{j=1}^{k})\Big) < +\infty$.

Then the $\varphi(u)$ defined in (2.5) *fulfills $\varphi \in C^1(S,R)$ and satisfies* (2.6).

Proof. We divide the proof into several parts.

1. Let $J_1(u) = \frac{1}{2}\int_0^T e^G(E|\dot{u}|^2)dt$. we will prove that $J_1(u) \in C^1(S,R)$.

For arbitrary $u, v \in S$, we have

$$J_1(u+v) = \frac{1}{2}\int_0^T e^G E|\dot{u}|^2 dt + \frac{1}{2}\int_0^T e^G E|\dot{v}|^2 dt + \int_0^T e^G (E\dot{u})(E\dot{v})dt.$$

Since $0 \leq \frac{1}{2}\int_0^T e^G E|\dot{v}|^2 dt \leq \frac{\mu_2}{2}\|v\|_S^2 \Rightarrow \lim_{\|v\|_S \to 0} \frac{\frac{1}{2}\int_0^T e^G E|\dot{v}|^2 dt}{\|v\|_S} = 0$. It follows that

$$(J_1'(u), v) = \int_0^T e^G (E\dot{u})(E\dot{v})dt.$$

For fixed u, $J_1'(u)$ is a linear functional with respect to v. By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\Big|\int_0^T e^G (E\dot{u})(E\dot{v})dt\Big| &\leq \Big(\int_0^T |e^G E\dot{u}|^2 dt\Big)^{\frac{1}{2}} \Big(\int_0^T |E\dot{v}|^2 dt\Big)^{\frac{1}{2}} \\
&\leq \Big(\int_0^T |e^G E\dot{u}|^2 dt\Big)^{\frac{1}{2}} \Big(\int_0^T E|\dot{v}|^2 dt\Big)^{\frac{1}{2}} \\
&\leq \Big(\int_0^T |e^G E\dot{u}|^2 dt\Big)^{\frac{1}{2}} \|v\|_S,
\end{aligned}$$

where $(\int_0^T |e^G E\dot{u}|^2 dt\Big)^{\frac{1}{2}}$ is independent of v, therefore $J_1'(u)$ is a bounded functional in S.

2. Let $\widetilde{J_2}(u) = \int_0^T e^G E|u|^2 dt$, $J_2(u) = \frac{\lambda}{2}\int_0^T e^G E|u|^2 dt$. We will prove that $J_2(u), \widetilde{J_2}(u) \in C^1(S,R)$. $\forall u, v \in S$,

$$\widetilde{J_2}(u+v) = \int_0^T e^G E|u|^2 dt + \int_0^T e^G E|v|^2 dt + 2\int_0^T e^G EuEvdt.$$

Since $0 \leq \int_0^T e^G E|v|^2 dt \leq \mu_2 \|v\|_S^2, \Rightarrow \lim_{\|v\|_S \to 0} \frac{\int_0^T e^G E|v|^2 dt}{\|v\|_S} = 0$, then

$$(\widetilde{J'}_2(u), v) = 2\int_0^T e^G EuEvdt \Rightarrow (J_2'(u), v) = \lambda \int_0^T e^G EuEvdt.$$

When u fixed, $J_2'(u)$ is linear functional w.r.t. v.

$$\begin{aligned}
|(J_2'(u), v)| = |\lambda \int_0^T e^G EuEvdt| &\leq |\lambda|\Big(\int_0^T |e^G Eu|^2 dt\Big)^{\frac{1}{2}} \Big(\int_0^T |Ev|^2 dt\Big)^{\frac{1}{2}} \\
&\leq |\lambda|\Big(\int_0^T |e^G Eu|^2 dt\Big)^{\frac{1}{2}} \Big(\int_0^T E|v|^2 dt\Big)^{\frac{1}{2}}
\end{aligned}$$

$$\leq |\lambda|\Big(\int_0^T |e^G Eu|^2 dt\Big)^{\frac{1}{2}} \|v\|_S,$$

where $|\lambda|\Big(\int_0^T |e^G Eu|^2 dt\Big)^{\frac{1}{2}}$ is independent of v. Then, $J_2'(u)$ is a bounded functional in S.

3. Let $J_3(u) = \sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k} E\Big(\frac{e^{G(\xi_j)}b_j(\tau_j)u^2(\xi_j)}{2}\Big)I_A(\{\xi_j\}_{j=1}^k)\Big)$. We will prove that $J_3(u) \in C^1(S, R)$. $\forall u, v \in S$,

$$\begin{aligned}
|J_3(u+v) - J_3(u) - \sum_{k=1}^{\infty}\sum_{j=1}^{k} E\Big(e^{G(\xi_j)}b_j(\tau_j)u(\xi_j)v(\xi_j)\Big)I_A(\{\xi_j\}_{j=1}^k)|
&= |\sum_{k=1}^{\infty}\sum_{j=1}^{k} E\Big(e^{G(\xi_j)}b_j(\tau_j)\frac{v^2(\xi_j)}{2}\Big)I_A(\{\xi_j\}_{j=1}^k)| \\
&\leq \frac{\mu_2 B}{2}\|v\|_\infty^2 \leq \frac{\mu_2 B C_2^2}{2}\|v\|_S^2
\end{aligned}$$

$$\Rightarrow (J_3'(u), v) = \sum_{k=1}^{\infty}\sum_{j=1}^{k} E\Big(e^{G(\xi_j)}b_j(\tau_j)u(\xi_j)v(\xi_j)\Big)I_A(\{\xi_j\}_{j=1}^k).$$

When u is fixed, $J_3'(u)$ is a linear functional with respect to v. And

$$|(J_3'(u), v)| \leq \mu_2 B\|u\|_\infty\|v\|_\infty \leq \mu_2 B C_2\|u\|_\infty\|v\|_S,$$

which implies that $J_3'(u)$ is a bounded functional in S.

4. Let $J_4(u) = \int_0^T E(F(t,u))dt$, where $F(t,u) = \int_0^u f(t,s)e^{G(t)}ds$. Now, we will show that $J_4(u) \in C^1(S, R)$, $(J_4'(u), v) = \int_0^T e^G f(t,u)v dt, v \in S$ and $J_4'(u) : S \to S^*$ is continuous compact operator by several steps:

Step (1). We first prove that J_4 has Gâteaux differential in a neighborhood of u.

$$DJ_4(u,v) = \lim_{x\to 0}\frac{J_4(u+xv) - J_4(u)}{x} = \lim_{x\to 0}\int_0^T E\Big(\frac{1}{x}[F(t,u+xv) - F(t,u)]\Big)dt$$

By mean value theorem, there exist a $\theta \in (0,1)$, where $\theta = \theta(u,v,t,x)$. Therefore,

$$DJ_4(u,v) = \lim_{x\to 0}\int_0^T E\Big(e^G f(t,u+\theta xv)v\Big)dt. \tag{3.1}$$

We may assume that $|x| < 1$, and by Young inequality, we have

$$\begin{aligned}
E\Big|e^G f(t,u+\theta xv)v\Big| &\leq E\Big|\mu_2(a + b|u+\theta xv|^r)|v|\Big| \\
&\leq E\Big|\mu_2\frac{r}{r+1}(a + b|u+\theta xv|^r)^{\frac{r+1}{r}} + \mu_2\frac{1}{r+1}|v|^{r+1}\Big| \\
&\leq E\Big|\mu_2\frac{r}{r+1}(a + b\cdot 2^r(|u|^r + |\theta x|^r|v|^r))^{\frac{r+1}{r}} + \mu_2\frac{1}{r+1}|v|^{r+1}\Big| \\
&\leq E\Big|\mu_2\frac{r}{r+1}\Big(a + b\cdot 2^r(|u|^r + |v|^r)\Big)^{\frac{r+1}{r}} + \mu_2\frac{1}{r+1}|v|^{r+1}\Big|,
\end{aligned}$$

which is a constant independent on x. Then put it into (3.1), and apply the Control convergence theorem, we obtain

$$\begin{aligned}(3.1) &= \int_0^T E\Big(\lim_{x\to 0} e^G f(t, u+\theta xv)v\Big)dt\\ &= \int_0^T E\Big(e^G f(t,u)v\Big)dt\\ &= \int_0^T e^G (Ef(t,u))(Ev),\ u, v \in S.\end{aligned}$$

This implies that its limit exists.

Step (2). we will prove $DJ_4(u, v)$ is a linear bounded functional with respect v when u is fixed.

Obviously, $DJ_4(u, v) = \int_0^T e^G (Ef(t,u))(Ev)dt$ is linear about v.

$$\begin{aligned}|DJ_4(u,v)| &= |\int_0^T e^G (Ef(t,u))(Ev)dt|\\ &\le \mu_2\Big(\int_0^T |Ef(t,u)|^2\Big)^{\frac{1}{2}}\Big(\int_0^T |Ev|^2 dt\Big)^{\frac{1}{2}}\\ &\le \mu_2\Big(\int_0^T |Ef(t,u)|^2\Big)^{\frac{1}{2}} \|v\|_S.\end{aligned}$$

Hence, $DJ_4(u, v)$ is bounded about v. Then $< DJ_4(u), v >= DJ_4(u, v),\ \forall v \in S$.

Step (3). Now, we only need to prove that $DJ_4(u) : S \to S^*$ is continuous in u and is a compact operator.

i) $\forall v, u, \phi \in S$, by Hölder inequality and Embedding theorem we have

$$\begin{aligned}| < DJ_4(u) - DJ_4(v), \phi > | &= \int_0^T e^G E|f(t,u) - f(t,v)|E|\phi|dt\\ &\le \mu_2\Big(\int_0^T E|f(t,u) - f(t,v)|^{\frac{r+1}{r}}\Big)^{\frac{r}{r+1}}\Big(\int_0^T (E|\phi|)^{r+1}dt\Big)^{\frac{1}{r+1}}\\ &\le \mu_2 C(r+1)\Big(\int_0^T E|f(t,u) - f(t,v)|^{\frac{r+1}{r}}\Big)^{\frac{r}{r+1}}\Big(\int_0^T E|\dot{\phi}|^{r+1}dt\Big)^{\frac{1}{r+1}}\\ &\le \mu_2 C(r+1)\|f(t,u) - f(t,v)\|_{\frac{r+1}{r}}\|\phi\|_S.\end{aligned}$$

Thus,

$$\|DJ_4(u) - DJ_4(v)\|_{S^*} := \sup_{\phi\in S, \|\phi\|_S\neq 0} \frac{| < DJ_4(u) - DJ_4(v), \phi > |}{\|\phi\|_S} \le \mu_2 C(r+1)\|f(t,u) - f(t,v)\|_{\frac{r+1}{r}}.$$

Let $T_1 : f(t, u) \to DJ_4(u)$. From above we can deduce that $T_1 : L^{\frac{r+1}{r}}([0, T] \times \Omega) \to S^*$ is continuous.

ii) Since $f(t, u)$ is continuous on $[0, T] \times R$ and satisfies Carathédory condition. By the Lemma 2.7 and Combining with assumption (H1), we know there exists $T_2 : u \to f(t, u)$, which is a bounded and continuous operator on $L^{r+1}([0, T] \times \Omega) \to L^{\frac{r+1}{r}}([0, T] \times \Omega)$.

iii) By Remark 2.4, we can get that there exists $T_3|_S : S \to L^{r+1}([0, T] \times \Omega)$ and is a continuous compact operator.

From i)~iii), we obtain that $DJ_4 = T_1 \circ T_2 \circ T_3|_S : S \to S^*$ is a continuous compact operator. From Step (1)~(3) and Lemma 2.5 we have $J_4 \in C^1(S, R)$. At last, from parts 1.~4., we can conclude that $\varphi \in C^1(S, R)$. This proof is completed. □

Theorem 3.2. *When $f(t, u)$ satisfies*

(H1-1) $\forall(t, u) \in [0, T] \times R$, $|f(t, u)| \le a + b|u|^r$ holds, where $a, b > 0, r > 0$, and we set $0 < r < 1$;

$b_j(\tau_j)$ satisfy

(H2-1) $B := E\Big(\sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k}|b_j(\tau_j)|\Big)I_A(\{\xi_j\}_{j=1}^k)\Big) < +\infty$, and $0 < B < \frac{\mu_1\lambda}{\mu_2 C_2^2(\lambda+1)}, \lambda > 0$.

Then $\varphi(u)$ satisfies P.-S. condition.

Proof. We divide into two steps to prove this result.

Step 1: we first show that $\{u_k\}$ is a bounded sequence in S, and $\varphi'(u_k) \to 0, k \to \infty$ in S^*, then $\{u_k\}$ is a sequential compact set on S.

Because of the boundedness of $\{u_k\}$, there exists a $M_0 > 0$ satisfying $\|u_k\|_S < M_0, k = 1, 2, \cdots$. From Theorem 3.1 we know $J_4'(u) : S \to S^*$ is a compact operator. Then $\{J_4'(u_k)\}$ is a sequential compact set on S^*. So $\exists\{u_{k_i}\} \subset \{u_k\}$, such that $J_4'(u_{k_i}) \to J'(u)$ on S^* as $i \to \infty$.

We know

$$\begin{aligned}\varphi(u) = &\frac{1}{2}\int_0^T e^G E|\dot{u}|^2 dt + \frac{\lambda}{2}\int_0^T e^G E|u|^2 dt\\ &+ \sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k} E\Big(\frac{e^{G(\xi_j)} b_j(\tau_j) u^2(\xi_j)}{2}\Big) I_A(\{\xi_j\}_{j=1}^k)\Big) - J_4(u),\end{aligned}$$

$$\begin{aligned}(\varphi'(u), v) = &\int_0^T e^G E\dot{u}E\dot{v}dt + \lambda\int_0^T e^G EuEvdt\\ &+ \sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k} E\Big(e^{G(\xi_j)} b_j(\tau_j) u(\xi_j) v(\xi_j)\Big) I_A(\{\xi_j\}_{j=1}^k)\Big) - < J_4'(u), v > .\end{aligned}$$

Consider

$$\begin{aligned}< \varphi'(u_{k_i}) - \varphi'(u_{k_j}), u_{k_i} - u_{k_j} > = &\int_0^T e^G E|(u_{k_i} - u_{k_j})'|^2 dt + \lambda\int_0^T e^G E|u_{k_i} - u_{k_j}|^2 dt\\ &+ \sum_{k=1}^{\infty}\Big(\sum_{l=1}^{k} E\Big(e^{G(\xi_l)} b_l(\tau_l)|u_{k_i}(\xi_l) - u_{k_j}(\xi_l)|^2\Big) I_A(\{\xi_l\}_{l=1}^k)\Big)\\ &- < J_4'(u_{k_i}) - J_4'(u_{k_j}), u_{k_i} - u_{k_j} > . \qquad (3.2)\end{aligned}$$

When $i, j \to \infty$,

$$| < \varphi'(u_{k_i}) - \varphi'(u_{k_j}), u_{k_i} - u_{k_j} > | \le \Big(\|\varphi'(u_{k_i})\|_{S^*} + \|\varphi'(u_{k_j})\|_{S^*}\Big) \cdot 2M_0 \to 0,$$

$$| < J_4'(u_{k_i}) - J_4'(u_{k_j}), u_{k_i} - u_{k_j} > | \le \|J_4'(u_{k_i}) - J_4'(u_{k_j})\|_{S^*} \cdot 2M_0 \to 0.$$

By Cauchy-Schwarz inequality,

$$\int_0^T e^G E|(u_{k_i} - u_{k_j})'|^2 dt + \lambda\int_0^T e^G E|u_{k_i} - u_{k_j}|^2 dt \ge \frac{\mu_1\lambda}{\lambda+1}\|u_{k_i} - u_{k_j}\|_S.$$

And

$$\Big|\sum_{k=1}^{\infty}\Big(\sum_{l=1}^{k}E\Big(e^{G(\xi_l)}b_l(\tau_l)|u_{k_i}(\xi_l)-u_{k_i}(\xi_l)|^2\Big)I_A(\{\xi_l\}_{l=1}^{k})\Big)\Big|$$
$$\leq \mu_2 B\|u_{k_i}-u_{k_j}\|_\infty^2 \leq \mu_2 BC_2^2\|u_{k_i}-u_{k_j}\|_S^2.$$

Substituting above formula into (3.2) and by $0 < B < \frac{\mu_1\lambda}{\mu_2 C_2^2(\lambda+1)}$, $\lambda > 0$, when $i, j \to \infty$, we get

$$\begin{aligned} 0 &\leq \Big(\frac{\mu_1\lambda}{\lambda+1} - \mu_2 BC_2^2\Big)\|u_{k_i}-u_{k_j}\|_S^2 \\ &\leq \Big[\|\varphi'(u_{k_i})\|_{S^*} + \|\varphi'(u_{k_j})\|_{S^*} + \|J_4'(u_{k_i}) - J_4'(u_{k_j})\|_{S^*}\Big]\cdot 2M_0 \to 0. \end{aligned}$$

From this we know $\{u_{k_i}\}$ is a Cauchy sequence in S, due to the completeness of S, then $\{u_{k_i}\}$ is convergent in S. Thus $\{u_k\}$ is a sequential compact set.

Step 2: Next, we will prove that $\{u_k\}$ is a bounded set in S provided $\{\varphi(u_k)\}$ is a bounded set and $\varphi'(u_k) \to 0$ in S^* as $k \to \infty$.

Since

$$\begin{aligned} \Big|\int_0^T F(t,u)dt\Big| &\leq \int_0^T |F(t,u)|dt \\ &\leq \mu_2\int_0^T\Big(\int_0^{|u|}|f(t,s)|ds\Big)dt \\ &\leq \mu_2\int_0^T dt\int_0^{|u|}\Big(a+b|s|^r\Big)ds \\ &\leq \mu_2\int_0^T\Big(a|u| + \frac{b}{r+1}|u|^{r+1}\Big)dt, \end{aligned}$$

then

$$\begin{aligned} \Big|\int_0^T E(F(t,u))dt\Big| &\leq \int_0^T E|F(t,u)|dt \\ &\leq \mu_2\int_0^T\Big(aE|u| + \frac{b}{r+1}E|u|^{r+1}\Big)dt \\ &= \mu_2 Ta\|u\|_\infty + \frac{bT}{r+1}\mu_2\|u\|_\infty^{r+1} \\ &\leq \mu_2 TaC_2\|u\|_S + \frac{bT\mu_2 C_2^{r+1}}{r+1}\|u\|_S^{r+1}, (0<r<1). \end{aligned}$$

And

$$\Big|\sum_{k=1}^{\infty}\Big(\sum_{l=1}^{k}E\Big(e^{G(\xi_l)}b_l(\tau_l)\frac{u^2(\xi_l)}{2}\Big)I_A(\{\xi_l\}_{l=1}^{k})\Big)\Big|$$
$$\leq \frac{\mu_2 B}{2}\|u\|_\infty^2 \leq \frac{\mu_2 BC_2^2}{2}\|u\|_S^2.$$

putting above formulas into $\varphi(u)$ yields

$$\begin{aligned}\varphi(u) \geq & \frac{1}{2}\int_0^T e^G E|\dot{u}|^2 dt + \frac{\lambda}{2}\int_0^T e^G E|u|^2 dt \\ & - \Big|\int_0^T E(F(t,u))dt\Big| - \Big|\sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k} E\Big(\frac{e^{G(\xi_j)}b_j(\tau_j)u^2(\xi_j)}{2}\Big) I_A(\{\xi_j\}_{j=1}^k)\Big)\Big| \\ \geq & \Big(\frac{\mu_1\lambda}{2(\lambda+1)} - \frac{\mu_2 BC_2^2}{2}\Big)\|u\|_S^2 - aT\mu_2 C_2\|u\|_S - \frac{bT\mu_2 C_2^{r+1}}{r+1}\|u\|_S^{r+1},\ 0<r<1.\end{aligned}$$

Let $K_1 := \frac{\mu_1\lambda}{2(\lambda+1)} - \frac{\mu_2 BC_2^2}{2} > 0$ and replace $\{u_k\}$ with u on the above formula. Because of the boundedness of $\{\varphi(u_k)\}$, we have

$$+\infty > |\varphi(u_k)| \geq K_1\|u_k\|_S^2 - aT\mu_2 C_2\|u_k\|_S - \frac{bT\mu_2 C_2^{r+1}}{r+1}\|u_k\|_S^{r+1},\ 0<r<1.$$

If $\{u_k\}$ is unbounded, then there is subsequence $\{u_{k_i}\} \subset \{u_k\}$, such that $\|u_{k_i}\|_S \to \infty$, so then the right end of the above formula tends to infinity, which is contradict with the boundedness of $\{\varphi(u_k)\}$.

From above, we can deduce φ satisfies P.-S. condition on S. □

Theorem 3.3. *Let all the hypotheses listed in Theorem 3.2 be fulfilled, and $f(t,u)$ satisfies:*

(H3) $uf(t,u) \leq 0, \forall (t,u) \in [0,T]\times R$.

By the Minimax principle, one can deduce $\varphi(u)$ has a critical point in S, i.e. equation (1.1) *has at least a mild solution.*

Proof. By the results in Theorem 3.1 and Theorem 3.2, we have known $\varphi(u) \in C^1(S,R)$ and fulfills P.-S. condition. Next, we only need to show $\varphi(u)$ has a lower bound on S.

By (H3): $\int_0^T dt\int_0^u e^{G(t)} f(t,s)ds \leq 0 \Rightarrow \int_0^T E(F(t,u))dt \leq 0$. $\forall u \in S$, we have

$$\begin{aligned}\varphi(u) = & \frac{1}{2}\int_0^T e^G E|\dot{u}|^2 dt + \frac{\lambda}{2}\int_0^T e^G E|u|^2 dt \\ & + \sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k} E\Big(\frac{e^{G(\xi_j)}b_j(\tau_j)u^2(\xi_j)}{2}\Big) I_A(\{\xi_j\}_{j=1}^k)\Big) - \int_0^T E(F(t,u))dt \\ \geq & \Big(\frac{\mu_1\lambda}{2(\lambda+1)} - \frac{\mu_2 BC_2^2}{2}\Big)\|u\|_S^2 \geq 0\ (by\ (H2)),\end{aligned}$$

thus $\varphi(u)$ has lower bound on S, and then $\varphi(u)$ has a critical point on S, i.e. equation (1.1) has at least a mild solution by using Minimax principle. □

Theorem 3.4. *Suppose that*

(H1-2) $f(t,u) \leq \hat{a} + \hat{b}|u|^r, \forall (t,u) \in [0,T]\times R$ hold, where $\hat{a},\hat{b} > 0, r > 1$, and $\hat{a},\hat{b}$ satisfies $\hat{a}+\hat{b} < \frac{\mu_1\lambda(r+1)^{\frac{1}{r}}}{2(\lambda+1)TC_2^2}$;

(H2-2) $B = E\Big(\sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k}|b_j(\tau_j)|\Big)I_A(\{\xi_j\}_{j=1}^k)\Big) < +\infty$, $b_j(\cdot) > 0, \forall j = 1,2,\cdots$;

(H4) There are $\beta > 2, r_1 > 0$. For $\forall t \in [0,T], |u| \geq r_1$, have $0 < \beta F(t,u) \leq uf(t,u)$.

Then $\varphi(u) \in C^1(S,R)$ and $\varphi(u)$ satisfies P.-S. condition in S. By Mountain pass lemma, we can get φ has a critical point, i.e. Equation (1.1) *has at least a mild solution in S.*

Proof. (1) By hypothesis (H1-2),(H2-2) and using similar approach with Theorem 3.1, we can prove that $\varphi(u) \in C^1(S, R)$.

(2) we next prove $\varphi(u)$ satisfies P.-S. condition in E:

1) We will prove if $\{\varphi(u_k)\}$ is a bounded set and $\varphi(u_k) \to 0, k \to \infty$ in S^*, then $\{u_k\}$ is a bounded set in S.

Since

$$\varphi(u_k) = \frac{1}{2}\int_0^T e^G E|\dot{u}_k|^2 dt + \frac{\lambda}{2}\int_0^T e^G E|u_k|^2 dt - \int_0^T E(F(t, u_k))dt + \sum_{l=1}^{\infty}\Big(\sum_{j=1}^{l} E\Big(\frac{e^{G(\xi_j)} b_j(\tau_j) u_k^2(\xi_j)}{2}\Big) I_A(\{\xi_j\}_{j=1}^l)\Big),$$

taking the above formula into (H5) yields

$$2\varphi(u_k) - 2\sum_{l=1}^{\infty}\Big(\sum_{j=1}^{l} E\Big(\frac{e^{G(\xi_j)} b_j(\tau_j) u_k^2(\xi_j)}{2}\Big) I_A(\{\xi_j\}_{j=1}^l)\Big) + \frac{2}{\beta}\int_0^T E\Big(f(t, u_k)u_k e^{G(t)} dt\Big) \geq \int_0^T e^G E|\dot{u}_k|^2 dt + \lambda\int_0^T e^G E|u_k|^2 dt. \tag{3.3}$$

$$(\varphi'(u_k), u_k) = \int_0^T e^G E|\dot{u}_k|^2 dt + \lambda\int_0^T e^G E|u_k|^2 dt + \sum_{l=1}^{\infty}\Big(\sum_{j=1}^{l} E\Big(e^{G(\xi_j)} b_j(\tau_j) u_k^2(\xi_j)\Big) I_A(\{\xi_j\}_{j=1}^l)\Big) - \int_0^T E(e^G f(t, u_k)u_k)dt. \tag{3.4}$$

Putting (3.4)$\times\frac{2}{\beta}$ into (3.3), we get

$$2\varphi(u_k) + \sum_{l=1}^{\infty}\Big(\sum_{j=1}^{l} E\Big(e^{G(\xi_j)} b_j(\tau_j) u_k^2(\xi_j)\Big) I_A(\{\xi_j\}_{j=1}^l)\Big)(\frac{2}{\beta} - 1) - \frac{2}{\beta}\Big(\varphi'(u_k), u_k\Big) \geq (1 - \frac{2}{\beta})\Big(\int_0^T e^G E|\dot{u}_k|^2 + \lambda\int_0^T e^G E|u_k|^2\Big).$$

By $\beta > 2$ and Cauchy-Schwarz inequality,

$$\Rightarrow 2\varphi(u_k) - \frac{2}{\beta}\Big(\varphi'(u_k), u_k\Big) \geq (1 - \frac{2}{\beta})\frac{\mu_1\lambda}{\lambda + 1}\|u_k\|_S^2 \geq 0.$$

Because $\{\varphi(u_k)\}$ is bounded and $\varphi'(u_k) \to 0 \ in \ S^*$ as $k \to \infty$, then $\{u_k\}$ is bounded in S.

2) Then we will prove $\{u_k\}$ is a bounded set in S. Since $\varphi'(u_k) \to 0 \ k \to \infty$ in S^*, then $\{u_k\}$ is a sequential compact set in S. We consider that

$$< \varphi'(u_{k_i}) - \varphi'(u_{k_j}), u_{k_i} - u_{k_j} > = \int_0^T e^G E|(u_{k_i} - u_{k_j})'|^2 dt + \lambda\int_0^T e^G E|u_{k_i} - u_{k_j}|^2 dt$$

$$+\sum_{k=1}^{\infty}\Big(\sum_{l=1}^{k}E\Big(e^{G(\xi_l)}b_l(\tau_l)|u_{k_i}(\xi_l)-u_{k_j}(\xi_l)|^2\Big)I_A(\{\xi_l\}_{l=1}^k)\Big)$$
$$-\int_0^T E\Big(e^{G(t)}(f(t,u_{k_i})-f(t,u_{k_j}))(u_{k_i}-u_{k_j})\Big)dt. \quad (3.5)$$

In addition, because $\{u_k\}$ is bounded in S, then there is $\{u_{k_i}\}\subset\{u_k\}$ satisfying

$$\Rightarrow\begin{cases} u_{k_i}\to u \text{ is strong converge in } C([0,T]);\\ u_{k_i}\rightharpoonup u \text{ is weak converge in } S.\end{cases}$$

$$\Rightarrow\int_0^T E\Big(e^{G(t)}(f(t,u_{k_i})-f(t,u_{k_j}))(u_{k_i}-u_{k_j})\Big)dt\to 0,\ as\ i,j\to\infty. \quad (3.6)$$

Rewrite (3.7) and note that $<\varphi'(u_{k_i})-\varphi'(u_{k_j}),u_{k_i}-u_{k_j}>\to 0,(i,j\to\infty)$.

$$\Rightarrow<\varphi'(u_{k_i})-\varphi'(u_{k_j}),u_{k_i}-u_{k_j}>+\int_0^T E\Big(e^{G}(f(t,u_{k_i})-f(t,u_{k_j}))(u_{k_i}-u_{k_j})dt$$
$$\geq\frac{\mu_1\lambda}{(\lambda+1)}\|u_{k_i}-u_{k_j}\|_S^2. \quad (3.7)$$

From this, when $i,j\to 0$, $\Rightarrow\|u_{k_i}-u_{k_j}\|_S\to 0$, we have $\{u_{k_i}\}$ is a Cauchy sequences in S, and by the completeness of S, we further get $\{u_{k_i}\}$ is convergent in S. Then $\{u_k\}$ is a sequential compact in S.

By 1),2) we know $\varphi(u)$ satisfies P.-S. condition on E.

(3) At last, we verify whether $\varphi(u)$ fulfills the conditions of Mountain pass lemma.

a) It is obvious that $\varphi(0)=0$.

$$\begin{aligned}\varphi(u)&=\frac{1}{2}\int_0^T e^G E|\dot{u}|^2dt+\frac{\lambda}{2}\int_0^T e^G E|u|^2dt\\&+\sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k}E\Big(\frac{e^{G(\xi_j)}b_j(\tau_j)u^2(\xi_j)}{2}\Big)I_A(\{\xi_j\}_{j=1}^k)\Big)-\int_0^T E\Big(F(t,u)\Big)dt\\&\geq\frac{\mu_1\lambda}{2(\lambda+1)}\|u\|_S^2-\hat{a}\int_0^T E|u|dt-\frac{\hat{b}}{r+1}\int_0^T E|u|^{r+1}dt\\&\geq\frac{\mu_1\lambda}{2(\lambda+1)}\|u\|_S^2-\hat{a}TC_2\|u\|_S-\frac{\hat{b}}{r+1}TC_2^{r+1}\|u\|_S^{r+1}.\end{aligned}$$

Take $\rho=\frac{(r+1)^{\frac{1}{r}}}{C_2}>0,u\in\partial B_\rho(0)$, then $\|u\|_S=\rho$, then

$$\begin{aligned}\varphi(u)&\geq\Big(\frac{\mu_1\lambda}{2(\lambda+1)}-\frac{\hat{a}TC_2}{\rho}\Big)\|u\|_S^2+\Big(\frac{\hat{a}TC_2}{\rho}\|u\|_S^2-\hat{a}TC_2\|u\|_S\Big)\\&-\frac{\hat{b}}{r+1}TC_2^{r+1}\|u\|_S^{r+1}\\&=\|u\|_S^2\Big[\Big(\frac{\mu_1\lambda}{2(\lambda+1)}-\frac{\hat{a}TC_2}{\rho}\Big)-\frac{\hat{b}}{r+1}TC_2^{r+1}\|u\|_S^{r-1}\Big]\end{aligned}$$

$$= \rho^2\Big[\frac{\mu_1\lambda}{2(\lambda+1)} - \frac{TC_2^2}{(r+1)^{\frac{1}{r}}}(\hat{a}+\hat{b})\Big] > 0.$$

Here we used the assumption (H1-2).

b) By assumption (H4), $\exists\beta > 2, r_1 > 0, 0 < \beta F(t,u) \le uf(t,u)$.

$\forall t \in [0,T]$, when $|u| \ge r_1$,

$$\Rightarrow \begin{cases} \frac{\beta}{u} \le \frac{f(t,u)}{F(t,u)}, \ u \ge r_1, \\ \frac{\beta}{u} \ge \frac{f(t,u)}{F(t,u)}, \ u \le -r_1. \end{cases}$$

Integrate both sides of the above two formulas on $[r_1, u]$ and $[u, -r_1]$ respectively,

$$\Rightarrow \begin{cases} \beta ln\frac{u}{r_1} \le ln\frac{F(t,u)}{F(t,r_1)}, \ u \ge r_1, \\ \beta ln\frac{-r_1}{u} \ge ln\frac{F(t,-r_1)}{F(t,u)}, \ u \le -r_1. \end{cases}$$

$$\Rightarrow \begin{cases} F(t,u) \ge F(t,r_1)\Big(\frac{u}{r_1}\Big)^\beta, \ u \ge r_1, \\ F(t,u) \ge F(t,-r_1)\Big(\frac{u}{-r_1}\Big)^\beta, \ u \le -r_1. \end{cases}$$

Let

$$K := |r_1|^{-\beta}\{\min_{t\in[0,T]}\{F(t,r_1)\}, \min_{t\in[0,T]}\{F(t,-r_1)\}\} > 0,$$

$$\Rightarrow F(t,u) \ge K|u|^\beta, \text{ when } |u| \ge r_1, \beta > 2.$$

Now, we consider

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\int_0^T e^G E|\dot{u}|^2 dt + \frac{\lambda}{2}\int_0^T e^G E|u|^2 dt \\ &+ \sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k} E\Big(\frac{e^{G(\xi_j)}b_j(\tau_j)u^2(\xi_j)}{2}\Big)I_A(\{\xi_j\}_{j=1}^k)\Big) - \int_0^T E\Big(F(t,u)\Big)dt \\ &\le \frac{\mu_2(\lambda+1)}{2}\|u\|_S^2 + \frac{\mu_2\beta}{2}\|u\|_\infty^2 - K\int_0^T E|u|^\beta dt \ (\beta > 2) \\ &\le \frac{\mu_2}{2}\Big((\lambda+1) + BC_2^2\Big)\|u\|_S^2 - K\int_0^T E|u|^\beta dt. \end{aligned}$$

Here, we fix $u \in S$ and take $\|u\|_S = 1$. Then consider when $e = tu$, we get

$$\varphi(tu) \le \frac{t^2\mu_2}{2}\Big((\lambda+1) + BC_2^2\Big)\|u\|_S^2 - t^\beta K\int_0^T E|u|^\beta dt \ (\beta > 2).$$

Thus, when $t \to +\infty$, $\varphi(tu) \to -\infty$. Then there is $t_0 > \rho$ such that $\varphi(t_0u) \le 0$ hold. Let $e = t_0u$, $\|e\|_S = \|t_0u\|_S = |t_0|\|u\|_S = t_0 > \rho$, so $e \in E\backslash\overline{B_\rho(0)}$, and $\varphi(e) \le 0$.

From a),b) and becase $\varphi(u) \in C^1(S,R)$ satisfies P.-S. condition, by using Mountain pass lemma, we can obtain φ has a critical point, i.e. equation (1.1) has at least a mild solution in S. This completes the proof. □

4. Conclusions

In this paper, we mainly study sufficient conditions for the existence of solutions of a class of damped random impulsive differential equations under Dirichlet boundary value conditions, in which the variational method plays a key role. We conclude that the solution of the RIDE is equivalent to that of the energy functional obtained by the variational method, thus transforming the problem into a critical point problem for solving the energy functional. Finally, sufficient conditions for the existence of mild solutions of the studied equations are obtained by using the Minimax principle and the Mountain pass lemma. Although our equations are relatively limited, it is a pioneering attempt to study RIDEs by using variational method and critical point theory. Furthermore, we can also consider using topological degree and Hamiltonian system to study the behavior of solutions of corresponding RIDEs. We will continue to delve into this fascinating field with newer and broader results in the future.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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