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Research article

Dual group inverses of dual matrices and their applications in solving systems of linear dual equations

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Abstract: In this paper, we study a kind of dual generalized inverses of dual matrices, which is called the dual group inverse. Some necessary and sufficient conditions for a dual matrix to have the dual group inverse are given. If one of these conditions is satisfied, then compact formulas and efficient methods for the computation of the dual group inverse are given. Moreover, the results of the dual group inverse are applied to solve systems of linear dual equations. The dual group-inverse solution of systems of linear dual equations is introduced. The dual analog of the real least-squares solution and minimal P-norm least-squares solution are obtained. Some numerical examples are provided to illustrate the results obtained.

Keywords: dual matrix; dual group inverse; linear dual equation; dual group-inverse solution; least-squares solution; minimal P-norm least-squares solution
Mathematics Subject Classification: 15A09, 15A10

1. Introduction

A *dual number* $\widehat{a}$ [1] is usually denoted in the form

$$\widehat{a} = a + \varepsilon a_0,$$

where the real numbers a and a_0 are respectively the prime part and the dual part of $\widehat{a}$, and ε is the *dual unit* satisfying $\varepsilon \neq 0$, $0\varepsilon = \varepsilon 0 = 0$, $1\varepsilon = \varepsilon 1 = \varepsilon$ and $\varepsilon^2 = 0$. Actually, dual numbers can be defined over both the real and the complex fields [2], and in many cases, real numbers will suffice. Since a pure dual number εa_0 has no inverse, the set of dual numbers is not a field, but a ring [3]. For more details of the dual numbers, such as representations of dual numbers, functions of dual numbers, mathematical expressions of dual operations, the reader is referred to [4, 5] and references therein. The dual number, originally introduced by Clifford in 1873 [6], was further developed by Study in 1901 [7] to represent the dual angle in spatial geometry. Because of conciseness of notation, dual numbers and their algebra have been powerful and convenient tools for the analysis of mechanical systems, and have attracted a lot of attention over the last three decades because of their applicability to various areas of engineering like kinematic analysis [1, 4], robotics [8, 9], screw motion [10] and rigid body motion analysis [11].

A matrix with dual number entries is called a *dual matrix*. If A and B are two $m \times n$ (or m-by-n) real matrices, then the $m \times n$ dual matrix $\widehat{A}$, is defined as $\widehat{A} = A + \varepsilon B$, where A and B are respectively called the prime part and the dual part of $\widehat{A}$. In particular, if A and B are $n \times n$ real matrices, then we say that $\widehat{A}$ is a square dual matrix. Many terminologies of dual matrices, such as matrix multiplication, inverse of a dual matrix, QR and SVD decompositions, can be defined on the analogy of those of real matrices. For example, for two dual matrices $\widehat{A_1} = A_1 + \varepsilon B_1$ and $\widehat{A_2} = A_2 + \varepsilon B_2$, $\widehat{A_1}\widehat{A_2} = A_1A_2 + \varepsilon(A_1B_2 + B_1A_2)$, where A_1, A_2, B_1, B_2 have appropriate dimensions. If a real $n \times n$ matrix A is nonsingular, then the $n \times n$ dual matrix $\widehat{A} = A + \varepsilon B$ is also nonsingular and the inverse of $\widehat{A}$ is given by $\widehat{A}^{-1} = A^{-1} - \varepsilon A^{-1}BA^{-1}$ [4]. For rectangular dual matrices and singular square dual matrices, it is natural to study their dual generalized inverses. Angeles [1] investigated the usefulness of dual generalized inverses in kinematic analyses based on dual numbers. However, it should be noted that many important properties of dual generalized inverses of dual matrices are much different from those of real matrices. For example, it is well-known that the Moore-Penrose generalized inverse of a real matrix exists, while it was shown in [12, 13] that the dual Moore-Penrose generalized inverse (DMPGI, for short) of a rank-deficient dual matrix may not exist and there are uncountably many dual matrices that do not have them. Hence, it is interesting to investigate the necessary and sufficient conditions for the existence of dual generalized inverses and find efficient methods to compute them when they exist. The existence, computations and applications of dual generalized inverses have been a topic of recent interest. de Falco et al. [14] discussed the mathematical conditions of existence for different types of dual generalized inverses. Moreover, solutions of some meaningful kinematic problems were discussed to demonstrate the usefulness and versatility of dual generalized inverses. Pennestrì et al. [15] proposed novel and computationally efficient algorithms/formulas for the computation of the MPDGI. Udwadia et al. [13] investigated the question of whether all dual matrices have dual Moore-Penrose generalized inverses and showed that there are uncountably many dual matrices that do not have them. Udwadia [12] studied properties of the DMPGI and used them to solve systems of linear dual equations. Wang [16] gave some necessary and sufficient conditions for a dual matrix to have the DMPGI, and a compact formula for the computation of the DMPGI was also given.

As showed in [12], dual generalized inverses are powerful tools to study the solutions and least-squares solutions of systems of linear dual equations. Many applications of dual algebra in kinematics require numerical solutions of systems of linear dual equations. A common approach is to split the system of dual linear equations into the real part and dual part and then form a system of real equations. The solutions of the real part and dual part of the system of linear dual equations are usually computed separately. However, the availability of a dual generalized inverse of the coefficient matrix allows the simultaneous computation of the dual equation in a single step, and thus improve the overall computational efficiency. For this reason, the existence and the availability of an efficient method to compute dual generalized inverses could be of interest for the researchers.

Motivated by the above results, in this paper, we consider another kind of dual generalized inverse of dual matrices, which is called the *dual group inverse*. The dual group inverses, although exist only for square dual matrices, have some properties of inverse matrix that the DMPGI does not possess, for instance, a square dual matrix commutes with its dual group inverse. Furthermore, the dual group inverses also provide solutions and least-squares solutions of linear dual equations. Hence, they more closely resemble the true inverse of a dual matrix. For real cases, there are applications of the group inverse in various fields such as Markov chains [17], stationary iteration [18], fuzzy linear systems [19]. It is known that the group inverse of a real square matrix A exists if and only if the index of A is 1. However, for a square dual matrix $\widehat{A}$ whose prime part has index 1, the dual group inverse of $\widehat{A}$ may not exist. Hence, it is also interesting to investigate the existence and computations of the dual group inverse.

In this paper, we study the existence and computations of the dual group inverse, and discuss the usefulness of the dual group inverse in solving systems of linear dual equations. The outline of the rest of this paper is as follows. In Section 2, we present some notations which will be used later and review some preliminaries briefly. In Section 3, we give some necessary and sufficient conditions for a dual matrix to have the dual group inverse, compact formulas for the computation of the dual group inverse are also given. In Section 4, we discuss the applications of the dual group inverse in solving systems of linear dual equations. The concept of the dual group-inverse solution is introduced. The dual analog of the real minimum P-norm least-squares solution is obtained through the dual group inverse. Some numerical examples are provided to illustrate the results obtained.

2. Notations and preliminaries

Throughout this paper the following notations and definitions are used. $\mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}$ and $\mathbb{D}^{m\times n}$ denote the set of all $m \times n$ complex matrices, real matrices and dual matrices respectively. $\mathbb{R}^n$ and $\mathbb{D}^n$ denote the set of all n-dimensional real column vectors and n-dimensional dual column vectors respectively. For a real matrix A, $\mathcal{R}(A)$ is the range of A and $\mathcal{N}(A)$ is the null space of A. The index of a matrix $A \in \mathbb{R}^{n\times n}$, is the smallest positive integer such that rank(A^k)=rank(A^{k+1}), and denoted by Ind(A). For a square dual matrix $\widehat{A} = A + \varepsilon B$, $\widehat{A}^T = A^T + \varepsilon B^T$, where A^T is the transpose of A. For a nonsingular real matrix P, $P^{-1}\widehat{A}P = P^{-1}AP + \varepsilon P^{-1}BP$.

We first give the definition of the dual Moore-Penrose generalized inverse of a dual matrix in the following, which is analogous to real matrices.

Definition 2.1. The dual Moore-Penrose generalized inverse of a dual matrix $\widehat{A} \in \mathbb{D}^{m\times n}$, denoted by $\widehat{A}^{\dagger}$, is the unique matrix $\widehat{X} \in \mathbb{D}^{n\times m}$ satisfying the following dual Penrose equations

$$\widehat{A}\widehat{X}\widehat{A} = \widehat{A}, \quad \widehat{X}\widehat{A}\widehat{X} = \widehat{X}, \quad (\widehat{A}\widehat{X})^T = \widehat{A}\widehat{X}, \quad (\widehat{X}\widehat{A})^T = \widehat{X}\widehat{A}.$$

The Drazin inverse is a generalized inverse which is defined only for square matrices, and has many applications in the theory of finite Markov chains, singular linear difference equations, cryptograph etc (see [20, 21]).

Definition 2.2. [22] Let $A \in \mathbb{R}^{n\times n}$ and $\text{Ind}(A) = k$. Then the matrix $X \in \mathbb{R}^{n\times n}$ satisfying

$$A^kXA = A^k, \quad XAX = X, \quad AX = XA$$

is called the Drazin inverse of A, and is denoted by $X = A^D$.

If Ind(A)=1, then this special case of the Drazin inverse is known as the group inverse.

Definition 2.3. [22] Let $A \in \mathbb{R}^{n\times n}$. If $X \in \mathbb{R}^{n\times n}$ satisfies

$$AXA = A, \quad XAX = X, \quad AX = XA,$$

then X is called the group inverse of A, and is denoted by $A^{\#}$. It is known that A has a group inverse if and only if $\mathrm{Ind}(A) = 1$. For example, diagonalizable matrices have index 1, thus, a singular diagonalizable matrix has group inverse.

The dual group inverse of a square dual matrix can be defined similarly as the group inverse of a square real matrix.

Definition 2.4. Let $\widehat{A} \in \mathbb{D}^{n\times n}$. If a dual matrix $\widehat{X} \in \mathbb{D}^{n\times n}$ satisfies

$$\widehat{A}\widehat{X}\widehat{A} = \widehat{A}, \quad \widehat{X}\widehat{A}\widehat{X} = \widehat{X}, \quad \widehat{A}\widehat{X} = \widehat{X}\widehat{A},$$

then $\widehat{X}$ is called the dual group inverse of $\widehat{A}$, and is denoted by $\widehat{A}^{\#}$.

The following lemma gives a block representation of a real square matrix that has the group inverse.

Lemma 2.1. [22] *Let $A \in \mathbb{R}^{n\times n}$. Then A has a group inverse if and only if there exist nonsingular matrices P and C such that*

$$A = P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1}. \tag{2.1}$$

In this case, it is easy to verify that

$$A^{\#} = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}. \tag{2.2}$$

The block representation (2.1) is the Jordan canonical form of A.

The following three lemmas play important roles in Section 3.

Lemma 2.2. [22] *Let $A, P \in \mathbb{C}^{n\times n}$ satisfy $P^2 = P$. Then*
(i) $PA = A$ if and only if $\mathcal{R}(A) \subset \mathcal{R}(P)$.
(ii) $AP = A$ if and only if $\mathcal{N}(P) \subset \mathcal{N}(A)$.

Lemma 2.3. [23] *If*

$$M = \begin{bmatrix} A & C \\ 0 & D \end{bmatrix},$$

then $M^{\#}$ exists if and only if $A^{\#}$ and $D^{\#}$ exist and $(I - AA^{\#})C(I - DD^{\#}) = 0$. If so,

$$M^{\#} = \begin{bmatrix} A^{\#} & X \\ 0 & D^{\#} \end{bmatrix},$$

where $X = -A^{\#}CD^{\#} + (A^{\#})^2C(I - DD^{\#}) + (I - AA^{\#})C(D^{\#})^2$.

Lemma 2.4. [24] *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times m}$, $C \in \mathbb{C}^{n\times n}$ with Ind(B)=k and Ind(C)=l. Then*

(i) $rank\begin{bmatrix} A & B^k \\ C^l & 0 \end{bmatrix} = rank(B^k) + rank(C^l) + rank[(I_m - BB^D)A(I_n - C^DC)]$.

(ii) $rank\begin{bmatrix} A & B^k \end{bmatrix} = rank(B^k) + rank(A - BB^DA)$.

(iii) $rank\begin{bmatrix} A \\ C^l \end{bmatrix} = rank(C^l) + rank(A - AC^DC)$.

3. Dual group inverses of dual matrices

In this section, we study the dual group inverses of dual matrices. Some necessary and sufficient conditions for the existence of the dual group inverse are given. Some efficient methods for the computation of the dual group inverse are also presented. We firstly give a necessary and sufficient condition for a dual matrix to be the dual group inverse of a given dual matrix $\widehat{A} = A + \varepsilon B$.

Lemma 3.1. *Let $\widehat{A} = A + \varepsilon B$ be a dual matrix with $A, B \in \mathbb{R}^{n\times n}$ and $Ind(A) = 1$. Then an $n \times n$ dual matrix $\widehat{G} = G + \varepsilon R$ is a dual group inverse of $\widehat{A}$ if and only if $G = A^{\#}$ and*

$$B = AA^{\#}B + ARA + BA^{\#}A, \tag{3.1}$$

$$R = A^{\#}AR + A^{\#}BA^{\#} + RAA^{\#}, \tag{3.2}$$

$$AR + BA^{\#} = RA + A^{\#}B. \tag{3.3}$$

Proof. By Definition 2.4, $\widehat{G} = G + \varepsilon R$ is a dual group inverse of $\widehat{A} = A + \varepsilon B$ if and only if

$$(A + \varepsilon B)(G + \varepsilon R)(A + \varepsilon B) = AGA + \varepsilon(AGB + ARA + BGA) = A + \varepsilon B,$$

$$(G + \varepsilon R)(A + \varepsilon B)(G + \varepsilon R) = GAG + \varepsilon(GAR + GBG + RAG) = (G + \varepsilon R),$$

$$(A + \varepsilon B)(G + \varepsilon R) = AG + \varepsilon(AR + BG) = GA + \varepsilon(GB + RA) = (G + \varepsilon R)(A + \varepsilon B).$$

Then, we can see from the prime parts of the above equalities that $AGA = A$, $GAG = G$ and $AG = GA$, i.e., $G = A^{\#}$. On the other hand, we can see from the dual parts of the above equalities that $B = AGB + ARA + BGA$, $R = GAR + GBG + RAG$ and $AR + BG = GB + RA$. Hence, the equalities (3.1)–(3.3) follow. □

It is well-known that for a real square matrix A, if $A^{\#}$ exists, then it is unique. We now show that it is also true for dual matrices.

Theorem 3.1. *Let $\widehat{A} = A + \varepsilon B$ be a dual matrix with $A, B \in \mathbb{R}^{n\times n}$ and $Ind(A) = 1$. If the dual group inverse of $\widehat{A}$ exists, then it is unique.*

Proof. According to Lemma 3.1, if the dual group inverse of $\widehat{A} = A + \varepsilon B$ exists, then it must be of the form $A^{\#} + \varepsilon R$. Let $\widehat{G_1} = A^{\#} + \varepsilon R_1$ and $\widehat{G_2} = A^{\#} + \varepsilon R_2$ be two dual group inverses of $\widehat{A}$. To show the uniqueness of $\widehat{A}^{\#}$, we need only to show that $R_1 = R_2$.

It follows from (3.1) that

$$B = AA^{\#}B + AR_1A + BA^{\#}A \tag{3.4}$$

and

$$B = AA^{\#}B + AR_2A + BA^{\#}A. \tag{3.5}$$

Subtracting (3.4) from (3.5) gives

$$A(R_1 - R_2)A = 0. \tag{3.6}$$

Similarly, we can observe from (3.2) that

$$R_1 = A^{\#}AR_1 + A^{\#}BA^{\#} + R_1AA^{\#} \tag{3.7}$$

and

$$R_2 = A^{\#}AR_2 + A^{\#}BA^{\#} + R_2AA^{\#}. \tag{3.8}$$

Comparing (3.7) and (3.8) we have

$$R_1 - R_2 = A^{\#}A(R_1 - R_2) + (R_1 - R_2)AA^{\#}. \tag{3.9}$$

Furthermore, it can be seen from (3.3) that

$$AR_1 + BA^{\#} = R_1A + A^{\#}B \tag{3.10}$$

and

$$AR_2 + BA^{\#} = R_2A + A^{\#}B. \tag{3.11}$$

Subtracting (3.10) from (3.11) gives

$$A(R_1 - R_2) = (R_1 - R_2)A. \tag{3.12}$$

Now, postmultiplying (3.6) by $A^{\#}$ yields $A(R_1 - R_2)AA^{\#} = 0$. Thus, by (3.12),

$$0 = A(R_1 - R_2)AA^{\#} = (R_1 - R_2)AAA^{\#} = (R_1 - R_2)A = A(R_1 - R_2). \tag{3.13}$$

Substituting (3.13) into (3.9) we get

$$R_1 - R_2 = A^{\#}A(R_1 - R_2) + (R_1 - R_2)AA^{\#} = 0,$$

i.e., $R_1 = R_2$, which completes the proof. □

Although Lemma 3.1 provides a necessary and sufficient condition for a dual matrix to be the dual group inverse of a given dual matrix, however, it is still not easy for us to see whether the dual group inverse of a given dual matrix exists. The following theorem gives some sufficient and necessary conditions for the existence of $\widehat{A}^{\#}$ under the assumption that the prime part of $\widehat{A}$ has index 1. If one of these conditions is satisfied, we give a compact formula for the computation of $\widehat{A}^{\#}$.

Theorem 3.2. *Let $\widehat{A} = A + \varepsilon B$ be a dual matrix with $A, B \in \mathbb{R}^{n\times n}$ and $Ind(A) = 1$. Then the following conditions are equivalent:*
(i) The dual group inverse of $\widehat{A}$ exists;
(ii) $\widehat{A} = P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}P^{-1}$, where C and P are nonsingular matrices;
(iii) $(I - AA^{\#})B(I - AA^{\#}) = 0$;

(iv) $\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^{\#}$ *exists;*

(v) rank $\begin{bmatrix} B & A \\ A & 0 \end{bmatrix} = 2rank(A)$*;*

(vi) $\widehat{A}^{\dagger}$ *exists.*

Furthermore, if the dual group inverse of $\widehat{A}$ *exists, then*

$$\widehat{A}^{\#} = A^{\#} + \varepsilon R, \tag{3.14}$$

where

$$R = -A^{\#}BA^{\#} + (A^{\#})^2 B(I - AA^{\#}) + (I - AA^{\#})B(A^{\#})^2. \tag{3.15}$$

Proof. $(i) \Longrightarrow (ii)$: If $\mathrm{Ind}(A) = 1$, then A and $A^{\#}$ have the block matrix forms (2.1) and (2.2), respectively. Let $B = P\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}P^{-1}$ and $R = P\begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}P^{-1}$. If the dual group inverse of $\widehat{A}$ exists, then by Lemma 3.1, the condition (3.1) is satisfied, i.e.,

$$\begin{aligned} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} + \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2B_1 + CR_1C & B_2 \\ B_3 & 0 \end{bmatrix}. \end{aligned}$$

It can be seen from the above equality that $B_4 = 0$.

$(ii) \Longrightarrow (iii)$: If

$$A = P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1}, \quad B = P\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}P^{-1},$$

then a direct calculation shows that

$$(I - AA^{\#})B(I - AA^{\#}) = P\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}P^{-1} = 0.$$

$(iii) \Longrightarrow (i)$: By Lemma 2.1, there exist nonsingular matrices P and C such that A and $A^{\#}$ are of the forms (2.1) and (2.2), respectively. Let $B = P\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}P^{-1}$. Then it follows from $(I - AA^{\#})B(I - AA^{\#}) = 0$ that

$$\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} = 0,$$

which implies that $B_4 = 0$.

Now, denote

$$\widehat{G} = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} -C^{-1}B_1C^{-1} & C^{-2}B_2 \\ B_3C^{-2} & 0 \end{bmatrix}P^{-1}.$$

Then

$$\widehat{A}\widehat{G} = \left(P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}P^{-1}\right)$$

$$
\begin{aligned}
&\times \left(P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} -C^{-1}B_1C^{-1} & C^{-2}B_2 \\ B_3C^{-2} & 0 \end{bmatrix}P^{-1}\right) \\
&= P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} 0 & C^{-1}B_2 \\ B_3C^{-1} & 0 \end{bmatrix}P^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\widehat{GA} &= \left(P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} -C^{-1}B_1C^{-1} & C^{-2}B_2 \\ B_3C^{-2} & 0 \end{bmatrix}P^{-1}\right) \\
&\times \left(P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}P^{-1}\right) \\
&= P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} 0 & C^{-1}B_2 \\ B_3C^{-1} & 0 \end{bmatrix}P^{-1}.
\end{aligned}
$$

Hence, $\widehat{AG} = \widehat{GA}$.

Moreover,

$$
\begin{aligned}
\widehat{AGA} &= \left(P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} 0 & C^{-1}B_2 \\ B_3C^{-1} & 0 \end{bmatrix}P^{-1}\right) \\
&\times \left(P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}P^{-1}\right) \\
&= P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}P^{-1} \\
&= \widehat{A}
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{GAG} &= \left(P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} 0 & C^{-1}B_2 \\ B_3C^{-1} & 0 \end{bmatrix}P^{-1}\right) \\
&\times \left(P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} -C^{-1}B_1C^{-1} & C^{-2}B_2 \\ B_3C^{-2} & 0 \end{bmatrix}P^{-1}\right) \\
&= P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} -C^{-1}B_1C^{-1} & C^{-2}B_2 \\ B_3C^{-2} & 0 \end{bmatrix}P^{-1} \\
&= \widehat{G}.
\end{aligned}
$$

Therefore, $\widehat{A}^{\#}$ exists and $\widehat{A}^{\#} = \widehat{G}$.

Since

$$
A^{\#}BA^{\#} = P\begin{bmatrix} C^{-1}B_1C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1},
$$

$$
(A^{\#})^2B(I - AA^{\#}) = P\begin{bmatrix} C^{-2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}P^{-1} = P\begin{bmatrix} 0 & C^{-2}B_2 \\ 0 & 0 \end{bmatrix}P^{-1},
$$

$$
(I - AA^{\#})B(A^{\#})^2 = P\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}\begin{bmatrix} C^{-2} & 0 \\ 0 & 0 \end{bmatrix}P^{-1} = P\begin{bmatrix} 0 & 0 \\ B_3C^{-2} & 0 \end{bmatrix}P^{-1},
$$

then $\widehat{G} = \widehat{A}^{\#} = A^{\#} + \varepsilon[-A^{\#}BA^{\#} + (A^{\#})^2B(I - AA^{\#}) + (I - AA^{\#})B(A^{\#})^2]$, which completes the proofs of (3.14) and (3.15).

Since Ind(A)=1, then by (i) of Lemma 2.4,

$$rank\begin{bmatrix} B & A \\ A & 0 \end{bmatrix} = 2rank(A) + rank[(I - AA^{\#})B(I - AA^{\#})]. \tag{3.16}$$

It can be seen from (3.16) that $(I - AA^{\#})B(I - AA^{\#}) = 0$ if and only if $rank\begin{bmatrix} B & A \\ A & 0 \end{bmatrix} = 2rank(A)$. Thus ($iii$) $\Leftrightarrow$(v) follows.

The equivalence of (iii) and (iv) follows directly from Lemma 2.3, and the equivalence of (v) and (vi) follows from the equivalence of (a) and (c) in [16, Theorem 2.2], in which the author showed that $\widehat{A}^{\dagger}$ exists if and only if $rank\begin{bmatrix} B & A \\ A & 0 \end{bmatrix} = 2rank(A)$. □

It should be noticed from Theorem 3.2 that for a square dual matrix $\widehat{A}$ whose prime part has index 1, $\widehat{A}^{\#}$ exists if and only if $\widehat{A}^{\dagger}$ exists. However, if the index of the prime part of a square dual matrix $\widehat{A}$ is not 1, then $\widehat{A}^{\#}$ does not exist, but $\widehat{A}^{\dagger}$ may exist. Moreover, it is easy to see that if $\widehat{A}^{\#}$ exists, then for any nonsingular real matrix P, $(P^{-1}\widehat{A}P)^{\#}$ also exists and $(P^{-1}\widehat{A}P)^{\#} = P^{-1}\widehat{A}^{\#}P$, whereas the DMPGI does not possess this property.

If the prime part of a dual matrix $\widehat{A} = A + \varepsilon B$ has index 1, then it is not difficult for us to see whether $\widehat{A}^{\#}$ exists. For example, we can calculate the rank of A and the rank of the 2×2 block matrix $\begin{bmatrix} B & A \\ A & 0 \end{bmatrix}$ to see if the condition (v) in Theorem 3.2 holds. On the other hand, it is an unexpected result that if the dual group inverse of $\widehat{A}$ exists, then by Lemma 2.3, the prime part and the dual part of $\widehat{A}^{\#}$ are respectively the (1,1)-entry and the (1,2)-entry of the group inverse of the 2×2 upper triangular block matrix $\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}$, where A and B are respectively the prime part and the dual part of $\widehat{A}$. In another word, $\widehat{A}^{\#}$ is completely determined by the group inverse of the block matrix $\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}$. Hence, in order to obtain $\widehat{A}^{\#}$, we need only to compute $\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^{\#}$, which is an efficient method to compute $\widehat{A}^{\#}$.

Corollary 3.1. *Let $\widehat{A} = A + \varepsilon B$ be a dual matrix. If $\widehat{A}^{\#}$ exists, then*

$$\widehat{A}^{\#} = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^{\#} \begin{bmatrix} I \\ \varepsilon I \end{bmatrix}.$$

Corollary 3.2. *Let $\widehat{A} = A + \varepsilon B$ be a dual matrix with $A, B \in \mathbb{R}^{n\times n}$ and Ind$(A) = 1$. Then the following statements are equivalent:*

(i) The dual group inverse of $\widehat{A}$ exists and $\widehat{A}^{\#} = A^{\#} - \varepsilon A^{\#}BA^{\#}$;

(ii) $AA^{\#}B = BAA^{\#} = B$;

(iii) $\mathcal{R}(B) \subset \mathcal{R}(A)$ and $\mathcal{N}(A) \subset \mathcal{N}(B)$;

(iv) $rank\begin{bmatrix} B & A \end{bmatrix} = rank\begin{bmatrix} B \\ A \end{bmatrix} = rank(A)$.

Proof. $(i) \Longrightarrow (ii)$: If $\widehat{A}^{\#}$ exists, then by (ii) of Theorem 3.2,

$$\widehat{A} = P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}P^{-1}.$$

Moreover, if $\widehat{A}^{\#} = A^{\#} - \varepsilon A^{\#}BA^{\#}$, then by (3.15) we have that $(A^{\#})^2B(I-AA^{\#}) + (I-AA^{\#})B(A^{\#})^2 = 0$, i.e.,

$$\begin{bmatrix} C^{-2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}\begin{bmatrix} C^{-2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & C^{-2}B_2 \\ B_3C^{-2} & 0 \end{bmatrix} = 0.$$

Thus, $C^{-2}B_2 = 0$ and $B_3C^{-2} = 0$, i.e., $B_2 = 0$ and $B_3 = 0$. Now, $B = P\begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$ and it is easy to see that $AA^{\#}B = BAA^{\#} = B$.

$(ii) \Longrightarrow (i)$: If $AA^{\#}B = BAA^{\#} = B$, then $(I - AA^{\#})B(I - AA^{\#}) = 0$. Thus, by Theorem 3.2, the dual group inverse of $\widehat{A}$ exists. It is also clear that $(A^{\#})^2B(I - AA^{\#}) + (I - AA^{\#})B(A^{\#})^2 = 0$. Therefore, it follows from (3.15) that $\widehat{A}^{\#} = A^{\#} - \varepsilon A^{\#}BA^{\#}$.

The equivalence of (ii) and (iii) follows immediately from Lemma 2.2.

$(ii) \Longleftrightarrow (iv)$: We can observe from (ii) and (iii) of Lemma 2.4 that

$$rank\begin{bmatrix} B & A \end{bmatrix} = rank(A) + rank(B - AA^{\#}B)$$

and

$$rank\begin{bmatrix} B \\ A \end{bmatrix} = rank(A) + rank(B - BAA^{\#}).$$

Hence, $AA^{\#}B = BAA^{\#} = B$ if and only if $rank\begin{bmatrix} B & A \end{bmatrix} = rank\begin{bmatrix} B \\ A \end{bmatrix} = rank(A)$. □

Example 3.1. We provide an example from kinematics in which three line-vectors from points p_i to q_i, $1 \leq i \leq 3$, are drawn on a flat plane that lies in the plane $y = 2$ in an inertial coordinate frame. The points have coordinates

$$p_1 = (1, 2, 2),\ p_2 = (3, 2, 2),\ p_3 = (5, 2, 4)$$

and

$$q_1 = (3, 2, 3),\ q_2 = (4, 2, 3),\ q_3 = (8, 2, 6).$$

Then the dual matrix of line-vectors is

$$\widehat{A} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \varepsilon\begin{bmatrix} 2 & 2 & 4 \\ 3 & -1 & 2 \\ -4 & -2 & -6 \end{bmatrix} := A + \varepsilon B,$$

where the i-th column of the prime part and the dual part of $\widehat{A}$ are respectively $q_i - p_i$ and $p_i \times q_i$, $1 \leq i \leq 3$.

Since $rank(A) = rank(A^2) = 2$, then $A^\#$ exists. Moreover, $rank\begin{bmatrix} B & A \\ A & 0 \end{bmatrix} = 4 = 2rank(A)$. Hence, by Theorem 3.2, $\widehat{A}^\dagger$ and $\widehat{A}^\#$ exist. It follows from [16] that

$$\begin{aligned}
\widehat{A}^\dagger &= A^\dagger - \varepsilon[A^\dagger BA^\dagger - (A^TA)^\dagger B^T(I - AA^\dagger) - (I - A^\dagger A)B^T(AA^T)^\dagger] \\
&= \begin{bmatrix} 1.0000 & 0.0000 & -1.3333 \\ -1.0000 & 0.0000 & 1.6667 \\ 0.0000 & 0.0000 & 0.3333 \end{bmatrix} + \varepsilon \begin{bmatrix} -2.6667 & 10.6667 & -2.0000 \\ 3.3333 & -12.3333 & 2.0000 \\ 0.6667 & -1.6667 & 0.0000 \end{bmatrix}.
\end{aligned}$$

On the other hand,

$$\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^\# = \left[\begin{array}{ccc|ccc} 2 & -5 & -3 & 27 & -78 & -51 \\ 0 & 0 & 0 & 13 & -35 & -22 \\ -1 & 3 & 2 & -21 & 60 & 39 \\ \hline 0 & 0 & 0 & 2 & -5 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & 2 \end{array}\right].$$

Therefore,

$$\widehat{A}^\# = \begin{bmatrix} 2 & -5 & -3 \\ 0 & 0 & 0 \\ -1 & 3 & 2 \end{bmatrix} + \varepsilon \begin{bmatrix} 27 & -78 & -51 \\ 13 & -35 & -22 \\ -21 & 60 & 39 \end{bmatrix}.$$

Example 3.2. Let $\widehat{A} = A + \varepsilon B$ be a dual matrix, where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & -3 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 2 & -4 & 3 \\ 0 & 0 & 0 \\ 1 & -5 & 6 \end{bmatrix}.$$

Then $rank(A) = rank(A^2) = 2$ implies that $A^\#$ exists and

$$A^\# = \begin{bmatrix} 1.0000 & -1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ -0.3333 & 0.1111 & 0.3333 \end{bmatrix}.$$

Moreover, it is not difficult to see that rank$\begin{bmatrix} B & A \end{bmatrix} = rank\begin{bmatrix} B \\ A \end{bmatrix} = 2$. Hence, by Corollary 3.2, $\widehat{A}^\#$ exists and

$$\widehat{A}^\# = A^\# - \varepsilon A^\# BA^\# = \begin{bmatrix} 1.0000 & -1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ -0.3333 & 0.1111 & 0.3333 \end{bmatrix} + \varepsilon \begin{bmatrix} 1.0000 & -1.6667 & 1.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ -0.6667 & 0.4444 & 0.3333 \end{bmatrix}.$$

4. Dual group-inverse solution of the system of linear dual equations $\widehat{A}\widehat{x} = \widehat{b}$

In this section, we consider the linear dual equation

$$\widehat{A}\widehat{x} = \widehat{b}, \tag{4.1}$$

where $\widehat{A} \in \mathbb{D}^{n\times n}$ and $\widehat{x}, \widehat{b} \in \mathbb{D}^{n}$.

It was shown in [12] that the DMPGI plays an important role in the solutions and least-squares solutions of systems of linear dual equations. Despite the DMPGI, it appears that the dual group inverse also has some kind of least-squares and minimal properties.

Define the range and the null space of a dual matrix $\widehat{A} = A + \varepsilon B \in \mathbb{D}^{n\times n}$ as follows.

$$\mathcal{R}(\widehat{A}) = \{\widehat{w} \in \mathbb{D}^n : \widehat{w} = \widehat{A}\widehat{z},\ \widehat{z} \in \mathbb{D}^n\} = \{Ax + \varepsilon(Ay + Bx) : x, y \in \mathbb{R}^n\}, \tag{4.2}$$

$$\mathcal{N}(\widehat{A}) = \{\widehat{z} \in \mathbb{D}^n : \widehat{A}\widehat{z} = 0\} = \{x + \varepsilon y : Ax = 0, Ay + Bx = 0, x, y \in \mathbb{R}^n\}. \tag{4.3}$$

It is not difficult to see that if $\widehat{A}^{\#}$ exists, then $\mathcal{R}(\widehat{A}^{\#}) = \mathcal{R}(\widehat{A})$ and $\mathcal{N}(\widehat{A}^{\#}) = \mathcal{N}(\widehat{A})$.

We firstly give a necessary and sufficient condition for the Eq (4.1) to be consistent under the assumption that $\widehat{A}^{\#}$ exists, which is analogous to the case when A and b are real. We omit the proof.

Theorem 4.1. *If the dual group inverse of $\widehat{A} \in \mathbb{D}^{n\times n}$ exists, then the linear dual equation (4.1) is consistent if and only if $\widehat{A}\widehat{A}^{\#}\widehat{b} = \widehat{b}$. In this case, the general solution to (4.1) is*

$$\widehat{x} = \widehat{A}^{\#}\widehat{b} + (I - \widehat{A}\widehat{A}^{\#})\widehat{z}, \tag{4.4}$$

where $\widehat{z} \in \mathbb{D}^n$ is an arbitrary dual vector.

Notice that the condition $\widehat{A}\widehat{A}^{\#}\widehat{b} = \widehat{b}$ in Theorem 4.1 is equivalent to $\widehat{b} \in \mathcal{R}(\widehat{A})$. If $\widehat{A}^{\#}\widehat{b}$ is a solution to (4.1), then we call it the *dual group-inverse solution* to the linear dual equation (4.1). Although, in the case $\widehat{b} \notin \mathcal{R}(\widehat{A})$, the dual vector $\widehat{A}^{\#}\widehat{b}$ is not a solution to (4.1), for convenience, we also call it the dual group-inverse solution.

Next, we present some characterizations of the dual group-inverse solution $\widehat{A}^{\#}\widehat{b}$.

Theorem 4.2. *If the dual group inverse of a dual matrix $\widehat{A} = A + \varepsilon B \in \mathbb{D}^{n\times n}$ exists, then $\widehat{A}^{\#}\widehat{b}$ is the unique solution in $\mathcal{R}(\widehat{A})$ of*

$$\widehat{A}^2\widehat{x} = \widehat{A}\widehat{b}. \tag{4.5}$$

Proof. Firstly, if $\widehat{A}^{\#}$ exists, then it is obvious that the Eq (4.5) is consistent and $\widehat{A}^{\#}\widehat{b}$ is a solution.

It is clear that $\widehat{A}^{\#}\widehat{b} \in \mathcal{R}(\widehat{A}^{\#}) = \mathcal{R}(\widehat{A})$. Suppose that $\widehat{u}$ is another solution in $\mathcal{R}(\widehat{A})$ of (4.5). Then $\widehat{u} - \widehat{A}^{\#}\widehat{b} \in \mathcal{R}(\widehat{A})$. On the other hand, since $\widehat{u}$ and $\widehat{A}^{\#}\widehat{b}$ are solutions of (4.5), then $\widehat{A}^2(\widehat{u} - \widehat{A}^{\#}\widehat{b}) = 0$, which implies that $\widehat{A}^{\#}\widehat{A}^2(\widehat{u} - \widehat{A}^{\#}\widehat{b}) = \widehat{A}(\widehat{u} - \widehat{A}^{\#}\widehat{b}) = 0$, i.e., $\widehat{u} - \widehat{A}^{\#}\widehat{b} \in \mathcal{N}(\widehat{A})$. Hence, $\widehat{u} - \widehat{A}^{\#}\widehat{b} \in \mathcal{R}(\widehat{A}) \bigcap \mathcal{N}(\widehat{A})$.

Recall that for a real square matrix A, if $A^{\#}$ exists, then $\mathcal{R}(A) \bigcap \mathcal{N}(A) = \{0\}$. Next, we conclude that $\mathcal{R}(\widehat{A}) \bigcap \mathcal{N}(\widehat{A}) = \{0\}$, thus $\widehat{u} = \widehat{A}^{\#}\widehat{b}$ and the uniqueness of $\widehat{A}^{\#}\widehat{b}$ in $\mathcal{R}(\widehat{A})$ follows.

Indeed, for any $\widehat{z} \in \mathcal{R}(\widehat{A}) \bigcap \mathcal{N}(\widehat{A})$, we can see from (4.2) that there exist $x, y \in \mathbb{R}^n$ such that $\widehat{z} = Ax + \varepsilon(Ay + Bx)$. Moreover, since $\widehat{A}\widehat{z} = 0$, then

$$(A + \varepsilon B)[Ax + \varepsilon(Ay + Bx)] = A^2x + \varepsilon[A^2y + (AB + BA)x] = 0.$$

Thus $A^2x = 0$ and $A^2y + (AB + BA)x = 0$.

It can be seen from $A^2x = A(Ax) = 0$ that $Ax \in \mathcal{R}(A) \bigcap \mathcal{N}(A) = \{0\}$. Therefore, $Ax = 0$. In this case, $0 = A^2y + (AB + BA)x = A^2y + ABx = A(Ay + Bx)$, i.e, $Ay + Bx \in \mathcal{N}(A)$. On the other hand, since $\widehat{A}^{\#}$ exists, then it follows from Theorem 3.2 that $(I - AA^{\#})B(I - AA^{\#}) = 0$, i.e.,

$$B = AA^{\#}B + BAA^{\#} - AA^{\#}BAA^{\#}. \tag{4.6}$$

Substituting (4.6) into $Ay + Bx$ we get

$$Ay + Bx = Ay + (AA^{\#}B + BAA^{\#} - AA^{\#}BAA^{\#})x = A(y + A^{\#}Bx) \in \mathcal{R}(A).$$

Now, $Ay + Bx \in \mathcal{R}(A) \bigcap \mathcal{N}(A) = \{0\}$, i.e., $Ay + Bx = 0$. Therefore, $\widehat{z} = Ax + \varepsilon(Ay + Bx) = 0$, which implies that $\mathcal{R}(\widehat{A}) \bigcap \mathcal{N}(\widehat{A}) = \{0\}$. □

Since (4.5) is analogous to the normal equation $\widehat{A}^T\widehat{A}\widehat{x} = \widehat{A}^T\widehat{b}$ of (4.1), we shall call the linear dual equation (4.5) *the group normal equation* of (4.1). It is obvious that each solution of (4.1) is also a solution of (4.5).

The P-norm of a real vector x is defined as $\| x \|_P = \| P^{-1}x \|_2$, where $\| \cdot \|_2$ is the Eulidean norm and P is a nonsingular matrix that transforms A into its Jordan canonical form (2.1) (see [25]). For $\widehat{x} = u + \varepsilon v \in \mathbb{D}^n$, Udwadia [12] defined a norm of $\widehat{x}$ as

$$< \widehat{x} >= \| u \|_2 + \| v \|_2. \tag{4.7}$$

In this paper, we define a norm of $\widehat{x} = u + \varepsilon v$ as

$$\| \widehat{x} \| = \sqrt{\| u \|_2^2 + \| v \|_2^2}. \tag{4.8}$$

We will show that the expression for the norm given in (4.8) indeed define a norm.

(i) $\| \widehat{x} \| \geq 0$, and $\| \widehat{x} \| = 0$ if and only if $\widehat{x} = 0$.

(ii) For a real scalar k, $\| k\widehat{x} \| = \sqrt{\| ku \|_2^2 + \| kv \|_2^2} = |k|\sqrt{\| u \|_2^2 + \| v \|_2^2} = |k|\| \widehat{x} \|$.

(iii) For two dual vectors $\widehat{x} = u_1 + \varepsilon v_1, \widehat{y} = u_2 + \varepsilon v_2 \in \mathbb{D}^n$,

$$\begin{aligned}
\| \widehat{x} + \widehat{y} \|^2 &= \| u_1 + u_2 \|_2^2 + \| v_1 + v_2 \|_2^2 \\
&= \| u_1 \|_2^2 + \| u_2 \|_2^2 + \| v_1 \|_2^2 + \| v_2 \|_2^2 + 2{u_1}^T u_2 + 2{v_1}^T v_2 \\
&\leq \| u_1 \|_2^2 + \| u_2 \|_2^2 + \| v_1 \|_2^2 + \| v_2 \|_2^2 + 2\| u_1 \|_2 \| u_2 \|_2 + 2\| v_1 \|_2 \| v_2 \|_2 \\
&\leq \| u_1 \|_2^2 + \| u_2 \|_2^2 + \| v_1 \|_2^2 + \| v_2 \|_2^2 + 2\sqrt{(\| u_1 \|_2^2 + \| v_1 \|_2^2)(\| u_2 \|_2^2 + \| v_2 \|_2^2)} \\
&= \left(\sqrt{\| u_1 \|_2^2 + \| v_1 \|_2^2} + \sqrt{\| u_2 \|_2^2 + \| v_2 \|_2^2} \right)^2 \\
&= (\| \widehat{x} \| + \| \widehat{y} \|)^2,
\end{aligned}$$

i.e., $\| \widehat{x} + \widehat{y} \| \leq \| \widehat{x} \| + \| \widehat{y} \|$.

Furthermore, in order to study the minimal properties of the dual group-inverse solution, we define the P-norm of a dual vector $\widehat{x} = u + \varepsilon v$ as

$$\| \widehat{x} \|_P = \| P^{-1}\widehat{x} \| = \sqrt{\| P^{-1}u \|_2^2 + \| P^{-1}v \|_2^2}. \tag{4.9}$$

For a dual vector $\widehat{x}$, by considering the square of the norms defined in (4.7) and (4.8), we can see that the norm of $\widehat{x}$ defined in (4.8) is not greater than the norm of $\widehat{x}$ defined in (4.7). Moreover, the norm defined in (4.9) is a generalization of the norm defined in (4.8), since the norm defined in (4.9) will be reduced to the norm defined in (4.8) if we replace the nonsingular matrix P by the identity matrix I.

When the matrix A (Ind(A)=1) and the vector b are real, it was shown in [24, 26] that $A^{\#}b$ is the unique minimal P-norm least-squares solution of the inconsistent equation $Ax = b$. Now we consider the problem of finding the dual vector $\widehat{x}$ that is analogous to looking for x that makes $\| Ax - b \|_P$ as small as possible when A and b are real. We will show in the following theorem that although the dual group-inverse solution may not be a least-squares solution to the linear dual equation (4.1), because we can not guarantee that $\| \widehat{A}\widehat{A^{\#}}\widehat{b} - \widehat{b} \|_P \leq \| \widehat{A}\widehat{x} - \widehat{b} \|_P$ for any $x \in \mathbb{D}^n$, but the dual group-inverse solution provides a small P-norm of the error $\widehat{e} = \widehat{A}\widehat{x} - \widehat{b}$.

Theorem 4.3. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}^{n\times n}$, $\widehat{b} \in \mathbb{D}^n$ be such that $\widehat{A^{\#}}$ exists. Then*
(i) The choices of $\widehat{x} = \widehat{A^{\#}}\widehat{b} + (I - \widehat{A}\widehat{A^{\#}})\widehat{z}$, where $\widehat{z} \in \mathbb{D}^n$, give a small P-norm of the error of the inconsistent equation $\widehat{A}\widehat{x} = \widehat{b}$. The norm of the error

$$\| \widehat{e} \|_P = \| \widehat{A}\widehat{x} - \widehat{b} \|_P = \| \widehat{A}\widehat{A^{\#}}\widehat{b} - \widehat{b} \|_P, \tag{4.10}$$

where $\widehat{x}$ satisfies the group normal equation (4.5).
(ii) The dual group-inverse solution $\widehat{A^{\#}}\widehat{b}$ has a small P-norm among $\widehat{x} = \widehat{A^{\#}}\widehat{b} + (I - \widehat{A}\widehat{A^{\#}})\widehat{z}$, where $\widehat{z} \in \mathbb{D}^n$. In addition, among the solutions of (4.5), $\widehat{A^{\#}}\widehat{b}$ is the unique dual vector which is orthogonal to the null space of $\widehat{A}^T$.

Proof. (i) Denoting $\widehat{w_1} = \widehat{A}\widehat{A^{\#}}\widehat{b} - \widehat{b} = u_1 + \varepsilon v_1$ and $\widehat{w_2} = \widehat{A}\widehat{x} - \widehat{A}\widehat{A^{\#}}\widehat{b} = u_2 + \varepsilon v_2$. Then

$$\| \widehat{e} \|_P = \| \widehat{A}\widehat{x} - \widehat{b} \|_P = \| \widehat{w_1} + \widehat{w_2} \|_P = \sqrt{\| u_1 + u_2 \|_P^2 + \| v_1 + v_2 \|_P^2}. \tag{4.11}$$

It can be seen from the block representations of $\widehat{A}$ and $\widehat{A^{\#}}$ in Theorem 3.2 that

$$\begin{aligned}
[P^{-1}(\widehat{A}\widehat{A^{\#}} - I)P]^T(P^{-1}\widehat{A}P) &= \left(\begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & (B_3C^{-1})^T \\ (C^{-1}B_2)^T & 0 \end{bmatrix} \right) \\
&\times \left(\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix} \right) \\
&= \varepsilon \begin{bmatrix} 0 & 0 \\ (C^{-1}B_2)^TC - B_3 & 0 \end{bmatrix} := \varepsilon M.
\end{aligned}$$

If we denote $P^{-1}\widehat{b} = x + \varepsilon y$ and $P^{-1}(\widehat{x} - \widehat{A^{\#}}\widehat{b}) = w + \varepsilon z$, then

$$\begin{aligned}
(P^{-1}\widehat{w_1})^T(P^{-1}\widehat{w_2}) &= [P^{-1}(\widehat{A}\widehat{A^{\#}}\widehat{b} - \widehat{b})]^T[P^{-1}(\widehat{A}\widehat{x} - \widehat{A}\widehat{A^{\#}}\widehat{b})] \\
&= [P^{-1}(\widehat{A}\widehat{A^{\#}} - I)PP^{-1}\widehat{b}]^T[P^{-1}\widehat{A}PP^{-1}(\widehat{x} - \widehat{A^{\#}}\widehat{b})] \\
&= (P^{-1}\widehat{b})^T[P^{-1}(\widehat{A}\widehat{A^{\#}} - I)P]^T(P^{-1}\widehat{A}P)[P^{-1}(\widehat{x} - \widehat{A^{\#}}\widehat{b})] \\
&= \varepsilon x^T M w,
\end{aligned}$$

i.e., the prime part of $(P^{-1}\widehat{w_1})^T(P^{-1}\widehat{w_2})$ is zero. Thus $(P^{-1}u_1)^T(P^{-1}u_2) = 0$.

Hence, $\| u_1 + u_2 \|_P^2 = \| P^{-1}(u_1 + u_2) \|_2^2 = (P^{-1}u_1 + P^{-1}u_2)^T(P^{-1}u_1 + P^{-1}u_2) = \| u_1 \|_P^2 + \| u_2 \|_P^2 + 2(P^{-1}u_1)^T(P^{-1}u_2) = \| u_1 \|_P^2 + \| u_2 \|_P^2$.

Substituting the above equality into (4.11) we get an upper bound for $\| \widehat{e} \|_P$, i.e,

$$\| \widehat{e} \|_P = \| \widehat{A}\widehat{x} - \widehat{b} \|_P \leq \sqrt{\| u_1 \|_P^2 + \| u_2 \|_P^2 + (\| v_1 \|_P + \| v_2 \|_P)^2} := \delta_1. \tag{4.12}$$

Notice that the dual vector $\widehat{w_1}$ depends only on $\widehat{A}$ and $\widehat{b}$, in order to obtain the smallest value of the upper bound δ_1 given in (4.12), we can choose $\widehat{x}$ to make $\widehat{w_2} = 0$ so that $\| u_2 \|_P = \| v_2 \|_P = 0$.

We remark that if $\widehat{A}^{\#}$ exists, then the group normal equation (4.5) is equivalent to $\widehat{A}\widehat{x} = \widehat{A}\widehat{A}^{\#}\widehat{b}$, and it is not difficult to see that the general solution of the group normal equation (4.5) is

$$\widehat{x} = \widehat{A}^{\#}\widehat{b} + (I - \widehat{A}\widehat{A}^{\#})\widehat{z},$$

where z is an arbitrary dual vector.

Hence, the choices of $\widehat{x}$ that satisfy (4.5) will cause $\widehat{w_2}$ to vanish. The P-norm of the error is given by

$$\| \widehat{e} \|_P = \| \widehat{w_1} \|_P = \sqrt{\| u_1 \|_P^2 + \| v_1 \|_P^2} = \| \widehat{A}\widehat{A}^{\#}\widehat{b} - \widehat{b} \|_P.$$

(ii) Denoting the dual vectors $\widehat{A}^{\#}\widehat{b} = \widehat{\mu_1} = \alpha_1 + \varepsilon\beta_1$, $(I - \widehat{A}\widehat{A}^{\#})\widehat{z} = \widehat{\mu_2} = \alpha_2 + \varepsilon\beta_2$. Then

$$\begin{aligned}(P^{-1}\widehat{\mu_1})^T(P^{-1}\widehat{\mu_2}) &= (P^{-1}\widehat{A}^{\#}\widehat{b})^T[P^{-1}(I - \widehat{A}\widehat{A}^{\#})\widehat{z}] \\ &= (P^{-1}\widehat{A}^{\#}PP^{-1}\widehat{b})^T[P^{-1}(I - \widehat{A}\widehat{A}^{\#})PP^{-1}\widehat{z}] \\ &= (P^{-1}\widehat{b})^T(P^{-1}\widehat{A}^{\#}P)^T[P^{-1}(I - \widehat{A}\widehat{A}^{\#})P](P^{-1}\widehat{z})\end{aligned}$$

and it can be seen from the block representations of $\widehat{A}$ and $\widehat{A}^{\#}$ that the prime part of $(P^{-1}\widehat{A}^{\#}P)^T[P^{-1}(I - \widehat{A}\widehat{A}^{\#})P]$ is zero. Thus the prime part of $(P^{-1}\widehat{\mu_1})^T(P^{-1}\widehat{\mu_2})$ is also zero, i.e., $(P^{-1}\alpha_1)^T(P^{-1}\alpha_2) = 0$. Therefore, $\| \alpha_1 + \alpha_2 \|_P^2 = \| \alpha_1 \|_P^2 + \| \alpha_2 \|_P^2$.

Thus, as before, we obtain an upper bound for the P-norm of the dual vector $\widehat{x}$ given by

$$\| \widehat{x} \|_P \leq \sqrt{\| \alpha_1 \|_P^2 + \| \alpha_2 \|_P^2 + (\| \beta_1 \|_P + \| \beta_2 \|_P)^2} := \delta_2. \tag{4.13}$$

To make δ_2 in (4.13) as small as possible, we can choose $\widehat{z} = 0$ such that $\widehat{\mu_2} = (I - \widehat{A}\widehat{A}^{\#})\widehat{z} = 0$. In this case,

$$\| \widehat{x} \|_P = \| \widehat{\mu_1} \|_P = \sqrt{\| \alpha_1 \|_P^2 + \| \beta_1 \|_P^2} = \| \widehat{A}^{\#}\widehat{b} \|_P.$$

By Theorem 4.1, $\mathcal{N}(\widehat{A}^T) = [I - \widehat{A}^T(\widehat{A}^T)^{\#}]\widehat{z}$, where $\widehat{z}$ is an arbitrary dual vector. For any $\widehat{y} \in \mathcal{R}(\widehat{A})$, there exists a dual vector $\widehat{x} \in \mathbb{D}^n$ such that $\widehat{y} = \widehat{A}\widehat{x}$. It follows that $\widehat{y}^T[I - \widehat{A}^T(\widehat{A}^T)^{\#}]\widehat{z} = \widehat{x}^T\widehat{A}^T[I - \widehat{A}^T(\widehat{A}^T)^{\#}]\widehat{z} = 0$, i.e., $\mathcal{R}(\widehat{A})$ is orthogonal to $\mathcal{N}(\widehat{A}^T)$. Moreover, by Theorem 4.2, $\widehat{A}^{\#}\widehat{b}$ is the unique solution in $\mathcal{R}(\widehat{A})$ of $\widehat{A}^2\widehat{x} = \widehat{A}\widehat{b}$, therefore $\widehat{A}^{\#}\widehat{b}$ is orthogonal to $\mathcal{N}(\widehat{A}^T)$. □

Example 4.1. Consider the inconsistent equation $\widehat{A}\widehat{x} = \widehat{b}$ given in [12], where

$$\widehat{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 14 \end{bmatrix} := A + \varepsilon B, \quad \widehat{b} = \begin{bmatrix} 8.2 \\ 7.3 \\ 15.1 \end{bmatrix} + \varepsilon \begin{bmatrix} 30.2 \\ 32.8 \\ 53.6 \end{bmatrix}.$$

Then A is diagonalizable, i.e.,

$$\begin{aligned} A &= \begin{bmatrix} 0.4082 & 0.7071 & -0.3015 \\ 0.4082 & -0.7071 & -0.3015 \\ 0.8165 & 0.0000 & 0.9045 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.4082 & 0.7071 & -0.3015 \\ 0.4082 & -0.7071 & -0.3015 \\ 0.8165 & 0.0000 & 0.9045 \end{bmatrix}^{-1} \\ &:= PDP^{-1}. \end{aligned}$$

It is obvious that $rank(A) = rank(A^2) = 2$. Hence, $A^{\#}$ exists. Moreover, since $rank\begin{bmatrix} B & A \\ A & 0 \end{bmatrix} = 4$, then by Theorem 3.2, $\widehat{A}^{\#}$ exists.

Therefore, by Corollary 3.1,

$$\begin{aligned} \widehat{A}^{\#} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^{\#} \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix} \\ &= \begin{bmatrix} -0.4400 & 0.5600 & 0.0400 \\ 0.5600 & -0.4400 & 0.0400 \\ 0.1200 & 0.1200 & 0.0800 \end{bmatrix} + \varepsilon \begin{bmatrix} -0.9200 & 0.1600 & -0.2000 \\ -0.8800 & 0.2000 & 0.1600 \\ -0.3600 & -1.2000 & -0.0800 \end{bmatrix}. \end{aligned}$$

The P-norm of the error is

$$\| \widehat{e} \|_P = \| \widehat{A}\widehat{A}^{\#}\widehat{b} - \widehat{b} \|_P = \sqrt{\| P^{-1}u \|_2^2 + \| P^{-1}v \|_2^2} = 0.7452,$$

where $u = [-0.0800, -0.0800, 0.2400]^T$ and $v = [-0.3800, -0.3000, -0.3000]^T$.

On the other hand,

$$\widehat{A}^{\#}\widehat{b} = \begin{bmatrix} 1.0840 \\ 1.9840 \\ 3.0680 \end{bmatrix} + \varepsilon \begin{bmatrix} -2.1720 \\ 1.2840 \\ -1.0720 \end{bmatrix} := x + \varepsilon y.$$

Then

$$\| \widehat{A}^{\#}\widehat{b} \|_P = \sqrt{\| P^{-1}x \|_2^2 + \| P^{-1}y \|_2^2} = 4.6795.$$

We will show in the following that if $\widehat{A}^{\#}$ exists and $\widehat{A}^{\#} = A^{\#} - \varepsilon A^{\#}BA^{\#}$, then the dual group-inverse solution $\widehat{A}^{\#}\widehat{b}$ is the minimal P-norm least-squares solution to the inconsistent equation $\widehat{A}\widehat{x} = \widehat{b}$.

Theorem 4.4. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}^{n\times n}$, $\widehat{b} \in \mathbb{D}^n$ be such that $\widehat{A}^{\#}$ exists and $\widehat{A}^{\#} = A^{\#} - \varepsilon A^{\#}BA^{\#}$. Then $\widehat{x}^*$ satisfies*

$$\| \widehat{b} - \widehat{A}\widehat{x}^* \|_P = \min_{\widehat{x}\in\mathbb{D}^n} \| \widehat{b} - \widehat{A}\widehat{x} \|_P$$

if and only if $\widehat{x}^*$ satisfies the group normal equation (4.5). Moreover, the dual group-inverse solution $\widehat{A}^{\#}\widehat{b}$ is the unique minimal P-norm solution of (4.5).

Proof. Write $\widehat{b} = \widehat{A}\widehat{A}^{\#}\widehat{b} + (I - \widehat{A}\widehat{A}^{\#})\widehat{b}$. Then

$$\begin{aligned} \| \widehat{b} - \widehat{A}\widehat{x} \|_P^2 &= \| \widehat{A}\widehat{A}^{\#}\widehat{b} - \widehat{A}\widehat{x} \|_P^2 + \| (I - \widehat{A}\widehat{A}^{\#})\widehat{b} \|_P^2 \\ &+ 2[P^{-1}(\widehat{A}^{\#}\widehat{b} - \widehat{x})]^T (P^{-1}\widehat{A}P)^T [P^{-1}(I - \widehat{A}\widehat{A}^{\#})P]P^{-1}\widehat{b}. \end{aligned} \tag{4.14}$$

If $\widehat{A}^{\#} = A^{\#} - \varepsilon A^{\#}BA^{\#}$, then $\widehat{A}^{\#}$ has the block representation

$$\widehat{A}^{\#} = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1} + \varepsilon P\begin{bmatrix} -C^{-1}B_1C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}. \tag{4.15}$$

It can be deduced from (2.1) and (4.15) that the third term of the right hand side of (4.14) vanishes. Hence,

$$\|\,\widehat{b} - \widehat{A}\widehat{x}\,\|_P^2 = \|\,\widehat{A}\widehat{A}^{\#}\widehat{b} - \widehat{A}\widehat{x}\,\|_P^2 + \|\,(I - \widehat{A}\widehat{A}^{\#})\widehat{b}\,\|_P^2 \geq \|\,(I - \widehat{A}\widehat{A}^{\#})\widehat{b}\,\|_P^2,$$

the equality holds if and only if $\widehat{A}\widehat{x} = \widehat{A}\widehat{A}^{\#}\widehat{b}$.

On the other hand, since $\widehat{A}\widehat{x} = \widehat{A}\widehat{A}^{\#}\widehat{b}$ is equivalent to (4.5) and the general solution to (4.5) is

$$\widehat{x} = \widehat{A}^{\#}\widehat{b} + (I - \widehat{A}\widehat{A}^{\#})\widehat{z}.$$

Then

$$\|\,\widehat{A}^{\#}\widehat{b} + (I - \widehat{A}\widehat{A}^{\#})\widehat{z}\,\|_P^2 = \|\,\widehat{A}^{\#}\widehat{b}\,\|_P^2 + \|\,(I - \widehat{A}\widehat{A}^{\#})\widehat{z}\,\|_P^2 \geq \|\,\widehat{A}^{\#}\widehat{b}\,\|_P^2.$$

Equality in the above relation holds if and only if $(I - \widehat{A}\widehat{A}^{\#})\widehat{z} = 0$, i.e., $\widehat{x} = \widehat{A}^{\#}\widehat{b}$. □

Corollary 4.1. *Let $\widehat{A} = A + \varepsilon B \in \mathbb{D}^{n\times n}$, $\widehat{b} \in \mathbb{D}^n$ be such that $\widehat{A}^{\#}$ exists and $\widehat{A}^{\#} = A^{\#} - \varepsilon A^{\#}BA^{\#}$. Then, if $\widehat{A}\widehat{x} = \widehat{b}$ is consistent, then $\widehat{A}^{\#}\widehat{b}$ is the unique minimal P-norm solution of $\widehat{A}\widehat{x} = \widehat{b}$; if $\widehat{A}\widehat{x} = \widehat{b}$ is inconsistent, then $\widehat{A}^{\#}\widehat{b}$ is the unique minimal P-norm least-squares solution of $\widehat{A}\widehat{x} = \widehat{b}$.*

5. Conclusions

This paper mainly studied the existence, computations and applications of the dual group inverse. We have shown some differences between the dual group inverse of square dual matrices and the group inverse of square real matrices, especially in the existence and computations. An interesting phenomenon is that for a dual matrix $\widehat{A}$ whose prime part has index 1, $\widehat{A}^{\#}$ exists if and only if $\widehat{A}^{\dagger}$ exists. If the dual group inverse of a dual matrix $\widehat{A} = A + \varepsilon B$ exists, then $\widehat{A}^{\#}$ can be easily obtained by computing the group inverse of the 2×2 upper triangular block matrix $\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}$.

We also discussed the applications of the dual group inverse in solving systems of linear dual equations. Some results which are analogous to the real matrices were obtained. For one thing, if the coefficient dual matrix of the linear dual equation $\widehat{A}\widehat{x} = \widehat{b}$ exists and $\widehat{A}^{\#} = A^{\#} + \varepsilon[-A^{\#}BA^{\#} + (A^{\#})^2B(I - AA^{\#}) + (I - AA^{\#})B(A^{\#})^2]$, then the least-squares and minimal properties of the linear dual equation $\widehat{A}\widehat{x} = \widehat{b}$ are somewhat different from those of the real case. For another, if the coefficient dual matrix of the linear dual equation $\widehat{A}\widehat{x} = \widehat{b}$ exists and $\widehat{A}^{\#} = A^{\#} - \varepsilon A^{\#}BA^{\#}$, then the least-squares and minimal properties of the linear dual equation $\widehat{A}\widehat{x} = \widehat{b}$ are almost the same as those of the real case.

We can see from Theorem 3.2 that the condition Ind(A)=1 is necessary for the existence of the dual group inverse of the dual matrix $\widehat{A} = A + \varepsilon B$. However, the indices of the prime parts of many dual matrices from kinematics and mechanisms may be larger than one. In this case, in order to deal with some problems in kinematics and mechanisms, we have to introduce some new dual generalized inverses. That will be our future work.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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