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Research article

Wong-Zakai approximations and long term behavior of second order non-autonomous stochastic lattice dynamical systems with additive noise

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Abstract: In this article, we investigate the Wong-Zakai approximations of a class of second order non-autonomous stochastic lattice systems with additive white noise. We first prove the existence and uniqueness of tempered pullback random attractors for the original stochastic system and its Wong-Zakai approximation. Then, we establish the upper semicontinuity of these attractors for Wong-Zakai approximations as the step-length of the Wiener shift approaches zero.

Keywords: Wong-Zakai approximation; lattice systems; random attractor; white noise; upper semicontinuity
Mathematics Subject Classification: 37L30, 37L55, 37L60

1. Introduction

Lattice dynamical systems arise from a variety of applications in electrical engineering, biology, chemical reaction, pattern formation and so on, see, e.g., [4, 7, 14, 19, 33]. Many researchers have discussed broadly the deterministic models in [6, 12, 34, 39], etc. Stochastic lattice equations, driven by additive independent white noise, was discussed for the first time in [2], followed by extensions in [8, 13, 15, 16, 21, 23, 27, 32, 35–38, 40].

In this paper, we will study the long term behavior of the following second order non-autonomous stochastic lattice system driven by additive white noise: for given $\tau \in \mathbb{R}$, $t > \tau$ and $i \in \mathbb{Z}$,

$$\begin{cases} \ddot{u} + \nu A\dot{u} + h(\dot{u}) + Au + \lambda u + f(u) = g(t) + a\dot{\omega}(t), \\ u(\tau) = (u_{\tau i})_{i\in\mathbb{Z}} = u_\tau, \ \ \dot{u}(\tau) = (u^1_{\tau i})_{i\in\mathbb{Z}} = u^1_\tau, \end{cases} \tag{1.1}$$

where $u = (u_i)_{i\in\mathbb{Z}}$ is a sequence in l^2, ν and λ are positive constants, $\dot{u} = (\dot{u}_i)_{i\in\mathbb{Z}}$ and $\ddot{u} = (\ddot{u}_i)_{i\in\mathbb{Z}}$ denote the fist and the second order time derivatives respectively, $Au = ((Au)_i)_{i\in\mathbb{Z}}$, $A\dot{u} = ((A\dot{u})_i)_{i\in\mathbb{Z}}$, A is a linear operators defined in (2.2), $a = (a_i)_{i\in\mathbb{Z}} \in l^2$, $f(u) = (f_i(u_i))_{i\in\mathbb{Z}}$ and $h(\dot{u}) = (h_i(\dot{u}_i))_{i\in\mathbb{Z}}$ satisfy certain conditions, $g(t) = \big(g_i(t)\big)_{i\in\mathbb{Z}} \in L^2_{loc}(\mathbb{R}, l^2)$ is a given time dependent sequence, and $\omega(t) = W(t,\omega)$ is a two-sided real-valued Wiener process on a probability space.

The approximation we use in the paper was first proposed in [18,22] where the authors investigated the chaotic behavior of random equations driven by $\mathcal{G}_\delta(\theta_t\omega)$. Since then, their work was extended by many scholars. To the best of my knowledge, there are three forms of Wong-Zakai approximations $\mathcal{G}_\delta(\theta_t\omega)$ used recenly, Euler approximation of Brownian [3, 10, 17, 20, 25, 28–30], Colored noise [5, 11, 26, 31] and Smoothed approximation of Brownian motion by mollifiers [9]. In this paper, we will focus on Euler approximation of Brownian and compare the long term behavior of system (1.1) with pathwise deterministic system given by

$$\begin{cases}\ddot{u}^\delta + \nu A\dot{u}^\delta + h(\dot{u}^\delta) + Au^\delta + \lambda u^\delta + f(u^\delta) = g(t) + a\mathcal{G}_\delta(\theta_t\omega),\\ u^\delta(\tau) = (u^\delta_{\tau i})_{i\in\mathbb{Z}} = u^\delta_\tau,\ \ \dot{u}^\delta(\tau) = (u^{\delta,1}_{\tau i})_{i\in\mathbb{Z}} = u^{\delta,1}_\tau,\end{cases} \tag{1.2}$$

for $\delta \in \mathbb{R}$ with $\delta \neq 0$, $\tau \in \mathbb{R}$, $t > \tau$ and $i \in \mathbb{Z}$, where $\mathcal{G}_\delta(\theta_t\omega)$ is defined in (3.2). Note that the solution of system (1.2) is written as u^δ to show its dependence on δ.

This paper is organized as follows. In Section 2, we prove the existence and uniqueness of random attractors of system (1.1). Section 3 is devoted to consider the the Wong-Zakai approximations associated with system (1.1). In Section 4, we establish the convergence of solutions and attractors for approximate system (1.2) when $\delta \to 0$.

Throughout this paper, the letter c and $c_i (i = 1, 2, \dots)$ are generic positive constants which may change their values from line to line.

2. Stochastic lattice system with additive white noise

In this section, we will define a continuous cocycle for second order non-autonomous stochastic lattice system (1.1), and then prove the existence and uniqueness of pullback attractors.

A standard Brownian motion or Wiener process $(W_t)_{t\in\mathbb{R}}$ (i.e., with two-sided time) in $\mathbb{R}$ is a process with $W_0 = 0$ and stationary independent increments satisfying $W_t - W_s \sim \mathcal{N}(0, |t-s|I)$. $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P is the corresponding Wiener measure on $(\Omega, \mathcal{F})$, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

the probability space $(\Omega, \mathcal{F}, P)$ is called Wiener space. Define the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega,\ t \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system (see [1]) and there exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\tilde{\Omega} \subseteq \Omega$ of full measure such that for each $\omega \in \Omega$,

$$\frac{\omega(t)}{t} \to 0 \ \text{ as } \ t \to \pm\infty. \tag{2.1}$$

For the sake of convenience, we will abuse the notation slightly and write the space $\tilde{\Omega}$ as Ω.

We denote by

$$l^p = \{u | u = (u_i)_{i\in\mathbb{Z}}, u_i \in \mathbb{R},\ \sum_{i\in\mathbb{Z}} |u_i|^p < +\infty\},$$

with the norm as

$$\|u\|_p^p = \sum_{i\in\mathbb{Z}} |u_i|^p.$$

In particular, l^2 is a Hilbert space with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ given by

$$(u, v) = \sum_{i\in\mathbb{Z}} u_i v_i, \qquad \|u\|^2 = \sum_{i\in\mathbb{Z}} |u_i|^2,$$

for any $u = (u_i)_{i\in\mathbb{Z}},\ v = (v_i)_{i\in\mathbb{Z}} \in l^2$.

Define linear operators B, B^*, and A acting on l^2 in the following way: for any $u = (u_i)_{i\in\mathbb{Z}} \in l^2$,

$$(Bu)_i = u_{i+1} - u_i, \qquad (B^*u)_i = u_{i-1} - u_i,$$

and

$$(Au)_i = 2u_i - u_{i+1} - u_{i-1}. \tag{2.2}$$

Then we find that $A = BB^* = B^*B$ and $(B^*u, v) = (u, Bv)$ for all $u, v \in l^2$.

Also, we let $F_i(s) = \int_0^s f_i(r)dr$, $h(\dot{u}) = (h_i(\dot{u}_i))_{i\in\mathbb{Z}}$, $f(u) = (f_i(u_i))_{i\in\mathbb{Z}}$ with $f_i, h_i \in C^1(\mathbb{R}, \mathbb{R})$ satisfy the following assumptions:

$$|f_i(s)| \le \alpha_1(|s|^p + |s|), \tag{2.3}$$

$$sf_i(s) \ge \alpha_2 F_i(s) \ge \alpha_3 |s|^{p+1}, \tag{2.4}$$

and

$$h_i(0) = 0,\ \ 0 < h_1 \le h_i'(s) \le h_2,\ \ \forall s \in \mathbb{R}, \tag{2.5}$$

where $p > 1$, α_i and h_j are positive constants for $i = 1, 2, 3$ and $j = 1, 2$.

In addition, we let

$$\beta = \frac{h_1\lambda}{4\lambda + h_2^2},\ \ \beta < \frac{1}{\nu}, \tag{2.6}$$

and

$$\sigma = \frac{h_1\lambda}{\sqrt{4\lambda + h_2^2}(h_2 + \sqrt{4\lambda + h_2^2})}. \tag{2.7}$$

For any $u, v \in l^2$, we define a new inner product and norm on l^2 by

$$(u, v)_\lambda = (1 - \nu\beta)(Bu, Bv) + \lambda(u, v),\ \ \|u\|_\lambda^2 = (u, u)_\lambda = (1 - \nu\beta)\|Bu\|^2 + \lambda\|u\|^2.$$

Denote

$$l^2 = (l^2, (\cdot,\cdot), \|\cdot\|), \qquad l_\lambda^2 = (l^2, (\cdot,\cdot)_\lambda, \|\cdot\|_\lambda).$$

Then the norms $\|\cdot\|$ and $\|\cdot\|_\lambda$ are equivalent to each other.

Let $E = l_\lambda^2 \times l^2$ endowed with the inner product and norm

$$(\psi_1, \psi_2)_E = (u^{(1)}, u^{(2)})_\lambda + (v^{(1)}, v^{(2)}), \qquad \|\psi\|_E^2 = \|u\|_\lambda^2 + \|v\|^2,$$

for $\psi_j = (u^{(j)}, v^{(j)})^T = ((u_i^{(j)}), (v_i^{(j)}))_{i\in\mathbb{Z}}^T \in E,\ j = 1, 2, \quad \psi = (u, v)^T = ((u_i), (v_i))_{i\in\mathbb{Z}}^T \in E$.

A family $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ of bounded nonempty subsets of E is called tempered (or subexponentially growing) if for every $\epsilon > 0$, the following holds:

$$\lim_{t\to-\infty} e^{\epsilon t}\|D(\tau + t, \theta_t\omega)\|^2 = 0,$$

where $\|D\| = \sup_{x\in D}\|x\|_E$. In the sequel, we denote by $\mathcal{D}$ the collection of all families of tempered nonempty subsets of E, i.e.,

$$\mathcal{D} = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ is tempered in } E\}.$$

The following conditions will be needed for g when deriving uniform estimates of solutions, for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^{\tau} e^{\gamma s}\|g(s)\|^2 ds < \infty, \tag{2.8}$$

and for any $\varsigma > 0$

$$\lim_{t\to-\infty} e^{\varsigma t}\int_{-\infty}^{0} e^{\gamma s}\|g(s + t)\|^2 ds = 0, \tag{2.9}$$

where $\gamma = \min\{\frac{\sigma}{2}, \frac{\alpha_2\beta}{p+1}\}$.

Let $\bar{v} = \dot{u} + \beta u$ and $\bar{\varphi} = (u, \bar{v})^T$, then system (1.1) can be rewritten as

$$\dot{\bar{\varphi}} + L_1(\bar{\varphi}) = H_1(\bar{\varphi}) + G_1(\omega), \tag{2.10}$$

with initial conditions

$$\bar{\varphi}_\tau = (u_\tau, \bar{v}_\tau)^T = (u_\tau, u_\tau^1 + \beta u_\tau)^T,$$

where

$$L_1(\bar{\varphi}) = \begin{pmatrix} \beta u - \bar{v} \\ (1 - \nu\beta)Au + \nu A\bar{v} + \lambda u + \beta^2 u - \beta\bar{v} \end{pmatrix} + \begin{pmatrix} 0 \\ h(\bar{v} - \beta u) \end{pmatrix},$$

$$H_1(\bar{\varphi}) = \begin{pmatrix} 0 \\ -f(u) + g(t) \end{pmatrix}, \quad G_1(\omega) = \begin{pmatrix} 0 \\ a\dot{\omega}(t) \end{pmatrix}.$$

Denote

$$v(t) = \bar{v}(t) - a\omega(t) \text{ and } \varphi = (u, v)^T.$$

By (2.10) we have

$$\dot{\varphi} + L(\varphi) = H(\varphi) + G(\omega), \tag{2.11}$$

with initial conditions

$$\varphi_\tau = (u_\tau, v_\tau)^T = (u_\tau, u_\tau^1 + \beta u_\tau - a\omega(\tau))^T,$$

where

$$L(\varphi) = \begin{pmatrix} \beta u - v \\ (1 - \nu\beta)Au + \nu Av + \lambda u + \beta^2 u - \beta v \end{pmatrix} + \begin{pmatrix} 0 \\ h(v - \beta u + a\omega(t)) \end{pmatrix},$$

$$H(\varphi) = \begin{pmatrix} 0 \\ -f(u) + g(t) \end{pmatrix}, \quad G(\omega) = \begin{pmatrix} a\omega(t) \\ \beta a\omega(t) - \nu Aa\omega(t) \end{pmatrix}.$$

Note that system (2.11) is a deterministic functional equation and the nonlinearity in (2.11) is locally Lipschitz continuous from E to E. Therefore, by the standard theory of functional differential equations, system (2.11) is well-posed. Thus, we can define a continuous cocycle $\Phi_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E \to E$ associated with system (2.10), where for $\tau \in \mathbb{R}$, $t \in \mathbb{R}^+$ and $\omega \in \Omega$

$$\begin{aligned}\Phi_0(t, \tau, \omega, \bar{\varphi}_\tau) &= \bar{\varphi}(t + \tau, \tau, \theta_{-\tau}\omega, \bar{\varphi}_\tau) \\ &= (u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau), \bar{v}(t + \tau, \tau, \theta_{-\tau}\omega, \bar{v}_\tau))^T \\ &= (u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau), v(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau) + a(\omega(t) - \omega(-\tau)))^T \\ &= \varphi(t + \tau, \tau, \theta_{-\tau}\omega, \varphi_\tau) + (0, a(\omega(t) - \omega(-\tau)))^T,\end{aligned}$$

where $v_\tau = \bar{v}_\tau + a\omega(-\tau)$.

Lemma 2.1. *Suppose that (2.3)–(2.8) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $T > 0$, there exists $c = c(\tau, \omega, T) > 0$ such that for all $t \in [\tau, \tau + T]$, the solution φ of system (2.11) satisfies*

$$\begin{aligned}\|\varphi(t, \tau, \omega, \varphi_\tau)\|_E^2 + \int_\tau^t \|\varphi(s, \tau, \omega, \varphi_\tau)\|_E^2 ds \leq & c \int_\tau^t (\|g(s)\|^2 + |\omega(s)|^2 + |\omega(s)|^{p+1}) ds \\ & + c(\|\varphi_\tau\|_E^2 + 2\sum_{i\in\mathbb{Z}} F_i(u_{\tau,i})).\end{aligned}$$

Proof. Taking the inner product $(\cdot,\cdot)_E$ on both side of the system (2.11) with φ, it follows that

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_E^2 + (L(\varphi), \varphi)_E = (H(\varphi), \varphi)_E + (G(\omega), \varphi)_E. \tag{2.12}$$

For the second term on the left-hand side of (2.12), we have

$$(L(\varphi), \varphi)_E = \beta\|u\|_\lambda^2 + \beta^2(u, v) - \beta\|v\|^2 + \nu(Av, v) + (h(v - \beta u + a\omega(t)), v).$$

By the mean value theorem and (2.5), there exists $\xi_i \in (0, 1)$ such that

$$\begin{aligned}&\beta^2(u, v) + (h(v - \beta u + a\omega(t)), v) \\ &= \beta^2(u, v) + \sum_{i\in\mathbb{Z}} h_i'(\xi_i(v_i - \beta u_i + a_i\omega(t)))(v_i - \beta u_i + a_i\omega(t))v_i \\ &\geq (\beta^2 - h_2\beta)\|u\|\|v\| + h_1\|v\|^2 - h_2|(a\omega(t), v)|.\end{aligned}$$

Then

$$\begin{aligned}(L(\varphi), \varphi)_E - \sigma\|\varphi\|_E^2 - \frac{h_1}{2}\|v\|^2 \geq & (\beta - \sigma)\|u\|_\lambda^2 + (\frac{h_1}{2} - \beta - \sigma)\|v\|^2 \\ & - \frac{\beta h_2}{\sqrt{\lambda}}\|u\|_\lambda\|v\| - h_2|(a\omega(t), v)|,\end{aligned}$$

which along with (2.6) and (2.7) implies that

$$(L(\varphi), \varphi)_E \geq \sigma\|\varphi\|_E^2 + \frac{h_1}{2}\|v\|^2 - \frac{\sigma + h_1}{6}\|v\|^2 - c|\omega(t)|^2\|a\|^2. \tag{2.13}$$

As to the first term on the right-hand side of (2.12), by (2.3) and (2.4) we get

$$\begin{aligned}(H(\varphi),\varphi)_E &= (-f(u),\dot{u}+\beta u-a\omega(t))+(g(t),v)\\ &\leq -\frac{d}{dt}(\sum_{i\in\mathbb{Z}}F_i(u_i))-\alpha_2\beta\sum_{i\in\mathbb{Z}}F_i(u_i)+\alpha_1\sum_{i\in\mathbb{Z}}(|u_i|^p+|u_i|)|a_i\omega(t)|+(g(t),v)\\ &\leq -\frac{d}{dt}(\sum_{i\in\mathbb{Z}}F_i(u_i))-\frac{\alpha_2\beta}{p+1}\sum_{i\in\mathbb{Z}}F_i(u_i)+c|\omega(t)|^{p+1}\|a\|^{p+1}\\ &\quad+\frac{\sigma\lambda}{4}\|u\|^2+c\|a\|^2|\omega(t)|^2+\frac{\sigma+h_1}{6}\|v\|^2+c\|g(t)\|^2.\end{aligned}\tag{2.14}$$

The last term of (2.12) is bounded by

$$\begin{aligned}(G(\omega),\varphi)_E &= \omega(t)(a,u)_\lambda+\beta\omega(t)(a,v)-\nu\omega(t)(Aa,v)\\ &\leq \frac{\sigma}{4}\|u\|_\lambda^2+\frac{1}{\sigma}\|a\|_\lambda^2|\omega(t)|^2+\frac{\sigma+h_1}{6}\|v\|^2+c|\omega(t)|^2\|a\|^2.\end{aligned}\tag{2.15}$$

It follows from (2.12)–(2.15) that

$$\begin{aligned}&\frac{d}{dt}\Big(\|\varphi\|_E^2+2\sum_{i\in\mathbb{Z}}F_i(u_i)\Big)+\gamma\Big(\|\varphi\|_E^2+2\sum_{i\in\mathbb{Z}}F_i(u_i)\Big)+\gamma\|\varphi\|_E^2\\ &\leq c\Big(\|g(t)\|^2+|\omega(t)|^2+|\omega(t)|^{p+1}\Big),\end{aligned}\tag{2.16}$$

where $\gamma=\min\{\frac{\sigma}{2},\frac{\alpha_2\beta}{p+1}\}$. Multiplying (2.16) by $e^{\gamma t}$ and then integrating over (τ,t) with $t\geq\tau$, we get for every $\omega\in\Omega$

$$\begin{aligned}&\|\varphi(t,\tau,\omega,\varphi_\tau)\|_E^2+\gamma\int_\tau^t e^{\gamma(s-t)}\|\varphi(s,\tau,\omega,\varphi_\tau)\|_E^2ds\\ &\leq e^{\gamma(\tau-t)}\Big(\|\varphi_\tau\|_E^2+2\sum_{i\in\mathbb{Z}}F_i(u_{\tau,i})\Big)+c\int_\tau^t e^{\gamma(s-t)}\Big(\|g(s)\|^2+|\omega(s)|^2+|\omega(s)|^{p+1}\Big)ds,\end{aligned}\tag{2.17}$$

which implies desired result. □

Lemma 2.2. *Suppose that (2.3)–(2.9) hold. Then the continuous cocycle Φ_0 associated with system (2.10) has a closed measurable $\mathcal{D}$-pullback absorbing set $K_0=\{K_0(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, where for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$*

$$K_0(\tau,\omega)=\{\bar\varphi\in E:\|\bar\varphi\|_E^2\leq R_0(\tau,\omega)\},\tag{2.18}$$

where $\bar\varphi_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$ and $R_0(\tau,\omega)$ is given by

$$R_0(\tau,\omega)=c+c|\omega(-\tau)|^2+c\int_{-\infty}^0 e^{\gamma s}\Big(\|g(s+\tau)\|^2+|\omega(s)-\omega(-\tau)|^2+|\omega(s)-\omega(-\tau)|^{p+1}\Big)ds,\tag{2.19}$$

where c is a positive constant independent of τ, ω and $\mathcal{D}$.

Proof. By (2.17), we get for every $\tau \in \mathbb{R}$, $t \in \mathbb{R}^+$ and $\omega \in \Omega$

$$
\begin{aligned}
&\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})\|_E^2 + \gamma \int_{\tau-t}^{\tau} e^{\gamma(s-\tau)} \|\varphi(s, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})\|_E^2 ds \\
&\leq e^{-\gamma t}\Big(\|\varphi_{\tau-t}\|_E^2 + 2\sum_{i\in\mathbb{Z}} F_i(u_{\tau-t,i})\Big) \\
&\quad + c\int_{\tau-t}^{\tau} e^{\gamma(s-\tau)}\Big(\|g(s)\|^2 + |\omega(s-\tau) - \omega(-\tau)|^2 + |\omega(s-\tau) - \omega(-\tau)|^{p+1}\Big) ds \\
&\leq e^{-\gamma t}\Big(\|\varphi_{\tau-t}\|_E^2 + 2\sum_{i\in\mathbb{Z}} F_i(u_{\tau-t,i})\Big) \\
&\quad + c\int_{-t}^{0} e^{\gamma s}\Big(\|g(s+\tau)\|^2 + |\omega(s) - \omega(-\tau)|^2 + |\omega(s) - \omega(-\tau)|^{p+1}\Big) ds.
\end{aligned}
\tag{2.20}
$$

By (2.1) and (2.8), the last integral on the right-hand side of (2.20) is well defined. For any $s \geq \tau - t$,

$$
\bar{\varphi}(s, \tau - t, \theta_{-\tau}\omega, \bar{\varphi}_{\tau-t}) = \varphi(s, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}) + (0, a(\omega(s-\tau) - \omega(-\tau)))^T,
$$

which along with (2.20) implies that

$$
\begin{aligned}
&\|\bar{\varphi}(\tau, \tau - t, \theta_{-\tau}\omega, \bar{\varphi}_{\tau-t})\|_E^2 + \gamma \int_{\tau-t}^{\tau} e^{\gamma(s-\tau)} \|\bar{\varphi}(s, \tau - t, \theta_{-\tau}\omega, \bar{\varphi}_{\tau-t})\|_E^2 ds \\
&\leq 2\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})\|_E^2 + 2\gamma \int_{\tau-t}^{\tau} e^{\gamma(s-\tau)} \|\varphi(s, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})\|_E^2 ds \\
&\quad + 2\|a\|^2\Big(|\omega(-\tau)|^2 + \gamma\int_{\tau-t}^{\tau} e^{\gamma(s-\tau)} |\omega(s-\tau) - \omega(-\tau)|^2 ds\Big) \\
&\leq 4e^{-\gamma t}\Big(\|\bar{\varphi}_{\tau-t}\|_E^2 + \|a\|^2|\omega(-t) - \omega(-\tau)|^2 + \sum_{i\in\mathbb{Z}} F_i(u_{\tau-t,i})\Big) + c|\omega(-\tau)|^2 \\
&\quad + c\int_{-\infty}^{0} e^{\gamma s}\Big(\|g(s+\tau)\|^2 + |\omega(s) - \omega(-\tau)|^2 + |\omega(s) - \omega(-\tau)|^{p+1}\Big) ds.
\end{aligned}
\tag{2.21}
$$

By (2.3) and (2.4) we have

$$
\sum_{i\in\mathbb{Z}} F_i(u_{\tau-t,i}) \leq \frac{1}{\alpha_2}\sum_{i\in\mathbb{Z}} f_i(u_{\tau-t,i})u_{\tau-t,i} \leq \frac{1}{\alpha_2}\max_{-\|u_{\tau-t}\|\leq s\leq\|u_{\tau-t}\|} |f_i'(s)|\|u_{\tau-t}\|^2. \tag{2.22}
$$

Using $\bar{\varphi}_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$, (2.1) and (2.22), we find

$$
\limsup_{t\to+\infty} 4e^{-\gamma t}\Big(\|\bar{\varphi}_{\tau-t}\|_E^2 + \|a\|^2|\omega(-t) - \omega(-\tau)|^2 + \sum_{i\in\mathbb{Z}} F_i(u_{\tau-t,i})\Big) = 0, \tag{2.23}
$$

which along with (2.21) implies that there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,

$$
\begin{aligned}
&\|\bar{\varphi}(\tau, \tau - t, \theta_{-\tau}\omega, \bar{\varphi}_{\tau-t})\|_E^2 + \gamma \int_{\tau-t}^{\tau} e^{\gamma(s-\tau)} \|\bar{\varphi}(s, \tau - t, \theta_{-\tau}\omega, \bar{\varphi}_{\tau-t})\|_E^2 ds \\
&\leq c + c|\omega(-\tau)|^2 + c\int_{-\infty}^{0} e^{\gamma s}\Big(\|g(s+\tau)\|^2 + |\omega(s) - \omega(-\tau)|^2 + |\omega(s) - \omega(-\tau)|^{p+1}\Big) ds,
\end{aligned}
\tag{2.24}
$$

where c is a positive constant independent of τ, ω and D. Note that K_0 given by (2.18) is closed measurable random set in E. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D \in \mathcal{D}$, it follows from (2.24) that for all $t \geq T$,

$$\Phi_0(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K_0(\tau, \omega), \tag{2.25}$$

which implies that K_0 pullback attracts all elements in $\mathcal{D}$. By (2.1) and (2.9), one can easily check that K_0 is tempered, which along with (2.25) completes the proof. □

Next, we will get uniform estimates on the tails of solutions of system (2.10).

Lemma 2.3. *Suppose that (2.3)–(2.9) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and $\varepsilon > 0$, there exist $T = T(\tau, \omega, D, \varepsilon) > 0$ and $N = N(\tau, \omega, \varepsilon) > 0$ such that for all $t \geq T$, the solution $\bar{\varphi}$ of system (2.10) satisfies*

$$\sum_{|i| \geq N} |\bar{\varphi}_i(\tau, \tau - t, \theta_{-\tau}\omega, \bar{\varphi}_{\tau - t, i})|_E^2 \leq \varepsilon,$$

where $\bar{\varphi}_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ and $|\bar{\varphi}_i|_E^2 = (1 - \nu\beta)|Bu|_i^2 + \lambda|u_i|^2 + |\bar{v}_i|^2$.

Proof. Let η be a smooth function defined on $\mathbb{R}^+$ such that $0 \leq \eta(s) \leq 1$ for all $s \in \mathbb{R}^+$, and

$$\eta(s) = \begin{cases} 0, & 0 \leq s \leq 1; \\ 1, & s \geq 2. \end{cases}$$

Then there exists a constant C_0 such that $|\eta'(s)| \leq C_0$ for $s \in \mathbb{R}^+$. Let k be a fixed positive integer which will be specified later, and set $x = (x_i)_{i\in\mathbb{Z}}$, $y = (y_i)_{i\in\mathbb{Z}}$ with $x_i = \eta(\frac{|i|}{k})u_i$, $y_i = \eta(\frac{|i|}{k})v_i$. Note $\psi = (x, y)^T = ((x_i), (y_i))_{i\in\mathbb{Z}}^T$. Taking the inner product of system (2.11) with ψ, we have

$$(\dot{\varphi}, \psi)_E + (L(\varphi), \psi)_E = (H(\varphi), \psi)_E + (G, \psi)_E. \tag{2.26}$$

For the first term of (2.26), we have

$$\begin{aligned} (\dot{\varphi}, \psi)_E &= (1 - \nu\beta) \sum_{i\in\mathbb{Z}} (B\dot{u})_i(Bx)_i + \lambda \sum_{i\in\mathbb{Z}} \dot{u}_i x_i + \sum_{i\in\mathbb{Z}} \dot{v}_i y_i \\ &= \frac{1}{2}\frac{d}{dt} \sum_{i\in\mathbb{Z}} \eta(\frac{|i|}{k})|\varphi_i|_E^2 + (1 - \nu\beta) \sum_{i\in\mathbb{Z}} (B\dot{u})_i\Big((Bx)_i - \eta(\frac{|i|}{k})(Bu)_i\Big) \\ &\geq \frac{1}{2}\frac{d}{dt} \sum_{i\in\mathbb{Z}} \eta(\frac{|i|}{k})|\varphi_i|_E^2 - \frac{(1 - \nu\beta)C_0}{k} \sum_{i\in\mathbb{Z}} |(B(v - \beta u + a\omega(t))_i||u_{i+1}| \\ &\geq \frac{1}{2}\frac{d}{dt} \sum_{i\in\mathbb{Z}} \eta(\frac{|i|}{k})|\varphi_i|_E^2 - \frac{c}{k}\|\varphi\|_E^2 - \frac{c}{k}|\omega(t)|^2\|a\|^2, \end{aligned} \tag{2.27}$$

where $|\varphi_i|_E^2 = (1 - \nu\beta)|Bu|_i^2 + \lambda|u_i|^2 + |v_i|^2$. As to the second term on the left-hand side of (2.26), we get

$$\begin{aligned} (L(\varphi), \psi)_E =& \beta(1 - \nu\beta)(Au, x) + (1 - \nu\beta)((Au, y) - (Av, x)) + \nu(Av, y) + \lambda\beta(u, x) \\ &+ \beta^2(u, y) - \beta(v, y) + (h(v - \beta u + a\omega(t)), y). \end{aligned}$$

It is easy to check that

$$(Au, x) = \sum_{i\in\mathbb{Z}}(Bu)_i\Big(\eta(\frac{|i|}{k})(Bu)_i + (Bx)_i - \eta(\frac{|i|}{k})(Bu)_i\Big) \geq \sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|Bu|_i^2 - \frac{2C_0}{k}\|u\|^2,$$

$$(Av, y) = \sum_{i\in\mathbb{Z}}(Bv)_i\Big(\eta(\frac{|i|}{k})(Bv)_i + (By)_i - \eta(\frac{|i|}{k})(Bv)_i\Big) \geq \sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|Bv|_i^2 - \frac{2C_0}{k}\|v\|^2,$$

and

$$(Au, y) - (Av, x) \geq -\frac{C_0}{k}\sum_{i\in\mathbb{Z}}|(Bu)_i||v_{i+1}| - \frac{C_0}{k}\sum_{i\in\mathbb{Z}}|(Bv)_i||u_{i+1}| \geq -\frac{2C_0}{k}(\|u\|^2 + \|v\|^2).$$

By the mean value theorem and (2.5), there exists $\xi_i \in (0, 1)$ such that

$$\begin{aligned}
&\beta^2(u, y) + (h(v - \beta u + a\omega(t)), y)\\
&= \beta^2\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})u_iv_i + \sum_{i\in\mathbb{Z}}h_i'(\xi_i(v_i - \beta u_i + a_i\omega(t)))(v_i - \beta u_i + a_i\omega(t))\eta(\frac{|i|}{k})v_i\\
&\geq \beta(\beta - h_2)\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|u_iv_i| + h_1\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|v_i|^2 - h_2\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|v_ia_i\omega(t)|.
\end{aligned}$$

Then

$$\begin{aligned}
&(L(\varphi), \varphi)_E - \sigma\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|\varphi_i|_E^2 - \frac{h_1}{2}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|v_i|^2\\
&\geq (\beta - \sigma)\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})\Big((1 - \nu\beta)|Bu|_i^2 + \lambda u_i^2\Big) + (\frac{h_1}{2} - \beta - \sigma)\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|v_i|^2\\
&\quad - \frac{\beta h_2}{\sqrt{\lambda}}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|v_i|\Big((1 - \nu\beta)(Bu)_i^2 + \lambda|u_i|^2\Big)^{\frac{1}{2}} - h_2\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|v_ia_i\omega(t)| - \frac{c}{k}\|\varphi\|_E^2,
\end{aligned}$$

which along with (2.6) and (2.7) implies that

$$\begin{aligned}
(L(\varphi), \varphi)_E &\geq \sigma\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|\varphi_i|_E^2 + \frac{h_1}{2}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|v_i|^2 - \frac{c}{k}\|\varphi\|_E^2 - h_2\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|v_ia_i\omega(t)|\\
&\geq \sigma\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|\varphi_i|_E^2 + \frac{h_1}{6}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|v_i|^2 - \frac{c}{k}\|\varphi\|_E^2 - c\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|a_i|^2|\omega(t)|^2.
\end{aligned} \tag{2.28}$$

As to the first term on the right-hand side of (2.26), by (2.3) and (2.4)we get

$$\begin{aligned}
(H(\varphi), \psi)_E &= -\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})f_i(u_i)(\dot{u}_i + \beta u_i - a_i\omega(t)) + \sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})g_i(t)v_i\\
&\leq -\frac{d}{dt}(\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})F_i(u_i)) - \frac{\alpha_2\beta}{p+1}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})F_i(u_i)\\
&\quad + c\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|\omega(t)|^{p+1}|a_i|^{p+1} + \frac{\sigma\lambda}{4}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|u_i|^2\\
&\quad + c\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|a_i|^2|\omega(t)|^2 + \frac{\sigma}{6}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|v_i|^2 + c\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|g_i(t)|^2.
\end{aligned} \tag{2.29}$$

For the last term of (2.26), we have

$$(G,\psi)_E = \omega(t)(a,x)_\lambda + \beta\omega(t)(a,y) - \nu\omega(t)(Aa,y) \\ = \omega(t)(1-\nu\beta)(Ba,Bx) - \nu\omega(t)(Ba,By) + \lambda\omega(t)(a,x) + \beta\omega(t)(a,y), \tag{2.30}$$

As to the first two terms on the right-hand side of (2.30), we get

$$\begin{aligned}\omega(t)(1-\nu\beta)(Ba,Bx) &= \omega(t)(1-\nu\beta)\sum_{i\in\mathbb{Z}}(a_{i+1}-a_i)\Big(\eta(\frac{|i+1|}{k})u_{i+1} - \eta(\frac{|i|}{k})u_i\Big) \\ &\le \Big(\sum_{i\in\mathbb{Z}}\eta(\frac{|i+1|}{k})u_{i+1}^2\Big)^{\frac{1}{2}}\Big(\sum_{i\in\mathbb{Z}}\eta(\frac{|i+1|}{k})(\omega(t)(1-\nu\beta)(a_{i+1}-a_i))^2\Big)^{\frac{1}{2}} \\ &\quad + \Big(\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})u_i^2\Big)^{\frac{1}{2}}\Big(\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})(\omega(t)(1-\nu\beta)(a_{i+1}-a_i))^2\Big)^{\frac{1}{2}} \\ &\le \frac{\sigma\lambda}{8}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})u_i^2 + c|\omega(t)|^2\sum_{|i|\ge k}a_i^2,\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}-\nu\omega(t)(Ba,By) &= -\nu\omega(t)\sum_{i\in\mathbb{Z}}(a_{i+1}-a_i)\Big(\eta(\frac{|i+1|}{k})v_{i+1} - \eta(\frac{|i|}{k})v_i\Big) \\ &\le \frac{\sigma}{6}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})v_i^2 + c|\omega(t)|^2\sum_{|i|\ge k}a_i^2.\end{aligned} \tag{2.32}$$

The last two terms of (2.30) is bounded by

$$\lambda\omega(t)(a,x) + \beta\omega(t)(a,y) \le \frac{\sigma\lambda}{8}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})u_i^2 + \frac{\sigma+h_1}{6}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})v_i^2 + c|\omega(t)|^2\sum_{|i|\ge k}a_i^2. \tag{2.33}$$

It follows from (2.26)–(2.33) that

$$\begin{aligned}&\frac{d}{dt}\Big(\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})(|\varphi_i|_E^2 + 2F_i(u_i))\Big) + \gamma\Big(\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})(|\varphi_i|_E^2 + 2F_i(u_i))\Big) + \gamma\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|\varphi|_E^2 \\ &\le \frac{c}{k}\|\varphi\|_E^2 + \frac{c}{k}|\omega(t)|^2 + c\sum_{|i|\ge k}|a_i|^{p+1}|\omega(t)|^{p+1} + c\sum_{|i|\ge k}|g_i(t)|^2 + c\sum_{|i|\ge k}|a_i|^2|\omega(t)|^2,\end{aligned} \tag{2.34}$$

where $\gamma = \min\{\frac{\sigma}{2}, \frac{\alpha_2\beta}{p+1}\}$. Multiplying (2.34) by $e^{\gamma t}$, replacing ω by $\theta_{-\tau}\omega$ and integrating on $(\tau-t,\tau)$ with $t\in\mathbb{R}^+$, we get for every $\omega\in\Omega$

$$\begin{aligned}&\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})\Big(|\varphi_i(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t,i})|_E^2 + 2F_i(u_i(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t,i}))\Big) \\ &\le e^{-\gamma t}\Big(\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})(|\varphi_{\tau-t,i}|_E^2 + 2F_i(u_{\tau-t,i}))\Big) + \frac{c}{k}\int_{\tau-t}^{\tau}e^{\gamma(s-\tau)}\|\varphi(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|_E^2 ds \\ &\quad + \frac{c}{k}\int_{-\infty}^{0}e^{\gamma s}|\omega(s)-\omega(-\tau)|^2 ds + c\sum_{|i|\ge k}|a_i|^{p+1}\int_{-\infty}^{0}e^{\gamma s}|\omega(s)-\omega(-\tau)|^{p+1}ds \\ &\quad + c\sum_{|i|\ge k}|a_i|^2\int_{-\infty}^{0}e^{\gamma s}|\omega(s)-\omega(-\tau)|^2 ds + c\int_{-\infty}^{0}e^{\gamma s}\sum_{|i|\ge k}|g_i(s+\tau)|^2 ds.\end{aligned} \tag{2.35}$$

For any $s \geq \tau - t$,

$$\bar{\varphi}(s,\tau-t,\theta_{-\tau}\omega,\bar{\varphi}_{\tau-t}) = \varphi(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}) + (0, a(\omega(s-\tau)-\omega(-\tau)))^T,$$

which along with (2.35) implies that

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}} \eta(\frac{|i|}{k})\Big(|\bar{\varphi}_i(\tau,\tau-t,\theta_{-\tau}\omega,\bar{\varphi}_{\tau-t,i})|_E^2 + 2F_i(u_i(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t,i}))\Big) \\
&\leq 4e^{-\gamma t}\Big(\sum_{i\in\mathbb{Z}} \eta(\frac{|i|}{k})(|\bar{\varphi}_{\tau-t,i}|_E^2 + |a_i|^2|\omega(-t)-\omega(-\tau)|^2 + F_i(u_{\tau-t,i}))\Big) \\
&\quad + \frac{c}{k}\int_{\tau-t}^{\tau} e^{\gamma(s-\tau)}\|\bar{\varphi}(s,\tau-t,\theta_{-\tau}\omega,\bar{\varphi}_{\tau-t})\|_E^2 ds + \frac{c}{k}\int_{-\infty}^0 e^{\gamma s}|\omega(s)-\omega(-\tau)|^2 ds \qquad (2.36)\\
&\quad + c\sum_{|i|\geq k}|a_i|^{p+1}\int_{-\infty}^0 e^{\gamma s}|\omega(s)-\omega(-\tau)|^{p+1}ds + c\sum_{|i|\geq k}|a_i|^2\int_{-\infty}^0 e^{\gamma s}|\omega(s)-\omega(-\tau)|^2 ds \\
&\quad + c\int_{-\infty}^0 e^{\gamma s}\sum_{|i|\geq k}|g_i(s+\tau)|^2 ds + 2\sum_{|i|\geq k}|a_i|^2|\omega(-\tau)|^2.
\end{aligned}$$

By (2.1) and (2.8), the last four integrals in (2.36) are well defined. By (2.3) and (2.4), we obtain

$$\sum_{i\in\mathbb{Z}} \eta(\frac{|i|}{k})F_i(u_{i,\tau-t}) \leq \frac{1}{\alpha_2}\sum_{i\in\mathbb{Z}} \eta(\frac{|i|}{k})f_i(u_{\tau-t,i})u_{\tau-t,i} \leq \frac{1}{\alpha_2}\max_{-\|u_{\tau-t}\|\leq s\leq\|u_{\tau-t}\|}|f_i'(s)|\|u_{\tau-t}\|^2,$$

which along with $\bar{\varphi}_{\tau-t} \in D(\tau-t,\theta_{-t}\omega)$ and (2.1) implies that

$$\limsup_{t\to+\infty} 4e^{-\gamma t}\Big(\sum_{i\in\mathbb{Z}} \eta(\frac{|i|}{k})(|\bar{\varphi}_{\tau-t,i}|_E^2 + |a_i|^2|\omega(-t)-\omega(-\tau)|^2 + F_i(u_{\tau-t,i}))\Big) = 0.$$

Then there exists $T_1 = T_1(\tau,\omega,D,\varepsilon) > 0$ such that for all $t \geq T_1$,

$$4e^{-\gamma t}\Big(\sum_{i\in\mathbb{Z}} \eta(\frac{|i|}{k})(|\bar{\varphi}_{\tau-t,i}|_E^2 + |a_i|^2|\omega(-t)-\omega(-\tau)|^2 + F_i(u_{\tau-t,i}))\Big) \leq \frac{\varepsilon}{4}. \qquad (2.37)$$

By (2.1) and (2.24), there exist $T_2 = T_2(\tau,\omega,D,\varepsilon) > T_1$ and $N_1 = N_1(\tau,\omega,\varepsilon) > 0$ such that for all $t \geq T_2$ and $k \geq N_1$

$$\frac{c}{k}\int_{\tau-t}^{\tau} e^{\gamma(s-\tau)}\|\bar{\varphi}(s,\tau-t,\theta_{-\tau}\omega,\bar{\varphi}_{\tau-t})\|_E^2 ds + \frac{c}{k}\int_{-\infty}^0 e^{\gamma s}|\omega(s)-\omega(-\tau)|^2 ds \leq \frac{\varepsilon}{4}. \qquad (2.38)$$

By (2.8), there exists $N_2 = N_2(\tau,\omega,\varepsilon) > N_1$ such that for all $k \geq N_2$,

$$2\sum_{|i|\geq k}|a_i|^2|\omega(-\tau)|^2 + c\int_{-\infty}^0 e^{\gamma s}\sum_{|i|\geq k}|g_i(s+\tau)|^2 ds \leq \frac{\varepsilon}{4}. \qquad (2.39)$$

By (2.1) again, we find that there exists $N_3 = N_3(\tau,\omega,\varepsilon) > N_2$ such that for all $k \geq N_3$,

$$c\sum_{|i|\geq k}|a_i|^{p+1}\int_{-\infty}^0 e^{\gamma s}|\omega(s)-\omega(-\tau)|^{p+1}ds + c\sum_{|i|\geq k}|a_i|^2\int_{-\infty}^0 e^{\gamma s}|\omega(s)-\omega(-\tau)|^2 ds \leq \frac{\varepsilon}{4}. \qquad (2.40)$$

Then it follows from (2.36)–(2.40) that for all $t \geq T_2$ and $k \geq N_3$

$$\sum_{|i|\geq 2k} |\bar{\varphi}_i(\tau, \tau - t, \theta_{-\tau}\omega, \bar{\varphi}_{\tau-t,i})|_E^2 \leq \sum_{i\in\mathbb{Z}} \eta(\frac{|i|}{k})|\bar{\varphi}_i(\tau, \tau - t, \theta_{-\tau}\omega, \bar{\varphi}_{\tau-t,i})|_E^2 \leq \varepsilon.$$

This concludes the proof. □

As a consequence of Lemma 2.2 and Lemma 2.3, we get the existence of $\mathcal{D}$-pullback attractors for Φ_0 immediately.

Theorem 2.1. *Suppose that (2.3)–(2.9) hold. Then the continuous cocycle Φ_0 associated with system (2.10) has a unique $\mathcal{D}$-pullback attractors $\mathcal{A}_0 = \{\mathcal{A}_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in E.*

3. Wong-Zakai approximation of second order lattice system

In this section, we will approximate the solutions of system (1.1) by the pathwise Wong-Zakai approximated system (1.2). Given $\delta \neq 0$, define a random variable $\mathcal{G}_\delta$ by

$$\mathcal{G}_\delta(\omega) = \frac{\omega(\delta)}{\delta}, \quad \text{for all } \omega \in \Omega. \tag{3.1}$$

From (3.1) we find

$$\mathcal{G}_\delta(\theta_t\omega) = \frac{\omega(t+\delta) - \omega(t)}{\delta} \quad \text{and} \quad \int_0^t \mathcal{G}_\delta(\theta_s\omega)ds = \int_t^{t+\delta} \frac{\omega(s)}{\delta}ds + \int_\delta^0 \frac{\omega(s)}{\delta}ds. \tag{3.2}$$

By (3.2) and the continuity of ω we get for all $t \in \mathbb{R}$,

$$\lim_{\delta\to 0} \int_0^t \mathcal{G}_\delta(\theta_s\omega)ds = \omega(t). \tag{3.3}$$

Note that this convergence is uniform on a finite interval as stated below.

Lemma 3.1. *([17]). Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$. Then for every $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon, \tau, \omega, T) > 0$ such that for all $0 < |\delta| < \delta_0$ and $t \in [\tau, \tau + T]$,*

$$\Big| \int_0^t \mathcal{G}_\delta(\theta_s\omega)ds - \omega(t)\Big| < \varepsilon.$$

By Lemma 3.1, we find that there exist $c = c(\tau, \omega, T) > 0$ and $\tilde{\delta}_0 = \tilde{\delta}_0(\tau, \omega, T) > 0$ such that for all $0 < |\delta| < \tilde{\delta}_0$ and $t \in [\tau, \tau + T]$,

$$\Big| \int_0^t \mathcal{G}_\delta(\theta_s\omega)ds\Big| \leq c. \tag{3.4}$$

By (3.3) we find that $\mathcal{G}_\delta(\theta_t\omega)$ is an approximation of the white noise in a sense. This leads us to consider system (1.2) as an approximation of system (1.1).

Let $\bar{v}^\delta = \dot{u}^\delta + \beta u^\delta$ and $\bar{\varphi}_\delta = (u^\delta, \bar{v}^\delta)$, the system (1.2) can be rewritten as

$$\dot{\bar{\varphi}}_\delta + L_{\delta,1}(\bar{\varphi}_\delta) = H_{\delta,1}(\bar{\varphi}_\delta) + G_{\delta,1}(\omega), \tag{3.5}$$

with initial conditions

$$\bar{\varphi}_{\delta,\tau} = (u_\tau^\delta, \bar{v}_\tau^\delta)^T = (u_\tau^\delta, u_\tau^{\delta,1} + \beta u_\tau^\delta)^T,$$

where

$$L_{\delta,1}(\bar{\varphi}) = \begin{pmatrix} \beta u^\delta - \bar{v}^\delta \\ (1-\nu\beta)Au^\delta + \nu A\bar{v}^\delta + \lambda u^\delta + \beta^2 u^\delta - \beta\bar{v}^\delta \end{pmatrix} + \begin{pmatrix} 0 \\ h(\bar{v}^\delta - \beta u^\delta) \end{pmatrix},$$

$$H_{\delta,1}(\bar{\varphi}_\delta) = \begin{pmatrix} 0 \\ -f(u^\delta) + g(t) \end{pmatrix}, \quad G_{\delta,1}(\omega) = \begin{pmatrix} 0 \\ a\mathcal{G}_\delta(\theta_t\omega) \end{pmatrix}.$$

Denote

$$v^\delta(t) = \bar{v}^\delta(t) - a\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds \text{ and } \varphi_\delta = (u^\delta, v^\delta)^T.$$

By (3.5) we have

$$\dot{\varphi}_\delta + L_\delta(\varphi_\delta) = H_\delta(\varphi_\delta) + G_\delta(\omega), \tag{3.6}$$

with initial conditions

$$\varphi_{\delta,\tau} = (u_\tau^\delta, v_\tau^\delta)^T = (u_\tau^\delta, u_\tau^{\delta,1} + \beta u_\tau^\delta - a\int_0^\tau \mathcal{G}_\delta(\theta_s\omega)ds)^T,$$

where

$$L_\delta(\varphi_\delta) = \begin{pmatrix} \beta u^\delta - v^\delta \\ (1-\nu\beta)Au^\delta + \nu Av^\delta + \lambda u^\delta + \beta^2 u^\delta - \beta v^\delta \end{pmatrix} + \begin{pmatrix} 0 \\ h(v^\delta - \beta u^\delta + a\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds) \end{pmatrix},$$

$$H_\delta(\varphi_\delta) = \begin{pmatrix} 0 \\ -f(u^\delta) + g(t) \end{pmatrix}, \quad G_\delta(\omega) = \begin{pmatrix} a\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds \\ \beta a\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds - \nu Aa\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds \end{pmatrix}.$$

Note that system (3.6) is a deterministic functional equation and the nonlinearity in (3.6) is locally Lipschitz continuous from E to E. Therefore, by the standard theory of functional differential equations, system (3.6) is well-posed. Thus, we can define a continuous cocycle $\Phi_\delta : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E \to E$ associated with system (3.5), where for $\tau \in \mathbb{R}$, $t \in \mathbb{R}^+$ and $\omega \in \Omega$

$$\begin{aligned}
\Phi_\delta(t,\tau,\omega,\bar{\varphi}_{\delta,\tau}) &= \bar{\varphi}_\delta(t+\tau,\tau,\theta_{-\tau}\omega,\bar{\varphi}_{\delta,\tau}) \\
&= (u^\delta(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau^\delta), \bar{v}^\delta(t+\tau,\tau,\theta_{-\tau}\omega,\bar{v}_\tau^\delta))^T \\
&= \Big(u^\delta(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau^\delta), v^\delta(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau^\delta) + a\int_{-\tau}^t \mathcal{G}_\delta(\theta_s\omega)ds\Big)^T \\
&= \varphi_\delta(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_{\delta,\tau}) + \Big(0, a\int_{-\tau}^t \mathcal{G}_\delta(\theta_s\omega)ds\Big)^T,
\end{aligned}$$

where $v_\tau^\delta = \bar{v}_\tau^\delta - a\int_{-\tau}^0 \mathcal{G}_\delta(\theta_s\omega)ds$.

For later purpose, we now show the estimates on the solutions of system (3.6) on a finite time interval.

Lemma 3.2. *Suppose that (2.3)–(2.8) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $T > 0$, there exist $\delta_0 = \delta_0(\tau, \omega, T) > 0$ and $c = c(\tau, \omega, T) > 0$ such that for all $0 < |\delta| < \delta_0$ and $t \in [\tau, \tau + T]$, the solution φ_δ of system (3.6) satisfies*

$$\|\varphi_\delta(t, \tau, \omega, \varphi_{\delta,\tau})\|_E^2 + \int_\tau^t \|\varphi_\delta(s, \tau, \omega, \varphi_{\delta,\tau})\|_E^2 ds \le c\Big(\|\varphi_{\delta,\tau}\|_E^2 + 2\sum_{i\in\mathbb{Z}} F_i(u_{\tau,i}^\delta)\Big)$$
$$+ c\int_\tau^t \Big(\|g(s)\|^2 + |\int_0^s \mathcal{G}_\delta(\theta_l\omega)dl|^2 + |\int_0^s \mathcal{G}_\delta(\theta_l\omega)dl|^{p+1}|\Big)ds.$$

Proof. Taking the inner product $(\cdot,\cdot)_E$ on both side of the system (3.6) with φ_δ, it follows that

$$\frac{1}{2}\frac{d}{dt}\|\varphi_\delta\|_E^2 + (L_\delta(\varphi_\delta), \varphi_\delta)_E = (H_\delta(\varphi_\delta), \varphi_\delta)_E + (G_\delta(\omega), \varphi_\delta)_E. \tag{3.7}$$

By the similar calculations in (2.13)–(2.15), we get

$$(L_\delta(\varphi_\delta), \varphi_\delta)_E \ge \sigma\|\varphi_\delta\|_E^2 + \frac{h_1}{2}\|v^\delta\|^2 - \frac{\sigma + h_1}{6}\|v^\delta\|^2 - c|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds|^2\|a\|^2, \tag{3.8}$$

$$\begin{aligned}(H_\delta(\varphi_\delta), \varphi_\delta)_E \le& -\frac{d}{dt}(\sum_{i\in\mathbb{Z}} F_i(u_i^\delta)) - \frac{\alpha_2\beta}{p+1}\sum_{i\in\mathbb{Z}} F_i(u_i^\delta) + c|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds|^{p+1}\|a\|^{p+1} \\ &+ \frac{\sigma\lambda}{4}\|u^\delta\|^2 + c\|a\|^2|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds|^2 + c\|g(t)\|^2 + \frac{\sigma + h_1}{6}\|v^\delta\|^2,\end{aligned} \tag{3.9}$$

and

$$(G_\delta(\omega), \varphi_\delta)_E \le \frac{\sigma}{4}\|u^\delta\|_\lambda^2 + c\|a\|^2|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds|^2 + \frac{\sigma + h_1}{6}\|v^\delta\|^2. \tag{3.10}$$

It follows from (3.7)–(3.10) that

$$\begin{aligned}&\frac{d}{dt}\Big(\|\varphi_\delta\|_E^2 + 2\sum_{i\in\mathbb{Z}} F_i(u_i^\delta)\Big) + \gamma\Big(\|\varphi_\delta\|_E^2 + 2\sum_{i\in\mathbb{Z}} F_i(u_i^\delta)\Big) + \gamma\|\varphi_\delta\|_E^2 \\ &\le c\Big(\|g(t)\|^2 + |\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds|^2 + |\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds|^{p+1}\Big),\end{aligned} \tag{3.11}$$

where $\gamma = \min\{\frac{\sigma}{2}, \frac{\alpha_2\beta}{p+1}\}$. Multiplying (3.11) by $e^{\gamma t}$ and integrating on (τ, t) with $t \ge \tau$, we get for every $\omega \in \Omega$

$$\|\varphi_\delta(t, \tau, \omega, \varphi_{\delta,\tau})\|_E^2 + \gamma\int_\tau^t e^{\gamma(s-t)}\|\varphi_\delta(s, \tau, \omega, \varphi_{\delta,\tau})\|_E^2 ds \le e^{\gamma(\tau-t)}\Big(\|\varphi_{\delta,\tau}\|_E^2 + 2\sum_{i\in\mathbb{Z}} F_i(u_{\tau,i}^\delta)\Big)$$
$$+ c\int_\tau^t e^{\gamma(s-t)}\Big(\|g(s)\|^2 + |\int_0^s \mathcal{G}_\delta(\theta_l\omega)dl|^2 + |\int_0^s \mathcal{G}_\delta(\theta_l\omega)dl|^{p+1}\Big)ds,$$

which implies the desired result. □

In what follows, we derive uniform estimates on the solutions of system (3.5) when t is sufficiently large.

Lemma 3.3. *Suppose that (2.3)–(2.8) hold. Then for every $\delta \neq 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D,\delta) > 0$ such that for all $t \geq T$, the solution $\bar{\varphi}_\delta$ of system (3.5) satisfies*

$$\|\bar{\varphi}_\delta(\tau,\tau-t,\theta_{-\tau}\omega,\bar{\varphi}_{\delta,\tau-t})\|_E^2 + \gamma\int_{\tau-t}^{\tau} e^{\gamma(s-\tau)}\|\bar{\varphi}_\delta(s,\tau-t,\theta_{-\tau}\omega,\bar{\varphi}_{\delta,\tau-t})\|_E^2 ds \leq R_\delta(\tau,\omega),$$

where $\bar{\varphi}_{\delta,\tau-t} \in D(\tau-t,\theta_{-t}\omega)$ and $R_\delta(\tau,\omega)$ is given by

$$\begin{aligned}R_\delta(\tau,\omega) =& c\int_{-\infty}^{0} e^{\gamma s}\Big(\|g(s+\tau)\|^2 + |\int_{-\tau}^{s}\mathcal{G}_\delta(\theta_l\omega)dl|^2 + |\int_{-\tau}^{s}\mathcal{G}_\delta(\theta_l\omega)dl|^{p+1}\Big)ds\\ &+ c + c|\int_{-\tau}^{0}\mathcal{G}_\delta(\theta_l\omega)dl|^2,\end{aligned} \tag{3.12}$$

where c is a positive constant independent of τ, ω and δ.

Proof. Multiplying (3.11) by $e^{\gamma t}$, replacing ω by $\theta_{-\tau}\omega$ and integrating on $(\tau-t,\tau)$ with $t \in \mathbb{R}^+$, we get for every $\omega \in \Omega$

$$\begin{aligned}&\|\varphi_\delta(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\delta,\tau-t})\|_E^2 + 2\sum_{i\in\mathbb{Z}} F_i(u_i^\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t,i}^\delta))\\ &\quad+\gamma\int_{\tau-t}^{\tau} e^{\gamma(s-\tau)}\|\varphi_\delta(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\delta,\tau-t})\|_E^2 ds\\ &\leq e^{-\gamma t}\Big(\|\varphi_{\delta,\tau-t}\|_E^2 + 2\sum_{i\in\mathbb{Z}} F_i(u_{\tau-t,i}^\delta)\Big)\\ &\quad+ c\int_{-\infty}^{0} e^{\gamma s}\Big(\|g(s+\tau)\|^2 + |\int_{-\tau}^{s}\mathcal{G}_\delta(\theta_l\omega)dl|^2 + |\int_{-\tau}^{s}\mathcal{G}_\delta(\theta_l\omega)dl|^{p+1}\Big)ds.\end{aligned} \tag{3.13}$$

By (2.1), (2.8) and (3.2), the last integral on the right-hand side of (3.13) is well defined. For any $s \geq \tau - t$,

$$\bar{\varphi}_\delta(s,\tau-t,\theta_{-\tau}\omega,\bar{\varphi}_{\delta,\tau-t}) = \varphi_\delta(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\delta,\tau-t}) + \Big(0, a\int_0^s \mathcal{G}_\delta(\theta_{l-\tau}\omega)dl\Big)^T,$$

which along with (3.13) shows that

$$\begin{aligned}&\|\bar{\varphi}_\delta(\tau,\tau-t,\theta_{-\tau}\omega,\bar{\varphi}_{\delta,\tau-t})\|_E^2 + \gamma\int_{\tau-t}^{\tau} e^{\gamma(s-\tau)}\|\bar{\varphi}_\delta(s,\tau-t,\theta_{-\tau}\omega,\bar{\varphi}_{\tau-t})\|_E^2 ds\\ &\leq 4e^{-\gamma t}\Big(\|\bar{\varphi}_{\delta,\tau-t}\|_E^2 + \|a\|^2|\int_{-\tau}^{-t}\mathcal{G}_\delta(\theta_l\omega)dl|^2 + \sum_{i\in\mathbb{Z}} F_i(u_{\tau-t,i})\Big) + c|\int_{-\tau}^{0}\mathcal{G}_\delta(\theta_l\omega)dl|^2\\ &\quad+ c\int_{-\infty}^{0} e^{\gamma s}\Big(\|g(s+\tau)\|^2 + |\int_{-\tau}^{s}\mathcal{G}_\delta(\theta_l\omega)dl|^2 + |\int_{-\tau}^{s}\mathcal{G}_\delta(\theta_l\omega)dl|^{p+1}\Big)ds,\end{aligned} \tag{3.14}$$

Note that (2.3) and (2.4) implies that

$$\sum_{i\in\mathbb{Z}} F_i(u_{\tau-t,i}^\delta) \leq \frac{1}{\alpha_2}\sum_{i\in\mathbb{Z}} f_i(u_{\tau-t,i}^\delta)u_{\tau-t,i}^\delta \leq \frac{1}{\alpha_2}\max_{-\|u_{\tau-t}^\delta\|\leq s\leq\|u_{\tau-t}^\delta\|}|f_i'(s)|\|u_{\tau-t}^\delta\|^2,$$

which along with $\bar{\varphi}_{\delta,\tau-t} \in D(\tau - t, \theta_{-t}\omega)$,(2.1) and (3.2) implies that

$$\limsup_{t\to+\infty} 4e^{-\gamma t}\Big(\|\bar{\varphi}_{\delta,\tau-t}\|_E^2 + \|a\|^2 \Big|\int_{-\tau}^{-t} \mathcal{G}_\delta(\theta_l\omega)dl\Big|^2 + \sum_{i\in\mathbb{Z}} F_i(u_{\tau-t,i})\Big) = 0. \tag{3.15}$$

Then (3.14) and (3.15) can imply the desired estimates. □

Next, we show that system (3.5) has a $\mathcal{D}$-pullback absorbing set.

Lemma 3.4. *Suppose that (2.3)–(2.9) hold. Then the continuous cocycle Φ_δ associated with system (3.5) has a closed measurable $\mathcal{D}$-pullback absorbing set $K_\delta = \{K_\delta(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, where for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$*

$$K_\delta(\tau,\omega) = \{\bar{\varphi}_\delta \in E : \|\bar{\varphi}_\delta\|_E^2 \le R_\delta(\tau,\omega)\}, \tag{3.16}$$

where $R_\delta(\tau,\omega)$ is given by (3.12). In addition, we have for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$

$$\lim_{\delta\to 0} R_\delta(\tau,\omega) = R_0(\tau,\omega), \tag{3.17}$$

where $R_0(\tau,\omega)$ is defined in (2.19).

Proof. Note K_δ given by (3.16) is closed measurable random set in E. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D \in \mathcal{D}$, it follows from Lemma 3.3 that there exists $T_0 = T_0(\tau,\omega,D,\delta)$ such that for all $t \ge T_0$,

$$\Phi_\delta(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K_\delta(\tau,\omega),$$

which implies that K_δ pullback attracts all elements in $\mathcal{D}$. By (2.1), (2.8) and (3.2), we can prove $K_\delta(\tau,\omega)$ is tempered. The convergence (3.17) can be obtained by Lebesgue dominated convergence as in [17]. □

We are now in a position to derive uniform estimates on the tail of solutions of system (3.5).

Lemma 3.5. *Suppose that (2.3)–(2.8) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\varepsilon > 0$, there exist $\delta_0 = \delta_0(\omega) > 0$, $T = T(\tau,\omega,\varepsilon) > 0$ and $N = N(\tau,\omega,\varepsilon) > 0$ such that for all $t \ge T$ and $0 < |\delta| < \delta_0$, the solution $\bar{\varphi}_\delta$ of system (3.5) satisfies*

$$\sum_{|i|\ge N} |\bar{\varphi}_{\delta,i}(\tau, \tau - t, \theta_{-\tau}\omega, \bar{\varphi}_{\delta,\tau-t,i})|_E^2 \le \varepsilon,$$

where $\bar{\varphi}_{\delta,\tau-t} \in K_\delta(\tau - t, \theta_{-t}\omega)$ and $|\bar{\varphi}_{\delta,i}|_E^2 = (1-\nu\beta)|Bu^\delta|_i^2 + \lambda|u_i^\delta|^2 + |\bar{v}_i^\delta|^2$.

Proof. Let η be a smooth function defined in Lemma 2.3, and set $x = (x_i)_{i\in\mathbb{Z}}$, $y = (y_i)_{i\in\mathbb{Z}}$ with $x_i = \eta(\frac{|i|}{k})u_i^\delta$, $y_i = \eta(\frac{|i|}{k})v_i^\delta$. Note $\psi = (x,y)^T = ((x_i),(y_i))_{i\in\mathbb{Z}}^T$. Taking the inner product of system (3.6) with ψ, we have

$$(\dot{\varphi}_\delta, \psi)_E + (L_\delta(\varphi_\delta), \psi)_E = (H_\delta(\varphi_\delta), \psi)_E + (G_\delta, \psi)_E. \tag{3.18}$$

For the first term of (3.18), we have

$$\begin{aligned}(\dot{\varphi}_\delta,\psi)_E &= (1-\nu\beta)\sum_{i\in\mathbb{Z}}(B\dot{u}^\delta)_i(Bx)_i+\lambda\sum_{i\in\mathbb{Z}}\dot{u}_i^\delta x_i+\sum_{i\in\mathbb{Z}}\dot{v}_i^\delta y_i\\ &= \frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|\varphi_{\delta,i}|_E^2+(1-\nu\beta)\sum_{i\in\mathbb{Z}}(B\dot{u}^\delta)_i\Big((Bx)_i-\eta(\frac{|i|}{k})(Bu^\delta)_i\Big)\\ &\geq \frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|\varphi_{\delta,i}|_E^2-\frac{(1-\nu\beta)C_0}{k}\sum_{i\in\mathbb{Z}}|B(v^\delta-\beta u^\delta+a\int_0^t\mathcal{G}_\delta(\theta_s\omega)ds)|_i|u_{i+1}^\delta|\\ &\geq \frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|\varphi_{\delta,i}|_E^2-\frac{c}{k}\|\varphi_\delta\|_E^2-\frac{c}{k}|\int_0^t\mathcal{G}_\delta(\theta_s\omega)ds|^2\|a\|^2,\end{aligned}\tag{3.19}$$

where $|\varphi_{\delta,i}|_E^2=(1-\nu\beta)|Bu^\delta|_i^2+\lambda|u_i^\delta|^2+|v_i^\delta|^2$. By the similar calculations in (2.28)–(2.33), we get

$$\begin{aligned}(L_\delta(\varphi_\delta),\psi)_E \geq&\sigma\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|\varphi_{\delta,i}|_E^2+\frac{h_1}{6}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|v_i^\delta|^2-\frac{c}{k}\|\varphi_\delta\|_E^2\\ &-c\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|a_i|^2|\int_0^t\mathcal{G}_\delta(\theta_s\omega)ds|^2,\end{aligned}\tag{3.20}$$

$$\begin{aligned}(H_\delta(\varphi_\delta),\psi)_E \leq& -\frac{d}{dt}(\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})F_i(u_i^\delta))-\frac{\alpha_2\beta}{p+1}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})F_i(u_i^\delta)\\ &+\frac{\sigma\lambda}{4}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|u_i^\delta|^2+\frac{\sigma}{6}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|v_i^\delta|^2+c\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|g_i(t)|^2\\ &+c\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|a_i|^2|\int_0^t\mathcal{G}_\delta(\theta_s\omega)ds|^2+c\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|a_i|^{p+1}|\int_0^t\mathcal{G}_\delta(\theta_s\omega)ds|^{p+1},\end{aligned}\tag{3.21}$$

and

$$\begin{aligned}(G_\delta,\psi)_E &= (1-\nu\beta)\int_0^t\mathcal{G}_\delta(\theta_s\omega)ds(Bx,Ba)_\lambda+\beta\int_0^t\mathcal{G}_\delta(\theta_s\omega)ds(y,a)\\ &\leq \frac{\sigma\lambda}{4}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|u_i^\delta|^2+\Big(\frac{h_1}{6}+\frac{\sigma}{3}\Big)\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|v_i^\delta|^2+c|\int_0^t\mathcal{G}_\delta(\theta_s\omega)ds|^2\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|a_i|^2.\end{aligned}\tag{3.22}$$

It follows from (3.18)–(3.22) that

$$\begin{aligned}&\frac{d}{dt}\Big(\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})(|\varphi_{\delta,i}|_E^2+2F_i(u_i^\delta))\Big)+\gamma\Big(\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})(|\varphi_{\delta,i}|_E^2+2F_i(u_i^\delta))\Big)+\gamma\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|\varphi_{\delta,i}|_E^2\\ &\leq \frac{c}{k}\|\varphi_\delta\|_E^2+\frac{c}{k}|\int_0^t\mathcal{G}_\delta(\theta_s\omega)ds|^2+c\sum_{|i|\geq k}|g_i(t)|^2+c\sum_{|i|\geq k}|a_i|^{p+1}|\int_0^t\mathcal{G}_\delta(\theta_s\omega)ds|^{p+1}\\ &\quad+c\sum_{|i|\geq k}|a_i|^2|\int_0^t\mathcal{G}_\delta(\theta_s\omega)ds|^2,\end{aligned}\tag{3.23}$$

where $\gamma=\min\{\frac{\sigma}{2},\frac{\alpha_2\beta}{p+1}\}$. Multiplying (3.23) by $e^{\gamma t}$, replacing ω by $\theta_{-\tau}\omega$ and integrating on $(\tau-t,\tau)$ with

$t \in \mathbb{R}^+$, we get for every $\omega \in \Omega$

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}} \eta(\frac{|i|}{k})\Big(|\varphi_{\delta,i}(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\delta,\tau-t,i})|_E^2 + 2F_i(u_i^\delta(\tau,\tau-t,\theta_{-\tau}\omega,u^\delta_{\tau-t,i}))\Big) \\
&\leq e^{-\gamma t}\sum_{i\in\mathbb{Z}} \eta(\frac{|i|}{k})\Big(|\varphi_{\delta,\tau-t,i}|_E^2 + 2F_i(u^\delta_{\tau-t,i})\Big) + \frac{c}{k}\int_{\tau-t}^{\tau} e^{\gamma(s-\tau)}\|\varphi_\delta(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\delta,\tau-t})\|_E^2 ds \\
&\quad + \frac{c}{k}\int_{-\infty}^0 e^{\gamma s}|\int_{-\tau}^s \mathcal{G}_\delta(\theta_l\omega)dl|^2 ds + c\int_{-\infty}^0 e^{\gamma s}\sum_{|i|\geq k}|g_i(s+\tau)|^2 ds \\
&\quad + c\sum_{|i|\geq k}|a_i|^2\int_{-\infty}^0 e^{\gamma s}|\int_{-\tau}^s \mathcal{G}_\delta(\theta_l\omega)dl|^2 ds + c\sum_{|i|\geq k}|a_i|^{p+1}\int_{-\infty}^0 e^{\gamma s}|\int_{-\tau}^s \mathcal{G}_\delta(\theta_l\omega)dl|^{p+1} ds.
\end{aligned} \tag{3.24}$$

For any $s \geq \tau - t$,

$$\bar{\varphi}_\delta(s,\tau-t,\theta_{-\tau}\omega,\bar{\varphi}_{\delta,\tau-t}) = \varphi_\delta(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\delta,\tau-t}) + \Big(0, a\int_0^s \mathcal{G}_\delta(\theta_{l-\tau}\omega)dl\Big)^T,$$

which along with (3.24) shows that

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}} \eta(\frac{|i|}{k})|\bar{\varphi}_{\delta,i}(\tau,\tau-t,\theta_{-\tau}\omega,\bar{\varphi}_{\delta,\tau-t,i})|_E^2 \\
&\leq 2\sum_{i\in\mathbb{Z}} \eta(\frac{|i|}{k})|\varphi_{\delta,i}(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\delta,\tau-t,i})|_E^2 + 2\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|a_i\int_0^\tau \mathcal{G}_\delta(\theta_{l-\tau}\omega)dl|^2 \\
&\leq 2\sum_{|i|\geq k}|a_i\int_{-\tau}^0 \mathcal{G}_\delta(\theta_l\omega)dl|^2 + 4e^{-\gamma t}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})\Big(|\bar{\varphi}_{\delta,\tau-t,i}|_E^2 + |a_i\int_{-\tau}^{-t}\mathcal{G}_\delta(\theta_l\omega)dl|^2 + F_i(u^\delta_{\tau-t,i})\Big) \\
&\quad + \frac{c}{k}\int_{\tau-t}^{\tau} e^{\gamma(s-\tau)}\|\bar{\varphi}_\delta(s,\tau-t,\theta_{-\tau}\omega,\bar{\varphi}_{\delta,\tau-t})\|_E^2 ds + \frac{c}{k}\int_{-\infty}^0 e^{\gamma s}|\int_{-\tau}^s \mathcal{G}_\delta(\theta_l\omega)dl|^2 ds \\
&\quad + c\int_{-\infty}^0 e^{\gamma s}\sum_{|i|\geq k}|g_i(s+\tau)|^2 ds + c\sum_{|i|\geq k}|a_i|^2\int_{-\infty}^0 e^{\gamma s}|\int_{-\tau}^s \mathcal{G}_\delta(\theta_l\omega)dl|^2 ds \\
&\quad + c\sum_{|i|\geq k}|a_i|^{p+1}\int_{-\infty}^0 e^{\gamma s}|\int_{-\tau}^s \mathcal{G}_\delta(\theta_l\omega)dl|^{p+1} ds.
\end{aligned} \tag{3.25}$$

By (2.1) and (2.8), the last four integrals on the right-hand side of (3.24) are well defined. Note that (2.3) and (2.4) implies that

$$\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})F_i(u^\delta_{\tau-t,i}) \leq \frac{1}{\alpha_2}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})f_i(u^\delta_{\tau-t,i})u^\delta_{\tau-t,i} \leq \frac{1}{\alpha_2}\max_{-\|u^\delta_{\tau-t}\|\leq s\leq\|u^\delta_{\tau-t}\|}|f_i'(s)|\|u^\delta_{\tau-t}\|^2.$$

Since $\bar{\varphi}_{\delta,\tau-t} \in K_\delta(\tau-t,\theta_{-t}\omega)$, we find

$$\limsup_{t\to+\infty} e^{-\gamma t}\sum_{i\in\mathbb{Z}}\eta(\frac{|i|}{k})|\bar{\varphi}_{\delta,\tau-t,i}|_E^2 \leq \limsup_{t\to+\infty} e^{-\gamma t}\|K_\delta(\tau-t,\theta_{-t}\omega)\|_E^2 = 0,$$

which along with (2.1) and (3.2) shows that there exist $T_1 = T_1(\tau, \omega, \varepsilon) > 0$ and $\delta_0 > 0$ such that for all $t \geq T_1$ and $0 < |\delta| < \delta_0$,

$$4e^{-\gamma t} \sum_{i \in \mathbb{Z}} \eta(\frac{|i|}{k}) \Big(|\bar{\varphi}_{\delta,\tau-t,i}|_E^2 + |a_i \int_{-\tau}^{-t} \mathcal{G}_\delta(\theta_l \omega) dl|^2 + F_i(u^\delta_{\tau-t,i}) \Big) \leq \frac{\varepsilon}{4}. \tag{3.26}$$

By Lemma 3.3, (2.1) and (3.2), there exist $T_2 = T_2(\tau, \omega, \varepsilon) > T_1$ and $N_1 = N_1(\tau, \varepsilon) > 0$ such that for all $t \geq T_2$, $k \geq N_1$ and $0 < |\delta| < \delta_0$

$$\frac{c}{k} \int_{\tau-t}^{\tau} e^{\gamma(s-\tau)} \|\bar{\varphi}_\delta(s, \tau - t, \theta_{-\tau}\omega, \bar{\varphi}_{\delta,\tau-t})\|_E^2 ds + \frac{c}{k} \int_{-\infty}^{0} e^{\gamma s} |\int_{-\tau}^{s} \mathcal{G}_\delta(\theta_l \omega) dl|^2 ds \leq \frac{\varepsilon}{4}. \tag{3.27}$$

By (2.8), there exists $N_2 = N_2(\tau, \varepsilon) > N_1$ such that for all $k \geq N_2$,

$$2 \sum_{|i| \geq k} |a_i \int_{-\tau}^{0} \mathcal{G}_\delta(\theta_l \omega) dl|^2 + c \int_{-\infty}^{0} e^{\gamma s} \sum_{|i| \geq k} |g_i(s + \tau)|^2 ds \leq \frac{\varepsilon}{4}. \tag{3.28}$$

By (2.1) and (3.2) again, we find that there exists $N_3 = N_3(\tau, \varepsilon) > N_2$ such that for all $k \geq N_3$ and $0 < |\delta| < \delta_0$,

$$c \sum_{|i| \geq k} |a_i|^{p+1} \int_{-\infty}^{0} e^{\gamma s} |\int_{-\tau}^{s} \mathcal{G}_\delta(\theta_l \omega) dl|^{p+1} ds + c \sum_{|i| \geq k} |a_i|^2 \int_{-\infty}^{0} e^{\gamma s} |\int_{-\tau}^{s} \mathcal{G}_\delta(\theta_l \omega) dl|^2 ds \leq \frac{\varepsilon}{4}. \tag{3.29}$$

Then it follows from (3.25)–(3.29) that for all $t \geq T_2$, $k \geq N_3$ and $0 < |\delta| < \delta_0$,

$$\sum_{|i| \geq 2k} |\bar{\varphi}_{\delta,i}(\tau, \tau - t, \theta_{-\tau}\omega, \bar{\varphi}_{\delta,\tau-t,i})|_E^2 \leq \sum_{i \in \mathbb{Z}} \eta(\frac{|i|}{k}) |\bar{\varphi}_{\delta,i}(\tau, \tau - t, \theta_{-\tau}\omega, \bar{\varphi}_{\delta,\tau-t,i})|_E^2 \leq \varepsilon.$$

This concludes the proof. □

By Lemma 3.4, Φ_δ has a closed $\mathcal{D}$-pullback absorbing set, and Lemma 3.5 shows that Φ_δ is asymptotically null in E with respect to $\mathcal{D}$. Therefore, we get the existence of $\mathcal{D}$-pullback attractors for Φ_δ .

Lemma 3.6. *Suppose that (2.3)–(2.9) hold. Then the continuous cocycle Φ_δ associated with (3.5) has a unique $\mathcal{D}$-pullback attractors $\mathcal{A}_\delta = \{\mathcal{A}_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in E.*

For the attractor $\mathcal{A}_\delta$ of Φ_δ, we have the uniform compactness as showed below.

Lemma 3.7. *Suppose that (2.3)–(2.9) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, there exists $\delta_0 = \delta_0(\omega) > 0$ such that $\bigcup_{0<|\delta|<\delta_0} \mathcal{A}_\delta(\tau, \omega)$ is precompact in E.*

Proof. Given $\varepsilon > 0$, we will prove that $\bigcup_{0<|\delta|<\delta_0} \mathcal{A}_\delta(\tau, \omega)$ has a finite covering of balls of radius less than ε. By (3.2) we have

$$\int_{s}^{0} \mathcal{G}_\delta(\theta_l \omega) dl = -\int_{s}^{s+\delta} \frac{\omega(l)}{\delta} dl + \int_{0}^{\delta} \frac{\omega(l)}{\delta} dl. \tag{3.30}$$

By $\lim_{\delta\to 0}\int_0^\delta \frac{\omega(r)}{\delta}dr = 0$, there exists $\delta_1 = \delta_1(\omega) > 0$ such that for all $0 < |\delta| < \delta_1$,

$$|\int_0^\delta \frac{\omega(l)}{\delta}dl| \le 1. \tag{3.31}$$

Similarly, there exists l_1 between s and $s+\delta$ such that $\int_s^{s+\delta}\frac{\omega(l)}{\delta}dl = \omega(l_1)$, which along with (2.1) implies that there exists $T_1 = T_1(\omega) < 0$ such that for all $s \le T_1$ and $|\delta| \le 1$,

$$|\int_s^{s+\delta} \frac{\omega(l)}{\delta}dl| \le 1 - s. \tag{3.32}$$

Let $\delta_2 = \min\{\delta_1, 1\}$. By (3.30)–(3.32) we get for all $0 < |\delta| < \delta_2$ and $s \le T_1$,

$$|\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl| < 2 - s. \tag{3.33}$$

By (3.4), there exist $\delta_0 = \delta_0(\omega) \in (0, \delta_2)$ and $c_1(\omega) > 0$ such that for all $0 < |\delta| \le \delta_0$ and $T_1 \le s \le 0$,

$$|\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl| \le c_1(\omega),$$

which along with (3.33) implies that for all $0 < |\delta| < \delta_0$ and $s \le 0$,

$$|\int_s^0 \mathcal{G}_\delta(\theta_l\omega)dl| \le -s + c_2(\omega), \tag{3.34}$$

where $c_2(\omega) = 2 + c_1(\omega)$. Denote by

$$B(\tau,\omega) = \{\bar\varphi_\delta \in E : \|\bar\varphi_\delta\|^2 \le R(\tau,\omega)\},$$

and

$$\begin{aligned} R(\tau,\omega) =& c\int_{-\infty}^0 e^{\gamma s}\Big(\|g(s+\tau)\|^2 + 2(c_2 - s)^2 + 2(|\tau| + c_2)^2 + 2^p(c_2 - s)^{p+1} + 2^p(|\tau| + c_2)^{p+1}\Big)ds \\ &+ c + 2c(|\tau| + c_2)^2, \end{aligned} \tag{3.35}$$

with c and c_2 being as in (3.12) and (3.34). By (3.12) and (3.35) we find that for all $0 < |\delta| < \delta_0$,

$$R_\delta(\tau,\omega) \le R(\tau,\omega). \tag{3.36}$$

By (3.35) and (3.36), we find that $K_\delta(\tau,\omega) \subseteq B(\tau,\omega)$ for all $0 < |\delta| < \delta_0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$. Therefore, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$,

$$\bigcup_{0<|\delta|<\delta_0} \mathcal{A}_\delta(\tau,\omega) \subseteq \bigcup_{0<|\delta|<\delta_0} K_\delta(\tau,\omega) \subseteq B(\tau,\omega). \tag{3.37}$$

By Lemma 3.5, there exist $T = T(\tau,\omega,\varepsilon) > 0$ and $N = N(\tau,\omega,\varepsilon) > 0$ such that for all $t \ge T$ and $0 < |\delta| < \delta_0$,

$$\sum_{|i|\ge N} |\bar\varphi_{\delta,i}(\tau, \tau - t, \theta_{-\tau}\omega, \bar\varphi_{\delta,\tau-t,i})|_E^2 \le \frac{\varepsilon}{4}, \tag{3.38}$$

for any $\bar{\varphi}_{\delta,\tau-t} \in K_\delta(\tau - t, \theta_{-t}\omega)$. By (3.38) and the invariance of $\mathcal{A}_\delta$, we obtain

$$\sum_{|i|\geq N} |\bar{\varphi}_i|_E^2 \leq \frac{\varepsilon}{4}, \quad \text{for all } \bar{\varphi} = (\bar{\varphi}_i)_{i\in\mathbb{Z}} \in \bigcup_{0<|\delta|<\delta_0} \mathcal{A}_\delta(\tau, \omega). \tag{3.39}$$

We find that (3.37) implies the set $\{(\bar{\varphi}_i)_{|i|<N} : \bar{\varphi} \in \bigcup_{0<|\delta|<\delta_0} \mathcal{A}_\delta(\tau, \omega)\}$ is bounded in a finite dimensional space and hence is precompact. This along with (3.39) implies $\bigcup_{0<|\delta|<\delta_0} \mathcal{A}_\delta(\tau, \omega)$ has a finite covering of balls of radius less than ε in E. This completes the proof. □

4. Upper semicontinuity of pullback attractors

In this section, we will study the limiting of solutions of (3.5) as $\delta \to 0$. Hereafter, we need an additional condition on f: For all $i \in \mathbb{Z}$ and $s \in \mathbb{R}$,

$$|f_i'(s)| \leq \alpha_4 |s|^{p-1} + \kappa_i, \tag{4.1}$$

where α_4 is a positive constant, $\kappa = (\kappa_i)_{i\in\mathbb{Z}} \in l^2$ and $p > 1$.

Lemma 4.1. *Suppose that (2.3)–(2.7) and (4.1) hold. Let $\bar{\varphi}$ and $\bar{\varphi}_\delta$ are the solutions of (2.10) and (3.5), respectively. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and $\varepsilon \in (0, 1)$, there exist $\delta_0 = \delta_0(\tau, \omega, T, \varepsilon) > 0$ and $c = c(\tau, \omega, T) > 0$ such that for all $t \in [\tau, \tau + T]$ and $0 < |\delta| < \delta_0$,*

$$\|\bar{\varphi}_\delta(t, \tau, \omega, \bar{\varphi}_{\delta,\tau}) - \bar{\varphi}(t, \tau, \omega, \bar{\varphi}_\tau)\|_E^2 \leq 2e^{c(t-\tau)}\|\bar{\varphi}_{\delta,\tau} - \bar{\varphi}_\tau\|_E^2 + c\varepsilon.$$

Proof. Let $\tilde{\varphi} = \varphi_\delta - \varphi$ and $\tilde{\varphi} = (\tilde{u}, \tilde{v})^T$, where $\tilde{u} = u^\delta - u$, $\tilde{v} = v^\delta - v$, φ and φ_δ are the solutions of (2.11) and (3.6), respectively. By (2.11) and (3.6) we get

$$\dot{\tilde{\varphi}} + \tilde{L}(\tilde{\varphi}) = \tilde{H}(\tilde{\varphi}) + \tilde{G}(\omega), \tag{4.2}$$

where

$$\tilde{L}(\tilde{\varphi}) = \begin{pmatrix} \beta\tilde{u} - \tilde{v} \\ (1 - \nu\beta)A\tilde{u} + \nu A\tilde{v} + \lambda\tilde{u} + \beta^2\tilde{u} - \beta\tilde{v} \end{pmatrix} + \begin{pmatrix} 0 \\ h(v^\delta - \beta u^\delta + a\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds) - h(v - \beta u + a\omega(t)) \end{pmatrix},$$

$$\tilde{H}(\tilde{\varphi}) = \begin{pmatrix} 0 \\ -f(u^\delta) + f(u) \end{pmatrix}, \quad \tilde{G}(\omega) = \begin{pmatrix} a(\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds - \omega(t)) \\ (\beta a - \nu Aa)(\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds - \omega(t)) \end{pmatrix}.$$

Taking the inner product of (4.2) with $\tilde{\varphi}$ in E, we have

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\varphi}\|_E^2 + (\tilde{L}(\tilde{\varphi}), \tilde{\varphi})_E = (\tilde{H}(\tilde{\varphi}), \tilde{\varphi})_E + (\tilde{G}(\omega), \tilde{\varphi})_E. \tag{4.3}$$

For the second term on the left-hand side of (4.3), using the similar calculations in (2.13) we have

$$\begin{aligned} (\tilde{L}(\tilde{\varphi}), \tilde{\varphi})_E &\geq \sigma\|\tilde{\varphi}\|_E^2 + \frac{h_1}{2}\|\tilde{v}\|^2 - h_2|(a(\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds - \omega(t)), \tilde{v})| \\ &\geq \sigma\|\tilde{\varphi}\|_E^2 + \frac{h_1}{4}\|\tilde{v}\|^2 - c|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds - \omega(t))|^2\|a\|^2. \end{aligned} \tag{4.4}$$

For the first term on the right-hand side of (4.3), by (4.1) we get

$$
\begin{aligned}
(f(u)-f(u^{\delta}),\tilde{v}) &= \sum_{i\in\mathbb{Z}}(f_i(u_i)-f_i(u_i^{\delta}))\tilde{v}_i = \frac{1}{h_1}\sum_{i\in\mathbb{Z}}|f_i(u_i)-f_i(u_i^{\delta})|^2+\frac{h_1}{4}\sum_{i\in\mathbb{Z}}|\tilde{v}_i|^2 \\
&\le c(\|\varphi\|_E^{2p-2}+\|\varphi_\delta\|_E^{2p-2})\|\tilde{\varphi}\|_E^2+\frac{h_1}{4}\|\tilde{v}\|^2+\frac{2\|\kappa\|^2}{h_1\lambda}\|\tilde{\varphi}\|_E^2.
\end{aligned}
\tag{4.5}
$$

As to the last term of (4.3), we have

$$
\begin{aligned}
&(a(\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds-\omega(t)),\tilde{u})_\lambda+((\beta a-\nu Aa)(\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds-\omega(t)),\tilde{v}) \\
&\le \sigma\|\tilde{u}\|_\lambda^2+\frac{1}{4\sigma}|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds-\omega(t)|^2\|a\|_\lambda^2+\sigma\|\tilde{v}\|^2+c|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds-\omega(t)|^2\|a\|^2.
\end{aligned}
\tag{4.6}
$$

It follows from (4.3)–(4.6) that

$$
\frac{d}{dt}\|\tilde{\varphi}\|_E^2 \le c(\|\varphi\|_E^{2p-2}+\|\varphi_\delta\|_E^{2p-2}+1)\|\tilde{\varphi}\|_E^2+c|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds-\omega(t)|^2. \tag{4.7}
$$

By Lemma 2.1 and Lemma 3.2, there exists $\delta_1=\delta_1(\tau,\omega,T)>0$ and $c_1=c_1(\tau,\omega,T)>0$ such that for all $0<|\delta|<\delta_1$ and $t\in[\tau,\tau+T]$,

$$
\|\varphi_\delta(t,\tau,\omega,\varphi_{\delta,\tau})\|_E^2+\|\varphi(t,\tau,\omega,\varphi_\tau)\|_E^2\le c_1,
$$

which along with (4.7) shows that for all $0<|\delta|<\delta_1$ and $t\in[\tau,\tau+T]$

$$
\frac{d}{dt}\|\tilde{\varphi}\|_E^2 \le c\|\tilde{\varphi}\|_E^2+c|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds-\omega(t)|^2. \tag{4.8}
$$

Applying Gronwall's inequality and Lemma 3.1 to (4.8), we see that for every $\varepsilon\in(0,1)$, there exists $\delta_0=\delta_0(\tau,\omega,T,\varepsilon)\in(0,\delta_1)$ such that for all $0<|\delta|<\delta_0$ and $t\in[\tau,\tau+T]$

$$
\|\tilde{\varphi}(t,\tau,\omega,\tilde{\varphi}_\tau)\|_E^2\le e^{c(t-\tau)}\|\tilde{\varphi}_\tau\|_E^2+c\varepsilon. \tag{4.9}
$$

On the other hand, we have

$$
\bar{\varphi}_\delta(t,\tau,\omega,\bar{\varphi}_{\delta,\tau})-\bar{\varphi}(t,\tau,\omega,\bar{\varphi}_\tau)=\tilde{\varphi}+(0,a(\int_0^t \mathcal{G}_\delta(\theta_s)ds-\omega(t)))^T,
$$

which along with (4.9) implies the desired result. □

Finally, we establish the upper semicontinuity of random attractors as $\delta\to 0$.

Theorem 4.1. *Suppose that (2.3)–(2.9) and (4.1) hold. Then for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$,*

$$
\lim_{\delta\to 0} d_E(\mathcal{A}_\delta(\tau,\omega),\mathcal{A}_0(\tau,\omega))=0, \tag{4.10}
$$

where $d_E(\mathcal{A}_\delta(\tau,\omega),\mathcal{A}_0(\tau,\omega))=\sup\limits_{x\in\mathcal{A}_\delta(\tau,\omega)}\inf\limits_{y\in\mathcal{A}_0(\tau,\omega)}\|x-y\|_E$.

Proof. Let $\delta_n \to 0$ and $\bar{\varphi}_{\delta_n,\tau} \to \bar{\varphi}_\tau$ in E. Then by Lemma 4.1, we find that for all $\tau \in \mathbb{R}$, $t \geq 0$ and $\omega \in \Omega$,

$$\Phi_{\delta_n}(t, \tau, \omega, \bar{\varphi}_{\delta_n,\tau}) \to \Phi_0(t, \tau, \omega, \bar{\varphi}_\tau) \text{ in } E. \tag{4.11}$$

By (3.16)–(3.17) we have, for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\delta \to 0} \|K_\delta(\tau, \omega)\|_E^2 \leq R_0(\tau, \omega). \tag{4.12}$$

Then by (4.11), (4.12) and Lemma 3.7, (4.10) follows from Theorem 3.1 in [24] immediately. □

5. Conclusions

In this paper we use similar idea in [30] but apply to second order non-autonomous stochastic lattice dynamical systems with additive noise. we establish the convergence of solutions of Wong-zakai approximations and the upper semicontinuity of random attractors of the approximate random system as the step-length of the Wiener shift approaches zero. In addition, as to the second order non-autonomous stochastic lattice dynamical systems with multiplicative noise, we can use the similar method in [29] to get the corresponding results.

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Conflict of interest

The authors declare no conflict of interest.

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