



Research article

Coupled fixed point theorems on C^* -algebra valued bipolar metric spaces

Gunaseelan Mani¹, Arul Joseph Gnanaprakasam², Absar Ul Haq³, Imran Abbas Baloch^{4,5,*} and Fahd Jarad^{6,7,8,*}

¹ Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602 105, Tamil Nadu, India

² Department of Mathematics, College of Engineering and Technology, Faculty of Engineering and Technology, SRM Institute of Science and Technology, SRM Nagar, Kattankulathur 603203, Kanchipuram, Chennai, Tamil Nadu, India

³ Department of Natural Sciences and Humanities, University of Engineering and Technology (Narowal Campus), Lahore 54000, Pakistan

⁴ Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan

⁵ Higher Education Department, Govt. Graduate College for Boys Gulberg Lahore, Punjab, Pakistan

⁶ Department of Mathematics, Çankaya University, Etimesgut 06790, Ankara, Turkey

⁷ Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia

⁸ Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan

* **Correspondence:** Email: iabbasbaloch@gmail.com, fahd@cankaya.edu.tr.

Abstract: In the present paper, we introduce the notion of a C^* -algebra valued bipolar metric space and prove coupled fixed point theorems. Some of the well-known outcomes in the literature are generalized and expanded by the results shown. An example and application to support our result is presented.

Keywords: C^* -algebra; C^* -algebra valued bipolar metric space; coupled fixed point

Mathematics Subject Classification: 47H10, 54H25, 54C30

1. Introduction

In 2016, Mutlu and Grdal [1] introduced the concepts of bipolar metric space and proved fixed point theorems.

Definition 1.1. [1] Let Γ and Ψ be two non-void sets and $\varphi : \Gamma \times \Psi \rightarrow \mathbb{R}^+$ be a function, such that

- (a) $\varphi(\sigma, \zeta) = 0$ iff $\sigma = \zeta$, for all $(\sigma, \zeta) \in \Gamma \times \Psi$.
- (b) $\varphi(\sigma, \zeta) = \varphi(\zeta, \sigma)$, for all $\sigma, \zeta \in \Gamma \cap \Psi$.
- (c) $\varphi(\sigma, \zeta) \leq \varphi(\sigma, \omega) + \varphi(\sigma_1, \omega) + \varphi(\sigma_1, \zeta)$, for all $\sigma, \sigma_1 \in \Gamma$ and $\omega, \zeta \in \Psi$.

The pair (Γ, Ψ, φ) is called a bipolar metric space.

In bipolar metric spaces, a lot of significant work has been done (see [2–10]). In 2014, Ma, Jiang and Sun [11] introduced the notion of a C^* -algebra-valued metric space and proved fixed point theorem. In 2015, Batul and Kamran [12] proved fixed theorems on C^* -algebra-valued metric space. In the present paper, we introduce a new notion of C^* -algebra valued bipolar metric space and proved coupled fixed point theorems. The details on C^* -algebra are available in [13–17].

An algebra \mathbb{A} , together with a conjugate linear involution map $\vartheta \mapsto \vartheta^*$, is called a \star -algebra if $(\vartheta\varpi)^* = \varpi^*\vartheta^*$ and $(\vartheta^*)^* = \vartheta$ for all $\vartheta, \varpi \in \mathbb{A}$. Moreover, the pair (\mathbb{A}, \star) is called a unital \star -algebra if \mathbb{A} contains the identity element $1_{\mathbb{A}}$. By a Banach \star -algebra we mean a complete normed unital \star -algebra (\mathbb{A}, \star) such that the norm on \mathbb{A} is submultiplicative and satisfies $\|\vartheta^*\| = \|\vartheta\|$ for all $\vartheta \in \mathbb{A}$. Further, if for all $\vartheta \in \mathbb{A}$, we have $\|\vartheta^*\vartheta\| = \|\vartheta\|^2$ in a Banach \star -algebra (\mathbb{A}, \star) , then \mathbb{A} is known as a C^* -algebra. A positive element of \mathbb{A} is an element $\vartheta \in \mathbb{A}$ such that $\vartheta = \vartheta^*$ and its spectrum $\sigma(\vartheta) \subset \mathbb{R}_+$, where $\sigma(\vartheta) = \{\nu \in \mathbb{R} : \nu 1_{\mathbb{A}} - \vartheta \text{ is noninvertible}\}$. The set of all positive elements will be denoted by \mathbb{A}_+ . Such elements allow us to define a partial ordering \geq on the elements of \mathbb{A} . That is,

$$\varpi \geq \vartheta \text{ if and only if } \varpi - \vartheta \in \mathbb{A}_+.$$

If $\vartheta \in \mathbb{A}$ is positive, then we write $\vartheta \geq 0_{\mathbb{A}}$, where $0_{\mathbb{A}}$ is the zero element of \mathbb{A} . Each positive element ϑ of a C^* -algebra \mathbb{A} has a unique positive square root. From now on, by \mathbb{A} we mean a unital C^* -algebra with identity element $1_{\mathbb{A}}$. Further, $\mathbb{A}_+ = \{\vartheta \in \mathbb{A} : \vartheta \geq 0_{\mathbb{A}}\}$ and $(\vartheta^*\vartheta)^{1/2} = |\vartheta|$.

2. Preliminaries

In this section, we extend Definition 1.1 to introduce the notion bipolar metric space in the setting of C^* -algebra as follows.

Definition 2.1. Let \mathbb{A} be a C^* -algebra, and Γ, Ψ be two non-void sets. A mapping $\varphi : \Gamma \times \Psi \rightarrow \mathbb{A}_+$ be a function such that

- (a) $\varphi(\sigma, \zeta) = 0$ iff $\sigma = \zeta$, for all $(\sigma, \zeta) \in \Gamma \times \Psi$.
- (b) $\varphi(\sigma, \zeta) = \varphi(\zeta, \sigma)$, for all $\sigma, \zeta \in \Gamma \cap \Psi$.
- (c) $\varphi(\sigma, \zeta) \leq \varphi(\sigma, \gamma) + \varphi(\sigma_1, \gamma) + \varphi(\sigma_1, \zeta)$, for all $\sigma, \sigma_1 \in \Gamma$ and $\gamma, \zeta \in \Psi$.

The 4-tuple $(\Gamma, \Psi, \mathbb{A}, \varphi)$ is called a C^* -algebra valued bipolar metric space.

Lemma 2.2 ([14, 17]). *Suppose that \mathbb{A} is a unital C^* -algebra with a unit I .*

(A1) *For any $\sigma \in \mathbb{A}_+$, we have $\sigma \leq I$ iff $\|\sigma\| \leq 1$.*

(A2) *If $\vartheta \in \mathbb{A}_+$ with $\|\vartheta\| < \frac{1}{2}$, then $(I - \vartheta)$ is invertible and $\|\vartheta(I - \vartheta)^{-1}\| < 1$.*

(A3) Suppose that $\vartheta, \varpi \in \mathbb{A}$ with $\vartheta\varpi \geq 0_{\mathbb{A}}$ and $\vartheta\varpi = \varpi\vartheta$, then $\varpi\vartheta \geq 0_{\mathbb{A}}$.

(A4) By \mathbb{A}' we denote the set $\{\vartheta \in \mathbb{A} : \vartheta\varpi = \varpi\vartheta, \forall \varpi \in \mathbb{A}\}$. Let $\vartheta \in \mathbb{A}'$, if $\varpi, c \in \mathbb{A}$ with $\varpi \geq c \geq 0_{\mathbb{A}}$, and $I - \vartheta \in \mathbb{A}'_+$ is an invertible operator, then

$$(I - \vartheta)^{-1}\varpi \geq (I - \vartheta)^{-1}c.$$

(A5) If $\varpi, c \in \mathbb{A}_b = \{\sigma \in \mathbb{A} : \sigma = \sigma^*\}$ and $\vartheta \in \mathbb{A}$, then $\varpi \leq c$ implies that $\vartheta^*\varpi\vartheta \leq \vartheta^*c\vartheta$.

(A6) $0_{\mathbb{A}} \leq \vartheta \leq \varpi$, then $\|\vartheta\| \leq \|\varpi\|$.

Notice that in a C^* -algebra, if $0_{\mathbb{A}} \leq \vartheta, \varpi$, one cannot conclude that $0_{\mathbb{A}} \leq \vartheta\varpi$.

Definition 2.3. Let $(\Gamma_1, \Psi_1, \mathbb{A}, \varphi)$ and $(\Gamma_2, \Psi_2, \mathbb{A}, \varphi)$ be pairs of sets and given a function $\Phi : \Gamma_1 \cup \Psi_1 \rightarrow \Gamma_2 \cup \Psi_2$.

(B1) If $\Phi(\Gamma_1) \subseteq \Gamma_2$ and $\Phi(\Psi_1) \subseteq \Psi_2$, then Φ is called a covariant map, or a map from $(\Gamma_1, \Psi_1, \mathbb{A}, \varphi_1)$ to $(\Gamma_2, \Psi_2, \mathbb{A}, \varphi_2)$ and this is written as

$$\Phi : (\Gamma_1, \Psi_1, \mathbb{A}, \varphi_1) \rightrightarrows (\Gamma_2, \Psi_2, \mathbb{A}, \varphi_2).$$

(B2) If $\Phi(\Gamma_1) \subseteq \Psi_2$ and $\Phi(\Psi_1) \subseteq \Gamma_2$, then Φ is called a contravariant map from $(\Gamma_1, \Psi_1, \mathbb{A}, \varphi_1)$ to $(\Gamma_2, \Psi_2, \mathbb{A}, \varphi_2)$ and this is denoted as

$$\Phi : (\Gamma_1, \Psi_1, \mathbb{A}, \varphi_1) \leftrightsquigarrow (\Gamma_2, \Psi_2, \mathbb{A}, \varphi_2).$$

Definition 2.4. Let $(\Gamma, \Psi, \mathbb{A}, \varphi)$ be a C^* -algebra valued bipolar metric space.

(C1) A point $\sigma \in \Gamma \cup \Psi$ is said to be a left point if $\sigma \in \Gamma$, a right point if $\sigma \in \Psi$ and a central point if both hold. Similarly, a sequence $\{\sigma_\alpha\}$ on the set Γ and a sequence $\{\zeta_n\}$ on the set Ψ are called a left and right sequence with respect to \mathbb{A} , respectively.

(C2) A sequence $\{\sigma_\alpha\}$ converges to a point ζ with respect to \mathbb{A} iff $\{\sigma_\alpha\}$ is a left sequence, ζ is a right point and $\lim_{\alpha \rightarrow \infty} \varphi(\sigma_\alpha, \zeta) = 0_{\mathbb{A}}$ or $\{\sigma_\alpha\}$ is a right sequence, ζ is a left point and $\lim_{\alpha \rightarrow \infty} \varphi(\zeta, \sigma_\alpha) = 0_{\mathbb{A}}$.

(C3) A bisequence $(\{\sigma_n\}, \{\zeta_n\})$ is a sequence on the set $\Gamma \times \Psi$. If the sequence $\{\sigma_n\}$ and $\{\zeta_n\}$ are convergent with respect to \mathbb{A} , then the bisequence $(\{\sigma_n\}, \{\zeta_n\})$ is said to be convergent with respect to \mathbb{A} . $(\{\sigma_n\}, \{\zeta_n\})$ is a Cauchy bisequence with respect to \mathbb{A} if $\lim_{\alpha, \beta \rightarrow \infty} \varphi(\sigma_\alpha, \zeta_\beta) = 0_{\mathbb{A}}$, hence biconvergent with respect to \mathbb{A} .

(C4) $(\Gamma, \Psi, \mathbb{A}, \varphi)$ is complete, if every Cauchy bisequence with respect to \mathbb{A} is convergent in $\Gamma \times \Psi$.

3. Main results

Theorem 3.1. Let $(\Gamma, \Psi, \mathbb{A}, \varphi)$ be a complete C^* -algebra valued bipolar metric space. Suppose

$$\Phi : (\Gamma^2, \Psi^2, \mathbb{A}, \varphi) \rightrightarrows (\Gamma, \Psi, \mathbb{A}, \varphi)$$

is a covariant mapping such that

$$\varphi(\Phi(\sigma, \zeta), \Phi(u, v)) \leq v^* \varphi(\sigma, u)v + v^* \varphi(\zeta, v)v \text{ for all } \sigma, \zeta \in \Gamma, u, v \in \Psi,$$

where $v \in \mathbb{A}$ with $2\|v\|^2 < 1$. Then the function

$$\Phi : \Gamma^2 \cup \Psi^2 \rightarrow \Gamma \cup \Psi$$

has a unique coupled fixed point.

Proof. Let $\sigma_0, \zeta_0 \in \Gamma$ and $u_0, v_0 \in \Psi$. For each $\alpha \in \mathbb{N}$, define

$$\Phi(\sigma_\alpha, \zeta_\alpha) = \sigma_{\alpha+1},$$

$$\Phi(\zeta_\alpha, \sigma_\alpha) = \zeta_{\alpha+1},$$

$$\Phi(u_\alpha, v_\alpha) = u_{\alpha+1}$$

and

$$\Phi(v_\alpha, u_\alpha) = v_{\alpha+1}.$$

Then $(\{\sigma_\alpha\}, \{\zeta_\alpha\})$ and $(\{u_\alpha\}, \{v_\alpha\})$ are bisequences on $(\Gamma, \Psi, \mathbb{A}, \varphi)$. Then, for each $\alpha \in \mathbb{N}$,

$$\begin{aligned} \varphi(\sigma_\alpha, u_{\alpha+1}) &= \varphi(\Phi(\sigma_{\alpha-1}, \zeta_{\alpha-1}), \Phi(u_\alpha, v_\alpha)) \\ &\leq v^* \varphi(\sigma_{\alpha-1}, u_\alpha)v + v^* \varphi(\zeta_{\alpha-1}, v_\alpha)v \\ &= v^* \mathcal{M}_\alpha v, \end{aligned}$$

where

$$\mathcal{M}_\alpha = \varphi(\sigma_{\alpha-1}, u_\alpha) + \varphi(\zeta_{\alpha-1}, v_\alpha).$$

Similarly, we get

$$\begin{aligned} \varphi(\zeta_\alpha, v_{\alpha+1}) &= \varphi(\Phi(\zeta_{\alpha-1}, \sigma_{\alpha-1}), \Phi(v_\alpha, u_\alpha)) \\ &\leq v^* \varphi(\zeta_{\alpha-1}, v_\alpha)v + v^* \varphi(\sigma_{\alpha-1}, u_\alpha)v \\ &= v^* \mathcal{M}_\alpha v. \end{aligned}$$

Now,

$$\begin{aligned} \mathcal{M}_{\alpha+1} &= \varphi(\sigma_\alpha, u_{\alpha+1}) + \varphi(\zeta_\alpha, v_{\alpha+1}) \\ &\leq v^* [\varphi(\sigma_{\alpha-1}, u_\alpha) + \varphi(\zeta_{\alpha-1}, v_\alpha)]v \\ &\quad + v^* [\varphi(\zeta_{\alpha-1}, v_\alpha) + \varphi(\sigma_{\alpha-1}, u_\alpha)]v \\ &\leq (\sqrt{2}v)^* \mathcal{M}_\alpha (\sqrt{2}v). \end{aligned}$$

By Lemma 2.2 (A5), we have

$$0_{\mathbb{A}} \leq \mathcal{M}_{\alpha+1} \leq (\sqrt{2}v)^* \mathcal{M}_\alpha (\sqrt{2}v) \leq \dots \leq ((\sqrt{2}v)^*)^\alpha \mathcal{M}_1 (\sqrt{2}v)^\alpha.$$

On the other hand,

$$\begin{aligned}\varphi(\sigma_{\alpha+1}, u_{\alpha}) &= \varphi(\Phi(\sigma_{\alpha}, \zeta_{\alpha}), \Phi(u_{\alpha-1}, v_{\alpha-1})) \\ &\leq v^{\star} \varphi(\sigma_{\alpha}, u_{\alpha-1})v + v^{\star} \varphi(\zeta_{\alpha}, v_{\alpha-1})v \\ &= v^{\star} \mathcal{S}_{\alpha} v,\end{aligned}$$

where

$$\mathcal{S}_{\alpha} = \varphi(\sigma_{\alpha}, u_{\alpha-1}) + \varphi(\zeta_{\alpha}, v_{\alpha-1}).$$

Similarly, we get

$$\begin{aligned}\varphi(\zeta_{\alpha+1}, v_{\alpha}) &= \varphi(\Phi(\zeta_{\alpha}, \sigma_{\alpha}), \Phi(v_{\alpha-1}, u_{\alpha-1})) \\ &\leq v^{\star} \varphi(\zeta_{\alpha}, v_{\alpha-1})v + v^{\star} \varphi(\sigma_{\alpha}, u_{\alpha-1})v \\ &= v^{\star} \mathcal{S}_{\alpha} v.\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{S}_{\alpha+1} &= \varphi(\sigma_{\alpha+1}, u_{\alpha}) + \varphi(\zeta_{\alpha+1}, v_{\alpha}) \\ &\leq v^{\star} [\varphi(\sigma_{\alpha}, u_{\alpha-1}) + \varphi(\zeta_{\alpha}, v_{\alpha-1})]v \\ &\quad + v^{\star} [\varphi(\zeta_{\alpha}, v_{\alpha-1}) + \varphi(\sigma_{\alpha}, u_{\alpha-1})]v \\ &\leq (\sqrt{2}v)^{\star} \mathcal{S}_{\alpha} (\sqrt{2}v).\end{aligned}$$

By Lemma 2.2 (A5), we have

$$0_{\mathbb{A}} \leq \mathcal{S}_{\alpha+1} \leq (\sqrt{2}v)^{\star} \mathcal{S}_{\alpha} (\sqrt{2}v) \leq \dots \leq ((\sqrt{2}v)^{\star})^{\alpha} \mathcal{S}_1 (\sqrt{2}v)^{\alpha}.$$

Moreover,

$$\begin{aligned}\varphi(\sigma_{\alpha}, u_{\alpha}) &= \varphi(\Phi(\sigma_{\alpha-1}, \zeta_{\alpha-1}), \Phi(u_{\alpha-1}, v_{\alpha-1})) \\ &\leq v^{\star} \varphi(\sigma_{\alpha-1}, u_{\alpha-1})v + v^{\star} \varphi(\zeta_{\alpha-1}, v_{\alpha-1})v \\ &= v^{\star} \mathcal{R}_{\alpha} v,\end{aligned}$$

where

$$\mathcal{R}_{\alpha} = \varphi(\sigma_{\alpha-1}, u_{\alpha-1}) + \varphi(\zeta_{\alpha-1}, v_{\alpha-1}).$$

Similarly, we get

$$\begin{aligned}\varphi(\zeta_{\alpha}, v_{\alpha}) &= \varphi(\Phi(\zeta_{\alpha-1}, \sigma_{\alpha-1}), \Phi(v_{\alpha-1}, u_{\alpha-1})) \\ &\leq v^{\star} \varphi(\zeta_{\alpha-1}, v_{\alpha-1})v + v^{\star} \varphi(\sigma_{\alpha-1}, u_{\alpha-1})v \\ &= v^{\star} \mathcal{R}_{\alpha} v.\end{aligned}$$

Now,

$$\begin{aligned}
\mathcal{R}_{\alpha+1} &= \varphi(\sigma_\alpha, u_\alpha) + \varphi(\zeta_\alpha, v_\alpha) \\
&\leq v^*[\varphi(\sigma_{\alpha-1}, u_{\alpha-1}) + \varphi(\zeta_{\alpha-1}, v_{\alpha-1})]v \\
&\quad + v^*[\varphi(\zeta_{\alpha-1}, v_{\alpha-1}) + \varphi(\sigma_{\alpha-1}, u_{\alpha-1})]v \\
&\leq (\sqrt{2}v)^* \mathcal{R}_\alpha(\sqrt{2}v).
\end{aligned}$$

By Lemma 2.2 (A5), we have

$$0_{\mathbb{A}} \leq \mathcal{R}_{\alpha+1} \leq (\sqrt{2}v)^* \mathcal{R}_\alpha(\sqrt{2}v) \leq \cdots \leq ((\sqrt{2}v)^*)^\alpha \mathcal{R}_1(\sqrt{2}v)^\alpha.$$

Now,

$$\begin{aligned}
\varphi(\sigma_\alpha, u_\beta) &\leq \varphi(\sigma_\alpha, u_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, u_{\alpha+1}) + \cdots + \varphi(\sigma_{\beta-1}, u_\beta), \\
\varphi(\zeta_\alpha, v_\beta) &\leq \varphi(\zeta_\alpha, v_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, v_{\alpha+1}) + \cdots + \varphi(\zeta_{\beta-1}, v_\beta),
\end{aligned}$$

and

$$\begin{aligned}
\varphi(\sigma_\beta, u_\alpha) &\leq \varphi(\sigma_\beta, u_{\beta-1}) + \varphi(\sigma_{\beta-1}, u_{\beta-1}) + \cdots + \varphi(\sigma_{\alpha+1}, u_\alpha), \\
\varphi(\zeta_\beta, v_\alpha) &\leq \varphi(\zeta_\beta, v_{\beta-1}) + \varphi(\zeta_{\beta-1}, v_{\beta-1}) + \cdots + \varphi(\zeta_{\alpha+1}, v_\alpha),
\end{aligned}$$

for each $\alpha, \beta \in \mathbb{N}$, $\alpha < \beta$. Then,

$$\begin{aligned}
\varphi(\sigma_\alpha, u_\beta) + \varphi(\zeta_\alpha, v_\beta) &\leq (\varphi(\sigma_\alpha, u_{\alpha+1}) + \varphi(\zeta_\alpha, v_{\alpha+1})) + (\varphi(\sigma_{\alpha+1}, u_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, v_{\alpha+1})) \\
&\quad + \cdots + (\varphi(\sigma_{\beta-1}, u_\beta) + \varphi(\zeta_{\beta-1}, v_\beta)) \\
&= \mathcal{M}_{\alpha+1} + \mathcal{R}_{\alpha+2} + \mathcal{M}_{\alpha+2} + \cdots + \mathcal{R}_\beta + \mathcal{M}_\beta \\
&\leq ((\sqrt{2}v)^*)^\alpha \mathcal{M}_1(\sqrt{2}v)^\alpha + ((\sqrt{2}v)^*)^{\alpha+1} \mathcal{R}_1(\sqrt{2}v)^{\alpha+1} + \cdots \\
&\quad + ((\sqrt{2}v)^*)^{\beta-1} \mathcal{R}_1(\sqrt{2}v)^{\beta-1} + ((\sqrt{2}v)^*)^{\beta-1} \mathcal{M}_1(\sqrt{2}v)^{\beta-1} \\
&= \sum_{i=\alpha}^{\beta-1} ((\sqrt{2}v)^*)^i \mathcal{M}_1(\sqrt{2}v)^i + \sum_{i=\alpha+1}^{\beta-1} ((\sqrt{2}v)^*)^i \mathcal{R}_1(\sqrt{2}v)^i \\
&= \sum_{i=\alpha}^{\beta-1} ((\sqrt{2}v)^*)^i \mathcal{M}_1^{\frac{1}{2}} \mathcal{M}_1^{\frac{1}{2}}(\sqrt{2}v)^i + \sum_{i=\alpha+1}^{\beta-1} ((\sqrt{2}v)^*)^i \mathcal{R}_1^{\frac{1}{2}} \mathcal{R}_1^{\frac{1}{2}}(\sqrt{2}v)^i \\
&= \sum_{i=\alpha}^{\beta-1} (\mathcal{M}_1^{\frac{1}{2}}(\sqrt{2}v)^i)^* (\mathcal{M}_1^{\frac{1}{2}}(\sqrt{2}v)^i) + \sum_{i=\alpha+1}^{\beta-1} (\mathcal{R}_1^{\frac{1}{2}}(\sqrt{2}v)^i)^* (\mathcal{R}_1^{\frac{1}{2}}(\sqrt{2}v)^i) \\
&= \sum_{i=\alpha}^{\beta-1} |\mathcal{M}_1^{\frac{1}{2}}(\sqrt{2}v)^i|^2 + \sum_{i=\alpha+1}^{\beta-1} |\mathcal{R}_1^{\frac{1}{2}}(\sqrt{2}v)^i|^2 \\
&\leq \left\| \sum_{i=\alpha}^{\beta-1} |\mathcal{M}_1^{\frac{1}{2}}(\sqrt{2}v)^i|^2 \right\| 1_{\mathbb{A}} + \left\| \sum_{i=\alpha+1}^{\beta-1} |\mathcal{R}_1^{\frac{1}{2}}(\sqrt{2}v)^i|^2 \right\| 1_{\mathbb{A}} \\
&\leq \sum_{i=\alpha}^{\beta-1} \|\mathcal{M}_1^{\frac{1}{2}}\|^2 \|(\sqrt{2}v)^i\|^2 \|1_{\mathbb{A}}\| + \sum_{i=\alpha+1}^{\beta-1} \|\mathcal{R}_1^{\frac{1}{2}}\|^2 \|(\sqrt{2}v)^i\|^2 \|1_{\mathbb{A}}\|
\end{aligned}$$

$$\begin{aligned}
&\leq \|\mathcal{M}_1^{\frac{1}{2}}\|^2 \sum_{i=\alpha}^{\beta-1} \|(\sqrt{2}v)^2\|^i \|1_{\mathbb{A}} + \|\mathcal{R}_1^{\frac{1}{2}}\|^2 \sum_{i=\alpha+1}^{\beta-1} \|(\sqrt{2}v)^2\|^i \|1_{\mathbb{A}} \\
&= \|\mathcal{M}_1^{\frac{1}{2}}\|^2 \sum_{i=\alpha}^{\beta-1} (2\|v\|^2)^i \|1_{\mathbb{A}} + \|\mathcal{R}_1^{\frac{1}{2}}\|^2 \sum_{i=\alpha+1}^{\beta-1} (2\|v\|^2)^i \|1_{\mathbb{A}} \\
&\rightarrow 0_{\mathbb{A}} \quad (\text{as } \beta, \alpha \rightarrow \infty)
\end{aligned}$$

and

$$\begin{aligned}
\varphi(\sigma_{\beta}, u_{\alpha}) + \varphi(\zeta_{\beta}, v_{\alpha}) &\leq (\varphi(\sigma_{\beta}, u_{\beta-1}) + \varphi(\zeta_{\beta}, v_{\beta-1})) + (\varphi(\sigma_{\beta-1}, u_{\beta-1}) + \varphi(\zeta_{\beta-1}, v_{\beta-1})) \\
&\quad + \cdots + (\varphi(\sigma_{\alpha+1}, u_{\alpha}) + \varphi(\zeta_{\alpha+1}, v_{\alpha})) \\
&= \mathcal{S}_{\beta} + \mathcal{R}_{\beta} + \mathcal{S}_{\beta-1} + \cdots + \mathcal{R}_{\alpha+2} + \mathcal{S}_{\alpha+1} \\
&\leq ((\sqrt{2}v)^{\star})^{\beta-1} \mathcal{S}_1(\sqrt{2}v)^{\beta-1} + ((\sqrt{2}v)^{\star})^{\beta-1} \mathcal{R}_1(\sqrt{2}v)^{\beta-1} + \cdots \\
&\quad + ((\sqrt{2}v)^{\star})^{\alpha+1} \mathcal{R}_1(\sqrt{2}v)^{\alpha+1} + ((\sqrt{2}v)^{\star})^{\alpha} \mathcal{S}_1(\sqrt{2}v)^{\alpha} \\
&= \sum_{i=\beta}^{\alpha+1} ((\sqrt{2}v)^{\star})^i \mathcal{S}_1(\sqrt{2}v)^i + \sum_{i=\beta}^{\alpha+2} ((\sqrt{2}v)^{\star})^i \mathcal{R}_1(\sqrt{2}v)^i \\
&= \sum_{i=\beta}^{\alpha+1} ((\sqrt{2}v)^{\star})^i \mathcal{S}_1^{\frac{1}{2}} \mathcal{S}_1^{\frac{1}{2}}(\sqrt{2}v)^i + \sum_{i=\beta}^{\alpha+2} ((\sqrt{2}v)^{\star})^i \mathcal{R}_1^{\frac{1}{2}} \mathcal{R}_1^{\frac{1}{2}}(\sqrt{2}v)^i \\
&= \sum_{i=\beta}^{\alpha+1} (\mathcal{S}_1^{\frac{1}{2}}(\sqrt{2}v)^i)^{\star} (\mathcal{S}_1^{\frac{1}{2}}(\sqrt{2}v)^i) + \sum_{i=\beta}^{\alpha+2} (\mathcal{R}_1^{\frac{1}{2}}(\sqrt{2}v)^i)^{\star} (\mathcal{R}_1^{\frac{1}{2}}(\sqrt{2}v)^i) \\
&= \sum_{i=\beta}^{\alpha+1} |\mathcal{S}_1^{\frac{1}{2}}(\sqrt{2}v)^i|^2 + \sum_{i=\beta}^{\alpha+2} |\mathcal{R}_1^{\frac{1}{2}}(\sqrt{2}v)^i|^2 \\
&\leq \left\| \sum_{i=\beta}^{\alpha+1} |\mathcal{S}_1^{\frac{1}{2}}(\sqrt{2}v)^i|^2 \right\| \|1_{\mathbb{A}} + \left\| \sum_{i=\beta}^{\alpha+2} |\mathcal{R}_1^{\frac{1}{2}}(\sqrt{2}v)^i|^2 \right\| \|1_{\mathbb{A}} \\
&\leq \sum_{i=\beta}^{\alpha+1} \|\mathcal{S}_1^{\frac{1}{2}}\|^2 \|(\sqrt{2}v)^i\|^2 \|1_{\mathbb{A}} + \sum_{i=\beta}^{\alpha+2} \|\mathcal{R}_1^{\frac{1}{2}}\|^2 \|(\sqrt{2}v)^i\|^2 \|1_{\mathbb{A}} \\
&\leq \|\mathcal{S}_1^{\frac{1}{2}}\|^2 \sum_{i=\beta}^{\alpha+1} \|(\sqrt{2}v)^2\|^i \|1_{\mathbb{A}} + \|\mathcal{R}_1^{\frac{1}{2}}\|^2 \sum_{i=\beta}^{\alpha+2} \|(\sqrt{2}v)^2\|^i \|1_{\mathbb{A}} \\
&= \|\mathcal{S}_1^{\frac{1}{2}}\|^2 \sum_{i=\beta}^{\alpha+1} (2\|v\|^2)^i \|1_{\mathbb{A}} + \|\mathcal{R}_1^{\frac{1}{2}}\|^2 \sum_{i=\beta}^{\alpha+2} (2\|v\|^2)^i \|1_{\mathbb{A}} \\
&\rightarrow 0_{\mathbb{A}} \quad (\text{as } \beta, \alpha \rightarrow \infty).
\end{aligned}$$

Therefore, $(\{\sigma_{\alpha}\}, \{u_{\alpha}\})$ and $(\{\zeta_{\alpha}\}, \{v_{\alpha}\})$ are Cauchy bisequences in $\Gamma \times \Psi$ with respect to \mathbb{A} . By completeness of $(\Gamma, \Psi, \mathbb{A}, \varphi)$, there exist $\sigma, \zeta \in \Gamma$ and $u, v \in \Psi$ with

$$\lim_{\alpha \rightarrow \infty} \sigma_{\alpha} = u, \quad \lim_{\alpha \rightarrow \infty} \zeta_{\alpha} = v, \quad \lim_{\alpha \rightarrow \infty} u_{\alpha} = \sigma \quad \text{and} \quad \lim_{\alpha \rightarrow \infty} v_{\alpha} = \zeta.$$

Since $(\{\sigma_\alpha\}, \{u_\alpha\})$ and $(\{\zeta_\alpha\}, \{v_\alpha\})$ are Cauchy bisequences, we derive that

$$\varphi(\sigma_\alpha, u_\alpha) < \frac{\epsilon}{2} \text{ and } \varphi(\zeta_\alpha, v_\alpha) < \frac{\epsilon}{2}.$$

Then,

$$\begin{aligned} \varphi(\Phi(\sigma, \zeta), u) &\leq \varphi(\Phi(\sigma, \zeta), u_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, u_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, u_\alpha) \\ &= \varphi(\Phi(\sigma, \zeta), \Phi(u_\alpha, v_\alpha)) + \varphi(\sigma_{\alpha+1}, u_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, u_\alpha) \\ &\leq \nu^* \varphi(\sigma, u_\alpha) \nu + \nu^* \varphi(\zeta, v_\alpha) \nu + \varphi(\sigma_{\alpha+1}, u_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, u_\alpha). \end{aligned}$$

As $\alpha \rightarrow \infty$, we have

$$\varphi(\Phi(\sigma, \zeta), u) < \epsilon.$$

Then,

$$\varphi(\Phi(\sigma, \zeta), u) = 0.$$

Hence, $\Phi(\sigma, \zeta) = u$. Similarly, we can derive $\Phi(\zeta, \sigma) = v$, $\Phi(u, v) = \sigma$ and $\Phi(v, u) = \zeta$. On the other hand, we derive that

$$\varphi(\sigma, u) = \varphi(\lim_{\alpha \rightarrow \infty} u_\alpha, \lim_{\alpha \rightarrow \infty} \sigma_\alpha) = \lim_{\alpha \rightarrow \infty} \varphi(\sigma_\alpha, u_\alpha) = 0$$

and

$$\varphi(\zeta, v) = \varphi(\lim_{\alpha \rightarrow \infty} v_\alpha, \lim_{\alpha \rightarrow \infty} \zeta_\alpha) = \lim_{\alpha \rightarrow \infty} \varphi(\zeta_\alpha, v_\alpha) = 0.$$

So, $\sigma = u$ and $\zeta = v$. Therefore, $(\sigma, \zeta) \in \Gamma^2 \cap \Psi^2$ is a coupled fixed point of Φ . Let $(e, f) \in \Gamma^2 \cup \Psi^2$ is a another coupled fixed point Φ . If $(e, f) \in \Gamma^2$, then

$$0_{\mathbb{A}} \leq \varphi(e, \sigma) = \varphi(\Phi(e, f), \Phi(\sigma, \zeta)) \leq \nu^* \varphi(e, \sigma) \nu + \nu^* \varphi(f, \zeta) \nu$$

and

$$0_{\mathbb{A}} \leq \varphi(f, \zeta) = \varphi(\Phi(f, e), \Phi(\zeta, \sigma)) \leq \nu^* \varphi(f, \zeta) \nu + \nu^* \varphi(e, \sigma) \nu,$$

which implies that

$$0_{\mathbb{A}} \leq \varphi(e, \sigma) + \varphi(f, \zeta) \leq (\sqrt{2}\nu)^*(\varphi(f, \zeta) + \varphi(e, \sigma))(\sqrt{2}\nu).$$

Then,

$$0 \leq \|\varphi(e, \sigma) + \varphi(f, \zeta)\| \leq \|\sqrt{2}\nu\|^2 \|\varphi(f, \zeta) + \varphi(e, \sigma)\|.$$

Since $2\|\nu\|^2 < 1$, we derive that

$$\varphi(f, \zeta) + \varphi(e, \sigma) = 0.$$

Therefore, $e = \sigma$ and $f = \zeta$. Similarly, if $(e, f) \in \Psi^2$, then $e = \sigma$ and $f = \zeta$. Then (σ, ζ) is a unique coupled fixed point of Φ . \square

Example 3.2. Let $\Gamma = [0, 1]$, $\Psi = \{0\} \cup \mathbb{N} - \{1\}$, $\mathbb{A}_+ = \mathcal{M}_2(\mathbb{C})$ and the map $\varphi : \Gamma \times \Psi \rightarrow \mathbb{A}_+$ is defined by

$$\varphi(\sigma, u) = \begin{bmatrix} |\sigma - u| & 0 \\ 0 & \mathbb{k}|\sigma - u| \end{bmatrix}$$

for all $\sigma \in \Gamma$ and $u \in \Psi$, where $\mathbb{k} \geq 0$ is a constant. Let \leq be the partial order on \mathbb{A} given by

$$(\vartheta_1, \varpi_1) \leq (\vartheta_2, \varpi_2) \Leftrightarrow \vartheta_1 \leq \vartheta_2 \text{ and } \varpi_1 \leq \varpi_2.$$

Then $(\Gamma, \Psi, \mathbb{A}, \varphi)$ is a complete C^* -algebra-valued bipolar metric space. Define

$$\Phi : \Gamma^2 \cup \Psi^2 \rightrightarrows \Gamma \cup \Psi$$

by

$$\Phi(\sigma, \zeta) = \frac{\sigma + \zeta}{3},$$

$\forall \sigma, \zeta \in \Gamma^2 \cup \Psi^2$. Then

$$\begin{aligned} \varphi(\Phi(\sigma, \zeta), \Phi(u, v)) &= \begin{bmatrix} |\Phi(\sigma, \zeta) - \Phi(u, v)| & 0 \\ 0 & \mathbb{k}|\Phi(\sigma, \zeta) - \Phi(u, v)| \end{bmatrix} \\ &= \begin{bmatrix} \left| \frac{\sigma + \zeta}{3} - \frac{u + v}{3} \right| & 0 \\ 0 & \mathbb{k} \left| \frac{\sigma + \zeta}{3} - \frac{u + v}{3} \right| \end{bmatrix} \\ &\leq \frac{1}{3} \left(\begin{bmatrix} |\sigma - u| & 0 \\ 0 & \mathbb{k}|\sigma - u| \end{bmatrix} + \begin{bmatrix} |\zeta - v| & 0 \\ 0 & \mathbb{k}|\zeta - v| \end{bmatrix} \right) \\ &= v^* \varphi(\sigma, u) v + v^* \varphi(\zeta, v) v, \end{aligned}$$

for all $\sigma, \zeta \in \Gamma$, $u, v \in \Psi$, where

$$v = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

and $\|v\| = \frac{1}{3} < \frac{1}{\sqrt{2}}$. All the conditions of Theorem 3.1 are fulfilled and Φ has a unique fixed point $(0, 0)$.

Theorem 3.3. Let $(\Gamma, \Psi, \mathbb{A}, \varphi)$ be a complete C^* -algebra valued bipolar metric space. Suppose

$$\Phi : (\Gamma \times \Psi, \Psi \times \Gamma, \mathbb{A}, \varphi) \rightrightarrows (\Gamma, \Psi, \mathbb{A}, \varphi)$$

is a covariant mapping such that

$$\varphi(\Phi(\sigma, u), \Phi(v, \zeta)) \leq v^* \varphi(\sigma, v) v + v^* \varphi(\zeta, u) v \text{ for all } \sigma, \zeta \in \Gamma, u, v \in \Psi,$$

where $v \in \mathbb{A}$ with $2\|v\|^2 < 1$. Then the function

$$\Phi : (\Gamma \times \Psi) \cup (\Psi \times \Gamma) \rightarrow \Gamma \cup \Psi$$

has a unique coupled fixed point.

Proof. Let $\sigma_0, \zeta_0 \in \Gamma$ and $u_0, v_0 \in \Psi$. For each $\alpha \in \mathbb{N}$, define $\Phi(\sigma_\alpha, u_\alpha) = \sigma_{\alpha+1}$, $\Phi(\zeta_\alpha, v_\alpha) = \zeta_{\alpha+1}$, $\Phi(u_\alpha, \sigma_\alpha) = u_{\alpha+1}$ and $\Phi(v_\alpha, \zeta_\alpha) = v_{\alpha+1}$. Then $(\{\sigma_\alpha\}, \{u_\alpha\})$ and $(\{\zeta_\alpha\}, \{v_\alpha\})$ are bisequences on $(\Gamma, \Psi, \mathbb{A}, \varphi)$. Then, for each $\alpha \in \mathbb{N}$,

$$\begin{aligned}\varphi(\sigma_\alpha, v_{\alpha+1}) &= \varphi(\Phi(\sigma_{\alpha-1}, u_{\alpha-1}), \Phi(v_\alpha, \zeta_\alpha)) \\ &\leq v^* \varphi(\sigma_{\alpha-1}, v_\alpha) v + v^* \varphi(\zeta_\alpha, u_{\alpha-1}) v,\end{aligned}$$

$$\begin{aligned}\varphi(\sigma_{\alpha+1}, v_\alpha) &= \varphi(\Phi(\sigma_\alpha, u_\alpha), \Phi(v_{\alpha-1}, \zeta_{\alpha-1})) \\ &\leq v^* \varphi(\sigma_\alpha, v_{\alpha-1}) v + v^* \varphi(\zeta_{\alpha-1}, u_\alpha) v,\end{aligned}$$

$$\begin{aligned}\varphi(\zeta_\alpha, u_{\alpha+1}) &= \varphi(\Phi(\zeta_{\alpha-1}, v_{\alpha-1}), \Phi(u_\alpha, \sigma_\alpha)) \\ &\leq v^* \varphi(\zeta_{\alpha-1}, u_\alpha) v + v^* \varphi(\sigma_\alpha, v_{\alpha-1}) v,\end{aligned}$$

$$\begin{aligned}\varphi(\zeta_{\alpha+1}, u_\alpha) &= \varphi(\Phi(\zeta_\alpha, v_\alpha), \Phi(u_{\alpha-1}, \sigma_{\alpha-1})) \\ &\leq v^* \varphi(\zeta_\alpha, u_{\alpha-1}) v + v^* \varphi(\sigma_{\alpha-1}, v_\alpha) v.\end{aligned}$$

Let

$$\mathcal{M}_\alpha = \varphi(\sigma_\alpha, v_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, u_\alpha),$$

for all $\alpha \in \mathbb{N}$. Then

$$\begin{aligned}\mathcal{M}_\alpha &= \varphi(\sigma_\alpha, v_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, u_\alpha) \\ &\leq v^* [\varphi(\sigma_{\alpha-1}, v_\alpha) + \varphi(\zeta_\alpha, u_{\alpha-1})] v \\ &\quad + v^* [\varphi(\zeta_\alpha, u_{\alpha-1}) + \varphi(\sigma_{\alpha-1}, v_\alpha)] v \\ &\leq (\sqrt{2}v)^* \mathcal{M}_{\alpha-1} (\sqrt{2}v).\end{aligned}$$

By Lemma 2.2 (A5), we have

$$0_{\mathbb{A}} \leq \mathcal{M}_\alpha \leq (\sqrt{2}v)^* \mathcal{M}_{\alpha-1} (\sqrt{2}v) \leq \cdots \leq ((\sqrt{2}v)^*)^\alpha \mathcal{M}_0 (\sqrt{2}v)^\alpha.$$

Let

$$\mathcal{S}_\alpha = \varphi(\sigma_{\alpha+1}, v_\alpha) + \varphi(\zeta_\alpha, u_{\alpha+1})$$

for all $\alpha \in \mathbb{N}$. Then

$$\begin{aligned}\mathcal{S}_\alpha &= \varphi(\sigma_{\alpha+1}, v_\alpha) + \varphi(\zeta_\alpha, u_{\alpha+1}) \\ &\leq v^* [\varphi(\sigma_\alpha, v_{\alpha-1}) + \varphi(\zeta_{\alpha-1}, u_\alpha)] v \\ &\quad + v^* [\varphi(\zeta_{\alpha-1}, u_\alpha) + \varphi(\sigma_\alpha, v_{\alpha-1})] v \\ &\leq (\sqrt{2}v)^* \mathcal{S}_{\alpha-1} (\sqrt{2}v).\end{aligned}$$

By Lemma 2.2 (A5), we have

$$0_{\mathbb{A}} \leq \mathcal{S}_{\alpha} \leq (\sqrt{2}v)^* \mathcal{S}_{\alpha-1} (\sqrt{2}v) \leq \cdots \leq ((\sqrt{2}v)^*)^{\alpha} \mathcal{S}_0 (\sqrt{2}v)^{\alpha}.$$

On the other hand,

$$\begin{aligned} \varphi(\sigma_{\alpha}, v_{\alpha+1}) &= \varphi(\Phi(\sigma_{\alpha-1}, u_{\alpha-1}), \Phi(v_{\alpha}, \zeta_{\alpha})) \\ &\leq v^* \varphi(\sigma_{\alpha-1}, v_{\alpha}) v + v^* \varphi(\zeta_{\alpha}, u_{\alpha-1}) v. \end{aligned}$$

$$\begin{aligned} \varphi(\sigma_{\alpha}, v_{\alpha}) &= \varphi(\Phi(\sigma_{\alpha-1}, u_{\alpha-1}), \Phi(v_{\alpha-1}, \zeta_{\alpha-1})) \\ &\leq v^* \varphi(\sigma_{\alpha-1}, v_{\alpha-1}) v + v^* \varphi(\zeta_{\alpha-1}, u_{\alpha-1}) v \end{aligned}$$

and

$$\begin{aligned} \varphi(\zeta_{\alpha}, u_{\alpha}) &= \varphi(\Phi(\zeta_{\alpha-1}, u_{\alpha-1}), \Phi(u_{\alpha-1}, \sigma_{\alpha-1})) \\ &\leq v^* \varphi(\zeta_{\alpha-1}, u_{\alpha-1}) v + v^* \varphi(\sigma_{\alpha-1}, v_{\alpha-1}) v \end{aligned}$$

for all $\alpha \in \mathbb{N}$. Let

$$\mathcal{R}_{\alpha} = \varphi(\sigma_{\alpha}, v_{\alpha}) + \varphi(\zeta_{\alpha}, u_{\alpha}),$$

for all $\alpha \in \mathbb{N}$. Then

$$\begin{aligned} \mathcal{R}_{\alpha} &= \varphi(\sigma_{\alpha}, v_{\alpha}) + \varphi(\zeta_{\alpha}, u_{\alpha}) \\ &\leq v^* [\varphi(\sigma_{\alpha-1}, v_{\alpha-1}) + \varphi(\zeta_{\alpha-1}, u_{\alpha-1})] v \\ &\quad + v^* [\varphi(\zeta_{\alpha-1}, u_{\alpha-1}) + \varphi(\sigma_{\alpha-1}, v_{\alpha-1})] v \\ &\leq (\sqrt{2}v)^* \mathcal{R}_{\alpha-1} (\sqrt{2}v). \end{aligned}$$

By Lemma 2.2 (A5), we have

$$0_{\mathbb{A}} \leq \mathcal{R}_{\alpha} \leq (\sqrt{2}v)^* \mathcal{R}_{\alpha-1} (\sqrt{2}v) \leq \cdots \leq ((\sqrt{2}v)^*)^{\alpha} \mathcal{R}_0 (\sqrt{2}v)^{\alpha}.$$

Now,

$$\begin{aligned} \varphi(\sigma_{\alpha}, v_{\beta}) &\leq \varphi(\sigma_{\alpha}, v_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, v_{\alpha+1}) + \cdots + \varphi(\sigma_{\beta-1}, v_{\beta}), \\ \varphi(\zeta_{\alpha}, u_{\beta}) &\leq \varphi(\zeta_{\alpha}, u_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, u_{\alpha+1}) + \cdots + \varphi(\zeta_{\beta-1}, u_{\beta}), \end{aligned}$$

and

$$\begin{aligned} \varphi(\sigma_{\beta}, v_{\alpha}) &\leq \varphi(\sigma_{\beta}, v_{\beta-1}) + \varphi(\sigma_{\beta-1}, v_{\beta-1}) + \cdots + \varphi(\sigma_{\alpha+1}, v_{\alpha}), \\ \varphi(\zeta_{\beta}, u_{\alpha}) &\leq \varphi(\zeta_{\beta}, u_{\beta-1}) + \varphi(\zeta_{\beta-1}, u_{\beta-1}) + \cdots + \varphi(\zeta_{\alpha+1}, u_{\alpha}), \end{aligned}$$

for each $\alpha, \beta \in \mathbb{N}$, $\alpha < \beta$. Then,

$$\begin{aligned}
\varphi(\sigma_\alpha, \mathfrak{v}_\beta) + \varphi(\zeta_\beta, \mathfrak{u}_\alpha) &\leq (\varphi(\sigma_\alpha, \mathfrak{v}_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, \mathfrak{u}_\alpha)) + (\varphi(\sigma_{\alpha+1}, \mathfrak{v}_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, \mathfrak{u}_{\alpha+1})) \\
&\quad + \cdots + (\varphi(\sigma_{\beta-1}, \mathfrak{v}_\beta) + \varphi(\zeta_\beta, \mathfrak{u}_{\beta-1})) \\
&= \mathcal{M}_\alpha + \mathcal{R}_{\alpha+1} + \mathcal{M}_{\alpha+1} + \cdots + \mathcal{R}_{\beta-1} + \mathcal{M}_{\beta-1} \\
&\leq ((\sqrt{2\nu})^\star)^\alpha \mathcal{M}_0 (\sqrt{2\nu})^\alpha + ((\sqrt{2\nu})^\star)^{\alpha+1} \mathcal{R}_0 (\sqrt{2\nu})^{\alpha+1} + \cdots \\
&\quad + ((\sqrt{2\nu})^\star)^{\beta-1} \mathcal{R}_0 (\sqrt{2\nu})^{\beta-1} + ((\sqrt{2\nu})^\star)^{\beta-1} \mathcal{M}_0 (\sqrt{2\nu})^{\beta-1} \\
&= \sum_{i=\alpha}^{\beta-1} ((\sqrt{2\nu})^\star)^i \mathcal{M}_0 (\sqrt{2\nu})^i + \sum_{i=\alpha+1}^{\beta-1} ((\sqrt{2\nu})^\star)^i \mathcal{R}_0 (\sqrt{2\nu})^i \\
&= \sum_{i=\alpha}^{\beta-1} ((\sqrt{2\nu})^\star)^i \mathcal{M}_0^{\frac{1}{2}} \mathcal{M}_0^{\frac{1}{2}} (\sqrt{2\nu})^i + \sum_{i=\alpha+1}^{\beta-1} ((\sqrt{2\nu})^\star)^i \mathcal{R}_0^{\frac{1}{2}} \mathcal{R}_0^{\frac{1}{2}} (\sqrt{2\nu})^i \\
&= \sum_{i=\alpha}^{\beta-1} (\mathcal{M}_0^{\frac{1}{2}} (\sqrt{2\nu})^i)^\star (\mathcal{M}_0^{\frac{1}{2}} (\sqrt{2\nu})^i) + \sum_{i=\alpha+1}^{\beta-1} (\mathcal{R}_0^{\frac{1}{2}} (\sqrt{2\nu})^i)^\star (\mathcal{R}_0^{\frac{1}{2}} (\sqrt{2\nu})^i) \\
&= \sum_{i=\alpha}^{\beta-1} |\mathcal{M}_0^{\frac{1}{2}} (\sqrt{2\nu})^i|^2 + \sum_{i=\alpha+1}^{\beta-1} |\mathcal{R}_0^{\frac{1}{2}} (\sqrt{2\nu})^i|^2 \\
&\leq \left\| \sum_{i=\alpha}^{\beta-1} |\mathcal{M}_0^{\frac{1}{2}} (\sqrt{2\nu})^i|^2 \right\| 1_{\mathbb{A}} + \left\| \sum_{i=\alpha+1}^{\beta-1} |\mathcal{R}_0^{\frac{1}{2}} (\sqrt{2\nu})^i|^2 \right\| 1_{\mathbb{A}} \\
&\leq \sum_{i=\alpha}^{\beta-1} \|\mathcal{M}_0^{\frac{1}{2}}\|^2 \|(\sqrt{2\nu})^i\|^2 1_{\mathbb{A}} + \sum_{i=\alpha+1}^{\beta-1} \|\mathcal{R}_0^{\frac{1}{2}}\|^2 \|(\sqrt{2\nu})^i\|^2 1_{\mathbb{A}} \\
&\leq \|\mathcal{M}_0^{\frac{1}{2}}\|^2 \sum_{i=\alpha}^{\beta-1} \|(\sqrt{2\nu})^2\|^i 1_{\mathbb{A}} + \|\mathcal{R}_0^{\frac{1}{2}}\|^2 \sum_{i=\alpha+1}^{\beta-1} \|(\sqrt{2\nu})^2\|^i 1_{\mathbb{A}} \\
&= \|\mathcal{M}_0^{\frac{1}{2}}\|^2 \sum_{i=\alpha}^{\beta-1} (2\|\nu\|^2)^i 1_{\mathbb{A}} + \|\mathcal{R}_0^{\frac{1}{2}}\|^2 \sum_{i=\alpha+1}^{\beta-1} (2\|\nu\|^2)^i 1_{\mathbb{A}} \\
&\rightarrow 0_{\mathbb{A}} \quad (\text{as } \beta, \alpha \rightarrow \infty)
\end{aligned}$$

and

$$\begin{aligned}
\varphi(\sigma_\beta, \mathfrak{v}_\alpha) + \varphi(\zeta_\alpha, \mathfrak{v}_\beta) &\leq (\varphi(\sigma_\beta, \mathfrak{v}_{\beta-1}) + \varphi(\zeta_{\beta-1}, \mathfrak{u}_\beta)) + (\varphi(\sigma_{\beta-1}, \mathfrak{v}_{\beta-1}) + \varphi(\zeta_{\beta-1}, \mathfrak{u}_{\beta-1})) \\
&\quad + \cdots + (\varphi(\sigma_{\alpha+1}, \mathfrak{v}_\alpha) + \varphi(\zeta_\alpha, \mathfrak{u}_{\alpha+1})) \\
&= \mathcal{S}_{\beta-1} + \mathcal{R}_{\beta-1} + \mathcal{S}_{\beta-1} + \cdots + \mathcal{R}_{\alpha+1} + \mathcal{S}_\alpha \\
&\leq ((\sqrt{2\nu})^\star)^{\beta-1} \mathcal{S}_0 (\sqrt{2\nu})^{\beta-1} + ((\sqrt{2\nu})^\star)^{\beta-1} \mathcal{R}_0 (\sqrt{2\nu})^{\beta-1} + \cdots \\
&\quad + ((\sqrt{2\nu})^\star)^{\alpha+1} \mathcal{R}_0 (\sqrt{2\nu})^{\alpha+1} + ((\sqrt{2\nu})^\star)^\alpha \mathcal{S}_0 (\sqrt{2\nu})^\alpha \\
&= \sum_{i=\alpha}^{\beta-1} ((\sqrt{2\nu})^\star)^i \mathcal{S}_0 (\sqrt{2\nu})^i + \sum_{i=\alpha+1}^{\beta-1} ((\sqrt{2\nu})^\star)^i \mathcal{R}_0 (\sqrt{2\nu})^i \\
&= \sum_{i=\alpha}^{\beta-1} ((\sqrt{2\nu})^\star)^i \mathcal{S}_0^{\frac{1}{2}} \mathcal{S}_0^{\frac{1}{2}} (\sqrt{2\nu})^i + \sum_{i=\alpha+1}^{\beta-1} ((\sqrt{2\nu})^\star)^i \mathcal{R}_0^{\frac{1}{2}} \mathcal{R}_0^{\frac{1}{2}} (\sqrt{2\nu})^i
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=\alpha}^{\beta-1} (\mathcal{S}_0^{\frac{1}{2}}(\sqrt{2}v)^i)^* (\mathcal{S}_0^{\frac{1}{2}}(\sqrt{2}v)^i) + \sum_{i=\alpha+1}^{\beta-1} (\mathcal{R}_0^{\frac{1}{2}}(\sqrt{2}v)^i)^* (\mathcal{R}_0^{\frac{1}{2}}(\sqrt{2}v)^i) \\
&= \sum_{i=\alpha}^{\beta-1} |\mathcal{S}_0^{\frac{1}{2}}(\sqrt{2}v)^i|^2 + \sum_{i=\alpha+1}^{\beta-1} |\mathcal{R}_0^{\frac{1}{2}}(\sqrt{2}v)^i|^2 \\
&\leq \left\| \sum_{i=\alpha}^{\beta-1} |\mathcal{S}_0^{\frac{1}{2}}(\sqrt{2}v)^i|^2 \right\|_{1_{\mathbb{A}}} + \left\| \sum_{i=\alpha+1}^{\beta-1} |\mathcal{R}_0^{\frac{1}{2}}(\sqrt{2}v)^i|^2 \right\|_{1_{\mathbb{A}}} \\
&\leq \sum_{i=\alpha}^{\beta-1} \|\mathcal{S}_0^{\frac{1}{2}}\|^2 \|(\sqrt{2}v)^i\|^2 \|1_{\mathbb{A}}\| + \sum_{i=\alpha+1}^{\beta-1} \|\mathcal{R}_0^{\frac{1}{2}}\|^2 \|(\sqrt{2}v)^i\|^2 \|1_{\mathbb{A}}\| \\
&\leq \|\mathcal{S}_0^{\frac{1}{2}}\|^2 \sum_{i=\alpha}^{\beta-1} \|(\sqrt{2}v)^2\|^i \|1_{\mathbb{A}}\| + \|\mathcal{R}_0^{\frac{1}{2}}\|^2 \sum_{i=\alpha+1}^{\beta-1} \|(\sqrt{2}v)^2\|^i \|1_{\mathbb{A}}\| \\
&= \|\mathcal{S}_0^{\frac{1}{2}}\|^2 \sum_{i=\alpha}^{\beta-1} (2\|v\|^2)^i \|1_{\mathbb{A}}\| + \|\mathcal{R}_0^{\frac{1}{2}}\|^2 \sum_{i=\alpha+1}^{\beta-1} (2\|v\|^2)^i \|1_{\mathbb{A}}\| \\
&\rightarrow 0_{\mathbb{A}} \quad (\text{as } \beta, \alpha \rightarrow \infty).
\end{aligned}$$

Therefore, $(\{\sigma_\alpha\}, \{v_\alpha\})$ and $(\{\zeta_\alpha\}, \{u_\alpha\})$ are Cauchy bisequences in $\Gamma \times \Psi$ with respect to \mathbb{A} . By completeness of $(\Gamma, \Psi, \mathbb{A}, \varphi)$, there exist $\sigma, \zeta \in \Gamma$ and $u, v \in \Psi$ with

$$\lim_{\alpha \rightarrow \infty} \sigma_\alpha = v, \quad \lim_{\alpha \rightarrow \infty} \zeta_\alpha = u, \quad \lim_{\alpha \rightarrow \infty} u_\alpha = \zeta \quad \text{and} \quad \lim_{\alpha \rightarrow \infty} v_\alpha = \sigma.$$

Then for given $\epsilon > 0$, there exists $\alpha_1 \in \mathbb{N}$ with $\varphi(\sigma_\alpha, v) < \frac{\epsilon}{2}$ for all $\alpha \geq \alpha_1$. Since $(\{\sigma_\alpha\}, \{v_\alpha\})$ and $(\{\zeta_\alpha\}, \{u_\alpha\})$ are Cauchy bisequences, we derive that

$$\varphi(\sigma_\alpha, v_\alpha) < \frac{\epsilon}{2}.$$

Then,

$$\begin{aligned}
\varphi(\Phi(\sigma, u), v) &\leq \varphi(\Phi(\sigma, u), v_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, v_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, v) \\
&= \varphi(\Phi(\sigma, u), \Phi(u_\alpha, \zeta_\alpha)) + \varphi(\sigma_{\alpha+1}, v_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, v) \\
&\leq v^* \varphi(\sigma, v_\alpha) v + v^* \varphi(\zeta_\alpha, u) v + \varphi(\sigma_{\alpha+1}, v_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, v).
\end{aligned}$$

As $\alpha \rightarrow \infty$, we have

$$\varphi(\Phi(\sigma, u), v) < \epsilon.$$

Then,

$$\varphi(\Phi(\sigma, u), v) = 0.$$

Hence, $\Phi(\sigma, u) = v$. Similarly, we can derive $\Phi(u, \sigma) = \zeta$, $\Phi(\zeta, v) = u$ and $\Phi(v, \zeta) = \sigma$. On the other hand, we derive that

$$\varphi(\sigma, v) = \varphi(\lim_{\alpha \rightarrow \infty} v_\alpha, \lim_{\alpha \rightarrow \infty} \sigma_\alpha) = \lim_{\alpha \rightarrow \infty} \varphi(\sigma_\alpha, v_\alpha) = 0$$

and

$$\varphi(\zeta, u) = \varphi(\lim_{\alpha \rightarrow \infty} u_\alpha, \lim_{\alpha \rightarrow \infty} \zeta_\alpha) = \lim_{\alpha \rightarrow \infty} \varphi(\zeta_\alpha, u_\alpha) = 0.$$

So, $\sigma = v$ and $\zeta = u$. Therefore, $(\sigma, u) \in (\Gamma \times \Psi) \cap (\Psi \times \Gamma)$ is a coupled fixed point of Φ . As in the proof of the Theorem 3.1, one can easily prove uniqueness part. \square

Example 3.4. Let $\Gamma = \{0, 1, 2, 7\}$, $\Psi = \{0, \frac{1}{4}, \frac{1}{2}, 3\}$, $\mathbb{A}_+ = \mathcal{M}_2(\mathbb{C})$ and the map $\varphi : \Gamma \times \Psi \rightarrow \mathbb{A}_+$ is defined by

$$\varphi(\sigma, u) = \begin{bmatrix} |\sigma - u| & 0 \\ 0 & \mathbb{k}|\sigma - u| \end{bmatrix},$$

for all $\sigma \in \Gamma$ and $u \in \Psi$, where $\mathbb{k} \geq 0$ is a constant. Let \leq be the partial order on \mathbb{A} given by

$$(\vartheta_1, \varpi_1) \leq (\vartheta_2, \varpi_2) \Leftrightarrow \vartheta_1 \leq \vartheta_2 \text{ and } \varpi_1 \leq \varpi_2.$$

Then $(\Gamma, \Psi, \mathbb{A}, \varphi)$ is a complete C^* -algebra-valued bipolar metric space. Define

$$\Phi : (\Gamma \times \Psi) \cup (\Psi \times \Gamma) \rightarrow \Gamma \cup \Psi$$

by

$$\Phi(\sigma, \zeta) = \frac{\sigma + \zeta}{5},$$

for all $\sigma, \zeta \in (\Gamma \times \Psi) \cup (\Psi \times \Gamma)$. Then

$$\begin{aligned} \varphi(\Phi(\sigma, u), \Phi(\zeta, v)) &= \begin{bmatrix} |\Phi(\sigma, u) - \Phi(\zeta, v)| & 0 \\ 0 & \mathbb{k}|\Phi(\sigma, u) - \Phi(\zeta, v)| \end{bmatrix} = \begin{bmatrix} |\frac{\sigma+u}{5} - \frac{\zeta+v}{5}| & 0 \\ 0 & \mathbb{k}|\frac{\sigma+u}{5} - \frac{\zeta+v}{5}| \end{bmatrix} \\ &\leq \frac{1}{5} \left(\begin{bmatrix} |\sigma - v| & 0 \\ 0 & \mathbb{k}|\sigma - v| \end{bmatrix} + \begin{bmatrix} |\zeta - u| & 0 \\ 0 & \mathbb{k}|\zeta - u| \end{bmatrix} \right) = v^* \varphi(\sigma, v) v + v^* \varphi(\zeta, u) v, \end{aligned}$$

for all $\sigma, \zeta \in \Gamma$ and $u, v \in \Psi$, where

$$v = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$$

and $\|v\| = \frac{1}{5} < \frac{1}{\sqrt{2}}$. All the conditions of Theorem 3.3 are fulfilled and Φ has a unique fixed point $(0, 0)$.

4. Applications

As an application of Theorem 3.1, we find an existence and uniqueness result for a type of following system of Fredholm integral equations.

Theorem 4.1. *Let us consider the system of Fredholm integral equations*

$$\begin{aligned} \sigma(\mu) &= \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \mathcal{G}(\mu, p, \sigma(p), \zeta(p)) dp + \delta(\mu), \quad \mu, p \in \mathcal{E}_1 \cup \mathcal{E}_2, \\ \zeta(\mu) &= \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \mathcal{G}(\mu, p, \zeta(p), \sigma(p)) dp + \delta(\mu), \quad \mu, p \in \mathcal{E}_1 \cup \mathcal{E}_2, \end{aligned} \quad (4.1)$$

where $\mathcal{E}_1 \cup \mathcal{E}_2$ is a Lebesgue measurable set. Suppose

(T1) $\mathcal{G} : (\mathcal{E}_1^2 \cup \mathcal{E}_2^2) \times \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ and $\delta \in L^\infty(\mathcal{E}_1) \cup L^\infty(\mathcal{E}_2)$.

(T2) There exists a continuous function $\kappa : \mathcal{E}_1^2 \times \mathcal{E}_2^2 \rightarrow \mathbb{R}$ and $\theta \in (0, 1)$, such that

$$|\mathcal{G}(\mu, p, \sigma(p), \zeta(p)) - \mathcal{G}(\mu, p, u(p), v(p))| \\ \leq \theta |\kappa(\mu, p)| (|\sigma(p) - u(p)| + |\zeta(p) - v(p)| + I - \theta^{-1}I),$$

for all $\mu, p \in \mathcal{E}_1 \cup \mathcal{E}_2$.

(T3) $\sup_{\mu \in \mathcal{E}_1 \cup \mathcal{E}_2} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} |\kappa(\mu, p)| dp \leq 1$.

Then the integral equation has a unique solution in $L^\infty(\mathcal{E}_1) \cup L^\infty(\mathcal{E}_2)$.

Proof. Let $\Gamma = L^\infty(\mathcal{E}_1)$ and $\Psi = L^\infty(\mathcal{E}_2)$ be two normed linear spaces, where $\mathcal{E}_1, \mathcal{E}_2$ are Lebesgue measurable sets and $m(\mathcal{E}_1 \cup \mathcal{E}_2) < \infty$. Let $\mathcal{H} = L^2(\mathcal{E}_1) \cup L^2(\mathcal{E}_2)$. Consider $\varphi : \Gamma \times \Psi \rightarrow L(\mathcal{H})$ defined by $\varphi(\sigma, \zeta) = \pi_{|\sigma - \zeta|}$, where $\pi_{\mathfrak{h}} : \mathcal{H} \rightarrow \mathcal{H}$ is the multiplication operator defined by $\pi_{\mathfrak{h}}(\omega) = \mathfrak{h}\omega$ for $\omega \in \mathcal{H}$. Then $(\Gamma, \Psi, \mathbb{A}, \varphi)$ is a complete C^* -algebra valued bipolar metric space.

Define the covariant mapping $\Phi : \Gamma^2 \cup \Psi^2 \rightarrow \Gamma \cup \Psi$ by

$$\Phi(\sigma, \zeta)(\mu) = \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \mathcal{G}(\mu, p, \sigma(p), \zeta(p)) dp + \delta(\mu), \quad \forall \mu, p \in \mathcal{E}_1 \cup \mathcal{E}_2.$$

Set $\tau = \theta I$, then $\tau \in L(\mathcal{H})_+$ and $\|\tau\| = \theta < 1$. For any $\omega \in \mathcal{H}$, we have

$$\begin{aligned} \|\varphi(\Phi(\sigma, \zeta), \Phi(u, v))\| &= \sup_{\|\omega\|=1} (\pi_{|\Phi(\sigma, \zeta) - \Phi(u, v)| + I} \omega, \omega) \\ &= \sup_{\|\omega\|=1} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} (|\Phi(\sigma, \zeta) - \Phi(u, v)| + I) \omega(\mu) \overline{\omega(\mu)} d\mu \\ &\leq \sup_{\|\omega\|=1} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} |\mathcal{G}(\mu, p, \sigma(p), \zeta(p)) \\ &\quad - \mathcal{G}(\mu, p, u(p), v(p))| dp |\omega(\mu)|^2 d\mu \\ &\quad + \sup_{\|\omega\|=1} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} dp |\omega(\mu)|^2 d\mu I \\ &\leq \sup_{\|\omega\|=1} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \left[\int_{\mathcal{E}_1 \cup \mathcal{E}_2} \theta |\kappa(\mu, p)| (|\sigma(p) - u(p)| \right. \\ &\quad \left. + |\zeta(p) - v(p)| + I - \theta^{-1}I) dp \right] |\omega(\mu)|^2 d\mu + I \\ &\leq \theta \sup_{\|\omega\|=1} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \left[\int_{\mathcal{E}_1 \cup \mathcal{E}_2} |\kappa(\mu, p)| dp \right] |\omega(\mu)|^2 d\mu (\|\sigma - u\|_\infty \\ &\quad + \|\zeta - v\|_\infty) \\ &\leq \theta \sup_{\mu \in \mathcal{E}_1 \cup \mathcal{E}_2} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} |\kappa(\mu, p)| dp \sup_{\|\omega\|=1} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} |\omega(\mu)|^2 d\mu (\|\sigma - u\|_\infty \\ &\quad + \|\zeta - v\|_\infty) \\ &\leq \theta [\|\sigma - u\|_\infty + \|\zeta - v\|_\infty] \\ &= \|\tau\| [\|\Phi(\sigma, u)\| + \|\Phi(\zeta, v)\|]. \end{aligned}$$

Therefore, all the conditions of Theorem 3.1 are fulfilled. Hence, the integral equation (4.1) has a unique solution. \square

5. Conclusions

In this paper, we introduced the notion of a C^* -algebra valued bipolar metric space and proved coupled fixed point theorems. An illustrative example is provided that show the validity of the hypothesis and the degree of usefulness of our findings.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions, which led to considerable improvement of the article. The work was supported by the Higher Education Commission of Pakistan.

Conflict of interest

The authors declare no conflicts of interest.

References

1. A. Mutlu, U. Gürdal, Bipolar metric spaces and some fixed point theorems, *J. Nonlinear Sci. Appl.*, **9** (2016), 5362–5373. <http://dx.doi.org/10.22436/jnsa.009.09.05>
2. U. Gürdal, A. Mutlu, K. Özkan, Fixed point results for $\alpha - \psi$ ontractive mappings in bipolar metric spaces, *J. Inequal. Spec. Funct.*, **11** (2020), 64–75.
3. G. N. V. Kishore, R. P. Agarwal, B. S. Rao, R. V. N. S. Rao, Caristi type cyclic contraction and common fixed point theorems in bipolar metric spaces with applications, *Fixed Point Theory Appl.*, **2018** (2018), 21. <https://doi.org/10.1186/s13663-018-0646-z>
4. G. N. V. Kishore, D. R. Prasad, B. S. Rao, V. S. Baghavan, Some applications via common coupled fixed point theorems in bipolar metric spaces, *J. Crit. Rev.*, **7** (2019), 601–607.
5. G. N. V. Kishore, K. P. R. Rao, A. Sombabu, R. V. N. S. Rao, Related results to hybrid pair of mappings and applications in bipolar metric spaces, *J. Math.*, **2019** (2019), 8485412. <https://doi.org/10.1155/2019/8485412>
6. B. S. Rao, G. N. V. Kishore, G. K. Kumar, Geraghty type contraction and common coupled fixed point theorems in bipolar metric spaces with applications to homotopy, *Int. J. Math. Trends Technol.*, **63** (2018), 25–34.
7. G. N. V. Kishore, K. P. R. Rao, H. Işık, B. S. Rao, A. Sombabu, Covarian mappings and coupled fixed point results in bipolar metric spaces, *Int. J. Nonlinear Anal. Appl.*, **12** (2021), 1–15. <https://dx.doi.org/10.22075/ijnaa.2021.4650>
8. A. Mutlu, K. Özkan, U. Gürdal, Locally and weakly contractive principle in bipolar metric spaces, *TWMS J. Appl. Eng. Math.*, **10** (2020), 379–388.
9. Y. U. Gaba, M. Aphane, H. Aydi, α , BK -contractions in bipolar metric spaces, *J. Math.*, **2021** (2021), 5562651. <https://doi.org/10.1155/2021/5562651>

10. A. Mutlu, K. Özkan, U. Gürdal, Coupled fixed point theorems on bipolar metric spaces, *Eur. J. Pure Math.*, **10** (2017), 655–667.
11. Z. H. Ma, L. N. Jiang, H. K. Sun, C^* -algebras-valued metric spaces and related fixed point theorems, *Fixed Point Theory Appl.*, **2014** (2014), 206. <https://doi.org/10.1186/1687-1812-2014-206>
12. S. Batul, T. Kamran, C^* -valued contractive type mappings, *Fixed Point Theory Appl.*, **2015** (2015), 142. <https://doi.org/10.1186/s13663-015-0393-3>
13. K. R. Davidson, *C^* -algebras by example*, Fields Institute Monographs, American Mathematical Society, 1996.
14. G. J. Murphy, *C^* -algebra and operator theory*, Waltham: Academic Press, 1990.
15. Q. H. Xu, T. E. D. Bieke, Z. Q. Chen, *Introduction to operator algebras and noncommutative L_p spaces*, Beijing: Science Press, 2010.
16. M. Gunaseelan, G. Arul Joseph, P. B. Khumalo, Y. U. Gaba, Common coupled fixed point theorems on C^* -algebra-valued partial metric spaces, *J. Func. Spaces*, **2022** (2022), 3336095. <https://doi.org/10.1155/2022/3336095>
17. R. G. Douglas, *Banach algebra techniques in operator theory*, Springer, Berlin, 1998. <https://doi.org/10.1007/978-1-4612-1656-8>



©2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)