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Research article

An efficient spectral-Galerkin method for a new Steklov eigenvalue problem in inverse scattering

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Abstract: An efficient spectral method is proposed for a new Steklov eigenvalue problem in inverse scattering. Firstly, we establish the weak form and the associated discrete scheme by introducing an appropriate Sobolev space and a corresponding approximation space. Then, according to the Fredholm Alternative, the corresponding operator forms of weak formulation and discrete formulation are derived. After that, the error estimates of approximated eigenvalues and eigenfunctions are proved by using the spectral approximation results of completely continuous operators and the approximation properties of orthogonal projection operators. We also construct an appropriate set of basis functions in the approximation space and derive the matrix form of the discrete scheme based on the tensor product. In addition, we extend the algorithm to the circular domain. Finally, we present plenty of numerical experiments and compare them with some existing numerical methods, which validate that our algorithm is effective and high accuracy.

Keywords: Steklov eigenvalue problem; spectral-Galerkin method; error estimations; tensor product; numerical experiments
Mathematics Subject Classification: 65N25, 65N35

1. Introduction

Steklov eigenvalue problems have significant physical background and wide applications in many fields of science and engineering [1–4]. Its theoretical analysis and numerical calculation have attracted the attention of many scholars, and a variety of finite element methods and spectral methods have been proposed [5–13], but these numerical methods are mainly based on the selfadjoint Steklov eigenvalue problem.

Recently, a new Steklov eigenvalue problem arising from inverse scattering attracts many researchers' interest. The corresponding weak formulation of the problem is non-selfadjoint and indefinite. We can not use Lax-Milgram theorem to determine the existence and uniqueness of the solution of the corresponding resource problems, which brings some difficulties to deduce the equivalent operator form of the eigenvalue problem. In addition, in order to prove the error estimation of approximated eigenvalues and eigenfunctions, we need not only to introduce conjugate eigenvalue problem and corresponding conjugate operators, but also to prove the error estimation between these operators and their approximated operators. However, there are still some numerical methods to solve the problem. For instance, Cakoni et al. discussed the conforming finite element approximation in [14], Liu et al. studied spectral indicator method in [15], Bi et al. discussed two-grid discretizations and a local finite element scheme in [16], Zhang et al. established a multigrid correction scheme in [17], Yang et al. used non-conforming Crouzeix-Raviart element solve it [18], Xu et al. discussed an asymptotically exact a posteriori error estimator for non-selfadjoint Steklov eigenvalue problem [19], Wang et al. studied a priori and a posteriori error estimates for a virtual element method for the non-self-adjoint Steklov eigenvalue problem [20]. To our knowledge, the study of spectral method for solving this problem has not been reported. Since the spectral methods have the characteristics of spectral accuracy [21–25], that is to say, we only need to spend less degrees of freedom to obtain higher accurate numerical solutions. Then, it is meaningful to propose an effective spectral method for solving a new Steklov eigenvalue problem in inverse scattering.

Hence, we shall study an effective spectral-Galerkin method for the new Steklov eigenvalue problem. Firstly, we establish the weak formulation and the associated discrete scheme by introducing an appropriate Sobolev space and a corresponding approximation space. Then, according to the Fredholm Alternative, the corresponding operator forms of weak formulation and discrete formulation are derived. After that, the error estimates of approximated eigenvalues and eigenfunctions are proved by using the spectral theory of compact operators and the approximation properties of orthogonal projection operators. We also construct an appropriate set of basis functions in the approximation space and derive the matrix form of the discrete scheme based on tensor product. In addition, we extend the algorithm to the circular domain and transform the original problem into an equivalent form under the polar coordinates. By using orthogonal polynomial approximation in the radial direction and Fourier basis function approximation in the θ direction, combining with pole conditions, we construct an appropriate approximation space and derive the corresponding matrix form of the discrete scheme. Finally, we provide plenty of numerical examples and compare them with some existing numerical methods, the numerical results confirm the effectiveness and high accuracy of our algorithm.

We derive the weak formulation and the corresponding discrete scheme and prove the error estimations of approximation eigenvalues and eigenfunctions in next section. In §3, we propose an efficient algorithm to solve the Steklov eigenvalue problem in the square domain. In §4, we extend our algorithm to the case of circular domain. In §5, we provide several numerical examples to validate the accuracy and efficiency of our algorithm. In §6, some concluding remarks are presented.

Throughout this article, a notation $a \lesssim b$ is used to mean that $a \leqslant Cb$, where C is a positive constant independent of any function or any discretization parameters.

2. The weak formulation, discrete scheme and the error estimates

2.1. The weak formulation and discrete scheme

Denote by $H^s(D)$ and $H^s(\partial D)$ the usual Sobolev spaces with integer order s in D and on ∂D, respectively. In particular, $H^0(D) = L^2(D)$ and $H^0(\partial D) = L^2(\partial D)$. The norm in $H^s(D)$ and $H^s(\partial D)$ are expressed as $\|\psi\|_{s,D}$ and $\|\psi\|_{s,\partial D}$, separately.

Consider the following Steklov eigenvalue problem:

$$\Delta\psi + k^2\beta(x)\psi = 0, \qquad \text{in } D, \tag{2.1}$$

$$\frac{\partial\psi}{\partial\nu} + \lambda\psi = 0, \qquad \text{on } \partial D, \tag{2.2}$$

where $D \subset \mathbb{R}^2$ (or $\mathbb{R}^3$) is a bounded polygon with Lipschitz boundary ∂D, ν is the unit outward normal on ∂D. Let k be the wavenumber and $\beta(x) = \beta_1(x) + i\frac{\beta_2(x)}{k}$ be the index of refraction that is a bounded complex value function with $\beta_1(x) > 0$ and $\beta_2(x) \geqslant 0$.

The weak formulation of the Eqs (2.1) and (2.2) is to find $(\lambda, \psi) \in \mathbb{C} \times H^1(D)$, $\psi \neq 0$ such that

$$\mathcal{A}(\psi, \phi) = -\lambda\mathcal{B}(\psi, \phi), \ \forall\ \phi \in H^1(D), \tag{2.3}$$

where

$$\mathcal{A}(\psi, \phi) = (\nabla\psi, \nabla\phi) - (k^2\beta(x)\psi, \phi),$$
$$(\psi, \phi) = \int_D \psi\overline{\phi}dx,$$
$$\mathcal{B}(\psi, \phi) = \int_{\partial D} \psi\overline{\phi}ds.$$

Refering to [15], we know that $\mathcal{A}(\cdot,\cdot)$ satisfies Garding's inequality, i.e., there exist constants $K < \infty$ and $\alpha_0 > 0$ such that

$$\mathrm{Re}\{\mathcal{A}(\phi, \phi)\} + K\|\phi\|^2_{0,D} \geq \alpha_0\|\phi\|^2_{1,D}, \forall\phi \in H^1(D).$$

Let K be a positive constant which is large enough, and the sesquilinear form is defined as follow:

$$\tilde{\mathcal{A}}(\psi, \phi) := \mathcal{A}(\psi, \phi) + K(\psi, \phi) = (\nabla\psi, \nabla\phi) - k^2(\beta\psi, \phi) + K(\psi, \phi), \ \ \psi, \phi \in H^1(D),$$

then it is easy to verify that $\tilde{\mathcal{A}}$ is $H^1(D)$-elliptic (see [15]).

We first focus on the case of $D = I^d (d = 2, 3)$ with $I = (-1, 1)$. Denote by $L_p(t)$ the Legendre polynomial of degree p. Let $V_N = \mathrm{span}\{L_0(t), L_1(t), \cdots, L_N(t)\}$, then the approximation space $X_N = (V_N)^d$.

The spectral-Galerkin approximation for the eigenvalue problem (2.3) reads: Find $(\lambda_N, \psi_N) \in \mathbb{C} \times X_N$, $\psi_N \neq 0$ such that

$$\mathcal{A}(\psi_N, \phi_N) = -\lambda_N\mathcal{B}(\psi_N, \phi_N), \ \forall\ \phi_N \in X_N. \tag{2.4}$$

2.2. Error estimation

We first consider the following source problem associated with (2.3): Given $g \in H^{-\frac{1}{2}}(\partial D)$, find $w \in H^1(D)$ such that

$$\mathcal{A}(w, \phi) = \mathcal{B}(g, \phi), \ \forall \phi \in H^1(D), \tag{2.5}$$

and the approximation source problem associated with (2.4): Find $w_N \in X_N$ such that

$$\mathcal{A}(w_N, \phi_N) = \mathcal{B}(g, \phi_N), \ \forall \phi_N \in X_N. \tag{2.6}$$

Further, we introduce Neumann eigenvalue problem as follows:

$$\Delta w + k^2\beta(x)w = 0 \ \ in\ D, \tag{2.7}$$

$$\frac{\partial w}{\partial \nu} = 0 \ \ on\ \partial D. \tag{2.8}$$

When k^2 is not a Neumann eigenvalue of (2.7) and (2.8), from the Fredholm Alternative (see, e.g., Section 5.3 of [26] or Lemma 1 in [18]), we know that for $g \in H^{-\frac{1}{2}}(\partial D)$, there exists a unique solution $w \in H^1(D)$ of (2.5) such that

$$\|w\|_{1,D} \leq C\|g\|_{-\frac{1}{2},\partial D}. \tag{2.9}$$

Define an operator $A : H^{-\frac{1}{2}}(\partial D) \to H^1(D)$ by

$$\mathcal{A}(Ag, \phi) = \mathcal{B}(g, \phi), \ \forall\ \phi \in H^1(D), \tag{2.10}$$

and a Neumann-to-Dirichlet mapping $\Gamma : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ by

$$\Gamma g = Ag|_{\partial D}.$$

We can similarly define a discrete operator $A_N : H^{-\frac{1}{2}}(\partial D) \to X_N$ such that

$$\mathcal{A}(A_N g, \phi_N) = \mathcal{B}(g, \phi_N), \ \forall\ \phi_N \in X_N, \tag{2.11}$$

and a discrete Neumann-to-Dirichlet operator $\Gamma_N : H^{-\frac{1}{2}}(\partial D) \to X_N^B$ satisfies

$$\Gamma_N g = A_N g|_{\partial D},$$

where $X_N^B = X_N|_{\partial D}$. Then from (2.9), we obtain $\|Ag\|_{1,D} \lesssim \|g\|_{-\frac{1}{2},\partial D}$.
Thus, the equivalent operator forms of (2.3) and (2.4) are:

$$A\psi = -\frac{1}{\lambda}\psi, \ A_N\psi_N = -\frac{1}{\lambda_N}\psi_N, \tag{2.12}$$

$$\Gamma\psi = -\frac{1}{\lambda}\psi, \ \Gamma_N\psi_N = -\frac{1}{\lambda_N}\psi_N. \tag{2.13}$$

In this paper, we always assume that k^2 is not a Neumann eigenvalue of (2.7) and (2.8).

Consider the dual problem of (2.3): Find $(\lambda^*, \psi^*) \in \mathbb{C} \times H^1(D)$, $\psi^* \neq 0$ such that

$$\mathcal{A}(\phi, \psi^*) = -\overline{\lambda^*}\mathcal{B}(\phi, \psi^*),\ \forall\ \phi \in H^1(D). \tag{2.14}$$

It's clearly that the primal and dual eigenvalues satisfy the relation: $\lambda = \overline{\lambda^*}$.

The discrete variational formulation associated with (2.14) is given by: Find $(\lambda_N^*, \psi_N^*) \in \mathbb{C} \times X_N$, $\psi_N^* \neq 0$ such that

$$\mathcal{A}(\phi_N, \psi_N^*) = -\overline{\lambda_N^*}\mathcal{B}(\phi_N, \psi_N^*),\ \forall\ \phi_N \in X_N. \tag{2.15}$$

Likewise, the primal and dual eigenvalues satisfy the relation: $\lambda_N = \overline{\lambda_N^*}$.

Similarly, define the dual operators $A^* : H^{-\frac{1}{2}}(\partial D) \to H^1(D)$ and $A_N^* : H^{-\frac{1}{2}}(\partial D) \to X_N$ such that

$$\mathcal{A}(\phi, A^*g) = \mathcal{B}(\phi, g),\ \forall\ \phi \in H^1(D), \tag{2.16}$$

$$\mathcal{A}(\phi_N, A_N^*g) = \mathcal{B}(\phi_N, g),\ \forall\ \phi_N \in X_N. \tag{2.17}$$

The corresponding Neumann-to-Dirichlet and discrete Neumann-to-Dirichlet dual operators are defined as follows:

$$\Gamma^* : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D),\ \Gamma_N^* : H^{-\frac{1}{2}}(\partial D) \to X_N^B.$$

Define the H^1 projection operators $\Pi_N^1 : H^1(D) \to X_N$ and ${\Pi_N^1}^* : H^1(D) \to X_N$ by

$$\mathcal{A}(w - \Pi_N^1 w, \phi_N) = 0,\ \forall\ \phi_N \in X_N. \tag{2.18}$$

$$\mathcal{A}(\phi_N, w^* - {\Pi_N^1}^* w^*) = 0,\ \forall\ \phi_N \in X_N. \tag{2.19}$$

Then for any $g \in H^{-\frac{1}{2}}(D)$, we have

$$\begin{aligned}&\mathcal{A}(A_N g - \Pi_N^1(Ag), \phi_N) = \mathcal{A}(A_N g - Ag + Ag - \Pi_N^1(Ag), \phi_N)\\ &= \mathcal{A}(A_N g - Ag, \phi_N) + \mathcal{A}(Ag - \Pi_N^1(Ag), \phi_N) = 0,\ \ \forall \phi_N \in X_N.\end{aligned}$$

Since the above equation admits a unique solution, we have $A_N = \Pi_N^1 A$. Similarly, we can obtain $A_N^* = {\Pi_N^1}^* A^*$.

There holds the following regular results, which will be used in the following theoretical analysis.

Lemma 1. *If $g \in L^2(\partial D)$, then $Ag \in H^{1+\frac{\kappa}{2}}(D)$ and*

$$\|Ag\|_{1+\frac{\kappa}{2},D} \leq C\|g\|_{0,\partial D}, \tag{2.20}$$

if $g \in H^{\frac{1}{2}}(\partial D)$, then $Ag \in H^{1+\kappa}(D)$ and

$$\|Ag\|_{1+\kappa,D} \leq C\|g\|_{\frac{1}{2},\partial D}, \tag{2.21}$$

where $\kappa = 1$ when the largest inner angle θ of D satisfying $\theta < \pi$, and $\kappa < \frac{\pi}{\theta}$ which can be arbitrarily close to $\frac{\pi}{\theta}$ when $\theta > \pi$.

Proof. see [27]. □

For the dual problem, we have the same regular results as the corresponding source problem. Defining an H^1-projection operator $P_N^1 : H^1(D) \to X_N$ by

$$(\nabla(w - P_N^1 w), \nabla\phi_N) + (w - P_N^1 w, \phi_N) = 0, \ \ \forall\ \phi_N \in X_N.$$

According to Theorem 8.4 in [28], we have the following lemma:

Lemma 2. *For any $w \in H^s(D)$ with $s \geq 1$ and $s \geqslant l \geqslant 0$,*

$$\|w - P_N^1 w\|_{l,D} \lesssim N^{l-s}\|w\|_{s,D}. \tag{2.22}$$

Denote

$$\eta_N = \sup_{g\in H^{\frac{1}{2}}(\partial D), \|g\|_{\frac{1}{2},\partial D}=1} \inf_{\phi_N\in X_N} \|Ag - \phi_N\|_{1,D},$$
$$\eta_N^* = \sup_{g\in H^{\frac{1}{2}}(\partial D), \|g\|_{\frac{1}{2},\partial D}=1} \inf_{\phi_N\in X_N} \|A^*g - \phi_N\|_{1,D}.$$

It is obvious that

$$\lim_{N\to\infty} \eta_N = \lim_{N\to\infty} \eta_N^* = 0.$$

Lemma 3. *Let w be the solution of* (2.5)*, if $w \in H^s(D)(s \geq 2)$, there hold:*

$$\|w - \Pi_N^1 w\|_{1,D} \lesssim N^{1-s}\|w\|_{s,D}, \tag{2.23}$$
$$\|w - \Pi_N^{1\,*} w\|_{1,D} \lesssim N^{1-s}\|w\|_{s,D}, \tag{2.24}$$
$$\|w - \Pi_N^1 w\|_{-\frac{1}{2},\partial D} \lesssim N^{-s}\|w\|_{s,D}. \tag{2.25}$$

Proof. According to the definition of the projection operator Π_N^1, we obtain

$$\mathcal{A}(w - \Pi_N^1 w, \phi_N) = 0,\ \forall\ \phi_N \in X_N.$$

From Theorem 3.1 in [15] and the inequality (2.22), we derive that

$$\|w - \Pi_N^1 w\|_{1,D} \lesssim \inf_{\phi_N\in X_N} \|w - \phi_N\|_{1,D}$$
$$\lesssim \|w - P_N^1 w\|_{1,D} \lesssim N^{1-s}\|w\|_{s,D}.$$

Similarly, we can arrive at

$$\|w - \Pi_N^{1\,*} w\|_{1,D} \lesssim N^{1-s}\|w\|_{s,D}.$$

From Lemma 2.2 in [16], for any $\phi_N \in X_N$ we derive that

$$\|w - \Pi_N^1 w\|_{-\frac{1}{2},\partial D} = \sup_{g\in H^{\frac{1}{2}}(\partial D), g\neq 0} \frac{|\mathcal{B}(w - \Pi_N^1 w, g)|}{\|g\|_{\frac{1}{2},\partial D}}$$

$$\leq \sup_{g\in H^{\frac{1}{2}}(\partial D),g\neq 0} \frac{\|w-\Pi_N^1 w\|_{1,D}\|Ag-\phi_N\|_{1,D}}{\|g\|_{\frac{1}{2},\partial D}}.$$

Taking $\phi_N = \Pi_N^1 Ag$ and using (2.21) and (2.23) we obtain that

$$\begin{aligned}\|w-\Pi_N^1 w\|_{-\frac{1}{2},\partial D} &\leq \sup_{g\in H^{\frac{1}{2}}(\partial D),g\neq 0} \frac{\|w-\Pi_N^1 w\|_{1,D}\|Ag-\Pi_N^1 Ag\|_{1,D}}{\|g\|_{\frac{1}{2},\partial D}}\\ &\lesssim N^{1-s}\|w\|_{s,D}\cdot N^{-1}\frac{\|Ag\|_{2,D}}{\|g\|_{\frac{1}{2},\partial D}}\\ &\lesssim N^{-s}\|w\|_{s,D}.\end{aligned}$$

The proof is completed. □

Consider the following auxiliary problem: Find $\xi_f \in H^1(D)$, such that

$$\mathcal{A}(\phi,\xi_f) = (\phi, f),\ \forall\ \phi\in H^1(D). \tag{2.26}$$

Referring to [16], we obtain the following result.

Lemma 4. *If $f \in L^2(D)$, then there exists a unique solution $\xi_f \in H^{1+\kappa}(D)$ to* (2.26) *and*

$$\|\xi_f\|_{1+\kappa,D} \lesssim \|f\|_{0,D}, \tag{2.27}$$

where the principle to determine κ see Lemma 1.

Remark 1. *If the regularity results, Lemmas 1 and 4 hold in the case of $D \in R^3$, we can prove the analysis and conclusion in this paper for $D \in R^3$. However, there are no such good results in R^3 (see, e.g., Remark 2.1 in [29] and Remark 1 in [18]). Our analysis is still valid in the case of $D \in R^3$, but the conclusions need minor modifications.*

Lemma 5. *For $\forall w \in H^1(D)$, we have*

$$\|w-\Pi_N^1 w\|_{0,D} \lesssim N^{1-s}\|w-\Pi_N^1 w\|_{1,D}. \tag{2.28}$$

Proof. According to the definition of the projection ${\Pi_N^1}^*$, for any $\xi \in H^s(D)(s \geq 2)$ we have

$$\mathcal{A}(\phi, \xi-{\Pi_N^1}^*\xi) = 0, \forall\phi\in X_N.$$

From Lemma 2.4 in [16], we can obtain that

$$\begin{aligned}\|w-\Pi_N^1 w\|_{0,D}^2 &= \mathcal{A}(w-\Pi_N^1 w, \xi_{w-\Pi_N^1 w} - {\Pi_N^1}^*\xi_{w-\Pi_N^1 w})\\ &\leq \|w-\Pi_N^1 w\|_{1,D}\|\xi_{w-\Pi_N^1 w} - {\Pi_N^1}^*\xi_{w-\Pi_N^1 w}\|_{1,D}.\end{aligned}$$

From the inequalities (2.24) and (2.27), we have

$$\begin{aligned}&\|\xi_{w-\Pi_N^1 w} - {\Pi_N^1}^*\xi_{w-\Pi_N^1 w}\|_{1,D}\\ &\lesssim N^{1-s}\|\xi_{w-\Pi_N^1 w}\|_{s,D}\end{aligned}$$

$$\lesssim N^{1-s}\|w-\Pi_N^1 w\|_{0,D}.$$

Thus, we obtain

$$\|w-\Pi_N^1 w\|_{0,D}\lesssim N^{1-s}\|w-\Pi_N^1 w\|_{1,D}.$$

This ends our proof. □

According to the classical spectral approximation theory: when $\lim_{N\to\infty}\|\Gamma-\Gamma_N\|_{H^{-\frac{1}{2}}(\partial D)\longrightarrow H^{-\frac{1}{2}}(\partial D)}=0$, we can obtain the error estimates of eigenvalue problem.

Lemma 6. *If* $s\geq 2$*, then* $\lim_{N\to\infty}\|\Gamma-\Gamma_N\|_{H^{-\frac{1}{2}}(\partial D)\longrightarrow H^{-\frac{1}{2}}(\partial D)}=0$ *and* Γ *is a compact operator.*

Proof. From Lemma 2.6 in [16], for any $\phi_N\in X_N$ we have

$$\|(\Gamma-\Gamma_N)\varphi\|_{-\frac{1}{2},\partial D}\leq\sup_{g\in H^{\frac{1}{2}}(\partial D),g\neq 0}\frac{\|(A-A_N)\varphi\|_{1,D}\|Ag-\phi_N\|_{1,D}}{\|g\|_{\frac{1}{2},\partial D}}.$$

Taking $\phi_N=\Pi_N^1 Ag$, from (2.23) and (2.21) we derive that

$$\begin{aligned}\|(\Gamma-\Gamma_N)\varphi\|_{-\frac{1}{2},\partial D}&\leq\sup_{g\in H^{\frac{1}{2}}(\partial D),g\neq 0}\frac{\|(A-A_N)\varphi\|_{1,D}\|Ag-\Pi_N^1 Ag\|_{1,D}}{\|g\|_{\frac{1}{2},\partial D}}\\&\leq\sup_{g\in H^{\frac{1}{2}}(\partial D),g\neq 0}\frac{\|(A-A_N)\varphi\|_{1,D}N^{-1}\|Ag\|_{2,D}}{\|g\|_{\frac{1}{2},\partial D}}\\&\lesssim N^{-1}\|(A-A_N)\varphi\|_{1,D}.\end{aligned}$$

From the above inequality, together with (2.23) and (2.9) we obtain that

$$\begin{aligned}\|\Gamma-\Gamma_N\|_{H^{-\frac{1}{2}}(\partial D)\to H^{-\frac{1}{2}}(\partial D)}&=\sup_{\varphi\in H^{-\frac{1}{2}}(\partial D),\varphi\neq 0}\frac{\|(\Gamma-\Gamma_N)\varphi\|_{-\frac{1}{2},\partial D}}{\|\varphi\|_{-\frac{1}{2},\partial D}}\\&\lesssim\sup_{\varphi\in H^{-\frac{1}{2}}(\partial D),\varphi\neq 0}N^{-1}\frac{\|(A-A_N)\varphi\|_{1,D}}{\|\varphi\|_{-\frac{1}{2},\partial D}}\\&\lesssim\sup_{\varphi\in H^{-\frac{1}{2}}(\partial D),\varphi\neq 0}\frac{N^{-1}\|A\varphi\|_{1,D}}{\|\varphi\|_{-\frac{1}{2},\partial D}}\\&\lesssim N^{-1}.\end{aligned}$$

Since $s\geqslant 2$, then we obtain $\lim_{N\to\infty}\|\Gamma-\Gamma_N\|_{H^{-\frac{1}{2}}(\partial D)\longrightarrow H^{-\frac{1}{2}}(\partial D)}=0$. Note that Γ_N is a finite rank operator. Then, Γ is a compact operator. This ends our proof. □

Let λ be the p-th eigenvalue of (2.3) with the algebraic multiplicity h and the ascent α. Since Γ_N converges to Γ, there are h eigenvalues $\lambda_{q,N}(q=\delta,\delta+1,\delta+2,\ldots,\delta+h-1)$ of (2.4) converging to λ. Let $M(\lambda)$ be the generalized eigenfunction space of (2.3) associated with the eigenvalue λ, and $M_N(\lambda)$ be the generalized eigenfunction space of (2.4) associated with the eigenvalue $\lambda_{q,N}$ ($q=\delta,\delta+1,\delta+2,\ldots,\delta+h-1$). As for the dual problems (2.14) and (2.15), we can also define $M^*(\lambda^*)$ and $M_N^*(\lambda^*)$.

Theorem 1. *Assume that ψ_N is the eigenfunction approximation of* (2.4)*, then there exists an eigenfunction $\psi \in M(\lambda)$ of* (2.3) *corresponding λ such that*

$$\|\psi - \psi_N\|_{-\frac{1}{2},\partial D} \lesssim N^{-\frac{s}{\alpha}}, \tag{2.29}$$

$$\|\psi - \psi_N\|_{1,D} \lesssim N^{\frac{2-2s}{\alpha}} + N^{-\frac{s}{\alpha}} + N^{1-s}, \tag{2.30}$$

$$\|\psi - \psi_N\|_{0,D} \lesssim N^{\frac{2-2s}{\alpha}} + N^{-\frac{s}{\alpha}}, \tag{2.31}$$

and

$$|\lambda - \lambda_N| \lesssim N^{\frac{2-2s}{\alpha}}. \tag{2.32}$$

Proof. From the Theorem 7.3 and Theorem 7.4 in [30] we know that there exists an eigenfunction ψ corresponding to λ and

$$|\lambda - \lambda_N| \lesssim \{\sum_{\tau,q=\delta}^{\delta+h-1} |\mathscr{B}((\Gamma - \Gamma_N)\varphi_\tau, \varphi_q^*)| + \|(\Gamma - \Gamma_N)|_{M(\lambda)}\|_{-\frac{1}{2},\partial D} \|(\Gamma^* - \Gamma_N^*)|_{M^*(\lambda^*)}\|_{-\frac{1}{2},\partial D}\}^{\frac{1}{\alpha}}, \tag{2.33}$$

$$\|\psi - \psi_N\|_{-\frac{1}{2},\partial D} \lesssim \|(\Gamma - \Gamma_N)|_{M(\lambda)}\|_{-\frac{1}{2},\partial D}^{\frac{1}{\alpha}}, \tag{2.34}$$

where $\varphi_\delta, \ldots, \varphi_{\delta+h-1}$ are the any basis for $M(\lambda)$ and $\varphi_\delta^*, \ldots, \varphi_{\delta+h-1}^*$ are the dual basis in $M^*(\lambda^*)$.

From Theorem 2.1 in [16], combining (2.23) and (2.24) we derive that

$$\begin{aligned} |\mathscr{B}((\Gamma - \Gamma_N)\varphi_\tau, \varphi_q^*)| &\leq \|(A - A_N)\varphi_\tau\|_{1,D} \|A^*\varphi_q^* - \Pi_N^{1\,*} A^*\varphi_q^*\|_{1,D} \\ &\lesssim N^{2-2s} \|A\varphi_\tau\|_s \|A^*\varphi_q^*\|_s \\ &\lesssim N^{2-2s}. \end{aligned} \tag{2.35}$$

We derive from Lemma 6, (2.22) and (2.23) that

$$\begin{aligned} \|(\Gamma - \Gamma_N)|_{M(\lambda)}\|_{-\frac{1}{2},\partial D} &= \sup_{f \in M(\lambda), \|f\|_{-\frac{1}{2},\partial D} = 1} \|(\Gamma - \Gamma_N) f\|_{-\frac{1}{2},\partial D} \\ &\lesssim N^{-s}. \end{aligned} \tag{2.36}$$

Similarly, we have

$$\|(\Gamma^* - \Gamma_N^*)|_{M^*(\lambda^*)}\|_{-\frac{1}{2},\partial D} \lesssim N^{-s}. \tag{2.37}$$

Combining the inequalities (2.34) and (2.36), we obtain (2.29). From (2.33), (2.35), (2.36) and (2.37), we obtain (2.32).

From Theorem 2.1 in [16], and the inequalities (2.9), (2.32), (2.29) and (2.23) , we deduce

$$\begin{aligned} &\|\psi_N - \psi\|_{1,D} \\ &\leq \|(\lambda - \lambda_N) A\psi\|_{1,D} + \|\lambda_N A(\psi - \psi_N)\|_{1,D} + \|\lambda_N (A - A_N)\psi_N\|_{1,D} \\ &\lesssim |\lambda - \lambda_N| + \|\psi_N - \psi\|_{-\frac{1}{2},\partial D} + \|(\Pi_N^1 A - A)(\psi_N - \psi)\|_{1,D} + \|(\Pi_N^1 A - A)\psi\|_{1,D} \\ &\lesssim N^{\frac{2-2s}{\alpha}} + N^{-\frac{s}{\alpha}} + N^{-\frac{s}{\alpha}} + N^{1-s} \|A\psi\|_{s,D} \end{aligned}$$

$$\lesssim N^{\frac{2-2s}{\alpha}} + N^{-\frac{s}{\alpha}} + N^{1-s}.$$

From Theorem 2.1 in [16], and the inequalities (2.28), (2.9), (2.32), (2.29) and (2.23), we obtain

$$\begin{aligned}
&\|\psi_N - \psi\|_{0,D} \\
\le\ & \|(\lambda - \lambda_N)A\psi\|_{0,D} + \|\lambda_N A(\psi_N - \psi)\|_{0,D} + \|\lambda_N(A_N - A)\psi_N\|_{0,D} \\
\le\ & \|(\lambda - \lambda_N)A\psi\|_{0,D} + \|\lambda_N A(\psi_N - \psi)\|_{0,D} + N^{1-s}\|(\Pi_N^1 A - A)\psi_N\|_{1,D} \\
\lesssim\ & |\lambda_N - \lambda| + \|\psi_N - \psi\|_{-\frac{1}{2},\partial D} + N^{1-s}[\|(\Pi_N^1 A - A)(\psi_N - \psi)\|_{1,D} + \|(\Pi_N^1 A - A)\psi\|_{1,D}] \\
\lesssim\ & N^{\frac{2-2s}{\alpha}} + N^{-\frac{s}{\alpha}} + N^{1-s}[N^{-\frac{s}{\alpha}} + N^{1-s}\|A\psi\|_{s,D}] \\
\lesssim\ & N^{\frac{2-2s}{\alpha}} + N^{-\frac{s}{\alpha}}.
\end{aligned}$$

This ends our proof. □

For the dual problems (2.14) and (2.15), we can also obtain the corresponding conclusion as Theorem 1.

3. Efficient implementation of the algorithm

In this section, we shall present an efficient algorithm to solve the discrete scheme (2.4). We first construct a group of basis functions of the approximation space X_N. Denote by $L_p(t)$ the Legendre polynomial of degree p. Let $\hat{\varphi}_l(t) = L_l(t) - L_{l+2}(t), l = 0, 1, \cdots, N-2$, $\hat{\varphi}_{N-1}(t) = L_0(t), \hat{\varphi}_N(t) = L_1(t)$. Then the approximation space $X_N = \text{span}\{\hat{\varphi}_l(x_1)\hat{\varphi}_j(x_2) : l, j = 0, 1, \ldots, N\}$.

Let $a_{lj} = \int_{-1}^{1} \hat{\varphi}_j'\hat{\varphi}_l' dt$, $b_{lj} = \int_{-1}^{1} \hat{\varphi}_j\hat{\varphi}_l dt$, $c_{lj} = \hat{\varphi}_j(-1)\hat{\varphi}_l(-1)$, $d_{lj} = \hat{\varphi}_j(1)\hat{\varphi}_l(1)$. By utilizing the orthogonal properties of the Legendre polynomials, we obtain that

(1)

$$\begin{cases}
a_{lj} = 4l + 6, b_{lj} = \frac{2}{2l+1} + \frac{2}{2l+5}, & l = j; \\
b_{lj} = b_{jl} = -\frac{2}{2l+5}, & j = l + 2; \\
a_{lj} = b_{lj} = c_{lj} = d_{lj} = 0, & \text{otherwise},
\end{cases}$$

where $l, j = 0, 1, \ldots, N-2$.

(2)

$$\begin{cases}
b_{lj} = b_{jl} = 2, l + j = N - 1; \\
b_{1N} = b_{N1} = \frac{2}{3}; \\
a_{lj} = a_{jl} = b_{lj} = b_{jl} = c_{lj} = c_{jl} = d_{lj} = d_{jl} = 0, \quad \text{otherwise},
\end{cases}$$

where $l = N-1, N$, $j = 0, 1, \ldots, N-2$.

(3)

$$\begin{cases}
a_{NN} = b_{N-1N-1} = 2; b_{NN} = \frac{2}{3}; \\
c_{NN} = d_{NN} = c_{N-1N-1} = d_{N-1N-1} = 1; \\
c_{N-1N} = c_{NN-1} = -1; d_{N-1N} = d_{NN-1} = 1; \\
a_{lj} = b_{lj} = 0, \quad \text{otherwise},
\end{cases}$$

where $l = N-1, N$, $j = N-1, N$.

Next, we shall derive the matrix form based on the tensor product of the discrete scheme (2.4). For simplicity, we only consider the case of $d = 2$. It can be derived similarly for the case of $d = 3$.

Let $\psi_N(x_1, x_2) = \sum\limits_{l,j=0}^{N} \psi_{lj}\hat{\varphi}_l(x_1)\hat{\varphi}_j(x_2)$ and

$$\Psi = \begin{pmatrix} \psi_{00} & \psi_{01} & \dots & \psi_{0N} \\ \psi_{10} & \psi_{11} & \dots & \psi_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{N0} & \psi_{N1} & \dots & \psi_{NN} \end{pmatrix}.$$

We denote by $\overline{\Psi}$ a column vectors with $(N+1)^2$ elements, which is consist of the $N+1$ columns of Ψ. Let $\phi_N(x_1, x_2) = \hat{\varphi}_m(x_1)\hat{\varphi}_n(x_2), m, n = 0, 1, \dots, N$, then we derive that

$$\begin{aligned}
\int_\Omega \nabla\psi_N \nabla\phi_N dx_1 dx_2 &= \sum_{l,j=0}^{N} \int_{-1}^{1}\int_{-1}^{1} \nabla[\hat{\varphi}_l(x_1)\hat{\varphi}_j(x_2)]\nabla[\hat{\varphi}_m(x_1)\hat{\varphi}_n(x_2)]dx_1 dx_2 \psi_{lj} \\
&= \sum_{l,j=0}^{N} \int_{-1}^{1}\int_{-1}^{1} \hat{\varphi}_l^{'}(x_1)\hat{\varphi}_m^{'}(x_1)\hat{\varphi}_j(x_2)\hat{\varphi}_n(x_2) \\
&\quad + \hat{\varphi}_l(x_1)\hat{\varphi}_m(x_1)\hat{\varphi}_j^{'}(x_2)\hat{\varphi}_n^{'}(x_2)dx_1 dx_2 \psi_{lj} \\
&= \sum_{l,j=0}^{N} [\int_{-1}^{1} \hat{\varphi}_l^{'}(x_1)\hat{\varphi}_m^{'}(x_1)dx_1 \int_{-1}^{1} \hat{\varphi}_j(x_2)\hat{\varphi}_n(x_2)dx_2 \\
&\quad + \int_{-1}^{1} \hat{\varphi}_l(x_1)\hat{\varphi}_m(x_1)dx_1 \int_{-1}^{1} \hat{\varphi}_j^{'}(x_2)\hat{\varphi}_n^{'}(x_2)dx_2]\psi_{lj} \\
&= \sum_{l,j=0}^{N} (a_{ml}b_{nj} + a_{nj}b_{ml})\psi_{lj} \\
&= \hat{A}(m,:)\Psi\hat{B}(n,:)^T + \hat{B}(m,:)\Psi\hat{A}(n,:)^T \\
&= \hat{B}(n,:)\otimes\hat{A}(m,:)\overline{\Psi} + \hat{A}(n,:)\otimes\hat{B}(m,:)\overline{\Psi},
\end{aligned}$$

$$\begin{aligned}
\int_{\partial D} \psi_N\phi_N ds &= \sum_{l,j=0}^{N} \{\int_{-1}^{1} [\hat{\varphi}_l(x_1)\hat{\varphi}_j(-1)\hat{\varphi}_m(x_1)\hat{\varphi}_n(-1) + \hat{\varphi}_l(x_1)\hat{\varphi}_j(1)\hat{\varphi}_m(x_1)\hat{\varphi}_n(1)]dx_1 \\
&\quad + \int_{-1}^{1} [\hat{\varphi}_l(-1)\hat{\varphi}_j(x_2)\hat{\varphi}_m(-1)\hat{\varphi}_n(x_2) + \hat{\varphi}_l(1)\hat{\varphi}_j(x_2)\hat{\varphi}_m(1)\hat{\varphi}_n(x_2)]dx_2\}\psi_{lj} \\
&= \sum_{l,j=0}^{N} [\hat{\varphi}_j(-1)\hat{\varphi}_n(-1) + \hat{\varphi}_j(1)\hat{\varphi}_n(1)] \int_{-1}^{1} \hat{\varphi}_l(x_1)\hat{\varphi}_m(x_1)dx_1 \psi_{lj} \\
&\quad + \sum_{l,j=0}^{N} [\hat{\varphi}_l(-1)\hat{\varphi}_m(-1) + \hat{\varphi}_l(1)\hat{\varphi}_m(1)] \int_{-1}^{1} \hat{\varphi}_j(x_2)\hat{\varphi}_n(x_2)dx_2 \psi_{lj} \\
&= \sum_{l,j=0}^{N} (c_{nj} + d_{nj})b_{ml}\psi_{lj} + \sum_{l,j=0}^{N} (c_{ml} + d_{ml})b_{nj}\psi_{lj}
\end{aligned}$$

$$= \hat{B}(m,:)\Psi(\hat{C}+\hat{D})(n,:)^T + (\hat{C}+\hat{D})(m,:)\Psi\hat{B}(n,:)^T$$
$$= (\hat{C}+\hat{D})(n,:)\otimes\hat{B}(m,:)\overline{\Psi} + \hat{B}(n,:)\otimes(\hat{C}+\hat{D})(m,:)\overline{\Psi},$$

where $\hat{A} = (a_{lj})_{l,j=0}^N$, $\hat{B} = (b_{lj})_{l,j=0}^N$, $\hat{C} = (c_{lj})_{l,j=0}^N$, $\hat{D} = (d_{lj})_{l,j=0}^N$. $\hat{A}(m,:)$ indicates the m-th row of the matrix $\hat{A}$, $\hat{B}(m,:)$, $\hat{C}(m,:)$, $\hat{D}(m,:)$ are similar to $\hat{A}(m,:)$. $\otimes$ is a notation of tensor product of matrix, then $\hat{A}\otimes\hat{B} = (a_{lj}\hat{B})_{l,j=0}^N$.

Let $\xi_\mu, w_\mu(\mu = 0,1,\dots,N_1)$ be the Labatto-Gauss points and weights, respectively. Then we have

$$\begin{aligned}\int_D \beta(x_1,x_2)\psi_N\phi_N dx_1dx_2 &= \sum_{l,j=0}^N \int_{-1}^1\int_{-1}^1 \beta(x_1,x_2)\hat{\varphi}_l(x_1)\hat{\varphi}_j(x_2)\hat{\varphi}_m(x_1)\hat{\varphi}_n(x_2)dx_1dx_2\psi_{lj}\\ &= \sum_{l,j=0}^N\sum_{\mu,\sigma=0}^{N_1}\beta(\xi_\mu,\xi_\sigma)\hat{\varphi}_l(\xi_\mu)\hat{\varphi}_j(\xi_\sigma)\hat{\varphi}_m(\xi_\mu)\hat{\varphi}_n(\xi_\sigma)\omega_\mu\omega_\sigma\psi_{lj}\\ &= \hat{P}\overline{\Psi},\end{aligned}$$

where

$$\hat{P} = \begin{pmatrix} p_{0000} & p_{0100} & \cdots & p_{0N00} & \cdots & p_{000N} & p_{010N} & \cdots & p_{0N0N}\\ p_{1000} & p_{1100} & \cdots & p_{1N00} & \cdots & p_{100N} & p_{110N} & \cdots & p_{1N0N}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ p_{N000} & p_{N100} & \cdots & p_{NN00} & \cdots & p_{N00N} & p_{N10N} & \cdots & p_{NN0N}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ p_{00N0} & p_{01N0} & \cdots & p_{0NN0} & \cdots & p_{00NN} & p_{01NN} & \cdots & p_{0NNN}\\ p_{10N0} & p_{11N0} & \cdots & p_{1NN0} & \cdots & p_{10NN} & p_{11NN} & \cdots & p_{1NNN}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ p_{N0N0} & p_{N1N0} & \cdots & p_{NNN0} & \cdots & p_{N0NN} & p_{N1NN} & \cdots & p_{NNNN} \end{pmatrix},$$

$$p_{mlnj} = \sum_{\mu,\sigma=0}^{N_1}\beta(\xi_\mu,\xi_\sigma)\hat{\varphi}_l(\xi_\mu)\hat{\varphi}_m(\xi_\mu)\hat{\varphi}_j(\xi_\sigma)\hat{\varphi}_n(\xi_\sigma)\omega_\mu\omega_\sigma.$$

From the above deduce, we obtain the matrix form of the discrete scheme (2.4) as follows:

$$(\hat{B}\otimes\hat{A} + \hat{A}\otimes\hat{B} - k^2\hat{P})\overline{\Psi} = -\lambda_N((\hat{C}+\hat{D})\otimes\hat{B} + \hat{B}\otimes(\hat{C}+\hat{D}))\overline{\Psi}. \tag{3.1}$$

Note that the matrix $\hat{P}$ can also be written as matrix form base on the tensor product and the stiffness matrix and mass matrix in (3.1) are all sparse when β is a constant, so we can solve (3.1) efficiently.

4. Extension to circular domain

We extend the above algorithm to a circular domain. Utilizing polar coordinate transformation $x = (x_1,x_2) = (r\cos\theta, r\sin\theta)$, the functions $\psi(x)$ and $\beta(x)$ are represented as $\widetilde{\psi}(r,\theta) = \psi(r\cos\theta, r\cos\theta)$ and $\tilde{\beta}(r,\theta) = \beta(r\cos\theta, r\cos\theta)$. Let $Lv = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}$, then the Eqs (2.1) and (2.2) can be rewritten as follows:

$$L\widetilde{\psi}(r,\theta) + k^2\tilde{\beta}(r,\theta)\widetilde{\psi}(r,\theta) = 0, \quad \text{in } \Omega = [0,R)\times[0,2\pi), \tag{4.1}$$

$$\frac{\partial \widetilde{\psi}}{\partial r}(R,\theta) = -\lambda \widetilde{\psi}(R,\theta),\ \theta \in [0,2\pi),\ \widetilde{\psi} \text{ periodic in } \theta. \tag{4.2}$$

Then from (2.3) we derive that the weak form of (4.1) and (4.2) is as follows:

$$a(\widetilde{\psi},\widetilde{\phi}) = -\lambda b(\widetilde{\psi},\widetilde{\phi}), \tag{4.3}$$

where

$$\begin{aligned} a(\widetilde{\psi},\widetilde{\phi}) &= \int_\Omega r\partial_r\widetilde{\psi}\partial_r\bar{\widetilde{\phi}}drd\theta + \int_\Omega \frac{1}{r}\partial_\theta\widetilde{\psi}\partial_\theta\bar{\widetilde{\phi}}drd\theta - \int_\Omega k^2\widetilde{\beta}\widetilde{\psi}\bar{\widetilde{\phi}}rdrd\theta,\\ b(\widetilde{\psi},\widetilde{\phi}) &= \int_0^{2\pi} R\widetilde{\psi}(R,\theta)\bar{\widetilde{\phi}}(R,\theta)d\theta. \end{aligned}$$

Since $\widetilde{\psi}$ is 2π periodic in θ, by using the expansion of the Fourier basis function we have

$$\widetilde{\psi}(r,\theta) = \sum_{|\tilde{m}|=0}^{\infty} u_{\tilde{m}}(r)e^{i\tilde{m}\theta}. \tag{4.4}$$

Substituting the expansion (4.4) into (4.3), and taking $\tilde{\phi}(r,\theta) = v_{\tilde{n}}(r)e^{i\tilde{n}\theta}$, we derive that

$$\begin{aligned} &\int_\Omega r\sum_{|\tilde{m}|=0}^{\infty} u'_{\tilde{m}}v'_{\tilde{n}}e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}drd\theta + \int_\Omega \sum_{|\tilde{m}|=0}^{\infty}\frac{\tilde{m}\tilde{n}}{r}u_{\tilde{m}}v_{\tilde{n}}e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}drd\theta\\ &-\int_\Omega\sum_{|\tilde{m}|=0}^{\infty} k^2\widetilde{\beta}(r,\theta)u_{\tilde{m}}v_{\tilde{n}}e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}rdrd\theta = -\lambda\int_0^{2\pi}\sum_{|\tilde{m}|=0}^{\infty} Ru_{\tilde{m}}(R)v_{\tilde{n}}(R)e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}d\theta. \end{aligned} \tag{4.5}$$

In order to make (4.5) well posed, we need to introduce the following polar condition (see [22])

$$\tilde{m}u_{\tilde{m}}(r)|_{r=0} = 0. \tag{4.6}$$

Let $r = \frac{R}{2}(t+1)$, $t \in (-1,1)$, $w_{\tilde{m}}(t) = u_{\tilde{m}}(r)$, $z_{\tilde{n}}(t) = v_{\tilde{n}}(r)$, $\gamma(t,\theta) = \widetilde{\beta}(r,\theta)$, then (4.5) can be rewritten as follows:

$$\begin{aligned} &\sum_{|\tilde{m}|=0}^{\infty}\int_0^{2\pi}\int_{-1}^{1}(t+1)w'_{\tilde{m}}z'_{\tilde{n}}e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}dtd\theta + \sum_{|\tilde{m}|=0}^{\infty}\int_0^{2\pi}\int_{-1}^{1}\frac{\tilde{m}\tilde{n}}{(t+1)}w_{\tilde{m}}z_{\tilde{n}}e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}dtd\theta\\ &-\sum_{|\tilde{m}|=0}^{\infty}\frac{R^2}{4}k^2\int_0^{2\pi}\int_{-1}^{1}(t+1)\gamma w_{\tilde{m}}z_{\tilde{n}}e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}dtd\theta = -\lambda\sum_{|\tilde{m}|=0}^{\infty}\int_0^{2\pi}Rw_{\tilde{m}}(1)z_{\tilde{n}}(1)e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}d\theta. \end{aligned} \tag{4.7}$$

Define the weighted Sobolev space:

$$\mathcal{H}^1_{\tilde{m}}(\Omega) = \{w_{\tilde{m}}(t)e^{i\tilde{m}\theta} : \int_{-1}^{1}(t+1)|w'_{\tilde{m}}|^2 + \frac{\tilde{m}^2}{t+1}|w_{\tilde{m}}|^2dt < \infty, \tilde{m}w_{\tilde{m}}(-1) = 0\},$$

which is endowed with the norm as follows:

$$\|w_{\tilde{m}}\|_{1,\tilde{m},\Omega} = [\int_{-1}^{1}(t+1)|w'_{\tilde{m}}|^2 + \frac{\tilde{m}^2}{t+1}|w_{\tilde{m}}|^2dt]^{\frac{1}{2}}.$$

Then the weak formulation of (4.5) is to find $w_{\tilde{m}}(t)e^{i\tilde{m}\theta} \in \mathcal{H}^1_{\tilde{m}}(\Omega)$, $\lambda \in \mathbb{C}$ such that

$$\sum_{|\tilde{m}|=0}^{\infty}\int_0^{2\pi}\int_{-1}^{1}(t+1)w'_{\tilde{m}}z'_{\tilde{n}}e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}dtd\theta + \sum_{|\tilde{m}|=0}^{\infty}\int_0^{2\pi}\int_{-1}^{1}\frac{\tilde{m}\tilde{n}}{(t+1)}w_{\tilde{m}}z_{\tilde{n}}e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}dtd\theta$$
$$-\sum_{|\tilde{m}|=0}^{\infty}\frac{R^2}{4}k^2\int_0^{2\pi}\int_{-1}^{1}(t+1)\gamma w_{\tilde{m}}z_{\tilde{n}}e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}dtd\theta = -\lambda\sum_{|\tilde{m}|=0}^{\infty}\int_0^{2\pi}Rw_{\tilde{m}}(1)z_{\tilde{n}}(1)e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}d\theta, \tag{4.8}$$

for $z_{\tilde{n}}(t)e^{-i\tilde{n}\theta} \in \mathcal{H}^1_{\tilde{n}}(\Omega)$.

Denote by P_N the space of polynomials of degree less than or equal to N, and set $X_N^M = \bigoplus_{|\tilde{m}|=0}^{M}\mathcal{H}^1_{\tilde{m}}(\Omega)\bigcap P_{\tilde{m}N}$, where $P_{\tilde{m}N} = \{p_N e^{i\tilde{m}\theta} : p_N \in P_N\}$.

Then the discrete form of (4.8) is to find $w_{\tilde{m}N}e^{i\tilde{m}\theta} \in X_N^M$, $\lambda_N \in \mathbb{C}$ such that

$$\sum_{|\tilde{m}|=0}^{M}\int_0^{2\pi}\int_{-1}^{1}(t+1)w'_{\tilde{m}N}z'_{\tilde{n}N}e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}dtd\theta + \sum_{|\tilde{m}|=0}^{M}\int_0^{2\pi}\int_{-1}^{1}\frac{\tilde{m}\tilde{n}}{(t+1)}w_{\tilde{m}N}z_{\tilde{n}N}e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}dtd\theta$$
$$-\frac{R^2}{4}k^2\sum_{|\tilde{m}|=0}^{M}\int_0^{2\pi}\int_{-1}^{1}(t+1)\gamma w_{\tilde{m}N}z_{\tilde{n}N}e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}dtd\theta$$
$$= -\lambda_N R\sum_{|\tilde{m}|=0}^{M}\int_0^{2\pi}w_{\tilde{m}N}(1)z_{\tilde{n}N}(1)e^{i\tilde{m}\theta}e^{-i\tilde{n}\theta}d\theta, \tag{4.9}$$

for $\forall\ z_{\tilde{n}N}e^{-i\tilde{n}\theta} \in X_N^M$.

4.1. Implementation of the algorithm

Let

$$\varphi_{\tilde{i}}(t) = L_{\tilde{i}}(t) - L_{\tilde{i}+2}(t), \tilde{i} = 0\ldots N-2, \varphi_{N-1}(t) = \frac{t+1}{2}, \varphi_N(t) = \frac{1-t}{2}.$$

It is clear that

$$\mathcal{H}^1_0(\Omega) \cap P_{0N} = \mathrm{span}\{\varphi^0_{\tilde{i}} : \varphi^0_{\tilde{i}} = \varphi_{\tilde{i}}, \tilde{i} = 0, 1, \ldots, N\},$$
$$\mathcal{H}^1_{\tilde{m}}(\Omega) \cap P_{\tilde{m}N} = \mathrm{span}\{\varphi^{\tilde{m}}_{\tilde{i}}e^{i\tilde{m}\theta} : \varphi^{\tilde{m}}_{\tilde{i}} = \varphi_{\tilde{i}}, \tilde{i} = 0, 1, \ldots, N-1\}, (|\tilde{m}| \neq 0).$$

Setting

$$f_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}} = \int_0^{2\pi}\int_{-1}^{1}(t+1)(\varphi^{\tilde{m}}_{\tilde{i}})'(\varphi^{\tilde{n}}_{\tilde{j}})'e^{i(\tilde{m}-\tilde{n})\theta}dtd\theta,\quad g_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}} = \int_0^{2\pi}\int_{-1}^{1}\frac{\tilde{m}\tilde{n}}{t+1}\varphi^{\tilde{m}}_{\tilde{i}}\varphi^{\tilde{n}}_{\tilde{j}}e^{i(\tilde{m}-\tilde{n})\theta}dtd\theta,$$
$$q_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}} = \int_0^{2\pi}\int_{-1}^{1}(t+1)\gamma(t,\theta)\varphi^{\tilde{m}}_{\tilde{i}}\varphi^{\tilde{n}}_{\tilde{j}}e^{i(\tilde{m}-\tilde{n})\theta}dtd\theta,\quad h_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}} = \int_0^{2\pi}\varphi^{\tilde{m}}_{\tilde{i}}(1)\varphi^{\tilde{n}}_{\tilde{j}}(1)e^{i(\tilde{m}-\tilde{n})\theta}d\theta.$$

Let $w_{\tilde{m}N} = \sum_{\tilde{i}=0}^{N-sign(|\tilde{m}|)} w^{\tilde{m}}_{\tilde{i}}\varphi^{\tilde{m}}_{\tilde{i}}(|\tilde{m}| = 0, \ldots, M)$, $z_{\tilde{n}N} = \varphi^{\tilde{n}}_{\tilde{j}}(|\tilde{n}| = 0, \ldots, M, \tilde{j} = 0, \ldots, N - sign(|\tilde{n}|))$, and inserting the expressions into (4.9), we obtain that

$$\sum_{|\tilde{m}|=0}^{M}\sum_{\tilde{i}=0}^{N-sign(|\tilde{m}|)} f_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}}w^{\tilde{m}}_{\tilde{i}} + \sum_{|\tilde{m}|=0}^{M}\sum_{\tilde{i}=0}^{N-sign(|\tilde{m}|)} g_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}}w^{\tilde{m}}_{\tilde{i}}$$

$$-\frac{R^2}{4}k^2\sum_{|\tilde{m}|=0}^{M}\sum_{\tilde{i}=0}^{N-sign(|\tilde{m}|)} q_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}}w_{\tilde{i}}^{\tilde{m}} = -\lambda_N R\sum_{|\tilde{m}|=0}^{M}\sum_{\tilde{i}=0}^{N-sign(|\tilde{m}|)} h_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}}w_{\tilde{i}}^{\tilde{m}}. \tag{4.10}$$

Then the corresponding matrix form of (4.10) can be derived as follows:

$$(F + G - \frac{R^2}{4}k^2 Q)\overline{W} = -\lambda_N RH\overline{W},$$

where

$$F = (f_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}}), G = (g_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}}), Q = (q_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}}), H = (h_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}}), W = (w_{\tilde{i}}^{\tilde{m}}),$$

$\overline{W}$ is a column vector with $N(2M + 1) + 1$ elements, which is consist of the column of W. By using the orthogonality of the Fourier system $e^{i\tilde{m}\theta}$, we obtain that

$$\begin{cases} f_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}} = 0, g_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}} = 0, h_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}} = 0, & \tilde{m} \neq \tilde{n}; \\ f_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}} = 2\pi\int_{-1}^{1}(t+1)(\varphi_{\tilde{i}}^{\tilde{m}})'(\varphi_{\tilde{j}}^{\tilde{m}})'dt, & \tilde{m} = \tilde{n}; \\ g_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}} = 2\pi\int_{-1}^{1}\frac{\tilde{m}^2}{t+1}\varphi_{\tilde{i}}^{\tilde{m}}\varphi_{\tilde{j}}^{\tilde{m}}dt, & \tilde{m} = \tilde{n}; \\ h_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}} = 2\pi\varphi_{\tilde{i}}^{\tilde{m}}(1)\varphi_{\tilde{j}}^{\tilde{m}}(1), & \tilde{m} = \tilde{n}. \end{cases}$$

It's obvious that the matrices F, G, and H are diagonal block matrices.

Let $t_{\tilde{s}},\ w_{\tilde{s}}(\tilde{s} = 0, 1, \ldots, N_1)$ be the Gauss-Lobatto points and the weights and $\theta_{\tilde{l}},\ \hat{\omega}_{\tilde{l}}(\tilde{l} = 0, 1, \ldots, M_1 - 1)$ be the Fourier points and the weights, respectively. Then we have

$$q_{\tilde{j}\tilde{i}\tilde{n}\tilde{m}} = \sum_{\tilde{s}=0}^{N_1}\sum_{\tilde{l}=0}^{M_1-1}(t_{\tilde{s}} + 1)\gamma(t_{\tilde{s}}, \theta_{\tilde{l}})\varphi_{\tilde{i}}(t_{\tilde{s}})\varphi_{\tilde{j}}(t_{\tilde{s}})e^{i(\tilde{m}-\tilde{n})\theta_{\tilde{l}}}\omega_{\tilde{s}}\hat{\omega}_{\tilde{l}}.$$

5. Numerical experiments

We shall present some numerical experiments to validate the effectiveness of the algorithm. We carry out our programs in Matlab 2018a.

5.1. Square domain

Example 1. Consider the problem (2.3) with $k = 1$ and $\beta(x) = 4$ on the domain $D = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^2$. The approximation eigenvalue $\lambda_N^i(i = 1, 2, 3, 4)$ for different N are shown in Table 1.

Table 1. The approximation eigenvalue $\lambda_N^i(i = 1, 2, 3, 4)$ for different N on the square.

N	λ_N^1	λ_N^2	λ_N^3	λ_N^4
10	2.202507126351584	−0.2122521695447584	−0.2122521695447588	−0.9080560857539495
15	2.202507126351587	−0.2122521695447581	−0.2122521695447585	−0.9080560857539485
20	2.202507126351584	−0.2122521695447591	−0.2122521695447593	−0.9080560857539485
25	2.202507126351590	−0.2122521695447577	−0.2122521695447578	−0.9080560857539492

We observe from Table 1 that the approximation eigenvalues reach at least fourteen-digit accuracy with $N \geq 20$. For comparison, we list in Table 2 the numerical results obtained by Multigrid Correction Scheme in [17]. However, the numerical results reported in Table 2 have at most six-digit accuracy despite utilizing a great quantity of degrees of freedom.

Next, we choose the numerical solutions of $N = 40$ as reference solutions, the corresponding error figures of the approximate eigenvalues $\lambda_N^i (i = 1, 2, 3, 4)$ with different N are listed in Figure 1. We know from Figure 1 that the approximation eigenvalues converge gradually with the increase of N.

Table 2. The eigenvalue approximations of (2.3) obtained by Multigrid Correction Scheme and direct method(square: $\beta(x) = 4$).

h	$\lambda^c_{1,h}$	$\lambda^c_{2,h}$	$\lambda^c_{3,h}$	$\lambda^c_{4,h}$
$\frac{2}{512}$	2.20250138679	−0.21225453108	−0.21225510721	−0.90806663225
$\frac{2}{1024}$	2.20250569143	−0.21225275994	−0.21225290397	−0.90805872239
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
$\frac{2}{512}$	2.20250138680	−0.21225453108	−0.21225510721	−0.90806663225
$\frac{2}{1024}$	2.20250569144	−0.21225275992	−0.21225290395	−0.90805872238

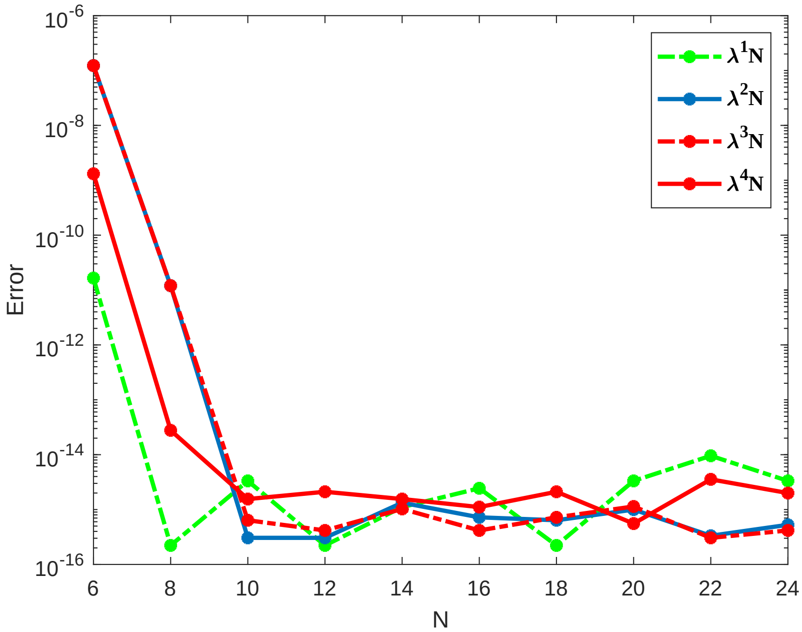

Figure 1. Errors curves between approximation solutions and the reference solution.

Example 2. We take $k = 1$, $\beta(x) = 4 + 4i$ and $D = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^2$ as our second example. The approximation eigenvalue $\lambda_N^i (i = 1, 2, 3, 4)$ of complex Steklov eigenvalues with largest imaginary parts on the square domain are shown in Table 3.

Table 3. The approximation eigenvalue $\lambda_N^i(i = 1, 2, 3, 4)$ for different N on the square.

N	λ_N^1	λ_N^2	λ_N^3	λ_N^4
10	0.6865518312509367	−0.3430465448633202	−0.3430465448633193	−0.9501102467115710
	+2.495293986064900i	+0.8507465067996103i	+0.8507465067996082i	+0.5400967091477815i
15	0.6865518312509322	−0.3430465448633182	−0.3430465448633182	−0.9501102467115667
	+2.495293986064902i	+0.8507465067996055i	+0.8507465067996055i	+0.5400967091477780i
20	0.6865518312509390	−0.3430465448633176	−0.3430465448633159	−0.9501102467115643
	+2.495293986064897i	+0.8507465067996055i	+0.8507465067996060i	+0.5400967091477757i
25	0.6865518312509413	−0.3430465448633178	−0.3430465448633153	−0.9501102467115623
	+2.495293986064906i	+0.8507465067996077i	+0.8507465067996044i	+0.5400967091477753i

We can see from Table 3 that the complex Steklov eigenvalues reach at least thirteen-digit accuracy with $N \geq 20$. The numerical solutions obtained by Multigrid Correction Scheme of [17] in Table 4 have at most seven-digit accuracy despite utilizing a great quantity of degrees of freedom.

Table 4. The eigenvalue approximations of (2.3) obtained by Multigrid Correction Scheme and direct method(square: $\beta(x) = 4 + 4i$).

h	$\lambda_{1,h}^c$	$\lambda_{2,h}^c$	$\lambda_{3,h}^c$	$\lambda_{6,h}^c$
$\frac{2}{512}$	0.6865580791	−0.3430478705	−0.3430446279	−0.9501192972
	+2.49529459i	+0.85074449i	+0.85074328i	+0.54009581i
$\frac{2}{1024}$	0.6865533933	−0.3430468763	−0.3430460656	−0.9501125093
	+2.49529414i	+0.850746i	+0.8507457i	+0.54009649i
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
$\frac{2}{512}$	0.6865580791	−0.3430478705	−0.3430446278	−0.9501192972
	+2.49529459i	+0.850744489i	+0.8507432795i	+0.540095814i
$\frac{2}{1024}$	-	-	-	-

Similarly, we also choose the numerical solutions of $N = 40$ as reference solutions, the corresponding error figures of the approximate eigenvalues $\lambda_N^i(i = 1, 2, 3, 4)$ with different N are listed in Figure 2. We observe from Figure 2 that the approximation eigenvalues also converge gradually with the increase of N.

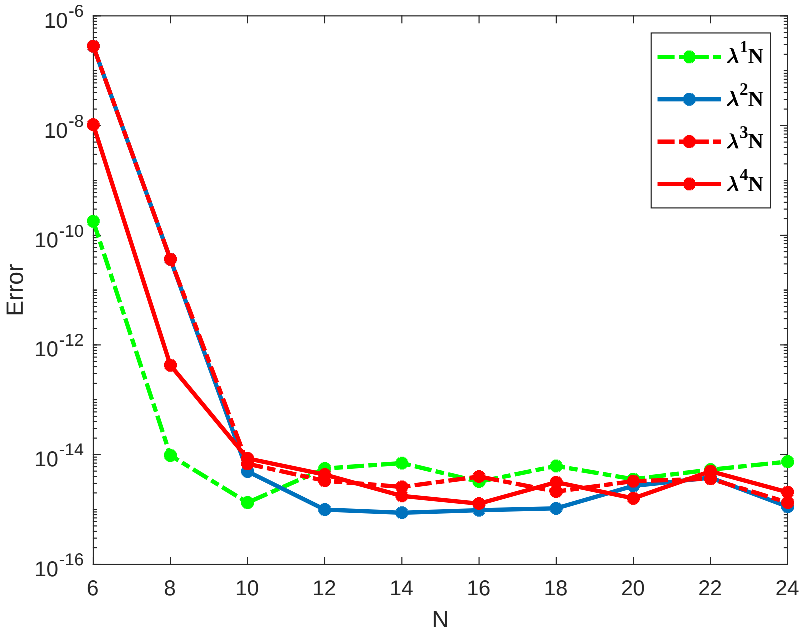

Figure 2. Errors curves between approximation solutions and the reference solution.

Example 3. When $\beta(x)$ is a variable coefficient, we take $k = 1, \beta(x_1, x_2) = [(x_1 + x_2)^2 + 1] + (x_1 - x_2)^2 i$ and $D = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^2$. The approximation eigenvalue $\lambda_N^i (i = 1, 2, 3, 4)$ of complex Steklov eigenvalues with largest imaginary parts on the square domain are shown in Table 5.

Table 5. The approximation eigenvalue $\lambda_N^i (i = 1, 2, 3, 4)$ for different N on the square.

N	λ_N^1	λ_N^2	λ_N^3	λ_N^4
10	−0.7865907038296051	0.5176843446683526	−1.221875087800295	−0.6465164817971206
	+0.1535916213694205i	+0.1276411469661209i	+0.07766877009084490i	+0.01723355206635077i
15	−0.7865907038296497	0.5176843446683336	−1.221875087800307	−0.6465164817971384
	+0.1535916213689529i	+0.1276411469658645i	+0.07766877009061117i	+0.01723355206591358i
20	−0.7865907038296479	0.5176843446683319	−1.221875087800304	−0.6465164817971389
	+0.1535916213689515i	+0.1276411469658636i	+0.07766877009060889i	+0.01723355206591307i
25	−0.7865907038296479	0.5176843446683351	−1.221875087800305	−0.6465164817971382
	+0.1535916213689520i	+0.1276411469658643i	+0.07766877009060946i	+0.01723355206591316i

We can see from Table 5 that the first four of complex Steklov eigenvalues reach at least thirteen-digit accuracy with $N \geq 20$. Again, we choose the numerical solutions of $N = 40$ as reference solutions, the corresponding error figures of the approximate eigenvalues $\lambda_N^i (i = 1, 2, 3, 4)$ with different N are listed in Figure 3. From Figure 3, we can see that the approximation eigenvalues also converge gradually with the increase of N.

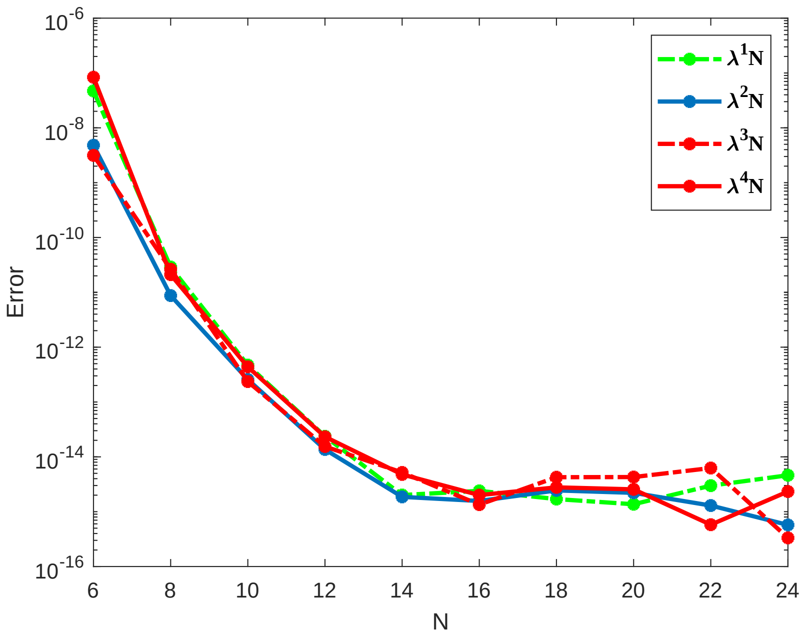

Figure 3. Errors curves between approximation solutions and the reference solution.

Example 4. We consider the problem (2.3) in a three-dimensional case, where we take $k = 2, \beta(x) = 1 + i$ and $\overline{D} = [0, 1]^3$. The approximation eigenvalue $\lambda_N^i(i = 1, 2, 3, 4)$ of complex Steklov eigenvalues with largest imaginary parts on $\overline{D} = [0, 1]^3$ are shown in Table 6.

Table 6. The approximation eigenvalue $\lambda_N^i(i = 1, 2, 3, 4)$ for different N on $\overline{D} = [0, 1]^3$.

N	λ_N^1	λ_N^2	λ_N^3	λ_N^4
5	0.6408976921157513	−0.6930457356247105	−0.6930457356247079	−0.6930457356247073
	+0.8336283636488195i	+0.4147709839135792i	+0.4147709839135787i	+0.4147709839135795i
10	0.6408976931254834	−0.6930457440435756	−0.6930457440435721	−0.6930457440435676
	+0.8336283608134208i	+0.4147710214947442i	+0.4147710214947438i	+0.4147710214947377i
15	0.6408976931254811	−0.6930457440435714	−0.6930457440435694	−0.6930457440435678
	+0.8336283608134180i	+0.4147710214947426i	+0.4147710214947404i	+0.4147710214947374i
20	0.6408976931254825	−0.6930457440435693	−0.6930457440435679	−0.6930457440435653
	+0.8336283608134184i	+0.4147710214947407i	+0.4147710214947393i	+0.4147710214947370i

We can see from Table 6 that the first four of complex Steklov eigenvalues reach at least thirteen-digit accuracy with $N \geq 15$.

5.2. Circular disk

Example 5. Consider the problem (2.3) with $k = 1$ in the unit disk centered at $(0, 0)$ with radius $R = 1$. We choose the index of refraction $\beta(x) = 4$. The three largest Steklov eigenvalues for different N and different M are listed in Tables 7–9, respectively.

Table 7. Numerical eigenvalues to λ_1 for different N and M in the circle.

N	$M = 4$	$M = 6$	$M = 8$	$M = 10$
10	5.151840642736440	5.151840642736451	5.151840642736432	5.151840642736449
15	5.151840642736440	5.151840642736460	5.151840642736455	5.151840642736447
20	5.151840642736454	5.151840642736441	5.151840642736452	5.151840642736441
25	5.151840642736454	5.151840642736440	5.151840642736422	5.151840642736442
30	5.151840642736442	5.151840642736443	5.151840642736438	5.151840642736472

Table 8. Numerical eigenvalues to λ_2 for different N and M in the circle.

N	$M = 4$	$M = 6$	$M = 8$	$M = 10$
10	0.2235784688644089	0.2235784688644088	0.2235784688644088	0.2235784688644094
15	0.2235784688644086	0.2235784688644084	0.2235784688644087	0.2235784688644084
20	0.2235784688644089	0.2235784688644082	0.2235784688644086	0.2235784688644084
25	0.2235784688644082	0.2235784688644083	0.2235784688644086	0.2235784688644084
30	0.2235784688644093	0.2235784688644087	0.2235784688644088	0.2235784688644090

Table 9. Numerical eigenvalues to λ_3 for different N and M in the circle.

N	$M = 4$	$M = 6$	$M = 8$	$M = 10$
10	0.2235784688644083	0.2235784688644084	0.2235784688644087	0.2235784688644091
15	0.2235784688644081	0.2235784688644082	0.2235784688644086	0.2235784688644080
20	0.2235784688644087	0.2235784688644075	0.2235784688644081	0.2235784688644083
25	0.2235784688644081	0.2235784688644079	0.2235784688644085	0.2235784688644084
30	0.2235784688644089	0.2235784688644086	0.2235784688644083	0.2235784688644088

We observe from Tables 7–9 that approximation eigenvalues to λ_1, λ_2, λ_3 reach at least fourteen-digit accuracy with $N \geq 15$ and $M \geq 4$. For comparison, we list the numerical results of [15] in Table 10. The numerical results to λ_1, λ_2, λ_3 reported in Table 10 have at most three-digit accuracy despite utilizing a great quantity of degrees of freedom.

Table 10. The largest six Steklov eigenvalues for the circle $\beta(x) = 4$.

h	1st	2nd	3rd	4th	5th	6th
0.2341	5.016606	0.206380	0.205917	−1.294039	−1.294339	−2.561531
0.1208	5.116979	0.219175	0.219048	−1.275370	−1.275440	−2.494866
0.0613	5.143045	0.222469	0.222436	−1.270670	−1.270687	−2.478245
0.0309	5.149636	0.223301	0.223292	−1.269493	−1.269497	−2.474088
0.0155	5.151289	0.223509	0.223507	−1.269198	−1.269199	−2.473049

Example 6. When $\beta(x)$ is complex, we take $\beta(x) = 4 + 4i$, $k = 1$ and Ω is a unit disk centered at $(0, 0)$

with radius $R = 1$. The numerical results of the first three complex Steklov eigenvalues with largest imaginary parts in a unit circle are shown in Tables 11–13, respectively.

Table 11. Numerical eigenvalues to λ_1 for different N and M in the circle.

N	$M = 4$	$M = 6$	$M = 8$	$M = 10$
10	−0.3205059883274507	−0.3205059883274481	−0.3205059883274489	−0.3205059883274535
	+3.124689326318511i	+3.124689326318512i	+3.124689326318508i	+3.124689326318517i
15	−0.3205059883274459	−0.3205059883274499	−0.3205059883274493	−0.3205059883274489
	+3.124689326318510i	+3.124689326318511i	+3.124689326318511i	+3.124689326318497i
20	−0.3205059883274491	−0.3205059883274469	−0.3205059883274485	−0.3205059883274508
	+3.124689326318506i	+3.124689326318513i	+3.124689326318509i	+3.124689326318506i
25	−0.3205059883274496	−0.3205059883274490	−0.3205059883274446	−0.3205059883274471
	+3.124689326318505i	+3.124689326318504i	+3.124689326318509i	+3.124689326318505i
30	−0.3205059883274480	−0.3205059883274495	−0.3205059883274517	−0.3205059883274509
	+3.124689326318509i	+3.124689326318511i	+3.124689326318508i	+3.124689326318516i

Table 12. Numerical eigenvalues to λ_2 for different N and M in the circle.

N	$M = 4$	$M = 6$	$M = 8$	$M = 10$
10	−0.1368609477039202	−0.1368609477039205	−0.1368609477039209	−0.1368609477039210
	+1.396737494788579i	+1.396737494788578i	+1.396737494788579i	+1.396737494788577i
15	−0.1368609477039197	−0.1368609477039195	−0.1368609477039205	−0.1368609477039195
	+1.396737494788579i	+1.396737494788580i	+1.396737494788578i	+1.396737494788576i
20	−0.1368609477039198	−0.1368609477039204	−0.1368609477039212	−0.1368609477039199
	+1.396737494788577i	+1.396737494788578i	+1.396737494788577i	+1.396737494788578i
25	−0.1368609477039201	−0.1368609477039194	−0.1368609477039191	−0.1368609477039208
	+1.396737494788579i	+1.396737494788577i	+1.396737494788578i	+1.396737494788579i
30	−0.1368609477039196	−0.1368609477039200	−0.1368609477039198	−0.1368609477039203
	+1.396737494788580i	+1.396737494788577i	+1.396737494788579i	+1.396737494788581i

Table 13. Numerical eigenvalues to λ_3 for different N and M in the circle.

N	$M = 4$	$M = 6$	$M = 8$	$M = 10$
10	−0.1368609477039208	−0.1368609477039200	−0.1368609477039202	−0.1368609477039211
	+1.396737494788581i	+1.396737494788579i	+1.396737494788579i	+1.396737494788579i
15	−0.1368609477039203	−0.1368609477039200	−0.1368609477039197	−0.1368609477039205
	+1.396737494788579i	+1.396737494788581i	+1.396737494788578i	+1.396737494788579i
20	−0.1368609477039197	−0.1368609477039202	−0.1368609477039205	−0.1368609477039204
	+1.396737494788578i	+1.396737494788579i	+1.396737494788578i	+1.396737494788580i
25	−0.1368609477039201	−0.1368609477039195	−0.1368609477039200	−0.1368609477039204
	+1.396737494788580i	+1.396737494788581i	+1.396737494788578i	+1.396737494788582i
30	−0.1368609477039198	−0.1368609477039193	−0.1368609477039201	−0.1368609477039199
	+1.396737494788581i	+1.39673749478858i	+1.396737494788581i	+1.396737494788582i

We observe from Tables 11–13 that approximation eigenvalues to λ_1, λ_2, λ_3 achieve at least thirteen-digit accuracy with $N \geq 15$ and $M \geq 4$. The results in Table 14 are obtained in [15], which are comparison to the results in Tables 11–13. The numerical eigenvalues to λ_1, λ_2, λ_3 reported in Table 14 have at most four-digit accuracy despite utilizing a great quantity of degrees of freedom.

Table 14. Table Eigenvalues for the circle $\beta(x) = 4 + 4i$.

h	1st	2nd	3rd	4th
0.2341	−0.298121	−0.134181	−0.133990	−1.371155
	+3.131620i	+1.375387i	+1.374565i	+0.786327i
0.1208	−0.314981	−0.136106	−0.136049	−1.357526
	+3.126494i	+1.391267i	+1.391044i	+0.790318i
0.0613	−0.319127	−0.136650	−0.136666	−1.354126
	+3.125146i	+1.395302i	+1.395359i	+0.79135i
0.0310	−0.320161	−0.136812	−0.136808	−1.353338
	+3.124804i	+1.396392i	+1.396378i	+0.791628i
0.0155	−0.320420	−0.136849	−0.136848	−1.353145
	+3.124718i	+1.396651i	+1.396647i	+0.791701i

6. Conclusions

In this paper, we propose an efficient spectral method for solving a new Steklov eigenvalue problem. First, we give an efficient spectral Galerkin approximation for the new Steklov eigenvalue problem in rectangular domain, and prove the error estimates of approximation eigenvalues and eigenfunctions. Secondly, we derive the matrix form of discrete scheme based on tensor product, and analyze the sparsity of mass matrix and stiffness matrix. In addition, we give an efficient spectral Galerkin approximation for the new Steklov eigenvalue problem in circular domain. Finally, we present ample numerical examples which validate the effectiveness and high accuracy of the algorithm.

The method proposed in this paper can be extended to some more complex problems, such as transmission eigenvalue problem, electromagnetic eigenvalue problem, nonlinear eigenvalue problem and so on, which will be our goal in the future.

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Conflict of interest

The authors declare that they have no competing interests.

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