AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 7507–7518.
DOI: 10.3934/math.2022421
Received: 14 April 2021
Revised: 06 December 2021
Accepted: 07 December 2021
Published: 14 February 2022

Research article

Characterization of extension map on fuzzy weakly cut-stable map

Nana Ma*, **Qingjun Luo and Geni Xu**

School of Statistics, Xi'an University of Finance and Economics, Xi'an 710100, China

* **Correspondence:** Email: manana@xaufe.edu.cn.

Abstract: In this paper, based on a complete residuated lattice, we propose the definition of fuzzy weakly cut-stable map and prove the extension property of the fuzzy weakly cut-stable map. Following this, it is explored the conditions under which the extension map to be fuzzy order isomorphism.

Keywords: quantitative domain; fuzzy weakly cut-stable map; extension map; isomorphism
Mathematics Subject Classification: 06B35, 08A72, 54C20

1. Introduction

Domain theory [1], which is a mathematical foundation of theoretical computer science, was put forward by Scott in the late 1960s. As an important content of domain theory, Dedekind-MacNeille completion (also called the normal completion), has been studied by many researchers [2–4].

Quantitative domain theory ([5–9]), which is regarded as the deep combination of denotational semantics of domain theory and the method of measurement, forms a new research field of domain theory. Its essence is to find a kind of model structure which can be calculated. As an important part of quantitative domain theory, the completions have also developed rapidly, and many researchers are devoted to studying the completions. Wagner [9] discussed the enriched Dedekind-MacNeille completion. Bělohlávek [10] discussed the Dedekind-MacNeille completion from the aspect of fuzzy concept lattice. Xie, Zhang and Fan [11] studied the Dedekind-MacNeille completion from the aspect of fuzzy poset. Wang and Zhao characterized the Dedekind-MacNeille completion from the aspect of the category [12]. Ma and Zhao studied the Dedekind-MacNeille completion based on a complete residuated lattice and obtain the full reflective subcategory of the category of fuzzy poset and fuzzy precontinuous in a certain morphism [13]. Halaš and Lihová characterized weakly cut-stable map by the 1st-order language [14]. In this paper, we study weakly cut-stable map based on a complete residuated lattice, that is, fuzzy weakly cut-stable map. Furthermore, we prove the extension property of fuzzy weakly cut-stable map. Following this, it is explored the conditions under which the extension map to be fuzzy order isomorphism.

The remainder of this paper is organized as follows. In Section 2, we review the basic definitions and results which are used in the rest of the paper. In Section 3, we propose the definitions of fuzzy weakly lower (upper) cut-stable map, fuzzy weakly lower (upper) cut-continuous map, and fuzzy weakly cut-preserving map. Furthermore, we discuss the extension property of fuzzy weakly cut-stable map. In Section 4, we explore the conditions under which the extension map to be fuzzy order isomorphism. Finally, we provide conclusions in Section 5.

2. Preliminaries

This section is devoted to providing a review of some key concepts of domain theory and fuzzy set theory.

Definition 2.1. ([15]) A *residuated lattice* is an algebraic structure $\mathcal{L} = (L;\ \wedge,\ \vee,\ *,\ \rightarrow,\ 0,\ 1)$ *of type* (2, 2, 2, 2, 0, 0) such that the following conditions hold.

(1) $(L;\ \wedge,\ \vee,\ 0,\ 1)$ is a bounded lattice with the least element 0 and the greatest element 1;
(2) $(L;\ *,\ 1)$ is a commutative monoid with the identity 1;
(3) $\mathcal{L}$ satisfies the *adjointness property*, *i.e.* $\forall\ x,\ y,\ z \in L,\ x * y \leq z$ *iff* $x \leq y \rightarrow z$.

A residuated lattice is called *complete* if the underlying lattice is complete. In this paper, if no otherwise specified, a complete residuated lattice is always denoted by L.

Definition 2.2. ([16,17]) A fuzzy poset is a pair (X, e) such that X is a set and $e : X \times X \longrightarrow L$ is a map, which satisfies the following conditions for all $x, y, z \in X$,

(1) $e(x, x) = 1$ (reflexivity);
(2) $e(x, y) * e(y, z) \leq e(x, z)$ (transitivity);
(3) $e(x, y) = e(y, x) = 1$ implies $x = y$ (antisymmetry).

For a set X, L^X denotes the set of all *fuzzy subsets* of X, that is, the set of all maps from X to L. For $\lambda \in L$, we use λ_X to denote the constant map with the value λ. For $C \subseteq X$,

$$\chi_C(x) = \begin{cases} 1, & x \in C, \\ 0, & \text{otherwises.} \end{cases}$$

Obviously, λ_X and $\chi_C(x)$ are the fuzzy subsets of X.

Definition 2.3. ([16,17]) Let (X, e) be a fuzzy poset and $A \in L^X$. An element $x_0 \in X$ is called a *supremum* (resp., *infimum*) of A, in symbols $x_0 = \sqcup A$ (resp., $x_0 = \sqcap A$), if the follwoing conditions hold.

(1) $\forall\ x \in X, A(x) \leq e(x, x_0)$ $(resp., \forall\ x \in X, A(x) \leq e(x_0, x))$;
(2) $\forall\ y \in X,\ \bigwedge_{x\in X}(A(x) \rightarrow e(x, y)) \leq e(x_0, y)$ $(resp., \forall\ y \in X,\ \bigwedge_{x\in X}(A(x) \rightarrow e(y, x)) \leq e(y, x_0))$.

Theorem 2.4. ([16,17]) *Let (X, e) be a fuzzy poset and $A \in L^X, x_0 \in X$. Then the following statements are equivalent*:

(1) $x_0 = \sqcup A$ $(x_0 = \sqcap A)$;
(2) $e(x_0, x) = \bigwedge_{y\in X}(A(y) \rightarrow e(y, x))$ $(e(x, x_0) = \bigwedge_{y\in X}(A(y) \rightarrow e(x, y)))$ *for all* $x \in X$.

Proposition 2.5. ([16,17]) *Let (X, e) be a fuzzy poset. Then (X, e) is a fuzzy complete lattice iff $\sqcup A$ exists for all $A \in L^X$ iff $\sqcap A$ exists for all $A \in L^X$.*

Definition 2.6. ([18]) Let $f : X \to Y$ be a map. The *Zadeh forward powerset operator* $f_L^{\to} : L^X \to L^Y$ and the *Zadeh backward powerset operator* $f_L^{\leftarrow} : L^Y \to L^X$ are, respectively, defined by $f_L^{\to}(A)(y) = \bigvee_{f(x)=y} A(x)$ for all $A \in L^X, y \in Y$ and $f_L^{\leftarrow}(B) = B \circ f$ for all $B \in L^Y$.

Definition 2.7. ([11]) Let (X, e) be a fuzzy poset and $A \in L^X$. Define A^l, A^u as follows: $\forall\ x \in X$, $A^l(x) = \bigwedge\limits_{x' \in X} (A(x') \to e(x, x'))$, $A^u(x) = \bigwedge\limits_{x' \in X} (A(x') \to e(x', x))$.

Proposition 2.8. ([10]) (The Dedekind-MacNeille completion) *Given a fuzzy poset (X, e), if $\forall A, B \in L^X$, $A = B^l, B = A^u$, then the pair (A, B) is called a fuzzy cut and A^u, B^l are usually called fuzzy upper cut and fuzzy lower cut, respectively. The collection of all fuzzy cuts, ordered by $e((A, B), (C, D)) = sub(A, C) = sub(D, B)$, is a fuzzy complete lattice, called the Dedekind-MacNeille completion of X. The Dedekind-MacNeille completion of X is also denoted by $DM_L(X)$ or $\mathcal{N}(X)$. That is to say, $DM_L(X) = \{(A, B) \mid \forall A, B \in L^X, A = B^l, B = A^u\}$.*

Since $\{(A, B) \mid \forall\ A, B \in L^X, A = B^l, B = A^u\}$ is isomorphic to $\{A \mid \forall\ A \in L^X, A = A^{ul}\}$, the Dedekind-MacNeille completion of X can also be denoted by $\{A \mid \forall\ A \in L^X, A = A^{ul}\}$, that is to say, $\mathcal{N}(X) = \{A \mid \forall\ A \in L^X, A = A^{ul}\}$.

Let (X, e) be a fuzzy poset and $x \in X$. Define two maps $\iota_x,\ \mu_x : X \to L$ by $\iota_x(y) = e(y, x)$, $\mu_x(y) = e(x, y)$ for all $y \in X$. Clearly, we have $\sqcup\, \iota_x = \sqcap\, \mu_x = x$ and the pair (ι_x, μ_x) is a fuzzy cut of X.

Lemma 2.9. ([10,13]) *Let (X, e) be a fuzzy poset and $A, B \in L^X$. Then*

(1) *$A \leq A^{ul}$ and $A \leq A^{lu}$;*
(2) *$sub(A, B) \leq sub(B^u, A^u)$ and $sub(A, B) \leq sub(B^l, A^l)$;*
(3) *$sub(A, B) \leq sub(A^{ul}, B^{ul})$ and $sub(A, B) \leq sub(A^{lu}, B^{lu})$;*
(4) *$A^u = A^{ulu}$ and $A^l = A^{lul}$.*

Theorem 2.10. ([14]) *Let P, Q be posets. For a mapping $f : P \to Q$ there exists a (unique) weakly complete lattice homomorphism $f^* : \mathcal{N}(P) \to \mathcal{N}(Q)$ extending f if and only if f is weakly cut-stable.*

3. Fuzzy weakly cut-stable map

In this section, we propose the definition of fuzzy weakly cut-stable map and discuss the extension theorem of fuzzy weakly cut-stable map.

Definition 3.1. Let (X, e_X) and (Y, e_Y) be fuzzy posets and $f : X \to Y$ be a map. Then f is called fuzzy weakly lower (upper) cut-stable if for all $A \in L^X$, when $A^u \neq 0_X, (f_L^{\to}(A^u))^l = (f_L^{\to}(A))^{ul}$ (when $A^l \neq 0_X$, $(f_L^{\to}(A^l))^u = (f_L^{\to}(A))^{lu}$), and when $A^u = 0_X, (f_L^{\to}(A))^u = (f_L^{\to}(1_X))^u$ (when $A^l = 0_X$, $(f_L^{\to}(A))^l = (f_L^{\to}(1_X))^l$). If f is both fuzzy weakly lower and upper cut-stable, then f is called fuzzy weakly cut-stable.

For simplicity, we denote $(f_L^{\to}(A))^-$ as $f_L^{\to}(A)^-$.

Definition 3.2. Let $(X, e_X), (Y, e_Y)$ be fuzzy posets and $f : X \to Y$ be a map. f is called fuzzy weakly lower (upper) cut-continuous if $\forall\, A \in L^X, A \neq 0_X$, $f_L^{\rightarrow}(A^{ul}) \leq f_L^{\rightarrow}(A)^{ul}(f_L^{\rightarrow}(A^{lu}) \leq f_L^{\rightarrow}(A)^{lu})$. If f is both fuzzy weakly lower and upper cut-continuous, then f is called fuzzy weakly cut-continuous.

Example 3.3. The identity mapping on a fuzzy poset is fuzzy weakly cut-continuous.

Definition 3.4. Let $(X, e_X), (Y, e_Y)$ be fuzzy posets, $f : X \to Y$ a map. f is called fuzzy weakly cut-preserving if for all fuzzy cut (A, B) of X, and $A, B \neq 0_X$, $(f_L^{\rightarrow}(B)^l, f_L^{\rightarrow}(A)^u)$ is a fuzzy cut of Y.

Example 3.5. For the fuzzy poset (L, e_L), define $f : L^L \to L$ as follows $\forall A \in L^L, f(A) = \sqcup A$, then f is fuzzy weakly cut-preserving.

Obversely, by Lemma 2.9, $(f_L^{\rightarrow}(B)^l, f_L^{\rightarrow}(A)^u)$ is a fuzzy cut of Y is equivalent to $f_L^{\rightarrow}(A)^{ul} = f_L^{\rightarrow}(B)^l$ or $f_L^{\rightarrow}(B)^{lu} = f_L^{\rightarrow}(A)^u$. Next, we explore the relationships among these maps.

Proposition 3.6. *If f is fuzzy weakly lower (upper) cut-stable, then f is fuzzy weakly cut-preserving. Conversely, if $f : X \to Y$ is fuzzy weakly (lower, upper) cut-continuous and fuzzy weakly cut-preserving, then it is fuzzy weakly (lower,upper) cut-stable.*

Proof. Suppose that $f : X \to Y$ is fuzzy weakly lower cut-stable map. If $\forall\, (A, B) \in DM_L(X), A, B \neq 0_X$, then $A^u = B, B^l = A$. Therefore, $f_L^{\rightarrow}(B)^l = f_L^{\rightarrow}(A^u)^l = f_L^{\rightarrow}(A)^{ul}$. Hence, f is fuzzy weakly cut-preserving.

Conversely, let f be fuzzy weakly lower cut-continuous and fuzzy weakly cut-preserving. $\forall\, 0_X \neq A \in L^X$, when $A^u \neq 0_X$, $(A^{ul}, A^u) \in DM_L(X)$, since f is fuzzy weakly cut-preserving, we have $(f_L^{\rightarrow}(A^u)^l, f_L^{\rightarrow}(A^{ul})^u) \in DM_L(Y)$, then $f_L^{\rightarrow}(A^u)^l = f_L^{\rightarrow}(A^{ul})^{ul}$. As $f_L^{\rightarrow}(A^{ul}) \leq f_L^{\rightarrow}(A)^{ul}$, we have $f_L^{\rightarrow}(A)^{ul} = f_L^{\rightarrow}(A)^{ulul} \geq f_L^{\rightarrow}(A^{ul})^{ul} = f_L^{\rightarrow}(A^u)^l$. $\forall\, y \in Y$,

$$
\begin{aligned}
f_L^{\rightarrow}(A^u)(y) &= \bigvee_{x\in X, f(x)=y} A^u(x)\\
&= \bigvee_{x\in X, f(x)=y}\ \bigwedge_{x_1\in X} (A(x_1) \to e(x_1, x))\\
&\leq \bigvee_{x\in X, f(x)=y}\ \bigwedge_{x_1\in X} (A(x_1) \to e(f(x_1), f(x)))\\
&= \bigwedge_{x_1\in X} (A(x_1) \to e(f(x_1), y))\\
&= \bigwedge_{y_1\in Y} (f_L^{\rightarrow}(A)(y_1) \to e(y_1, y))\\
&= f_L^{\rightarrow}(A)^u(y).
\end{aligned}
$$

Hence, $f_L^{\rightarrow}(A^u) \leq f_L^{\rightarrow}(A)^u$, $f_L^{\rightarrow}(A^u)^l \geq f_L^{\rightarrow}(A)^{ul}$. Therefore, $f_L^{\rightarrow}(A^u)^l = f_L^{\rightarrow}(A)^{ul}$.

When $A^u = 0_X$, $f_L^{\rightarrow}(A) \leq f_L^{\rightarrow}(1_X) = f_L^{\rightarrow}(A^{ul}) \leq f_L^{\rightarrow}(A)^{ul}$. Hence, $f_L^{\rightarrow}(A)^u \geq f_L^{\rightarrow}(1_X)^u = f_L^{\rightarrow}(A^{ul})^u \geq f_L^{\rightarrow}(A)^{ulu} = f_L^{\rightarrow}(A)^u$, that is, $f_L^{\rightarrow}(A)^u = f_L^{\rightarrow}(1_X)^u$.

Therefore, $f : X \to Y$ is fuzzy weakly lower cut-stable map. Other cases can be proved in a similar manner.

Proposition 3.7. *If f is fuzzy weakly cut-preserving or fuzzy weakly cut-continuous, then f is fuzzy order-preserving.*

Proof. Let $f : X \to Y$ be fuzzy weakly cut-preserving. Since $\forall\, x \in X, (\iota_x, \mu_x)$ is a fuzzy cut of X and $\iota_x, \mu_x \neq 0_X$, it holds that $(f_L^{\rightarrow}(\mu_x)^l, f_L^{\rightarrow}(\iota_x)^u)$ is a fuzzy cut of Y. Hence, $f_L^{\rightarrow}(\mu_x) \leq f_L^{\rightarrow}(\mu_x)^{lu} = f_L^{\rightarrow}(\iota_x)^u$, we have $\mu_x \leq f_L^{\leftarrow}(f_L^{\rightarrow}(\iota_x)^u)$. $\forall\ x, y \in X$, $e_X(x, y) = \mu_x(y) \leq f_L^{\leftarrow}(f_L^{\rightarrow}(\iota_x)^u)(y) = f_L^{\rightarrow}(\iota_x)^u(f(y)) = \bigwedge_{z\in Y} (f_L^{\rightarrow}(\iota_x)(z) \to e_Y(z, f(y))) =$

$\bigwedge_{z\in Y}\big((\bigvee_{f(x')=z} e_X(x',x)) \to e_Y(z,f(y))\big) = \bigwedge_{x'\in X}\big(e_X(x',x) \to e_Y(f(x'),f(y))\big) \le e_Y(f(x),f(y))$. Therefore, we have that f is fuzzy order-preserving. Next we prove if f is fuzzy weakly cut-continuous, then f is fuzzy order-preserving.

Firstly, $\forall\ z \in Y, (\chi_{f(x)})^{ul}(z) = \bigwedge_{z_1\in Y}((\chi_{f(x)})^u(z_1) \to e(z,z_1)) = \bigwedge_{z_1\in Y}((\bigwedge_{z_2\in Y}\chi_{f(x)}(z_2) \to e(z_2,z_1)) \to e(z,z_1)) = \bigwedge_{z_1\in Y}(f(x),z_1) \to e(z,z_1)) = e(z,f(x)) = \iota_{f(x)}(z)$.

Secondly, $\chi_{f(x)} = f_L^{\to}(\chi_x)$, $\forall\ z \in Y$, if $z = f(x)$, then $(\chi_{f(x)})(z) = 1 = \bigvee_{x_1\in X, f(x_1)=z}\chi_x(x_1) = f_L^{\to}(\chi_x)(z)$. If $z \ne f(x)$, then $(\chi_{f(x)})(z) = 0 = \bigvee_{x_1\in X, f(x_1)=z}\chi_x(x_1) = f_L^{\to}(\chi_x)(z)$.

Finally, $\forall\ x, y \in X$,

$$\begin{aligned} e(f(x),f(y)) &= \iota_{f(y)}(f(x)) \\ &= (\chi_{f(y)})^{ul}(f(x)) \\ &= (f_L^{\to}(\chi_y))^{ul}(f(x)) \\ &\ge f_L^{\to}((\chi_y)^{ul})(f(x)) \\ &= f_L^{\to}(\iota_y)(f(x)) \\ &= \bigvee_{z\in X, f(z)=f(x)}\iota_y(z) \\ &\ge e(x,y). \end{aligned}$$

By Propositions 3.6 and 3.7, we have every fuzzy weakly lower (upper) cut-stable map is fuzzy order-preserving. Next, we show the extension property of the fuzzy weakly cut-stable map.

Let X, Y be fuzzy posets, $f : X \to Y$ is a map. If $\forall\ 0_X \ne A \in L^X$, $f(\sqcup A) = \sqcup f_L^{\to}(A), f(\sqcap A) = \sqcap f_L^{\to}(A)$, then f is called a fuzzy weakly complete homomorphism.

Theorem 3.8. *Let X and Y be fuzzy posets. Then $f : X \to Y$ is fuzzy weakly cut-stable if and only if there exists a unique fuzzy weakly complete homomorphism f^*: $\mathcal{N}(X) \to \mathcal{N}(Y)$ extending f.*

Proof. (*Necessity*) Suppose that $f : X \to Y$ is fuzzy weakly cut-stable. Define $f^* : \mathcal{N}(X) \to \mathcal{N}(Y)$ by $\forall\ A \in \mathcal{N}(X)$. If $A \ne 0_X$, then $f^*(A^{ul}) = f_L^{\to}(A)^{ul}$. If $A = 0_X$, then $f^*(A^{ul}) = f_L^{\to}(1_X)^l$.

(1) f^* is well-defined. $\forall\ A, B \in L^X$, if $A, B \ne 0_X$, $A^{ul} = B^{ul}$, then $A^u = B^u$. When $A^u = B^u \ne 0_X$, $f_L^{\to}(A)^{ul} = f_L^{\to}(A^u)^l = f_L^{\to}(B^u)^l = f_L^{\to}(B)^{ul}$. When $A^u = B^u = 0_X$, $f_L^{\to}(A)^{ul} = f_L^{\to}(1_X)^{ul} = f_L^{\to}(B)^{ul}$. If $A \ne 0_X, B = 0_X$, then $A^{ul} = 0_X^{ul}$. Therefore, $A^u = 1_X$, $f_L^{\to}(A)^{ul} = f_L^{\to}(A^u)^l = f_L^{\to}(1_X)^l$. Hence $f^*(A^{ul}) = f^*(B^{ul})$. Similarly, $B \ne 0_X, A = 0_X$, $f^*(A^{ul}) = f^*(B^{ul})$. If $A = B = 0_X$, clearly we have $f^*(A^{ul}) = f^*(B^{ul})$.

(2) f^* is a fuzzy weakly complete homomorphism. $\forall\ \Phi \in L^{\mathcal{N}(X)}$, $\Phi \ne 0_{\mathcal{N}(X)}$. Firstly, we show $(\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi) * f_L^{\to}(\phi))^{ul} = (\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi) * (f_L^{\to}(\phi))^{ul})^{ul}$. $\forall\ \varphi \in \mathcal{N}(Y)$,

$$\begin{aligned}
sub((\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi)*f_L^{\rightarrow}(\phi))^{ul},\varphi) &= sub(\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi)*f_L^{\rightarrow}(\phi),\varphi)\\
&= \bigwedge_{y\in Y}(\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi)*f_L^{\rightarrow}(\phi)(y)\rightarrow\varphi(y))\\
&= \bigwedge_{\phi\in\mathcal{N}(X)}\bigwedge_{y\in Y}(\Phi(\phi)*f_L^{\rightarrow}(\phi)(y)\rightarrow\varphi(y))\\
&= \bigwedge_{\phi\in\mathcal{N}(X)}\bigwedge_{y\in Y}(\Phi(\phi)\rightarrow(f_L^{\rightarrow}(\phi)(y)\rightarrow\varphi(y)))\\
&= \bigwedge_{\phi\in\mathcal{N}(X)}(\Phi(\phi)\rightarrow sub(f_L^{\rightarrow}(\phi),\varphi))\\
&= \bigwedge_{\phi\in\mathcal{N}(X)}(\Phi(\phi)\rightarrow sub((f_L^{\rightarrow}(\phi))^{ul},\varphi))\\
&= sub((\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi)*f_L^{\rightarrow}(\phi)^{ul}),\varphi)\\
&= sub((\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi)*f_L^{\rightarrow}(\phi)^{ul})^{ul},\varphi).
\end{aligned}$$

Therefore, $(\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi)*f_L^{\rightarrow}(\phi))^{ul}=(\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi)*(f_L^{\rightarrow}(\phi))^{ul})^{ul}$.

By [17], we have that $\mathcal{N}(X)$ is complete lattice, and $\bigsqcup\Phi=\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi)*\phi=(\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi)*\phi)^{ul}$.

$$\begin{aligned}
f^*(\bigsqcup\Phi) &= f^*((\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi)*\phi)^{ul})\\
&= (f_L^{\rightarrow}(\bigvee_{\phi\in\mathcal{N}(x)}\Phi(\phi)*\phi))^{ul}\\
&= (\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi)*f_L^{\rightarrow}(\phi))^{ul}\\
&= (\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi)*(f_L^{\rightarrow}(\phi))^{ul})^{ul}\\
&= (\bigvee_{\phi\in\mathcal{N}(X)}\Phi(\phi)*f^*(\phi))^{ul}\\
&= (\bigvee_{\psi\in\mathcal{N}(Y)}(\bigvee_{\phi\in\mathcal{N}(X),f^*(\phi)=\psi}\Phi(\phi)*\psi))^{ul}\\
&= (\bigvee_{\psi\in\mathcal{N}(Y)}((f^*)_L{}^{\rightarrow}(\Phi)(\psi)*\psi))^{ul}\\
&= \bigsqcup(f^*)_L{}^{\rightarrow}(\Phi).
\end{aligned}$$

So, $f^*(\bigsqcup\Phi)=\bigsqcup(f^*)_L{}^{\rightarrow}(\Phi)$. Similarly we can show $\forall\ \Phi\in L^{\mathcal{N}(X)}$, $\Phi\neq 0_{\mathcal{N}(X)}$, $f^*(\sqcap\Phi)=\sqcap(f^*)_L{}^{\rightarrow}(\Phi)$. Hence, f^* is a fuzzy weakly complete homomorphism.

(3) Commutativity. $\forall\ x\in X$,

$$\begin{aligned}
f^*\circ\iota_X(x) &= f^*(\iota_x)\\
&= f^*(\iota_x^{ul})\\
&= f_L^{\rightarrow}(\iota_x)^{ul}\\
&= \iota_{f(x)}\\
&= \iota_Y\circ f(x).
\end{aligned}$$

(4) f^* is unique. Assume that there is $g:\mathcal{N}(X)\rightarrow\mathcal{N}(Y)$ such that $g\circ\iota_X=\iota_Y\circ f$.

$\forall\ \phi \in \mathcal{N}(X)$,

$$\begin{aligned} g(\phi) &= g(\sqcup(\iota_X)_L^{\to}(\phi)) \\ &= \sqcup g_L^{\to}((\iota_X)_L^{\to}(\phi)) \\ &= \sqcup(\iota_Y \circ f)_L^{\to}(\phi) \\ &= \sqcup(f^*)_L{}^{\to}((\iota_X)_L^{\to}(\phi)) \\ &= f^*(\sqcup(\iota_X)_L^{\to}(\phi)) \\ &= f^*(\phi). \end{aligned}$$

(*Sufficiency*) Assume that there is a fuzzy weakly complete homomorphism $f^* : X \to Y$ such that $f^* \circ \iota_X = \iota_Y \circ f$. $\forall\ 0_X \neq A \in L^X$. If $A^u \neq 0_X$, then $(\iota_X)_L^{\to}(A^u) \neq 0_{\mathcal{N}(X)}$.

$$\begin{aligned} f_L^{\to}(A^u)^l &= \sqcap\iota_Y^{\to}(f_L^{\to}(A^u)) \\ &= \sqcap(\iota_Y \circ f)_L^{\to}(A^u) \\ &= \sqcap(f^* \circ \iota_X)_L^{\to}(A^u) \\ &= \sqcap(f^*)_L{}^{\to}((\iota_X)_L^{\to}(A^u)) \\ &= f^*(\sqcap(\iota_X)_L^{\to}(A^u)) \\ &= f^*(\sqcup(\iota_X)_L^{\to}(A)) \\ &= \sqcup(f^* \circ \iota_X)_L^{\to}(A) \\ &= \sqcup(\iota_Y \circ f)_L^{\to}(A) \\ &= \sqcup(\iota_Y)_L^{\to} \circ f_L^{\to}(A) \\ &= f_L^{\to}(A)^{ul}. \end{aligned}$$

If $A^u = 0_X$, $f_L^{\to}(A)^{ul} = \sqcup(\iota_Y)_L^{\to}(f_L^{\to}(A)) = \sqcap(f^*)_L{}^{\to}((\iota_X)_L^{\to}(A)) = f^*(\sqcup(\iota_X)_L^{\to}(A)) = f^*(A^{ul}) = f^*(0_X^l) = f^*(1_X) = f^*(\sqcup(\iota_X)_L^{\to}(1_X)) = \sqcup(f^* \circ \iota_X)_L^{\to}(1_X) = \sqcup(\iota_Y \circ f)_L^{\to}(1_X) = \sqcup(\iota_Y)_L^{\to} \circ f_L^{\to}(1_X) = f_L^{\to}(1_X)^{ul}$.

Hence, $f_L^{\to}(A)^{ul} = f_L^{\to}(1_X)^{ul}$. Therefore, f is fuzzy weakly lower cut-stable. Similarly, f is also fuzzy weakly upper cut-stable.

4. Characterization of extension map

In this section, we explore the conditions under which the extension map to be fuzzy order isomorphism.

Definition 4.1. Let (X, e) be a fuzzy poset, $A \subseteq X$. If $\forall\ x \in X$, there exists $\phi \in L^A$ such that $\mu_x = \phi^u$, then we call (X, e) fuzzy A-dense.

Next we give an example of fuzzy A-dense.

Example 4.2. Let $X = \{0, a, b, c\}$, $L = \{0, \frac{1}{2}, 1\}$ and $* = \wedge$, the fuzzy order e_X is defined as follows:

$(e_{ij})_X$	0	a	b	c
0	1	1	1	1
a	0	1	$\frac{1}{2}$	1
b	0	$\frac{1}{2}$	1	1
c	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Let $A = \{0, a, b\} \subseteq X$, we can show $0 \in X$, there exists $\phi = \chi_0 \in L^A$ such that $\mu_0 = \phi^u$. $a \in X$, there exists $\phi = \chi_{\{0,a\}} \in L^A$ such that $\mu_a = \phi^u$. $b \in X$, there exists $\phi = \chi_{\{0,b\}} \in L^A$ such that $\mu_b = \phi^u$. $c \in X$, there exists $\phi = \chi_{\{0,c\}} \in L^A$ such that $\mu_c = \phi^u$. This shows that (X, e_X) is fuzzy A-dense.

Proposition 4.3. *Let (X, e) and (Y, e) be fuzzy posets. If f is fuzzy weakly cut-stable, then f^* is surjective iff Y is fuzzy $f(X)$-dense and $(f_L^{\rightarrow}(1_X))^l = 1_Y^l$.*

Proof. (*Necessity*) Suppose that f^* is surjective.

(1) By the definition of f^*, $(f_L^{\rightarrow}(1_X))^l = f^*(0_X^{ul})$.

(2) $(f_L^{\rightarrow}(1_X))^l = 1_Y^l$. Since $\forall\, y \in Y$,

$$\begin{aligned} (f_L^{\rightarrow}(1_X))^l(y) &= \bigwedge_{z\in Y}(f_L^{\rightarrow}(1_X)(z) \rightarrow e(y,z)) \\ &= \bigwedge_{z\in Y}((\bigvee_{x\in X, f(x)=z} 1_X(x)) \rightarrow e(y,z)) \\ &= \bigwedge_{x\in X} e(y, f(x)). \end{aligned}$$

Thus,

$$\begin{aligned} 1_Y^l(y) &= \bigwedge_{z\in Y}(1_Y(z) \rightarrow e(y,z)) \\ &= \bigwedge_{z\in Y} e(y,z) \\ &\leq \bigwedge_{x\in X} e(y, f(x)) \\ &= (f_L^{\rightarrow}(1_X))^l(y). \end{aligned}$$

Conversely, we need to show $(f_L^{\rightarrow}(1_X))^l \leq 1_Y^l$. $\forall\, z \in Y, \iota_z \in \mathcal{N}(Y)$, as f^* is surjective, there exists $A \in \mathcal{N}(X)$ such that $f^*(A^{ul}) = \iota_z, \iota_z = f^*(A^{ul}) \geq f^*(0_X^{ul}) = (f_L^{\rightarrow}(1_X))^l$. Thus $1_Y^l(y) = \bigwedge_{z\in Y} 1_Y(z) \rightarrow e(y,z) = \bigwedge_{z\in Y} \iota_z(y) \geq (f_L^{\rightarrow}(1_X))^l$. Therefore, $(f_L^{\rightarrow}(1_X))^l = 1_Y^l$.

$\forall\, y \in Y$, since f^* is surjective, there exists $A \in \mathcal{N}(X)$ such that $f^*(A^{ul}) = \iota_y = (\iota_y)^{ul}$. If $A \neq 0_X$, then $f^*(A^{ul}) = (f_L^{\rightarrow}(A))^{ul} = (f_L^{\rightarrow}(A) \mid_{f(X)})^{ul}$, we have $\mu_y = (f_L^{\rightarrow}(A) \mid_{f(X)})^u$. If $A = 0_X$, then $f^*(A^{ul}) = (f_L^{\rightarrow}(1_X))^l = (1_Y)^l$, we have $\mu_y = 1_Y = 0_Y^u = 0_Y \mid_{f(Y)}^u$. Thus Y is fuzzy $f(X)$-dense.

(*Sufficiency*) Let Y be fuzzy $f(X)$-dense and $(f_L^{\rightarrow}(1_X))^l = (1_Y)^l$. First we need to show $\forall\, y \in Y, \iota_y$ has preimage. Since Y is fuzzy $f(X)$-dense, there exists $B \in L^{f(X)}$ such that $\mu_y = B^u$. By $B \in L^{f(X)}$, $f_L^{\leftarrow}(B) \in L^X$. If $f_L^{\leftarrow}(B) \neq 0_X$, notice that $\forall\, y \in Y, f_L^{\rightarrow}(f_L^{\leftarrow}(B))(y) = \bigvee_{x\in X, f(x)=y} f_L^{\leftarrow}(B)(x) = \bigvee_{x\in X, f(x)=y} B(f(x)) = B(y)$, we have $f_L^{\rightarrow}(f_L^{\leftarrow}(B)) = B$, then $f^*(f_L^{\leftarrow}(B)^{ul}) = (f_L^{\rightarrow}(f_L^{\leftarrow}(B)))^{ul} = B^{ul} = \mu_y^l = \iota_y$.

If $f_L^{\leftarrow}(B) = 0_X$, notice that $\forall\, y \in Y$,

$$\begin{aligned} f_L^{\rightarrow}(0_X)^u(y) &= \bigwedge_{z\in Y} f_L^{\rightarrow}(0_X)(z) \rightarrow e(z,y) \\ &= \bigwedge_{x\in X}(\bigvee_{x\in X, f(x)=z} 0_X(x)) \rightarrow e(z,y) \\ &= 1 \\ &= 1_Y(y). \end{aligned}$$

We have $(1_Y)^l = f_L^{\rightarrow}(0_X)^{ul}$, then

$$\begin{array}{rcl} f^*(f_L^{\leftarrow}(B)^{ul}) & = & f^*(1_X^l) \\ & = & f_L^{\rightarrow}(1_X^l) \\ & = & (1_Y)^l \\ & = & f_L^{\rightarrow}(0_X)^{ul} \\ & = & \mu_y^l \\ & = & \iota_y. \end{array}$$

Therefore, $(f_L^{\leftarrow}(B))^{ul}$ is the preimage of ι_y.

Suppose that $B \in \mathcal{N}(Y)$. The above tells us that $\forall\ y \in Y$, there exists $B_y \in L^{f(X)}$ such that$f^*((f_L^{\leftarrow}(B_y))^{ul}) = \iota_y$. Having $B = \bigvee_{y\in Y} B(y) * \chi_y$ implies that $B^{ul} = (\bigvee_{y\in Y} B(y) * \chi_y)^{ul}$. Define $\Phi \in L^{\mathcal{N}(X)}, \forall\ y \in Y$.

$$\Phi(\psi) = \begin{cases} B(y), & \psi = (f_L^{\leftarrow}(B_y))^{ul}, \\ 0, & \text{otherwises} . \end{cases}$$

Notice that $(\bigvee_{y\in Y} B(y) * \iota_y)^{ul} = (\bigvee_{y\in Y} B(y) * \chi_y)^{ul}$. $\forall\ z \in Y$,

$$\begin{array}{rcl} (\bigvee_{y\in Y} B(y) * \iota_y)^u(z) & = & \bigwedge_{z_1\in Y} ((\bigvee_{y\in Y} B(y) * \iota_y(z_1)) \to e(z_1, z)) \\ & = & \bigwedge_{y\in Y} (B(y) \to e(y, z)) \\ & = & B^u(z) \\ & = & \bigwedge_{z_1\in Y} (B(z_1) \to e(z_1, z)) \\ & = & \bigwedge_{z_1\in Y} (\bigvee_{y\in Y} (B(y) * \chi_y(z_1)) \to e(z_1, z)) \\ & = & (\bigvee_{y\in Y} B(y) * \chi_y)^u(z). \end{array}$$

$$\begin{array}{rcl} f^*(\bigsqcup \Phi) & = & \bigsqcup (f^*)_L{}^{\rightarrow}(\Phi) \\ & = & (\bigvee_{\phi\in\mathcal{N}(Y)} (f^*)_L{}^{\rightarrow}(\Phi)(\phi) * \phi)^{ul} \\ & = & (\bigvee_{\phi\in\mathcal{N}(Y)} (\bigvee_{\psi\in\mathcal{N}(X), f^*(\psi)=\phi} \Phi(\psi)) * \phi)^{ul} \\ & = & (\bigvee_{\psi\in\mathcal{N}(X)} \Phi(\psi) * f^*(\psi))^{ul} \\ & = & (\bigvee_{y\in Y} B(y) * (f_L^{\leftarrow}(B_y))^{ul})^{ul} \\ & = & (\bigvee_{y\in Y} B(y) * \iota_y)^{ul} \\ & = & (\bigvee_{y\in Y} B(y) * \chi_y)^{ul} \\ & = & B^{ul} \\ & = & B. \end{array}$$

Theorem 4.4. *Let $f : (X, e) \to (Y, e)$ be fuzzy weakly cut-stable. Then f^* is fuzzy order isomorphism if and only if $e(f(x), f(y)) \leq e(x, y)$ for all $x, y \in X$, Y is fuzzy $f(X)$-dense and $(f_L^{\rightarrow}(1_X))^l = 1_Y^l$.*

Proof. (*Necessity*) Let f^* be fuzzy order isomorphism. $\forall\, x, y \in X, e(f(x), f(y)) = sub(\iota_{f(x)}, \iota_{f(y)})$ $= sub(f^*(\iota_x), f^*(\iota_y)) \leq sub(\iota_x, \iota_y) = e(x, y)$.

(*Sufficiency*) By Proposition 4.3, we have that f^* is surjective, we only need to show f^* is order embedding. That is $\forall\, A, B \in L^X$, $sub(f^*(A^{ul}), f^*(B^{ul})) \leq sub(A^{ul}, B^{ul})$.

If $A = 0_X$, obviously. If $A \neq 0_X$, when $B \neq 0_X$,

$$\begin{aligned}
sub(f^*(A^{ul}), f^*(B^{ul})) &= sub(f_L^{\rightarrow}(A)^{ul}, f_L^{\rightarrow}(B)^{ul}) \\
&\leq sub(f_L^{\rightarrow}(B)^u, f_L^{\rightarrow}(A)^u) \\
&= (\bigwedge_{y\in Y} f_L^{\rightarrow}(B)^u(y) \rightarrow f_L^{\rightarrow}(A)^u(y) \\
&= \bigwedge_{y\in Y}(\bigwedge_{z_1\in Y} f_L^{\rightarrow}(B)(z_1) \rightarrow e(z_1, y)) \rightarrow (\bigwedge_{z_2\in Y} f_L^{\rightarrow}(A)(z_2) \rightarrow e(z_2, y)) \\
&= \bigwedge_{y\in Y}(\bigwedge_{x_1\in Y} B(x_1) \rightarrow e(f(x_1), y)) \rightarrow (\bigwedge_{x_2\in X} A(x_2) \rightarrow e(f(x_2), y)) \\
&\leq \bigwedge_{x\in X}(\bigwedge_{x_1\in X} B(x_1) \rightarrow e(f(x_1), f(x))) \rightarrow (\bigwedge_{x_2\in X} A(x_2) \rightarrow e(f(x_2), f(x))) \\
&\leq \bigwedge_{x\in X}(\bigwedge_{x_1\in X} B(x_1) \rightarrow e(x_1, x)) \rightarrow (\bigwedge_{x_2\in X} A(x_2) \rightarrow e(x_2, x)) \\
&= sub(B^u, A^u) \\
&\leq sub(A^{ul}, B^{ul}).
\end{aligned}$$

When $B = 0_X$, notice that

$$\begin{aligned}
sub(1_Y, f_L^{\rightarrow}(A)^u) &= \bigwedge_{y\in Y}(1 \rightarrow f_L^{\rightarrow}(A)^u(y)) \\
&= \bigwedge_{y\in Y}(\bigwedge_{z\in Y} f_L^{\rightarrow}(A)(z) \rightarrow e(z, y)) \\
&= \bigwedge_{y\in Y}(\bigwedge_{x\in X} A(x) \rightarrow e(f(x), y)) \\
&\leq \bigwedge_{x_1\in X}(\bigwedge_{x\in X} A(x) \rightarrow e(f(x), f(x_1))) \\
&\leq \bigwedge_{x_1\in X}(1_X(x_1) \rightarrow (\bigwedge_{x\in X} A(x) \rightarrow e(x, x_1))) \\
&= \bigwedge_{x_1\in X}(1_X(x_1) \rightarrow A^u(x_1)) \\
&= sub(1_X, A^u) \\
&\leq sub(A^{ul}, 1_X^l).
\end{aligned}$$

Therefore,

$$\begin{aligned}
sub(f^*(A^{ul}), f^*(B^{ul})) &= sub(f_L^{\rightarrow}(A)^{ul}, f_L^{\rightarrow}(1_X)^l) \\
&= sub(f_L^{\rightarrow}(A)^{ul}, 1_Y^l) \\
&\leq sub(1_Y, f_L^{\rightarrow}(A)^u) \\
&\leq sub(A^{ul}, 1_X^l) \\
&= sub(A^{ul}, B^{ul}).
\end{aligned}$$

5. Conclusions

In this paper, we proposed the definitions of fuzzy weakly lower (upper) cut-stable map, fuzzy weakly lower (upper) cut-continuous map, and fuzzy weakly cut-preserving map. Furthermore, we obtained the equivalent characterization and the extension property of fuzzy weakly cut-stable map. In addition, we showed that the extension map of the fuzzy weakly cut-stable map was fuzzy order isomorphism under some conditions.

Acknowledgments

The authors express their gratitude to the anonymous reviewers and editor for their careful reading of the manuscript and for many valuable remarks and suggestions.

This work was supported by the National Natural Science Foundation of China (Grant Nos.12001413), Natural Science Basic Research Program of Shaanxi (No.2019JQ-869 and No.2021JQ763) and Scientific Research Project of Education Department of Shaanxi Province (No.20JK0640 and No.19JK0317)

Conflict of interest

The authors declare that there is no conflicts of interest regarding this manuscript.

References

1. G. Giers, K. H. Hofmann K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott, *Continuous lattices and domains*, Cambridge University Press, 2003.
2. A. Bishop, A universal mapping characterization of the completion by cuts, *Algebr. Univ.*, **8** (1978), 349–353. https://doi.org/10.1007/BF02485405
3. M. Erne, The Dedekind-MacNeille completion as a reflector, *Order*, **8** (1991), 159–173. https://doi.org/10.1007/BF00383401
4. M. Erne, Ideal completions and compactifications, *Appl. Categor. Struct.*, **9** (2001), 217–243. https://doi.org/10.1023/A:1011260817824
5. M. Liu, B. Zhao, A non-frame valued cartesian closed category of liminf complete fuzzy orders, *Fuzzy Sets Syst.*, **321** (2017), 50–54. https://doi.org/10.1016/j.fss.2016.07.008
6. N. N. Ma, B. Zhao, An equivalent of algebraic Ω-categories, *J. Math. Res. Appl.*, **37** (2017), 148–162.
7. H. L. Lai, D. X. Zhang, Closedness of the category of liminf complete fuzzy orders, *Fuzzy Sets Syst.*, **282** (2016), 86–98. https://doi.org/10.1016/j.fss.2014.10.031
8. W. Li, H. L. Lai, D. X. Zhang, Yoneda completeness and flat completeness of ordered fuzzy sets, *Fuzzy Sets Syst.*, **313** (2017), 1–24. https://doi.org/10.1016/j.fss.2016.06.009
9. K. R. Wagner, *Solving recursive domain equations with enriched categories*, Pittsburgh: Carnegie Mellon University, School of Computer Science, 1994.
10. R. Belohlavek, Concept lattices and order in fuzzy logic, *Ann. Pure Appl. Logic*, **128** (2004), 277–298. https://doi.org/10.1016/j.apal.2003.01.001
11. W. X. Xie, Q. Y. Zhang, L. Fan, The Dedekind-MacNeille completions for fuzzy posets, *Fuzzy Sets Syst.*, **160** (2009), 2292–2316. https://doi.org/10.1016/j.fss.2008.12.002
12. K. Y. Wang, B. Zhao, Join-completions of L-ordered sets, *Fuzzy Sets Syst.*, **199** (2012), 92–107. https://doi.org/10.1016/j.fss.2011.12.011
13. N. N. Ma, B. Zhao, Fuzzy cut-stable maps and its extension property, *J. Intell. Fuzzy Syst.*, **30** (2016), 2213–2221. https://doi.org/10.3233/IFS-151990

14. R. Halas, J. Lihova, On weakly cut-stable maps, *Inform. Sci.*, **180** (2010), 971–983. https://doi.org/10.1016/j.ins.2009.11.025

15. R. Belohlavek, Some properties of residuated lattices, *Czech. Math. J.*, **53** (2003), 161–171. https://doi.org/10.1023/A:1022935811257

16. Q. Y. Zhang, W. X. Xie, L. Fan, Fuzzy complete lattices, *Fuzzy Sets Syst.*, **160** (2009), 2275–2291. https://doi.org/10.1016/j.fss.2008.12.001

17. N. N. Ma, B. Zhao, Some results on fuzzy Z_L-continuous (algebraic) poset, *Soft Comput.*, **22** (2018), 4549–4559. https://doi.org/10.1007/s00500-017-2924-9

18. U. Hohle, S. E. Rodabaugh, *Mathematical of fuzzy sets: Logic, topology, and measure theory*, Kluwer Academic Publishers, Boston/Dordrecht/London, 1999.