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*Research article*

# A quintuple integral involving the product of Hermite polynomial $H_n(\beta x)$ and parabolic cylinder function $D_v(\alpha t)$: derivation and evaluation

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**Abstract:** In this paper, we derive an integral transform involving the product of Hermite polynomial $H_n(\beta x)$ and parabolic cylinder function $D_v(\alpha t)$. These integral transforms will be evaluated in terms of Lerch function. Various formulae are also evaluated in terms of special functions to complete this paper. All the results in this paper are new.

**Keywords:** parabolic cylinder function; quintuple integral; Lerch function; Cauchy integral
**Mathematics Subject Classification:** 30E20, 33-01, 33-03, 33-04, 33-33B

## 1. Significance statement

In 1932, Goldstein [1] published a paper on the operational representations of functions related to Whittaker's confluent hypergeometric function $W_{k,m}(x)$, and Weber's parabolic cylinder function $D_n(x)$ and that same year Varma [2] published a paper on the solution of Weber's differential equation. In 1934, Mitra [3] published a paper on the squares of Weber's parabolic cylinder functions and certain integrals connected with them. In 2001, Daniel [4] produced work involving the inner product of Weber's parabolic cylinder function and a Hermite polynomial which was defined and evaluated as a new class of definite integrals. In 1983, Martynov [5] published a paper on the derivation of exact equations and the theory of liquids, where a quintuple integral was evaluated.

Based upon previous literature one sees there is a demand for the Weber function and the Hermite polynomial, coupled with the definite integral of their product. In this current work we look to extend these previous works by deriving a closed form solution for the quintuple integral involving the product of the Hermite polynomial and parabolic cylinder function and express this integral in terms of a special function. This new integral transform will then be used to form a theorem from which special cases will be evaluated.

## 2. Introduction

In this paper we derive the quintuple definite integral given by

$$\begin{aligned}\int_{\mathbb{R}_+^5} & t^{-m/2}u^{-m}x^{m-1}z^{m-n}y^{-\frac{m}{2}-v+1}H_n(x\beta)D_v(t\alpha)\\ & \log^k\left(\frac{axz}{\sqrt{tu}\sqrt{y}}\right)E_{\alpha,\beta,b}(x,y,z,t,u)dxdydzdtdu\end{aligned} \tag{2.1}$$

where

$$E_{\alpha,\beta,b}(x,y,z,t,u) = \exp\left(-b\left(u^2+y^2+z^2\right)-\frac{1}{4}\alpha^2t^2-\beta^2x^2\right)$$

and the parameters $k, a, \alpha, \beta, n, v, m$ are general complex numbers and $Re(b) > 0, Re(n) < Re(m), Re(v) < Re(m), Re(m) < 1$. This definite integral will be used to derive special cases in terms of special functions and fundamental constants. The derivations follow the method used by us in [6]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw. \tag{2.2}$$

where $C$ is in general an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. We then multiply both sides by a function of $x$, $y$, $z$, $t$ and $u$, then take a definite quintuple integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Eq (2.2) by another function of $y$ and take the infinite sum of both sides such that the contour integral of both equations are the same.

## 3. Definite integral of the contour integral

We use the method in [6]. The variable of integration in the contour integral is $r = w + m$. The cut and contour are in the first quadrant of the complex $r$-plane. The cut approaches the origin from the interior of the first quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we form the quintuple integral by replacing $y$ by

$$\log\left(\frac{axz}{\sqrt{tu}\sqrt{y}}\right)$$

and multiplying by

$$t^{-m/2}u^{-m}x^{m-1}z^{m-n}y^{-\frac{m}{2}-v+1}H_n(x\beta)D_v(t\alpha)E_{\alpha,\beta,b}(x,y,z,t,u)$$

then taking the definite integral with respect to $x \in [0,\infty)$, $y \in [0,\infty)$, $z \in [0,\infty)$, $t \in [0,\infty)$ and $u \in [0,\infty)$ to obtain

$$\frac{1}{\Gamma(k+1)}\int_{\mathbb{R}_+^5} t^{-m/2}u^{-m}x^{m-1}z^{m-n}y^{-\frac{m}{2}-v+1}$$

$$H_n(x\beta)D_v(t\alpha)\log^k\left(\frac{axz}{\sqrt{tu}\sqrt{y}}\right)E_{\alpha,\beta,b}(x,y,z,t,u)dxdydzdtdu$$

$$=\frac{1}{2\pi i}\int_{\mathbb{R}_+^5}\int_C a^w w^{-k-1}t^{\frac{1}{2}(-m-w)}u^{-m-w}x^{m+w-1}$$
$$H_n(x\beta)z^{m-n+w}y^{-\frac{m}{2}-v-\frac{w}{2}+1}D_v(t\alpha)E_{\alpha,\beta,b}(x,y,z,t,u)dwdxdydzdtdu$$

$$=\frac{1}{2\pi i}\int_C\int_{\mathbb{R}_+^5} a^w w^{-k-1}t^{\frac{1}{2}(-m-w)}u^{-m-w}x^{m+w-1}$$
$$H_n(x\beta)z^{m-n+w}y^{-\frac{m}{2}-v-\frac{w}{2}+1}D_v(t\alpha)E_{\alpha,\beta,b}(x,y,z,t,u)dxdydzdtdudw$$

$$=\frac{1}{2\pi i}\int_C \pi^{5/2}a^w w^{-k-1}\alpha^{\frac{1}{2}(m+w-2)}\beta^{-m-w}\csc(\pi(m+w))$$
$$2^{\frac{1}{4}(m+4n+2v+w-14)}b^{\frac{1}{4}(m+2(n+v-4)+w)}dw \quad (3.1)$$

from equation (3.22.2.2) and (3.9.1.3) in [7] and equation (3.326.2) in [8] where $0<Re(w+m)$, $|\arg\alpha|<\pi/4$ and using the reflection formula (8.334.3) in [8] for the Gamma function. We are able to switch the order of integration over $x$, $y$, $z$, $t$ and $u$ using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C}\times[0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty)$.

## 4. The Hurwitz-Lerch zeta function and infinite sum of the contour integral

In this section we use Eq (2.2) to derive the contour integral representations for the Hurwitz-Lerch zeta function.

### 4.1. The Hurwitz-Lerch zeta function

The Hurwitz-Lerch zeta function (25.14) in [10] has a series representation given by

$$\Phi(z,s,v)=\sum_{n=0}^{\infty}(v+n)^{-s}z^n \quad (4.1)$$

where $|z|<1, v\neq 0,-1,..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-(v-1)t}}{e^t-z}dt \quad (4.2)$$

where $Re(v)>0$, and either $|z|\leq 1, z\neq 1, Re(s)>0$, or $z=1, Re(s)>1$.

### 4.2. Infinite sum of the contour integral

Using Eq (2.2) and replacing $y$ by

$$v=\log(a)+\frac{\log(\alpha)}{2}+\frac{\log(b)}{4}-\log(\beta)+i\pi(2y+1)+\frac{\log(2)}{4}$$

where $4iv = 4i\log(a) + 2i\log(2b) - 4i\log(\beta) - 4\pi$, then multiplying both sides by

$$-i\pi^{5/2}\alpha^{\frac{m}{2}-1}\beta^{-m}e^{i\pi m(2y+1)}2^{\frac{1}{4}(m+4n+2(v-7))+1}b^{\frac{1}{4}(m+2n+2v-8)}$$

taking the infinite sum over $l \in [0, \infty)$ and simplifying in terms of the Hurwitz-Lerch zeta function we obtain

$$\begin{aligned}
&-\frac{1}{\Gamma(k+1)}i\pi^{k+\frac{5}{2}}e^{\frac{1}{2}i\pi(k+2m)}\alpha^{\frac{m}{2}-1}\beta^{-m}b^{\frac{1}{4}(m+2(n+v-4))}2^{k+\frac{1}{4}(m+2(2n+v-5))}\\
&\Phi\left(e^{2im\pi}, -k, \frac{-4i\log(a) - i\log(2b) - 2i\log(\alpha) + 4i\log(\beta) + 4\pi}{8\pi}\right)\\
&= -\frac{\pi^{3/2}}{2\pi i}\sum_{l=0}^{\infty}\int_C a^w w^{-k-1}\alpha^{\frac{m}{2}-1}\beta^{-m}2^{\frac{1}{4}(m+2(2n+v-5))}b^{\frac{1}{4}(m+2(n+v-4))}\\
&\quad\cdot e^{(v-\log(a)+\pi i)w}e^{i\pi(2l+1)(2m+w)}dw\\
&= -\frac{\pi^{3/2}}{2\pi i}\int_C a^w w^{-k-1}\alpha^{\frac{m}{2}-1}\beta^{-m}2^{\frac{1}{4}(m+2(2n+v-5))}b^{\frac{1}{4}(m+2(n+v-4))}\\
&\quad\cdot e^{(v-\log(a)+\pi i)w}\left[\sum_{l=0}^{\infty}e^{i\pi(2l+1)(2m+w)}\right]dw\\
&= \frac{\pi^{3/2}}{2i}\int_C a^w w^{-k-1}\alpha^{\frac{1}{2}(m+w-2)}\beta^{-m-w}\csc(\pi(m+w))\\
&2^{\frac{1}{4}(m+4n+2v+w-14)}b^{\frac{1}{4}(m+2(n+v-4)+w)}dw
\end{aligned} \tag{4.3}$$

from equation (1.232.3) in [8] where $Im(m + w) > 0$ in order for the sum to converge.

## 5. Definite integral in terms of the Lerch function

**Theorem 5.1.** *For all $k, a, \alpha, \beta, u, v, m, Re(b) > 0, Re(n) < Re(m), Re(v) < Re(m), Re(m) < 1$*

$$\begin{aligned}
&\int_{\mathbb{R}_+^5} t^{-m/2}u^{-m}x^{m-1}z^{m-n}y^{-\frac{m}{2}-v+1}H_n(x\beta)D_v(t\alpha)\\
&\log^k\left(\frac{axz}{\sqrt{tu}\sqrt{y}}\right)E_{\alpha,\beta,b}(x, y, z, t, u)dxdydzdtdu\\
&= -i\pi^{k+\frac{5}{2}}e^{\frac{1}{2}i\pi(k+2m)}\alpha^{\frac{m}{2}-1}\beta^{-m}b^{\frac{1}{4}(m+2(n+v-4))}2^{k+\frac{1}{4}(m+2(2n+v-5))}\\
&\Phi\left(e^{2im\pi}, -k, \frac{-4i\log(a) - i\log(2b) - 2i\log(\alpha) + 4i\log(\beta) + 4\pi}{8\pi}\right)
\end{aligned} \tag{5.1}$$

*Proof.* The right-hand sides of relations (3.1) and (4.3) are identical; hence, the left-hand sides of the same are identical too. Simplifying with the Gamma function yields the desired conclusion. □

**Example 5.2.** The degenerate case.

$$\int_{\mathbb{R}_+^5} t^{-m/2}u^{-m}x^{m-1}z^{m-n}y^{-\frac{m}{2}-v+1}H_n(x\beta)D_v(t\alpha)$$

$$E_{\alpha,\beta,b}(x,y,z,t,u)dxdydzdtdu$$

$$= \pi^{5/2}\alpha^{\frac{m}{2}-1}\beta^{-m}\csc(\pi m)2^{\frac{1}{4}(m+2(2n+v-7))}b^{\frac{1}{4}(m+2(n+v-4))} \quad (5.2)$$

*Proof.* Use equation (5.1) and set $k = 0$ and simplify using entry (2) in Table below (64:12:7) in [11]. □

**Example 5.3.** For the Hurwitz zeta function $\zeta(s,v)$ see [9],

$$\begin{aligned}
&\int_{\mathbb{R}_+^5} \frac{1}{\sqrt[4]{t}\sqrt{u}\sqrt{x}} z^{\frac{1}{2}-n}y^{\frac{3}{4}-v}H_n(x\beta)D_v(t\alpha)\log^k\left(\frac{axz}{\sqrt{tu}\sqrt{y}}\right)E_{\alpha,\beta,b}(x,y,z,t,u)dxdydzdtdu \\
&= -\frac{1}{\alpha^{3/4}\sqrt{\beta}} ie^{\frac{1}{2}i\pi(k+1)}\pi^{k+\frac{5}{2}}b^{\frac{1}{4}\left(2(n+v-4)+\frac{1}{2}\right)}2^{k+\frac{1}{4}\left(2(2n+v-5)+\frac{1}{2}\right)} \\
&\left(2^k\zeta\left(-k, \frac{-4i\log(a)-i\log(2b)-2i\log(\alpha)+4i\log(\beta)+4\pi}{16\pi}\right)\right. \\
&\qquad \left.-2^k\zeta\left(-k, \frac{1}{2}\left(\frac{-4i\log(a)-i\log(2b)-2i\log(\alpha)+4i\log(\beta)+4\pi}{8\pi}+1\right)\right)\right)
\end{aligned} \tag{5.3}$$

*Proof.* Use Eq (5.1) set $m = 1/2$ and simplify in terms of the Hurwitz zeta function $\zeta(s,v)$ using entry (4) in Table below (64:12:7) in [11]. □

**Example 5.4.** The digamma function $\psi^{(0)}(x)$,

$$\begin{aligned}
&\int_{\mathbb{R}_+^5} \frac{1}{\sqrt[4]{t}\sqrt{u}\sqrt{x}\log\left(\frac{axz}{u\sqrt{t}\sqrt{y}}\right)} z^{\frac{1}{2}-n}y^{\frac{3}{4}-v}H_n(x\beta)D_v(t\alpha) \\
&E_{\alpha,\beta,b}(x,y,z,t,u)dxdydzdtdu \\
&= \frac{1}{\alpha^{3/4}\sqrt{\beta}} i\pi^{3/2}2^{n+\frac{v}{2}-\frac{35}{8}}b^{\frac{1}{8}(4n+4v-15)} \\
&\left(\psi^{(0)}\left(\frac{-4i\log(a)-i\log(2b)-2i\log(\alpha)+4i\log(\beta)+4\pi}{16\pi}\right)\right. \\
&\left.-\psi^{(0)}\left(\frac{-4i\log(a)-i\log(2b)-2i\log(\alpha)+4i\log(\beta)+12\pi}{16\pi}\right)\right)
\end{aligned} \tag{5.4}$$

*Proof.* Use Eq (5.3) and apply l'Hopital's rule as $k \to -1$ and simplify using equation (64:4:1) in [11]. □

**Example 5.5.** The fundamental constant $\log(2)$,

$$\begin{aligned}
&\int_{\mathbb{R}_+^5} \frac{z^{\frac{1}{2}-n}y^{\frac{3}{4}-v}H_n(x)D_v(t)E_{\alpha,\beta,b}(x,y,z,t,u)}{\sqrt[4]{t}\sqrt{u}\sqrt{x}\log\left(-\frac{xz}{u\sqrt{t}\sqrt{y}}\right)}dxdydzdtdu \\
&= -i\pi^{3/2}2^{\frac{n-5}{2}}\log(4)
\end{aligned} \tag{5.5}$$

*Proof.* Use Eq (5.4) and set $a = -1, \alpha = \beta = 1, b = 1/2$ and simplify. □

## 6. Conclusions

In this paper, we have presented a novel method for deriving a new integral transform involving the product of Hermite polynomial $H_n(\beta x)$ and Parabolic Cylinder function $D_v(\alpha t)$ with some interesting definite integrals, using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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**Conflict of interest**

There are no conflicts of interest.

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